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Research Article

Computing Edge Weights of Magic Labeling on Rooted Products of Graphs

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Labeling of graphs with numbers is being explored nowadays due to its diverse range of applications in the fields of civil, software, electrical, and network engineering. For example, in network engineering, any systems interconnected in a network can be converted into a graph and specific numeric labels assigned to the converted graph under certain rules help us in the regulation of data traffic, connectivity, and bandwidth as well as in coding/decoding of signals. Especially, both antimagic and magic graphs serve as models for surveillance or security systems in urban planning. In 1998, Enomoto et al. introduced the notion of super $(a, 0)$ edge-antimagic labeling of graphs. In this article, we shall compute super $(a, 0)$ edge-antimagic labeling of the rooted product of P_n and the complete bipartite graph $(K_{2,m})$ combined with the union of path, copies of paths, and the star. We shall also compute a super $(a, 0)$ edge-antimagic labeling of rooted product of P_n with a special type of pancyclic graphs. The labeling provided here will also serve as super $(a', 2)$ edge-antimagic labeling of the aforesaid graphs. All the structures discussed in this article are planar. Moreover, our findings have also been illustrated with examples and summarized in the form of a table and $3D$ plots.

1. Introduction

The antimagic and magic labelings on graphs are designed due to their wide applicability in various branches of engineering. In the literature, many results have appeared regarding numeric labelings on several operations of graphs such as graphs obtained from cartesian, corona, rooted, and strong products of various connected graphs; for instance, see [1–5]. In this article, we will provide super $(a, 0)$ edge-antimagic labeling of the rooted product of P_n and the complete bipartite graph $(K_{2,m})$ taking its disjoint union with path, copies of paths, and star graph. Further, we will provide super $(a, 0)$ edge-antimagic labeling of the rooted product of path P_n with specifically designed pancyclic graphs. We shall, in particular, target the planar graphs that are obtained as a result of our rooted products. These planar graphs minimize the possibility of overlapping of various entities in practical purposes, which is a major cause of inefficiency in organizations. The super $(a, 0)$ edge-antimagic labeling provided in this article on the specified graphs can be used as test-ready labeling in any engineering, networking, or industrial project where the scheme of design of connections is similar to the graphs obtained in this note.

1.1. Applications of Graph Labeling in Engineering

1.1.1. Software Engineering. In software engineering, the role of graph labeling is getting improved in the encryption of the security codes in order to halt the attacks of hackers on precious data and also in coding of data to transmit it to different networks and devices alike. Similarly, test-ready labels and reference labels are improving the configurations of software in designing the updated versions. A two-scan algorithm for the connected components in binary images involves labeling and is significantly helpful in making the graphics better and clearer (see [6]). In data mining, the concept of magic labeling is becoming useful with the passage of time. It makes the data collection for deriving new information easier by indicating the data of same weightage as one entity. In this way, magic labeling is making the task of data mining more easy, error-free, and less time- and effort-consuming in various organizations.

1.1.2. Networking. A network consists of nodes (vertices) and links between these nodes (edges). More technically, a network, say N, is an ordered 2-tuple consisting of two sets, set of nodes $V(N)$ and links between nodes called set of edges $E(N)$ such that $(E(N) \subset V(N) \times V(N))$. Thus, a graph is directly a representative of a network. In network engineering, the optimization and functioning of the networks are the primary hallmarks that require solid planning, construction, and management of the network at its core. Two basic types of networking are wired networking and wireless networking. One cannot deny the existence, importance, and major usage of wired networking in many principal instances. But, due to more usefulness, the apparent increase in the use of the wireless networks demands the application of robust tools, like graph labeling, to get more accuracy in the engineering of wireless networking (see [7]). We are living in an era of communication, in which radio transmission is playing an extremely important part. These wireless radio networks face a major challenge in the form of interference which makes the task of channel assignment harder. The main reason for this unwanted interruption is constraint-free transmission of the concurrent networks admitting same instance appearance [8, 9]. Such networks are first converted into graphs and then magic labeling helps in assigning constant weights to the concurrent networks. This whole procedure minimizes and even eliminates the interference in networks. Moreover, the radio labeling of graphs is tremendously helpful in the minimization of the problem of interference in wireless networks and has been playing a very vital role in the last few years. The $(a, 0)$ edge-antimagic labeling is particularly used for automatic routing in a network. A static network is represented first as a specific graph by connecting nodes in some topology to form a connected graph and then magic labeling is applied for automatic routing of data in the network. This labeling is designed with a constant edge weight, which helps routing to automatically detect the next node within that network (see [10]).

1.1.3. Telecommunication. The most commercially successful application of graph labeling appears in telecom engineering these days [11]. In telecommunication, a service coverage area is divided into a quadrilateral or a hexagon, termed as a cell in cellular networking. Furthermore, each cell works as a station. The base cell has the ability to communicate with mobile stations using its radio transceiver. The challenge for base cell here is to provide maximum channel reuse without violating the constraints in order to minimize the blocking. To tackle this challenge, a label is assigned to each user and the communication link of this user receives a distinct label. In this way, the numbers assigned to any two communicating terminals automatically specify the link label of the connecting path by simply using magic, antimagic, or graceful labeling. Conversely, the path label uniquely specifies the pair of users which it interconnects (see [12]).

1.1.4. Civil Engineering and Urban Planning. As a particular example, consider the wheel W_6, helm H_6, and prisms D_5 and D_6 in Figure 1. The edges of the graphs W_6, H_6, and D_6 are labeled with consecutive labels ranging from 1 up to the size of the graph such that the labels appearing on all the vertices are distinct. That is, the vertex antimagic labeling of these graphs is being provided here. Meanwhile, on D_5, edge-magic labeling with constant edge weight 29 is provided [13, 14]. Now, for instance, in a surveillance design of highly sensitive office, the rooms can be represented by vertices and legitimate or specified passages to reach those rooms can be represented by edges. If someone tries to violate even a single legitimate passage, a complete disruption in the labeling will occur. As in case of structure like D_5, the magic constant will get disturbed abruptly due to violation of passage. This will promptly indicate to the concerned security through a computer programme that someone has violated the passage and the security team will get alert. Now, these particular magic and antimagic schemes, once designed, can be used wherever they are needed in a similar security pattern. Both magic labeling and antimagic labeling are equally important in this regard. This is a major use of magic labeling and antimagic labeling in urban planning. Thus, these labelings play their part as model for surveillance or security for various types of buildings or areas [15].

Moreover, the functioning and routing of robots installed in restaurants and in factories as production lines and other such units come under light using one of the labeling functions. As in robotics, they help to decide which function to be used and which to be skipped at a certain instance for making the robots or certain robotic components moving or keeping them static. The idea of robotic routing with the help of the tools like labeling functions and distance-based dimensions not only helps in minimizing the time for a robot for taking a decision but also maximizes its accuracy [16]. Such benefits are the cause of massive reduction of cost in industry.

We have organized this article into six main sections. Section 1 presents the introduction, followed by Section 2 of the preliminary definitions. Section 3 contains our main findings. Section 4 discusses the illustration of our findings through examples and proposed open problems. Section 5 focuses on the synopsis and 3D plots of our results. Conclusion is given in the last section.

2. Preliminary Definitions

In this section, we will discuss some definitions and results that are useful in the presentation of our theorems in the next sections. Also, salient works previously done in this field shall be mentioned here.

Let $G = (V(G), E(G))$ be a simple, connected, and nonempty graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V(G)| = p$ and $|E(G)| = q$, respectively. We call G in this case a (p, q)- graph. Throughout this article, we will use (p, q)- graphs. For more insight into the graph theoretic terminologies, we refer the reader to [17].

A mapping that maps nonzero positive integers onto the vertices, edges, or both of a graph under certain conditions is called a labeling. It is called total labeling if we include both sets of vertices and edges in its domain. Some labelings carry the

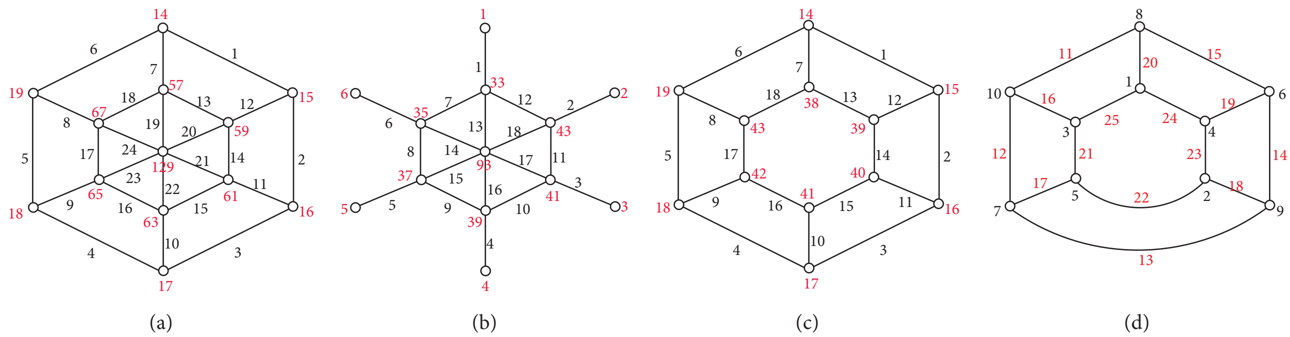

FIGURE 1: Vertex-antimagic labeling of wheel W_6, prism D_6, and helm H_6 and edge-magic labeling of prism D_5 [13, 14].

vertex set only or the edge set only in the domain and they are termed as vertex labelings or edge labelings, respectively. Two main types of labeling are magic labeling and antimagic labeling. In simple terms, magic labeling refers to equal vertex/edge weights and antimagic labeling refers to unequal vertex/edge weights.

Definition 1. For a (p,q)- graph $G=(V(G),E(G))$, the bijection $f\colon V(G)\cup E(G)\longrightarrow\{1,2,\dots,p+q\}$ is called (a,d) edge-antimagic labeling (or (a,d) edge-antimagic total labeling) if the edge weights $(f(x)+f(xy)+f(y))$, for each $xy\in E(G)$, form a sequence of consecutive positive integers with minimum edge weight $(a>0)$ and common difference d. If such a labeling exists, then G is said to be an (a,d) edge-antimagic graph.

Definition 2. An (a,d) edge-antimagic labeling f is called a super (a,d) edge-antimagic labeling of (p,q)- graph G if $f(V(G))=\{1,2,\dots,p\}$. Thus, a super (a,d) edge-antimagic graph is one that admits a super (a,d) edge-antimagic labeling.

In the above definitions, if we have $d=0$, then the minimum edge weight a becomes constant for all edges $xy\in E(G)$ known as magic constant or magic sum of the graph G.

Definition 3. A simple graph G with $|V(G)|=p$ is called a pancyclic graph if it contains cycle of every order from 3 to p.

Definition 4. The rooted product of two graphs G_1 and G_2 is obtained by taking $|V(G_1)|$ copies of G_2 and then, for every vertex v_i of G_1, identifying v_i with the root node of the i_{th} copy of G_2. It is denoted by $(G_1\circ G_2)$.

In 1963, Sadlácek defined the concept of magic labeling of graphs [18]. Later on, Hartsfield and Ringel [19] presented the idea of antimagic labeling for vertex-sums of a graph. The concept of $(a,0)$ edge-antimagic labeling of graphs was studied for the first time in [20] by Kotzig and Rosa who identified it by the name of magic valuation. In 1996, Ringel and Llado [21] studied this concept using a different terminology, that is, $(a,0)$ edge-antimagic labeling. Motivated by this concept, Enomoto et al., in 1998, defined the notion of super $(a,0)$ edge-antimagic labeling of graphs in [22]. They studied this concept with the term super edge-magic labeling of graphs. In the year 2000, Simanjantuk et al. introduced the idea of (a,d) edge-antimagic labeling of graphs in [23].

The historical background of the $(a,0)$ edge-antimagic labeling of graphs includes the following important and interesting conjectures on trees.

Conjecture 1. *Every tree is $(a,0)$ edge-antimagic [20].*

Conjecture 2. *Every tree is super $(a,0)$ edge-antimagic [22].*

In support of Conjecture 2, many particular classes of trees have been studied by various authors. Lee and Shah [24] verified this conjecture for trees with at most 17 vertices with the help of a programming software. In particular, the results can be found for stars, subdivided stars [25–29], W-trees [30–32], banana trees [33], caterpillars [34], subdivided caterpillars [35], and disjoint union of stars and books [36]. Further related studies can be seen in [37–39]. However, this conjecture is still open for working. In [22], it is proven that if a nontrivial (p,q)- graph G is super $(a,0)$ edge-antimagic, then $(q\leq 2p-3)$. In the same article, the authors proved that a complete bipartite graph $K_{m,n}$ is super $(a,0)$ edge-antimagic if and only if $m=1$ or $n=1$. In [40], it is proven that $K_{1,m}\cup K_{1,n}$ is super $(a,0)$ edge-antimagic if either m is a multiple of $n+1$ or n is a multiple of $m+1$. Enomoto et al. [22] proved that C_n is super $(a,0)$ edge-antimagic if and only if n is odd. In [41], it has been proven that $C_3\cup C_n$ is super $(a,0)$ edge-antimagic if and only if $n\geq 6$ and n is even (also see [42]). In [43], Figueroa-Centeno et al. showed that the generalized prism $C_n\times P_m$ is super $(a,0)$ edge-antimagic for every odd integer n. In [3], Baig et al. presented the super $(a,0)$ edge-antimagic labeling of a class of pancyclic graphs. The following lemma regarding super $(a,0)$ edge-antimagic graphs is very useful.

Lemma 1 (see [43]). *A (p,q)-graph G is super $(a,0)$ edge-antimagic total if and only if there exists a bijective function $f\colon V(G)\longrightarrow\{1,2,\dots,p\}$ such that the set $S=\{f(x)+f(y)\mid xy\in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super $(a,0)$ edge-antimagic total labeling of G with magic constant $(c=p+q+s)$, where $s=\min(S)$ and $S=\{c-(p+1),c-(p+2),\dots,c-(p+q)\}$.*

The sum $(f(x)+f(y))$ in Lemma 1 is termed as edge sum for each edge $(xy\in E(G))$. In fact, the set of all edge sums $S=\{f(x)+f(y)\mid xy\in E(G)\}$ in Lemma 1 constituting an arithmetic progression $\{c-(p+1),c-(p+2),\dots,c-(p+q)\}$ forms a super $(a,1)$ edge-antimagic vertex labeling of the graph G [23]. That is, f constitutes a super $(a,1)$ edge-antimagic vertex labeling of G. It means Lemma 1 states that whenever the set of edge sums forms a super $(a,1)$ edge-antimagic vertex labeling of G, it extends to a super $(a,0)$ edge-antimagic labeling of G.

We will frequently use the above lemma in the proofs of our results. Another very useful relevant result is as follows.

Theorem 1 (see [45]). *If a (p,q)-graph G is super $(a,0)$ edge-antimagic, then it is super $(a-q+1,2)$ edge-antimagic always.*

3. Main Results

This section consists of two further sections, in which we will present our main results. In Section 3.1, we will study the super $(a,0)$ edge-antimagic labeling of the disjoint union of the rooted product of P_n and the complete bipartite graph $K_{2,m}$ with path, copies of paths, and the star. Meanwhile, in Section 3.2, we will provide the super $(a,0)$ edge-antimagic labeling of rooted product of P_n with certain pancyclic graphs H_1 and H_2. It is pertinent to mention here that all our graphs obtained as the result of the rooted products are planar.

3.1. Super $(a,0)$ Edge-Antimagic Labeling of the Disjoint Union of the Rooted Product of P_n and $K_{2,m}$ with Path, Copies of Paths, and Star. The following open problem proposed by Ngurah et al. in [46] is our main motivation to study super $(a,0)$ edge-antimagic labeling of the graph containing copies of complete bipartite graph $K_{2,m}$.

3.1.1. Open Problem. For $n\geq 2$ and $m\geq 3$, can you determine any super $(a,0)$ edge-antimagic labeling of $(nK_{2,m})$?

The rooted product $(P_n\circ K_{2,m})$, in fact, contains n copies of the complete bipartite graph $K_{2,m}$. We swiftly move to our theorems now.

Theorem 2. *For even $(m\geq 2)$ and odd $(n\geq 3)$, the graph $(P_n\circ K_{2,m})\cup K_{1,n}\cup(n-1/2)K_1$ admits a super $(a,0)$ edge-antimagic labeling with magic constant $(3mn+8n+2)$.*

Proof. Consider the graph $G_1\cong(P_n\circ K_{2,m})\cup K_{1,n}\cup(n-1/2)K_1$ with vertex and edge sets as follows:

$$\begin{aligned} V(G_1) &= \left\{x_j^i\colon 1\leq i\leq n, 1\leq j\leq m\right\}\cup\{y_i,z_i\colon 1\leq i\leq n\}\\ &\quad\cup\{p_i\colon 1\leq i\leq n\}\cup\left\{c_i\colon 1\leq i\leq\frac{n-1}{2}\right\}\cup\{c\},\\ E(G_1) &= \left\{y_ix_j^i\colon 1\leq i\leq n, 1\leq j\leq m\right\}\\ &\quad\cup\left\{z_ix_j^i\colon 1\leq i\leq n, 1\leq j\leq m\right\}\\ &\quad\cup\{cp_i\colon 1\leq i\leq n\}\cup\{y_iy_{i+1}\colon 1\leq i\leq n-1\}, \end{aligned}\tag{1}$$

where $|V(G_1)|=((7n+2mn+1)/2)$ and $(|E(G_1)|=2n(m+1)-1)$. We define a labeling $(f_1\colon V(G_1)\longrightarrow\{1,2,\ldots,((7n+2mn+1)/2)\})$ as follows:

$$\begin{aligned} f_1\left(x_j^i\right) &= \begin{cases} \dfrac{5n+i}{2}\colon & j=\dfrac{m}{2}, 1\leq i\leq n, i\equiv 1 \pmod 2,\\ \dfrac{4n+i}{2}\colon & j=\dfrac{m}{2}, 2\leq i\leq n-1, i\equiv 0 \pmod 2,\\ n(m+2)-(i-1)\colon & j=\dfrac{m}{2}+1, 1\leq i\leq n,\\ n(j+3)-(i-1)\colon & 1\leq i\leq n, 1\leq j\leq\dfrac{m}{2}-1,\\ nj-(i-1)+n\colon & 1\leq i\leq n, \dfrac{m}{2}+2\leq j\leq m, \end{cases}\\ f_1(y_i) &= \begin{cases} \dfrac{i+2n+1}{2}\colon & 1\leq i\leq n, i\equiv 1 \pmod 2,\\ \dfrac{i+3n+1}{2}\colon & 2\leq i\leq n-1, i\equiv 0 \pmod 2, \end{cases}\\ f_1(z_i) &= \begin{cases} n(m+1)+\dfrac{i+2n+1}{2}\colon & 1\leq i\leq n, i\equiv 1 \pmod 2,\\ n(m+1)+\dfrac{i+3n+1}{2}\colon & 2\leq i\leq n-1, i\equiv 0 \pmod 2. \end{cases}\\ f_1(p_i) &= i\colon 1\leq i\leq n,\\ f_1(c) &= \frac{7n+2mn+1}{2},\\ f_1(c_i) &= mn+3n+i\colon 1\leq i\leq\frac{n-1}{2}. \end{aligned}\tag{2}$$

The set of all edge sums generated by the labeling scheme f_1 forms a sequence of consecutive integers $((5n+5/2), (5n+7/2), \ldots, ((4mn+9n+1)/2))$, which is a super $(a,1)$ edge-antimagic vertex labeling of G_1. Therefore, by Lemma 1, f_1 extends to a super $(a,0)$ edge-antimagic labeling of the graph G_1 with magic constant $(a = 3mn+8n+2)$.

Theorem 3. *For odd $m, n \geq 3$, the graph $(P_n \circ K_{2,m}) \cup K_{1,n} \cup (n-1/2)K_1$ admits a super $(a,0)$ edge-antimagic labeling with magic constant $(3mn+8n+2)$.*

Proof. Consider the graph $(G_2 \cong (P_n \circ K_{2,m}) \cup K_{1,n} \cup (n-1/2)K_1)$ with odd m as follows:

$$\begin{aligned} V(G_2) &= \left\{x_j^i\colon 1 \leq i \leq n, 1 \leq j \leq m\right\} \cup \{y_i, z_i\colon 1 \leq i \leq n\} \\ &\quad \cup \{p_i\colon 1 \leq i \leq n\} \cup \left\{c_i\colon 1 \leq i \leq \frac{n-1}{2}\right\} \cup \{c\}, \\ E(G_2) &= \left\{y_i x_j^i\colon 1 \leq i \leq n, 1 \leq j \leq m\right\} \\ &\quad \cup \left\{z_i x_j^i\colon 1 \leq i \leq n, 1 \leq j \leq m\right\} \cup \{c p_i\colon 1 \leq i \leq n\} \\ &\quad \cup \{y_i y_{i+1}\colon 1 \leq i \leq n-1\}, \end{aligned} \tag{3}$$

where $|V(G_2)| = ((7n+2mn+1)/2)$ and $(|E(G_2)| = 2n(m+1)-1)$. We define a labeling $(f_2\colon V(G_2) \longrightarrow \{1, 2, \ldots, ((7n+2mn+1)/2)\})$ as follows:

$$\begin{aligned} f_2\left(x_j^i\right) &= \begin{cases} \dfrac{5n+i}{2}\colon & j = \dfrac{m+1}{2}, 1 \leq i \leq n, i \equiv 1 \pmod 2, \\ \dfrac{4n+i}{2}\colon & j = \dfrac{m+1}{2}, 2 \leq i \leq n-1, i \equiv 0 \pmod 2, \\ n(m+2)-(i-1)\colon & j = \dfrac{m+3}{2}, 1 \leq i \leq n, \\ n(j+3)-(i-1)\colon & 1 \leq i \leq n, 1 \leq j \leq \dfrac{m}{2}-1, \\ nj-(i-1)+n\colon & 1 \leq i \leq n, \dfrac{m+5}{2} \leq j \leq m, \end{cases} \\ f_2(y_i) &= \begin{cases} \dfrac{i+2n+1}{2}\colon & 1 \leq i \leq n, i \equiv 1 \pmod 2, \\ \dfrac{i+3n+1}{2}\colon & 2 \leq i \leq n-1, i \equiv 0 \pmod 2, \end{cases} \\ f_2(z_i) &= \begin{cases} n(m+1) + \dfrac{i+2n+1}{2}\colon & 1 \leq i \leq n, i \equiv 1 \pmod 2, \\ n(m+1) + \dfrac{i+3n+1}{2}\colon & 2 \leq i \leq n-1, i \equiv 0 \pmod 2, \end{cases} \\ f_2(p_i) &= i\colon 1 \leq i \leq n, \\ f_2(c) &= \frac{7n+2mn+1}{2}, \\ f_2(c_i) &= mn+3n+i\colon 1 \leq i \leq \frac{n-1}{2}. \end{aligned} \tag{4}$$

The set of all edge sums generated by the above labeling scheme f_2 forms a sequence of consecutive integers $((5n+5/2), (5n+7/2), \ldots, ((4mn+9n+1)/2))$, which is a super $(a,1)$ edge-antimagic vertex labeling of G_2. Therefore, by Lemma 1, f_2 extends to a super $(a,0)$ edge-antimagic labeling of the graph G_2 with magic constant $(a = 3mn+8n+2)$.

Theorem 4. *For even $m \geq 2$ and odd $n \geq 3$, the graph $((P_n \circ K_{2,m}) \cup nP_2)$ admits a super $(a,0)$ edge-antimagic labeling with magic constant $((6mn+17n+3)/2)$.*

Proof. Consider the graph $(G_3 \cong ((P_n \circ K_{2,m}) \cup nP_2))$, for odd $n \geq 3$, constructed as

$$\begin{aligned} V(G_3) &= \left\{x_j^i\colon 1 \leq i \leq n, 1 \leq j \leq m\right\} \cup \{y_i, z_i\colon 1 \leq i \leq n\} \\ &\quad \cup \{p_i, q_i\colon 1 \leq i \leq n\}, \\ E(G_3) &= \left\{y_i x_j^i\colon 1 \leq i \leq n, 1 \leq j \leq m\right\} \\ &\quad \cup \left\{z_i x_j^i\colon 1 \leq i \leq n, 1 \leq j \leq m\right\} \cup \{p_i q_i\colon 1 \leq i \leq n\} \\ &\quad \cup \{y_i y_{i+1}\colon 1 \leq i \leq n-1\}, \end{aligned} \tag{5}$$

where we have $(|V(G_3)| = n(m+4))$ and $(|E(G_3)| = 2n(m+1)-1)$. Now, we define a labeling $(f_3\colon V(G_3) \longrightarrow \{1,2,\ldots,n(m+4)\})$ as follows:

$$f_3(x_j^i) = \begin{cases} \frac{1}{2}(5n+i)\colon & j=\frac{m}{2}, 1\le i\le n, i\equiv 1 \pmod 2,\\ \frac{1}{2}(4n+i)\colon & j=\frac{m}{2}, 2\le i\le n-1, i\equiv 0 \pmod 2,\\ n(m+2)-(i-1)\colon & j=\frac{m}{2}+1, 1\le i\le n,\\ n(j+3)-(i-1)\colon & 1\le i\le n, 1\le j\le \frac{m}{2}-1,\\ nj-(i-1)+n\colon & 1\le i\le n, \frac{m}{2}+2\le j\le m,\end{cases}$$

$$f_3(y_i) = \begin{cases} \frac{i+2n+1}{2}\colon & 1\le i\le n, i\equiv 1 \pmod 2,\\ \frac{i+3n+1}{2}\colon & 2\le i\le n-1, i\equiv 0 \pmod 2,\end{cases} \tag{6}$$

$$f_3(z_i) = \begin{cases} n(m+1)+\frac{i+2n+1}{2}\colon & 1\le i\le n, i\equiv 1 \pmod 2,\\ n(m+1)+\frac{i+3n+1}{2}\colon & 2\le i\le n-1, i\equiv 0 \pmod 2,\end{cases}$$

$$f_3(q_i) = \begin{cases} n(m+1)+\frac{5n-i+2}{2}\colon & 1\le i\le n, i\equiv 1 \pmod 2,\\ n(m+1)+\frac{6n-i+2}{2}\colon & 2\le i\le n-1, i\equiv 0 \pmod 2,\end{cases}$$

$$f_3(p_i) = i\colon\ 1\le i\le n.$$

The set of all edge sums generated by the above labeling scheme f_3 constitutes a sequence of consecutive integers $((5n+5/2), (5n+7/2), \ldots, ((4mn+9n+1)/2))$, which is a super $(a, 1)$ edge-antimagic vertex labeling of G_3. Therefore, by Lemma 1, f_3 extends to a super $(a,0)$ edge-antimagic labeling of the graph G_3 with magic constant $a = ((6mn+17n+3)/2)$.

Theorem 5. *For odd $(m, n\ge 3)$, the graph $((P_n \circ K_{2,m}) \cup nP_2)$ admits a super $(a,0)$ edge-antimagic labeling with magic constant $((6mn+17n+3)/2)$.*

Proof. Consider the graph $G_4 \cong ((P_n \circ K_{2,m}) \cup nP_2)$, for odd m, with the construction:

$$\begin{aligned} V(G_4) &= \left\{x_j^i\colon 1\le i\le n, 1\le j\le m\right\} \cup \{y_i, z_i\colon 1\le i\le n\}\\ &\quad \cup \{p_i, q_i\colon 1\le i\le n\},\\ E(G_4) &= \left\{y_i x_j^i\colon 1\le i\le n, 1\le j\le m\right\}\\ &\quad \cup \left\{z_i x_j^i\colon 1\le i\le n, 1\le j\le m\right\} \cup \{p_i q_i\colon 1\le i\le n\}\\ &\quad \cup \{y_i y_{i+1}\colon 1\le i\le n-1\},\end{aligned} \tag{7}$$

where $(|V(G_4)| = n(m+4))$ and $(|E(G_4)| = 2n(m+1)-1)$. Now, consider a labeling $(f_4\colon V(G_4) \longrightarrow \{1,2,\ldots,n(m+4)\})$ as follows:

$$f_4\left(x_j^i\right)=\begin{cases}\dfrac{1}{2}(5n+i)\colon & j=\dfrac{m+1}{2}, 1\le i\le n, i\equiv 1 \pmod 2,\\[2ex] \dfrac{1}{2}(4n+i)\colon & j=\dfrac{m+1}{2}, 2\le i\le n-1, i\equiv 0 \pmod 2,\\[2ex] n(m+2)-(i-1)\colon & j=\dfrac{m+3}{2}, 1\le i\le n,\\[2ex] n(j+3)-(i-1)\colon & 1\le i\le n, 1\le j\le \dfrac{m-1}{2},\\[2ex] nj-(i-1)+n\colon & 1\le i\le n, \dfrac{m+5}{2}\le j\le m,\end{cases}$$

$$f_4(y_i)=\begin{cases}\dfrac{i+2n+1}{2}\colon & 1\le i\le n, i\equiv 1 \pmod 2,\\[2ex] \dfrac{i+3n+1}{2}\colon & 2\le i\le n-1, i\equiv 0 \pmod 2,\end{cases} \tag{8}$$

$$f_4(z_i)=\begin{cases}n(m+1)+\dfrac{i+2n+1}{2}\colon & 1\le i\le n, i\equiv 1 \pmod 2,\\[2ex] n(m+1)+\dfrac{i+3n+1}{2}\colon & 2\le i\le n-1, i\equiv 0 \pmod 2,\end{cases}$$

$$f_4(q_i)=\begin{cases}n(m+1)+\dfrac{5n-i+2}{2}\colon & 1\le i\le n, i\equiv 1 \pmod 2,\\[2ex] n(m+1)+\dfrac{6n-i+2}{2}\colon & 2\le i\le n-1, i\equiv 0 \pmod 2,\end{cases}$$

$$f_4(p_i)=i\colon\ 1\le i\le n.$$

The set of all edge sums generated by the above labeling scheme f_4 constitutes a sequence of consecutive integers $((5n+5/2), (5n+7/2), \ldots, ((4mn+9n+1)/2))$, which is a super $(a, 1)$ edge-antimagic vertex labeling of G_4. Therefore, by Lemma 1, f_4 extends to a super $(a,0)$ edge-antimagic labeling of the graph G_4 with magic constant $a=((6mn+17n+3)/2)$.

Theorem 6. *For even $m\ge 2$ and odd $n\ge 3$, the graph $(P_n\circ K_{2,m})\cup P_{n+1}$ admits a super $(a,0)$ edge-antimagic labeling with magic constant $((6mn+13n+7)/2)$.*

Proof. Consider the graph $G_5\cong (P_n\circ K_{2,m})\cup P_{n+1}$, for odd $n\ge 3$, with the construction:

$$\begin{aligned}V(G_5)&=\left\{x_j^i\colon 1\le i\le n, 1\le j\le m\right\}\\ &\quad\cup\{y_i, z_i\colon 1\le i\le n\}\\ &\quad\cup\{p_i\colon 1\le i\le n+1\},\\ E(G_5)&=\left\{y_ix_j^i\colon 1\le i\le n, 1\le j\le m\right\}\cup\left\{z_ix_j^i\colon 1\le i\le n, 1\le j\le m\right\}\\ &\quad\cup\{p_ip_{i+1}\colon 1\le i\le n\}\cup\{y_iy_{i+1}\colon 1\le i\le n-1\},\end{aligned} \tag{9}$$

where we have $(|V(G_5)|=mn+3n+1)$ and $(|E(G_5)|=2n(m+1)-1)$. We define a labeling $(f_5\colon V(G_5)\longrightarrow\{1,2,\ldots,mn+3n+1\})$ as follows:

$$f_5(x_j^i) = \begin{cases} \frac{1}{2}(4n+i+1): & j = \frac{m}{2}, 1 \le i \le n, i \equiv 1 \pmod 2, \\ \frac{1}{2}(3n+i+1): & j = \frac{m}{2}, 2 \le i \le n-1, i \equiv 0 \pmod 2, \\ \frac{1}{2}(2mn+3n-2i+3): & j = \frac{m}{2}+1, 1 \le i \le n, \\ \frac{1}{2}(2nj+5n-2i+3): & 1 \le i \le n, 1 \le j \le \frac{m}{2}-1, \\ \frac{1}{2}(2nj+n-2i+3): & 1 \le i \le n, \frac{m}{2}+2 \le j \le m, \end{cases}$$

$$f_5(y_i) = \begin{cases} \frac{n+i+2}{2}: & 1 \le i \le n, i \equiv 1 \pmod 2, \\ \frac{2n+i+2}{2}: & 2 \le i \le n-1, i \equiv 0 \pmod 2, \end{cases} \tag{10}$$

$$f_5(z_i) = \begin{cases} n(m+1) + \frac{n+i+2}{2}: & 1 \le i \le n, i \equiv 1 \pmod 2, \\ n(m+1) + \frac{2n+i+2}{2}: & 2 \le i \le n-1, i \equiv 0 \pmod 2, \end{cases}$$

$$f_5(p_i) = \begin{cases} \frac{i+1}{2}: & 1 \le i \le n, i \equiv 1 \pmod 2, \\ \frac{1}{2}(2mn+5n+i+1): & 2 \le i \le n+1, i \equiv 0 \pmod 2. \end{cases}$$

The set of all edge sums generated by the above labeling scheme f_5 forms a sequence of consecutive integers $((3n+7/2), (3n+9/2), \ldots, ((4mn+7n+3)/2))$, which is a super $(a, 1)$ edge-antimagic vertex labeling of G_5. Therefore, by Lemma 1, f_5 extends to a super $(a,0)$ edge-antimagic labeling of the graph G_5 with magic constant $a = ((6mn+13n+7)/2)$.

Theorem 7. *For odd $m, n \ge 3$, the graph $(P_n \circ K_{2,m}) \cup P_{n+1}$ admits a super $(a,0)$ edge-antimagic labeling with magic constant $((6mn+13n+7)/2)$.*

Proof. Consider the graph $(G_6 \cong (P_n \circ K_{2,m}) \cup P_{n+1})$ for both odd m and $n \ge 3$ with vertex and edge sets as follows.

$$\begin{aligned} V(G_6) &= \{x_j^i : 1 \le i \le n, 1 \le j \le m\} \cup \{y_i, z_i : 1 \le i \le n\} \\ &\quad \cup \{p_i : 1 \le i \le n+1\}, \\ E(G_6) &= \{y_i x_j^i : 1 \le i \le n, 1 \le j \le m\} \\ &\quad \cup \{z_i x_j^i : 1 \le i \le n, 1 \le j \le m\} \\ &\quad \cup \{p_i p_{i+1} : 1 \le i \le n\} \cup \{y_i y_{i+1} : 1 \le i \le n-1\}, \end{aligned} \tag{11}$$

where $(|V(G_6)| = mn+3n+1)$ and $(|E(G_6)| = 2n(m+1)-1)$. Now, we define a labeling $(f_6 : V(G_6) \longrightarrow \{1, 2, \ldots, mn+3n+1\})$ as follows:

$$f_6(x_j^i) = \begin{cases} \frac{1}{2}(4n+i+1): & j = \frac{m+1}{2}, 1 \le i \le n, i \equiv 1 \pmod 2, \\ \frac{1}{2}(3n+i+1): & j = \frac{m+1}{2}, 2 \le i \le n-1, i \equiv 0 \pmod 2, \\ \frac{1}{2}(2mn+3n-2i+3): & j = \frac{m+3}{2}, 1 \le i \le n, \\ \frac{1}{2}(2nj+5n-2i+3): & 1 \le i \le n, 1 \le j \le \frac{m-1}{2}, \\ \frac{1}{2}(2nj+n-2i+3): & 1 \le i \le n, \frac{m+5}{2} \le j \le m, \end{cases}$$

$$f_6(y_i) = \begin{cases} \frac{n+i+2}{2}: & 1 \le i \le n, i \equiv 1 \pmod 2, \\ \frac{2n+i+2}{2}: & 2 \le i \le n-1, i \equiv 0 \pmod 2, \end{cases} \tag{12}$$

$$f_6(z_i) = \begin{cases} n(m+1) + \frac{n+i+2}{2}: & 1 \le i \le n, i \equiv 1 \pmod 2, \\ n(m+1) + \frac{2n+i+2}{2}: & 2 \le i \le n-1, i \equiv 0 \pmod 2, \end{cases}$$

$$f_6(p_i) = \begin{cases} \frac{i+1}{2}: & 1 \le i \le n, i \equiv 1 \pmod 2, \\ \frac{1}{2}(2mn+5n+i+1): & 2 \le i \le n+1, i \equiv 0 \pmod 2. \end{cases}$$

The set of all edge sums generated by the above labeling scheme f_6 forms a sequence of consecutive integers $((3n+7/2), (3n+9/2), \ldots, ((4mn+7n+3)/2))$, which is a super $(a,1)$ edge-antimagic vertex labeling of G_6. Therefore, by Lemma 1, f_6 extends to a super $(a,0)$ edge-antimagic labeling of the graph G_6 with magic constant $a = ((6mn+13n+7)/2)$.

3.1.2. Observations

(1) In Theorems 2, 4, and 6, the labels $(f_k(x_j^i))$ for $(1 \le j \le (m/2)-1)$ & $((m/2)+2 \le j \le m)$ will receive the values only when $m > 2$. Similarly, in Theorems 3, 5, and 7, the labels $(f_k(x_j^i))$ for $(m+5/2) \le j \le m$ will receive the values when $m > 3$. This happens because $K_{2,2} \cong C_4$ and $K_{2,3}$ has 3 vertices in one partitioned set. However, the labeling of our graphs $G_1, \ldots, G_6$ never gets disturbed anyway (here, $k \in \{1, \ldots, 6\}$).

(2) The following results from Theorems 2–7 are direct consequences of Theorem 1.

(3) As a cycle C_n is super $(a,0)$ edge-antimagic if and only if n is odd [22], this means that C_4 is not super $(a,0)$ edge-antimagic. We have observed an interesting substructure in Theorems 2, 4, and 6. That is, if we fix $m = 2$ in these theorems, we obtain a cyclic family of graphs, in which only cycle C_4 appears n times (n is odd). Obviously, these families are also super $(a,0)$ edge-antimagic, as per the proofs of our results. See super $(a,0)$ edge-antimagic labeling of $((P_5 \circ C_4) \cup K_{1,5} \cup 2K_1)$, $((P_5 \circ C_4) \cup 5P_2)$, and $((P_5 \circ C_4) \cup P_6)$ in Figure 2.

Theorem 8. *For all $m \ge 2$ and odd $n \ge 3$, the graph $((P_n \circ K_{2,m}) \cup K_{1,n} \cup (n-1/2)K_1)$ admits a super $(mn+6n+4, 2)$ edge-antimagic labeling.*

Theorem 9. *For all $m \ge 2$ and odd $n \ge 3$, the graph $((P_n \circ K_{2,m}) \cup nP_2)$ admits a super $(((2mn+13n+7)/2), 2)$ edge-antimagic labeling.*

Theorem 10. *For all $m \ge 2$ and odd $n \ge 3$, the graph $((P_n \circ K_{2,m}) \cup P_{n+1})$ admits a super $(((2mn+9n+11)/2), 2)$ edge-antimagic labeling.*

3.2. Super $(a,0)$ Edge-Antimagic Labeling of Rooted Product of P_n with Certain Pancyclic Graphs. In networking, the networks that are acyclic and networks containing a range of cycles admit same level of importance. In graph theoretic terms, the former corresponds to trees and the latter corresponds to multicyclic graphs. What about a network that

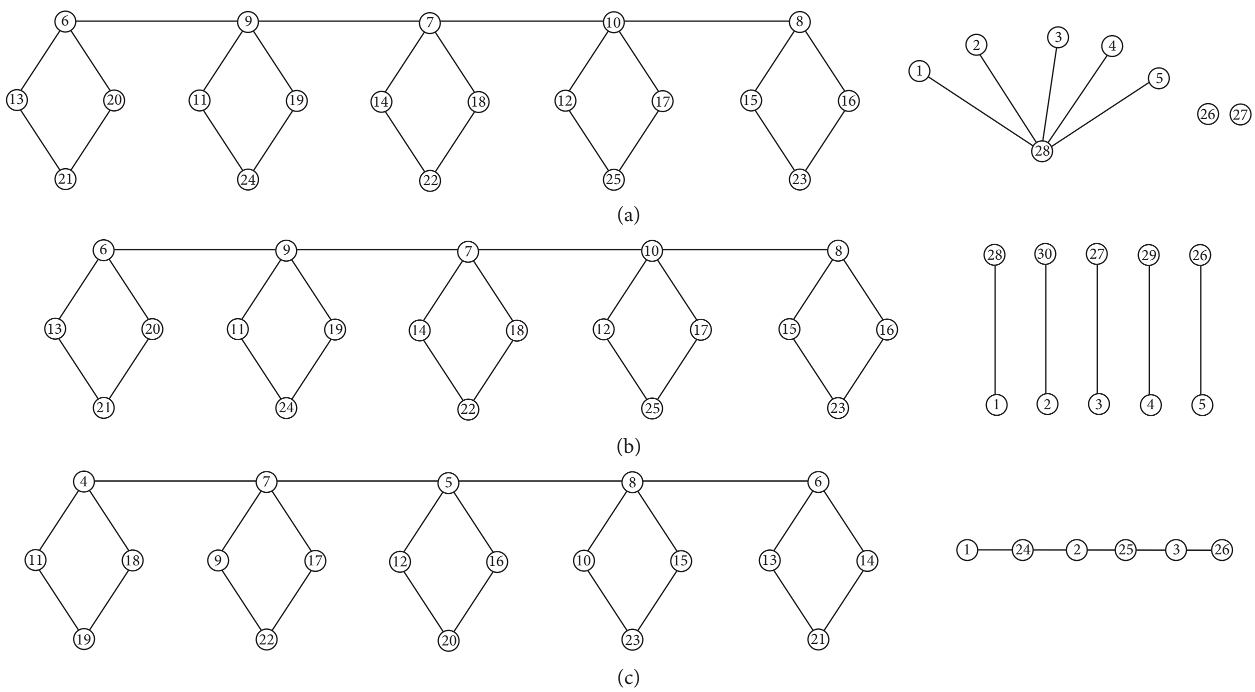

FIGURE 2: A super $(a, 0)$ edge-antimagic labeling of the cyclic graphs (a) $((P_5 \circ C_4) \cup K_{1,5} \cup 2K_1)$, (b) $((P_5 \circ C_4) \cup 5P_2)$, and (c) $((P_5 \circ C_4) \cup P_6)$ with magic constants 72, 74, and 66, respectively.

contains cyclic connection of all orders starting from 3 up to the number of systems connected within that network? In such situations, the task of the programmers gets more tough, as they have to work on encrypting powerful codes to keep their data safe by halting the attacks of hackers. It is because every system in such network is connected within a cycle. This kind of network corresponds to a pancyclic graph (see Definition 3).

We define here a specific pancyclic graph H_1 as follows.

Definition 5. Consider a pancyclic graph H_1 with the construction:

$$V(H_1) = \{x_1, x_2, x_3, x_4, x_5, x_6\} \cup \{y, z\},$$
$$E(H_1) = \{x_1x_3, x_3x_5, x_2x_4, x_4x_6, x_1x_2, x_3x_4, x_5x_6, x_2x_3, x_4x_5\} \cup \{yx_1, yx_2, zx_5, zx_6\}. \tag{13}$$

In the next result, we will show that the rooted product $(P_n \circ H_1)$ is super $(a, 0)$ edge-antimagic for odd values of n.

Theorem 11. *For odd n, the rooted product $(P_n \circ H_1)$ admits a super $(a, 0)$ edge-antimagic labeling with magic constant $(a = (45n + 3/2))$.*

Proof

(1) For $n = 1$, $(P_1 \circ H_1 \cong H_1)$. The vertex labeling $\{x_1, x_2, x_3, x_4, x_5, x_6, y, z\colon 1, 4, 3, 6, 5, 8, 2, 7\}$ extends to a super $(24, 0)$ edge-antimagic labeling of $(P_1 \circ H_1)$ by Lemma 1.

(2) For $n \geq 3$, consider the graph $(P_n \circ H_1)$ with $(|V(P_n \circ H_1)| = 8n)$ and $(|E(P_n \circ H_1)| = 14n - 1)$ consisting of the following vertex and edge sets:

$$V(P_n \circ H_1) = \{x_i, v_i\colon 1 \leq i \leq n\} \cup \{y_i, z_i, w_i\colon 1 \leq i \leq 2n\},$$
$$\begin{aligned} E(P_n \circ H_1) = {} & \{y_iy_{i+1}, z_iz_{i+1}, w_iw_{i+1}\colon 1 \leq i \leq 2n-1, i \equiv 1\,(\mathrm{mod}2)\} \\ & \cup \{z_iy_{i+1}, w_iz_{i+1}\colon 1 \leq i \leq 2n-1, i \equiv 1 \quad (\mathrm{mod}2)\} \\ & \cup \{x_iy_{2i-1}, x_iy_{2i}, v_iw_{2i-1}, v_iw_{2i}\colon 1 \leq i \leq n\} \\ & \cup \{x_ix_{i+1}\colon 1 \leq i \leq n-1\} \cup \{y_iz_i, z_iw_i\colon 1 \leq i \leq 2n\}. \end{aligned} \tag{14}$$

Consider a labeling $(\psi_1 : V(P_n \circ H_1) \longrightarrow \{1, 2, \ldots, |V(P_n \circ H_1)| = 8n\})$ defined as

$$\psi_1(x_i) = \begin{cases} \dfrac{i+1}{2}: & 1 \le i \le n, i \equiv 1 \pmod 2, \\ \dfrac{i+n+1}{2}: & 2 \le i \le n-1, i \equiv 0 \pmod 2, \end{cases}$$

$$\psi_1(y_i) = \begin{cases} \dfrac{1}{2}(6n-i+2): & 2 \le i \le 2n, i \equiv 0 \pmod 2, \\ \dfrac{1}{4}(6n+i+1): & 1 \le i \le 2n-1, i \equiv 1 \pmod 4, \\ \dfrac{1}{4}(4n+i+1): & 3 \le i \le 2n-3, i \equiv 3 \pmod 4, \end{cases}$$

$$\psi_1(z_i) = \begin{cases} \dfrac{1}{4}(16n+i): & 4 \le i \le 2n-2, i \equiv 0 \pmod 4, \\ \dfrac{1}{4}(12n+i+3): & 1 \le i \le 2n-1, i \equiv 1 \pmod 4, \\ \dfrac{1}{4}(18n+i): & 2 \le i \le 2n, i \equiv 2 \pmod 4, \\ \dfrac{1}{4}(14n+i+3): & 3 \le i \le 2n-3, i \equiv 3 \pmod 4, \end{cases}$$

$$\psi_1(w_i) = \begin{cases} \dfrac{1}{2}(12n-i+1): & 1 \le i \le 2n-1, i \equiv 1 \pmod 2, \\ \dfrac{1}{4}(26n+i+2): & 4 \le i \le 2n-2, i \equiv 0 \pmod 4, \\ \dfrac{1}{4}(24n+i+2): & 2 \le i \le 2n, i \equiv 2 \pmod 4, \end{cases}$$

$$\psi_1(v_i) = \begin{cases} \dfrac{15n+i}{2}: & 1 \le i \le n, i \equiv 1 \pmod 2, \\ \dfrac{14n+i}{2}: & 2 \le i \le n-1, i \equiv 0 \pmod 2. \end{cases} \tag{15}$$

The set of all edge sums generated by the above labeling scheme ψ_1 forms a sequence of consecutive integers $((n+5/2), (n+7/2), \ldots, (29n+1/2))$, which is a super $(a,1)$ edge-antimagic vertex labeling of $(P_n \circ H_1)$. Therefore, by Lemma 1, ψ_1 extends to a super $(a,0)$ edge-antimagic labeling of the graph $(P_n \circ H_1)$ with magic constant $(45n + 3/2)$.

Definition 6. Consider a pancyclic graph H_2 (non-isomorphic to H_1) with the vertex-edge connection as follows:

$$\begin{aligned} V(H_2) &= \{x_1, x_2, x_3, x_4, x_5, x_6\} \cup \{y, z\}, \\ E(H_2) &= \{x_1x_3, x_3x_5, x_2x_4, x_4x_6, x_1x_2, x_5x_6, x_2x_3, x_4x_5\} \\ &\quad \cup \{yx_1, yx_2, zx_5, zx_6, yz\}. \end{aligned} \tag{16}$$

Theorem 12. *For odd n, the rooted product $(P_n \circ H_2)$ admits a super $(a,0)$ edge-antimagic labeling with magic constant $(a = (45n + 3/2))$.*

Proof. Consider the rooted product $(P_n \circ H_2)$ with $(|V(P_n \circ H_2)| = 8n)$ and $(|E(P_n \circ H_2)| = 14n - 1)$ with vertex and edge sets as follows:

$$\begin{aligned} V(P_n \circ H_2) &= \{x_i, v_i : 1 \le i \le n\} \cup \{y_i, z_i, w_i : 1 \le i \le 2n\}, \\ E(P_n \circ H_2) &= \{y_iy_{i+1}, z_iz_{i+1}, w_iw_{i+1} : 1 \le i \le 2n-1, i \equiv 1 \pmod 2\} \\ &\quad \cup \{z_iy_{i+1}, w_iz_{i+1} : 1 \le i \le 2n-1, i \equiv 1 \pmod 2\} \\ &\quad \cup \{x_iy_{2i-1}, x_iy_{2i}, v_iw_{2i-1}, v_iw_{2i} : 1 \le i \le n\} \\ &\quad \cup \{x_ix_{i+1} : 1 \le i \le n-1\} \cup \{y_iz_i, z_iw_i : 1 \le i \le 2n\}. \end{aligned} \tag{17}$$

And the labeling scheme is the same as ψ_1 designed in Theorem 11.

The following result is a direct consequence of Theorem 1.

Theorem 13. *For odd n, the rooted products $(P_n \circ H_1)$ and $(P_n \circ H_2)$ are super $((17n + 7/2), 2)$ edge-antimagic.*

4. Illustration through Examples and Proposed Open Problems

4.1. Examples. As examples of Theorems 2 and 3, the super $(a,0)$ edge-antimagic labeling of $((P_5 \circ K_{2,6}) \cup K_{1,5} \cup 2K_1)$ and super $(a,0)$ edge-antimagic labeling of $((P_5 \circ K_{2,7}) \cup K_{1,5} \cup 2K_1)$ are presented in Figures 3 and 4, respectively. It is evident that the magic constant of the labeling in Figure 3 is 132 (corresponding to the parameters $n = 5$ and $m = 6$) and the magic constant of the labeling in Figure 4 is 147 (corresponding to the parameters $n = 5$ and $m = 7$). These are as per the magic constants depicted in the proofs of Theorems 2 and 3.

Similarly, Figures 5–8 illustrate Theorems 4–7, respectively, for fixed parameters mentioned in the caption of each. Here, the magic constants are as per our depiction as well.

Figures 9 and 10 illustrate super $(114, 0)$ edge-antimagic labeling of the rooted products appearing in Theorems 11 and 12 with parameter $n = 5$.

(1) In all of the above figures, Lemma 1 has helped us in the way that we do not require to label the edges. We only have observed that the edge sums are consecutive positive integers. So, applying the remaining labels $\{p+1, p+2, \ldots, q\}$ to the edges of the graph in descending or ascending sequence of edge sums will give rise to super $(a,0)$ or $(a',2)$ edge-antimagic labeling of the graph, respectively, for appropriate values of a and a'.

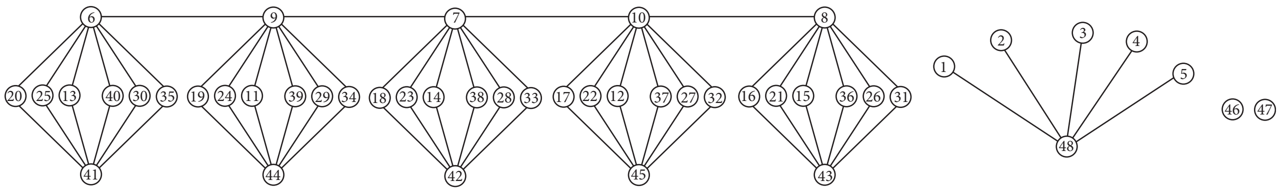

Figure 3: A super $(132, 0)$ edge-antimagic labeling of $((P_5 \circ K_{2,6}) \cup K_{1,5} \cup 2K_1)$.

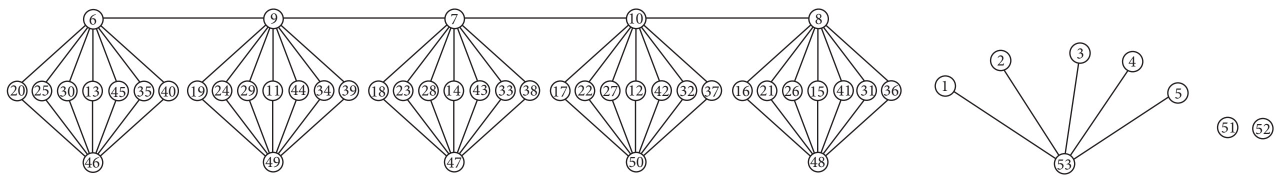

Figure 4: A super $(147, 0)$ edge-antimagic labeling of $((P_5 \circ K_{2,6}) \cup K_{1,5} \cup 2K_1)$.

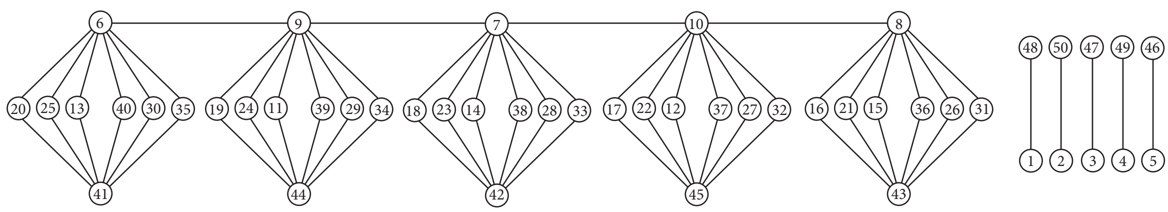

Figure 5: A super $(134, 0)$ edge-antimagic labeling of $((P_5 \circ K_{2,6}) \cup 5P_2)$.

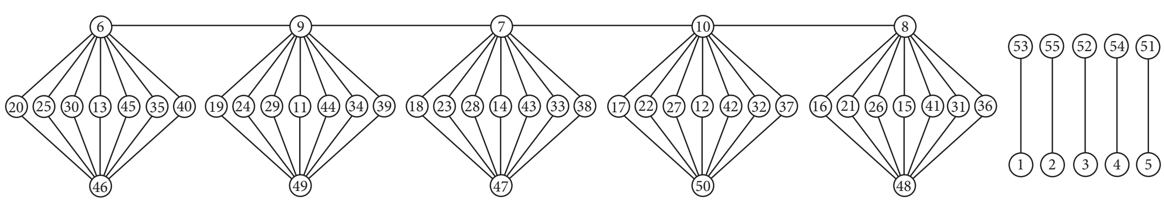

Figure 6: A super $(149, 0)$ edge-antimagic labeling of $((P_5 \circ K_{2,7}) \cup 5P_2)$.

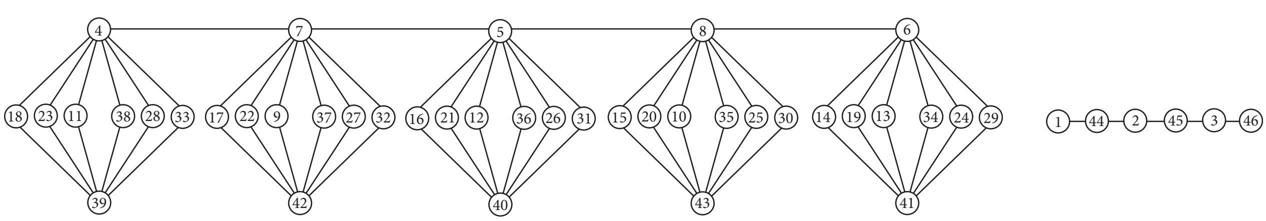

Figure 7: A super $(126, 0)$ edge-antimagic labeling of $((P_5 \circ K_{2,6}) \cup P_6)$.

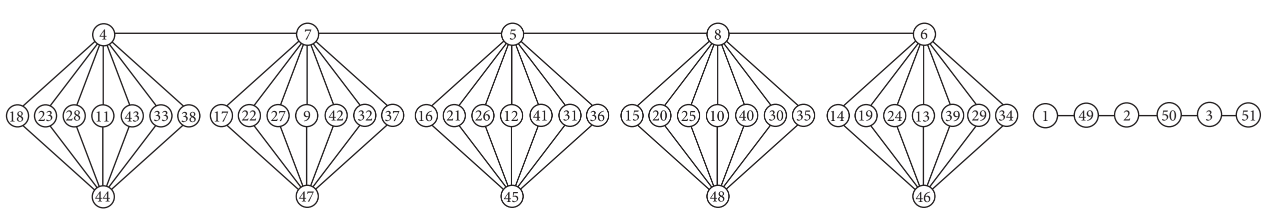

Figure 8: A super $(141, 0)$ edge-antimagic labeling of $((P_5 \circ K_{2,7}) \cup P_6)$.

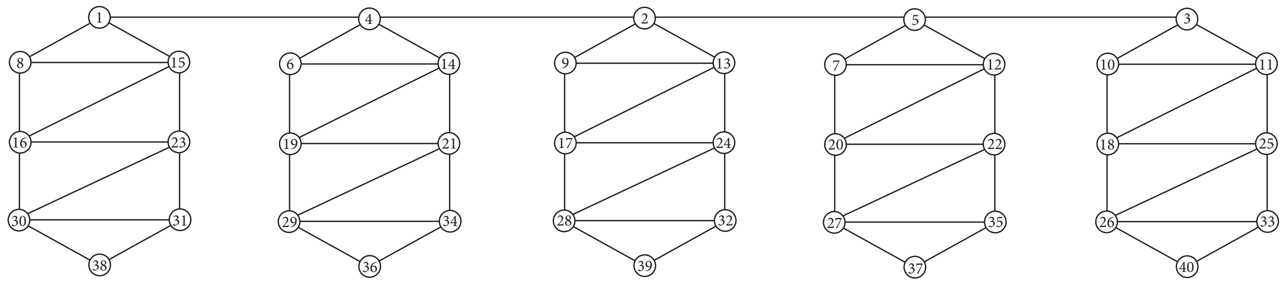

FIGURE 9: A super $(a, 0)$ edge-antimagic labeling of $(P_5 \circ H_1)$ with magic constant $a = 114$.

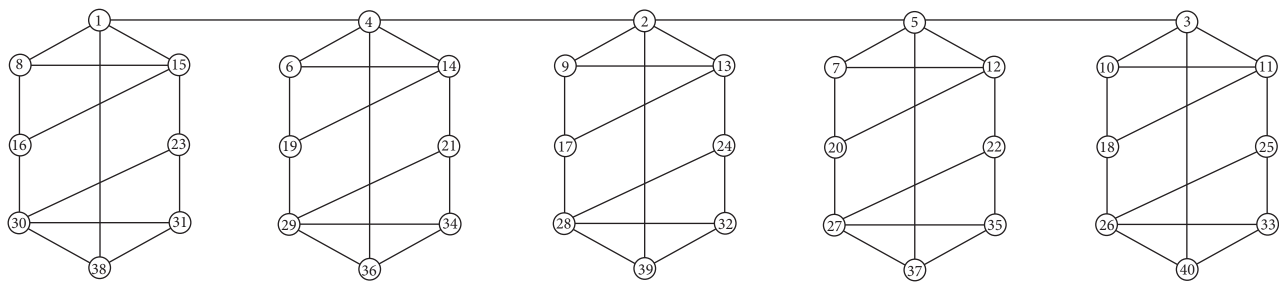

FIGURE 10: A super $(a, 0)$ edge-antimagic labeling of $P_5 \circ H_2$ with magic constant $a = 114$.

TABLE 1: Summary of the main findings.

Graph	Parameters	Magic constant $(a\,(d = 0))$	Minimum edge weight $(a'\,(d = 2))$	Structure
$(P_n \circ K_{2,m}) \cup K_{1,n} \cup (n - 1/2)K_1$	$(m \geq 2, 3 \leq n \equiv 1 \pmod 2)$	$(3mn + 8n + 2)$	$(mn + 6n + 4)$	Planar
$((P_n \circ K_{2,m}) \cup nP_2)$	$(m \geq 2, 3 \leq n \equiv 1 \pmod 2)$	$((6mn + 17n + 3)/2)$	$((2mn + 13n + 7)/2)$	Planar
$((P_n \circ K_{2,m}) \cup P_{n+1})$	$(m \geq 2, 3 \leq n \equiv 1 \pmod 2)$	$((6mn + 13n + 7)/2)$	$((2mn + 9n + 11)/2)$	Planar
$(P_n \circ H_1)$	$(n \equiv 1 \pmod 2)$	$((45n + 3)/2)$	$((17n + 7)/2)$	Planar
$(P_n \circ H_2)$	$(n \equiv 1 \pmod 2)$	$((45n + 3)/2)$	$((17n + 7)/2)$	Planar

4.2. Open Problems. Here, we propose few open problems related to the results (Theorems 2–7) presented in Section 3.1 as follows:

(1) For even $n \geq 2$, can you determine any super $(a, 0)$ edge-antimagic labeling of $((P_n \circ K_{2,m}) \cup K_{1,n} \cup (n - 1/2)K_1)$?

(2) For even $n \geq 2$, can you determine any super $(a, 0)$ edge-antimagic labeling of $((P_n \circ K_{2,m})) \cup nP_2)$?

(3) For even $n \geq 2$, can you determine any super $(a, 0)$ edge-antimagic labeling of $((P_n \circ K_{2,m}) \cup P_{n+1})$?

(4) For odd $n \geq 3$, can you determine super $(a, 0)$ edge-antimagic labeling of the graphs G_1, G_2, G_3, G_4, G_5, and G_6 for a magic constant (i.e., for any other value of a) other than those obtained in this article?

(5) For $l, m, n \in \mathbb{N}$, find any super $(a, 0)$ edge-antimagic labeling of the following graphs:

(1) $((P_n \circ K_{2,m}) \cup K_{1,\ell})$
(2) $((P_n \circ K_{2,m}) \cup lP_2)$
(3) $((P_n \circ K_{2,m}) \cup P_l)$

The open problems related to the results (Theorems 11 and 12) presented in Section 3.2 are as follows:

(1) For even $n \geq 2$, can you determine any super $(a, 0)$ edge-antimagic labeling of the rooted product $(P_n \circ H_1)$?

(2) For even $n \geq 2$, can you determine any super $(a, 0)$ edge-antimagic labeling of the rooted product $(P_n \circ H_2)$?

(3) For odd $n \geq 3$, can you determine any super $(a, 0)$ edge-antimagic labeling of the rooted products $(P_n \circ H_1)$ and $(P_n \circ H_2)$ with a magic constant that is different from those obtained here, that is, for any other value of a

5. Synopsis and 3D Graphical Trends and Relative Comparison of the Magic Constants

5.1. Synopsis. The computational results of our findings have been summarized in Table 1. The "Parameters" column represents those parameters for which we have been able to find super $(a, 0)$ and $(a', 2)$ edge-antimagic labeling of the corresponding graph.

5.2. 3D Graphical Comparison. The following are the 3D plot representations of the magic constants obtained in our

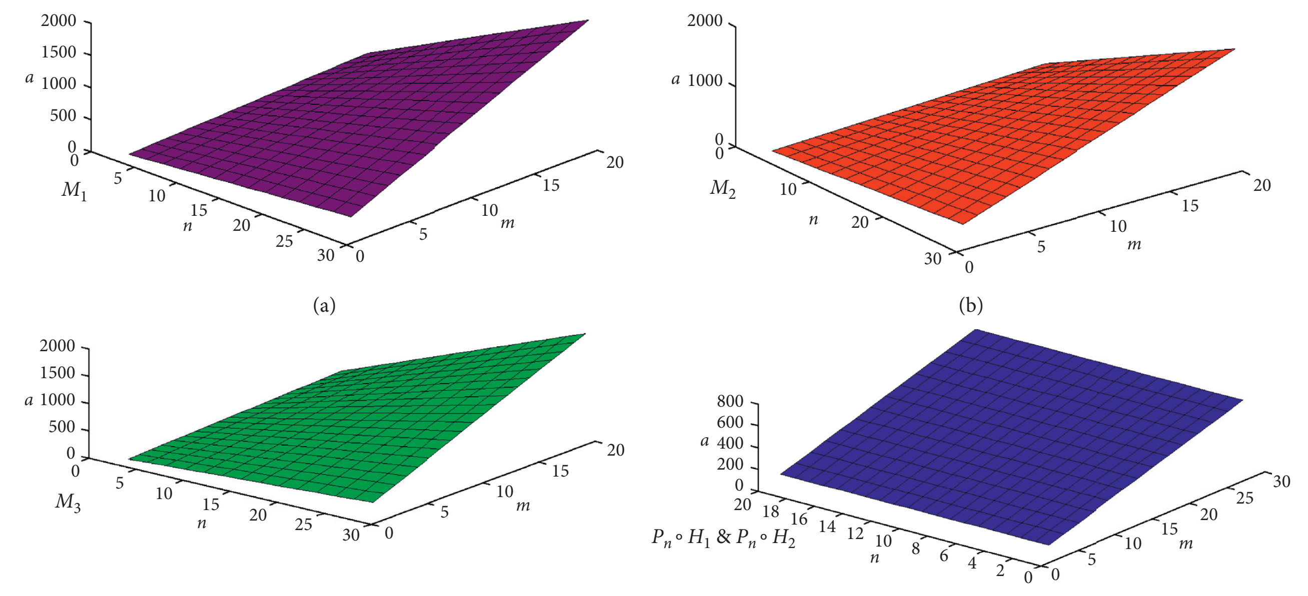

Figure 11: The 3D plots of the magic constants of $M_1 \cong (P_n \circ K_{2,m}) \cup K_{1,n} \cup (n-1/2)K_1$, $M_2 \cong (P_n \circ K_{2,m}) \cup nP_2$, ($M_3 \cong (P_n \circ K_{2,m}) \cup P_{n+1}$), and $(P_n \circ H_1)$ and $(P_n \circ H_2)$.

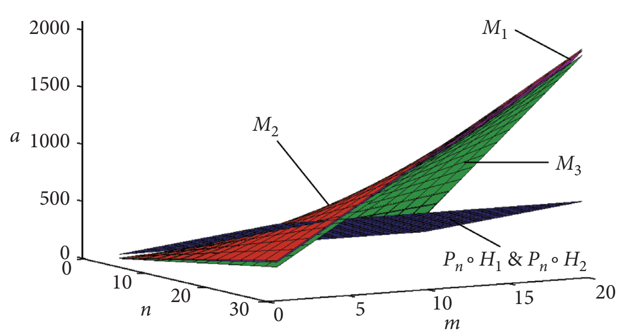

Figure 12: The comparison of the values of magic constants of M_1, M_2, and M_3, as well as $(P_n \circ H_1)$ and $(P_n \circ H_2)$.

results against the parameters $n \leq 30$ and $m \leq 20$, shown in Figure 11 (a represents magic constant in the plots). Meanwhile, Figure 12 shows the comparison of these magic constants against the said values of the parameters. Among these magic constants, the most dominant magic constant, especially in the comparison of M_1, M_2, and M_3, is of the graph M_2 (as red layer is the dominant one). It means that, among the magic constants of the networks discussed here, the disjoint union of the rooted product $(P_n \circ K_{2,m})$ and nP_2 is having the highest values.

6. Conclusion

In this note,

(1) we have obtained a super $(a,0)$ edge-antimagic labeling of the rooted product of P_n with the complete bipartite graph $(K_{2,m})$ taking its union with path, copies of paths, and star for possible values of n. The obtained results partially point towards the problem proposed in [46] regarding $nK_{2,m}$;

(2) we have presented a super $(a,0)$ edge-antimagic labeling of the rooted product of P_n with special pancyclic graphs H_1 and H_2;

(3) importantly, all the resultant graphs of the rooted products here are planar graphs;

(4) several open problems have also been proposed in this article for further working in this area;

(5) The obtained schemes are now all set to serve as test-ready labelings for engineers, programmers, and networking professionals to utilize them where they find them suitable.

Data Availability

The whole data are included within this article. However, more details on the data can be obtained from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

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