Research Article

Electromechanical Model of Coupling Spring Piezoelectric Oscillator with Loading

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The paper studies the influence of piezoelectric materials and structural parameters on the electromechanical conversion characteristics of the piezoelectric vibrator. Employing the mechanical theory of Euler–Bernoulli beam, the mechanical vibration and circuit balance equation are obtained about the single-step variable cross-section bimorph piezoelectric vibrator under the condition of external resistance, and the analytical equations are derived for the displacement, output voltage, and conversion efficiency of the piezoelectric vibrator excited by near natural frequency simple harmonic support. The validity of the theoretical model is verified by vibration test. Theoretical analysis and experimental results show that the mechanical and electrical conversion efficiency of the piezoelectric vibrator is the highest when the circuit impedance is matched, and the mechanical and electrical conversion efficiency of the piezoelectric vibrator is determined by the resistance ratio caused by mechanical and electrical coupling after resistance optimization.

1. Introduction

Oscillating float wave force structure can obtain energy by means of the oscillating motion of the floating body on the sea surface under the action of waves, which is easy to be combined with various forms of transmission and power generation systems. There are many energy conversion modes, and the oscillating float is the most suitable type of wave energy device to combine with piezoelectric power generation to realize continuous power supply for offshore equipment. The pendulum device is more suitable for wave energy utilization in shallow water near shore. Many offshore buoys and sensor and transmitting network nodes adopt the float structure. The research on the oscillating float wave piezoelectric power generation device can better promote the application of the existing offshore equipment technology. At the same time, the device has the characteristics of simple structure, low cost, and easy modularization.

There is a lot of research on oscillatory piezoelectric generation technology. In order to harvest ambient renewable or recovery energy, Hu et al. [1] presented a novel piezoelectric energy harvester which mainly includes vibrators, a rotor, and a cylindrical outer casing. Kumar et al. [2] focused on the flexible hybrid piezoelectric-thermolectric generator for harnessing electrical energy from mechanical and thermal energy, and the research is to integrate thermoelectric and piezoelectric in a small flexible device to harvest electrical energy from both thermal and mechanical energies. Krech et al. [3] investigated the effect of compliant layers between piezoelectric discs and made conclusions for the effect of compliant layers within piezoelectric composites on power generation providing electrical stimulation in low-frequency applications. In the study, piezoelectric energy generators based on spring and inertial mass, inertial mass-based piezoelectric energy generators with and without a spring were designed and tested [4]. Liu et al. [5] studied development of the environmental-friendly BZT-BCT/P(VDF-TrFE) composite film for the piezoelectric generator. Coelho et al. [6] evaluated the output load effect on a piezoelectric energy harvester and presented the study of a vibration energy harvesting system using the piezoelectric
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2.1. Equation of Motion. The cantilever bimorph piezoelectric oscillator with single step and variable cross section is shown in Figure 1. The piezoelectric oscillator is composed of a layer of the substrate (cantilever beam) and two layers of the piezoelectric film. The latter is pasted on the upper and lower surfaces of the former, respectively. The base plate and the piezoelectric film are aligned at the root of the cantilever beam with equal width and unequal length. The substrates are metal plates or epoxy resin and other polymer plates. Metal films are deposited on the upper and lower surfaces of piezoelectric films, and the thickness of the films is very small. The influence of the metal film is not considered in the study, and it is assumed that it has ideal conductivity. A wire is drawn from the metal film and connected to the resistance of the external circuit.

As shown in Figure 1, the origin O is located at the center of the clamping surface of the piezoelectric oscillator. x-axis extends to the free end along the length direction of the piezoelectric oscillator and y-axis along the thickness direction of the piezoelectric oscillator. Piezoelectric oscillator width is b, piezoelectric film length is \( l_p \), thickness is \( h_p \), substrate length is l, thickness is \( h_s \), and external resistance value is \( R \). It is assumed that the piezoelectric oscillator has small amplitude and ignores the influence of shear deformation and moment of inertia during vibration. Based on the vibration equation of Euler–Bernoulli beam, the vibration equation of the piezoelectric oscillator under external force \( F_{\text{beam}}(x,t) \) is obtained [22]:

\[
F_{\text{beam}}(x,t) = -\frac{\partial M^2(x,t)}{\partial x^2} + c_i \frac{\partial w(x,t)}{\partial t} + \mu(x) \frac{\partial w^2(x,t)}{\partial t^2},
\]

where \( t \) is the time, \( M(x,t) \) is the mass of the piezoelectric oscillator which is considered.

Unit length mass of the piezoelectric oscillator is

\[
\mu(x) = \mu_1 [H(x) - H(x - l_p)] + \mu_2 [H(x - l_p) - H(x - l)],
\]

where \( H(x) \) is a unit step function whose function expression is

\[
H(x - x_0) = \begin{cases} 0, & x < x_0, \\ 1, & x \geq x_0. \end{cases}
\]

\[
\mu_1 = b(p_h h_s + 2p_p h_p) \quad \text{and} \quad \mu_2 = b p_h h_s,
\]

are the unit length mass of the piezoelectric film covered by the piezoelectric oscillator and the uncovered section, respectively. \( p_h \) and \( p_p \) are the material density of the substrate and the piezoelectric film, respectively.

Cross-section bending moment of the piezoelectric oscillator is

\[
M(x,t) = b \int_{(-h/2)}^{(h/2)} \Gamma^1 y dy + b \left[ \int_{(-h/2)}^{(h/2)} \Gamma^p y dy + \int_{(-h/2)}^{(h/2)+h_s} \Gamma^p y dy \right] [H(x) - H(x - l_p)],
\]

where \( \Gamma^1 \) is the tensile stress along the x-axis of the cross section of the piezoelectric oscillator.
The superscripts \( s \) and \( p \) correspond to the cross sections of the substrate and the piezoelectric film, respectively. \( S_1 \) is the tensile strain along the \( x \)-axis of the cross section of the piezoelectric oscillator, \( Y_s \) is the elastic modulus of the substrate, \( Y_p \) is the elastic modulus of the piezoelectric film, \( e_{31} \) is the stress constant of the piezoelectric material, and \( E_3 \) is the electric field strength along the thickness direction of the piezoelectric film.

Under the assumption of Euler–Bernoulli beam and small deformation, the cross-sectional strain of the piezoelectric oscillator, \( \Delta x \) is the electric field strength along the thickness direction of the piezoelectric film.

\[
\begin{align*}
\Gamma_i^s &= S_i^s, \\
\Gamma_i^p &= Y_p S_i^p - e_{31} E_3.
\end{align*}
\]

where the voltage of the upper \( v_t \) and lower \( v_b \) about piezoelectric thin films is, respectively, the voltage of the two ends.

Equations (5) and (6) are combined, and joint equation (7) is substituted for equation (4). After integration, the expression of cross-section bending moment \( M(x,t) \) is substituted for equation (1) for partial differential calculation, and the structural vibration control equation of the piezoelectric oscillator is obtained after sorting out.

\[
\begin{align*}
\mu(x) \frac{\partial^2 w(x,t)}{\partial t^2} + c_v \frac{\partial w(x,t)}{\partial t} + YI(x) \frac{\partial^4 w(x,t)}{\partial x^4} - \varrho [v_t(t) + v_b(t)] \left[ \frac{d\delta(x)}{dx} - \frac{d\delta(x-l_p)}{dx} \right] &= F_{\text{beam}}(x,t),
\end{align*}
\]

where \( YI(x) \) is the cross-section flexural rigidity of the piezoelectric oscillator.

\[
YI(x) = YI_1 \left[ H(x) - H(x-l_p) \right] + YI_2 \left[ H(x-l_p) - H(x-l) \right],
\]

where \( \varrho \) is the coefficient of the circuit coupling term

\[
\varrho = \frac{e_{31} b}{2h_p} \left[ \left( \frac{h_p + \frac{h_1}{2}}{2} \right)^2 - \frac{h_2^2}{4} \right],
\]

where \( \delta(x) \) is the Dirac delta function or unit impulse function.

The unit impulse function \( \delta(x) \) is the derivative of the unit step function \( H(x) \). The integral of the product of the \( N \)-order derivative and the arbitrary continuous function \( g(x) \) has the following properties:
2.2. Circuit Equilibrium Equation. In the process of vibration and deformation of the piezoelectric oscillator, dielectric polarization and potential shift occur in the piezoelectric film:

\[ D_3 = e_{31}S_{31}^p + e_{33}E_3, \]

where \( D_3 \) is the potential shift on the surface of the piezoelectric thin film and \( e_{33} \) is the dielectric constant of piezoelectric thin films.

By Gauss's law, the amount of electricity flowing out of the piezoelectric film at both ends per unit time, i.e., instantaneous current, is

\[ i = \frac{d}{dt} \left( \int_0^l D_3 v(x,b) \, dx \right). \]  

(14)

The expression of electric field intensity \( E_3 \), equation (7), and piezoelectric film strain \( S_{31}^p \), equation (6), are substituted in equation (13). The expression of electric field displacement \( D_3 \) is obtained, and the instantaneous output current of the piezoelectric film on the upper and lower surfaces of the substrate is obtained by integration:

\[
\begin{cases}
    i_v(t) = -\frac{e_{33}bL}{h_p} \frac{dv_v(t)}{dt} - e_{31}h_{pc}b \int_0^l \frac{\partial^3 w(x,t)}{\partial x^2 \partial t} \, dx, \\
    i_b(t) = -\frac{e_{33}bL}{h_p} \frac{dv_b(t)}{dt} + e_{31}h_{pc}b \int_0^l \frac{\partial^3 w(x,t)}{\partial x^2 \partial t} \, dx,
\end{cases}
\]

(15)

where \( h_{pc} = \frac{h_p + h_s}{2} \) is the vertical distance between the central plane of the upper and lower piezoelectric thin film and the neutral plane of the composite beam, and the current output direction of the upper and lower piezoelectric thin film is opposite.

According to equation (15), without considering the influence of the current output direction, the piezoelectric thin film can be compared to the parallel circuit of current source \( i_p \) and capacitor \( C_p \) as shown in Figure 2(a), where the equivalent current source strength is considered.

\[ i_b(t) = e_{31}h_{pc}b \int_0^l \frac{\partial^3 w(x,t)}{\partial x^2 \partial t} \, dx. \]

(16)

Equivalent capacitance

\[ C_p = \frac{e_{33}bL}{h_p}. \]

(17)

When the piezoelectric film on the upper and lower surfaces of the substrate is connected in parallel or in series to the two ends of the external resistance, the corresponding equivalent circuit diagram is shown in Figure 2(b) and Figure 2(c).

According to Kirchhoff’s law, the corresponding circuit equilibrium equation is as follows:

For series connection:

\[ i_v(t) = -i_b(t) = \frac{v(t)}{R}, \]

(18a)

\[ v_v(t) + v_b(t) = v(t). \]

Parallel connection:

\[ -i_b(t) + i_v(t) = \frac{v(t)}{R}, \]

(18b)

\[ v_v(t) = v_b(t) = v(t). \]

According to equations (15), (18a), and (18b), the voltage \( v_v(t) \) and \( v_b(t) \) at both ends of the piezoelectric film on the upper and lower surfaces of the bicrystalline piezoelectric oscillator substrate with the same material and structure size and the correlation between the voltage \( v(t) \) at both ends of the resistance of the external circuit and the vertical displacement \( w(x,t) \) of the oscillator are obtained.

Series connection:

\[
\begin{align}
    v_v(t) &= v_b(t) = \frac{1}{2} v(t), \\
    \frac{1}{2} C_p \frac{dv(t)}{dt} + \frac{1}{R} v(t) &= -e_{31}h_{pc}b \int_0^l \frac{\partial^3 w(x,t)}{\partial x^2 \partial t} \, dx.
\end{align}
\]

(19a)

Parallel connection:

\[
\begin{align}
    v_v(t) &= v_b(t) = v(t), \\
    C_p \frac{dv(t)}{dt} + \frac{1}{2R} v(t) &= -e_{31}h_{pc}b \int_0^l \frac{\partial^3 w(x,t)}{\partial x^2 \partial t} \, dx.
\end{align}
\]

(19b)

2.3. Modal Analysis. The vertical displacement function \( w(x,t) \) of the piezoelectric oscillator is expanded by the modal method, and \( N_w \) terms are intercepted [22]:

\[ w(x,t) = \sum_{j=1}^{N_w} \phi_j(x)q_j(t), \]

(20)

where \( q_j \) is the \( j \)-th order modal coordinate and \( \phi_j \) is \( j \)-th order modal of the piezoelectric oscillator under the short-circuit boundary condition, which satisfies the equation

\[ YI(x) + \int_{-\infty}^{\infty} \frac{\partial^4 \phi_j(x)}{\partial x^4} \, dx = -\mu(x)\omega_j^2 \phi_j(x) = 0, \]

(21)

where the \( \omega_j \) state is the \( j \)-th order natural frequency of the piezoelectric oscillator under the short-circuit boundary condition, and the expression of mode \( \phi_j(x) \) is as follows:
Figure 2: (a) Equivalent circuit of the single-layer piezoelectric thin film; (b) equivalent circuit of the parallel piezoelectric thin film; (c) equivalent circuit of the series piezoelectric thin film.

\[
\begin{align*}
\phi_1(x) &= \phi_{1i}(x) [H(x) - H(x - l_p)] + \phi_{2i}(x) [H(x - l_p) - H(x - l)], \\
\phi_{1j}(x) &= C_{11j} \cosh(\sigma_{1j} x) + C_{12j} \sinh(\sigma_{1j} x) + C_{13j} \cos(\sigma_{1j} x) + C_{14j} \sin(\sigma_{1j} x), \\
\phi_{2j}(x) &= C_{21j} \cosh(\sigma_{2j} x) + C_{22j} \sinh(\sigma_{2j} x) + C_{23j} \cos(\sigma_{2j} x) + C_{24j} \sin(\sigma_{2j} x).
\end{align*}
\]

Modal eigenvalue

\[
\sigma_{ij} = \sqrt{\nu_{ij} \omega_{ij}^2 YI_1}, \quad \sigma_{1j} = \sqrt{\nu_{1j} \omega_{1j}^2 YI_1}, \quad \sigma_{2j} = \sqrt{\nu_{2j} \omega_{2j}^2 YI_2}.
\]

The modal coefficients \(C_{11j} \sim C_{24j}\) are determined by the boundary condition formula, section deformation compatibility formula, and mass normalization condition:

\[
\begin{align*}
\phi_{1j}(0) &= 0, \\
\frac{d\phi_{1j}(x)}{dx} \bigg|_{x=0} &= 0, \\
\frac{d^2\phi_{1j}(x)}{dx^2} \bigg|_{x=l} &= 0, \\
\frac{d^3\phi_{2j}(x)}{dx^3} \bigg|_{x=l} &= 0,
\end{align*}
\]

\[
\begin{align*}
\phi_{1j}(l_p) &= \phi_{2j}(l_p), \\
\frac{d\phi_{1j}(x)}{dx} \bigg|_{x=l_p} &= \frac{d\phi_{2j}(x)}{dx} \bigg|_{x=l_p}, \\
YI_1 \frac{d^3\phi_{1j}(x)}{dx^3} \bigg|_{x=l_p} &= YI_2 \frac{d^3\phi_{2j}(x)}{dx^3} \bigg|_{x=l_p},
\end{align*}
\]

\[
\sum_{j=1}^{m} \mu(x) \phi_j(x)^2 \, dx = 1.
\]

It can be proved that mode \(\phi_j(x)\) has the following orthogonal properties:

\[
\begin{align*}
\int_0^l \phi_i(x) \mu(x) \phi_j(x) \, dx &= \delta_{ij}, \\
\int_0^l \phi_i(x) YI(x) \frac{d^4 \phi_j(x)}{dx^4} \, dx &= \omega_j^2 \delta_{ij},
\end{align*}
\]

where \(\delta_{ij}\) is the Kronecker delta function, while \(i = j, \delta_{ij} = 1\); otherwise, \(\delta_{ij} = 0\).

Modal expansion equation (20) of the vertical displacement function \(w(x, t)\) of the piezoelectric oscillator is substituted into the vibration control equation of the piezoelectric oscillator. Equation (8) multiplies the items on both sides of the equation by \(\phi_j\), and integrates them along the \(x\)-axis from 0 to \(l\) according to the integral characteristics of Dirac delta function and the orthogonality of the modes, equations (12) and (3)–(27) are applied. The ordinary differential equation satisfied by \(j\)-th order modal coordinate \(q_j(t)\) is obtained as follows:

\[
\begin{align*}
\frac{d^2 q_j(t)}{dt^2} + 2 \xi \omega_j \frac{dq_j(t)}{dt} + \omega_j^2 q_j(t) - \theta_j v(t) &= 0, \\
\int_0^l F_{beam}(x, t) \phi_j(x) \, dx &= 0,
\end{align*}
\]

where \(\theta_j\) is the \(j\)-th coefficient of the modal electromechanical coupling term.

When the piezoelectric oscillator is output in series, \(\theta_j\) is obtained as follows:

\[
\theta_j = \frac{d \phi_j(x)}{dx} \bigg|_{x=l_p}. 
\]
The voltage \( v(t) \) and modal coordinates \( q_j(t) \) of the resistors at both ends of the external circuit satisfy the equation

\[
\frac{1}{2} C_p \frac{dv(t)}{dt} + \frac{1}{R} v(t) + \sum_{j=1}^{N_v} \chi_j \frac{dq_j(t)}{dt} = 0,
\]

where the modal coupling coefficient \( \chi_j \) of the circuit equation

\[
\chi_j = e_{31} h_{pc} b \int_0^l \frac{d^2 \phi_j(x)}{dx^2} dx = e_{31} h_{pc} b \frac{d\phi_j(x)}{dx} \bigg|_{x=i_p}.
\]

When the piezoelectric oscillator is output in parallel, the derivative of modal coupling coefficient of modal equation (31) is

\[
\theta_j = 2 \phi_j \frac{d\phi_j(x)}{dx} \bigg|_{x=i_p}.
\]

The voltage \( v(t) \) and modal coordinates \( q_j(t) \) of the resistors at both ends of the external circuit satisfy the equation

\[
C_p \frac{dv(t)}{dt} + \frac{1}{2R} v(t) + \sum_{j=1}^{N_v} \chi_j \frac{dq_j(t)}{dt} = 0.
\]

The expression of modal coupling coefficient \( \chi_j \) for the equation of the parallel output circuit is the same as equation (31).

### 2.4. Analytical Solution of Simple Harmonic Vibration of Near Natural Frequency Bearing

Under the simple harmonic excitation of the bearing whose vibration amplitude is \( \omega_0 \) and frequency is \( \omega_0 \),

\[
F_{beam}(x,t) = F_0(x) \sin(\omega_0 t),
\]

\[
F_0(x) = a_0 \omega_0^2 \mu(x).
\]

When the vibration frequency \( \omega_0 \) is close to the \( j \)-th order short-circuit natural frequency \( \omega_j \) of the piezoelectric oscillator, the secondary mode shapes other than the \( j \)-th order modes of modal equation (20) are neglected:

\[
w(x,t) = \phi_j(x) q_j(t),
\]

and there are two forms of the circuit equilibrium equations, one is series connection, the other is parallel connection.

**Series connection:**

\[
\frac{1}{2} C_p \frac{dv(t)}{dt} + \frac{1}{R} v(t) + \sum_{j=1}^{N_v} \chi_j \frac{dq_j(t)}{dt} = 0.
\]

**Parallel connection:**

\[
C_p \frac{dv(t)}{dt} + \frac{1}{2R} v(t) + \sum_{j=1}^{N_v} \chi_j \frac{dq_j(t)}{dt} = 0.
\]

Simultaneous equation, equations (28), (36a), and (36b) are solved to obtain the modal coordinates of the piezoelectric oscillator:

\[
\begin{align*}
q_j(t) &= \frac{a_0 \omega_0^2 \sqrt{l^2 + \Omega^2}}{\sqrt{\Omega^2 (1 - \Omega^2 + 2(\lambda + \gamma)^2 + (\lambda - 2\Omega^2 - \lambda\Omega^2)^2)}} \sin(\omega_j t + \theta_j), \\
\theta_j &= \arctan\left(\frac{\lambda - 2\Omega^2 - \lambda\Omega^2}{\Omega^2 (1 - \Omega^2 + 2(\lambda + \gamma)^2 + (\lambda - 2\Omega^2 - \lambda\Omega^2)^2)}\right) - \frac{\pi}{2}.
\end{align*}
\]

Voltage at both ends of resistance \( R \) follows as

\[
\begin{align*}
\gamma &= \frac{\lambda}{\omega_j R C_p}, \\
\lambda &= \frac{\omega_j RC_p}{\omega_0}, \\
\omega_0 &= \omega_j R C_p
\end{align*}
\]

where \( \gamma \) are connection mode parameters.

When the piezoelectric oscillator is output in series, \( \gamma_1 = 2 \) and \( \gamma_2 = 2 \); when output in parallel, \( \gamma_0 = 0.5 \). The conversion efficiency \( \varepsilon \) of the piezoelectric oscillator is the ratio of the output electric energy per unit period to the mechanical work done by the external force,

\[
\varepsilon = \frac{\int_0^T R^{-1} v(t)^2 dt}{\int_0^T F_{beam}(x,t) \dot{w}(x,t) dx dt} = \frac{\gamma \lambda}{\gamma \lambda + 2\lambda^2 + \Omega^2}.
\]

### 2.5. Conversion Efficiency Optimization

Let the conversion efficiency \( \varepsilon \) derive the oscillation period \( \lambda \) of the dimensionless circuit and obtain the resistance value of the external circuit which maximizes the conversion efficiency

\[
R_{opt} = \frac{\gamma_0}{\omega_0 C_p}.
\]

Equation (41) shows that the maximum conversion efficiency which occurs when the external resistance matches the impedance of the piezoelectric oscillator circuit. The conversion efficiency of the piezoelectric oscillator under the action of impedance matching external circuit resistance is as follows:
oscillator are helpful to improve the electromechanical damping. Optimizing the material and structural parameters including the electro-induced damping and the viscous characterization to the total damping of the structural vibration, depends on the ratio of the structural vibration damping conversion efficiency of the piezoelectric oscillator mainly satisfies impedance matching, the electromechanical conversion efficiency of the piezoelectric oscillator.

Equation (42) shows that when the piezoelectric oscillator satisfies impedance matching, the electromechanical conversion efficiency of the piezoelectric oscillator mainly depends on the ratio of the structural vibration damping caused by the mechanical-electrical coupling of the characterization to the total damping of the structural vibration, including the electro-induced damping and the viscous damping. Optimizing the material and structural parameters of the piezoelectric oscillator and improving the electromechanical coupling characteristics of the piezoelectric oscillator are helpful to improve the electromechanical conversion efficiency of the piezoelectric oscillator.

3. Model Validation

3.1. Test Equipment. The piezoelectric oscillator is composed of aluminized piezoelectric polymer (PVDF) and phosphorus bronze substrate. The piezoelectric film on the substrate is connected in parallel with pure resistance. The fixed end of the piezoelectric oscillator is connected with the exciter. Firstly, the signal generator generates a set of frequency excitation signal, then amplifies the signal power through the power amplifier, and finally transmits it to the exciter to excite the piezoelectric oscillator. The excitation displacement signal generated by the exciter is picked up by the current sensor installed above the excitation end and displayed by the oscilloscope. By adjusting the power amplification factor, the excitation amplitude at the excitation end is kept constant at different excitation frequencies. Laser displacement sensor is fixed above the free end of the piezoelectric oscillator to collect the displacement response of the end of the piezoelectric oscillator. Voltage at both ends of the external resistance is collected by the oscilloscope. The oscilloscope is an analog signal channel with a maximum acquisition frequency of 100 MHz and a resolution of 1 ns.

3.2. Theoretical and Experimental Results. Under the harmonic excitation of 0.12 mm displacement of the support seat of the piezoelectric oscillator, the relationship between the amplitude of structural vibration response and the voltage amplitude at both ends of the external resistance and the excitation frequency was tested under the condition of external resistance value 9.9 kΩ, 90.9 kΩ, 1.06 MΩ, 3.37 MΩ, and 10 MΩ. As shown in Figure 3(a), for the tested piezoelectric oscillator, the change of resistance has little effect on the structural vibration response of the piezoelectric oscillator, and the structural vibration response amplitude under each excitation value almost overlap. With the increase of vibration frequency, the structural response increases first and then decreases and reaches resonance at about 29 Hz. However, as shown in Figure 3(b), the output voltage at both ends of the external resistance increases with the increase of the value of the external resistance. With the increase of vibration frequency, the output voltage increases first and then decreases. The relationship between output power and external resistance is shown in Figure 4. As can be seen from the figure, with the increase of external resistance, the output power increases first and then decreases. There is an optimal load to maximize the output power. The theoretical and experimental comparison is shown in Figures 3 and 4. As can be seen from Figures 3 and 4, the theoretical calculation is in good agreement with the experimental results. The comparison results fully illustrate the accuracy of the theoretical model.

4. Electromechanical Coupling Optimization

The simultaneous equations (11), (29), (31), and (32) are substituted into the expression of dimensionless electromagnetic coupling coefficient $\gamma$. After expansion, the following results can be obtained:

$$
\gamma = \frac{\varepsilon_{33}}{\varepsilon_{31}} \frac{h_p (h_s + h_p)}{2 \omega^2 l_p} \left( \frac{d\phi_s(x)}{dx} \right)^2 |_{x=d_f}^2.
$$

(43)

From equation (23), there is

$$
\omega^2 = \frac{\sigma_{33}^2 Y_I}{\mu_1}.
$$

(44)

By substituting equation (44) into the latter (43), the latter can be obtained after sorting out.

$$
\gamma = \frac{\varepsilon_{31}}{\varepsilon_{33}} \frac{h_p (h_s + h_p)}{2 \omega^2 l_p} \left( \frac{d\phi_s(x)}{dx} \right)^2 |_{x=d_f}^2 = k_{31} \cdot \left( \frac{Y_p}{Y_s} \mu_1 \frac{\rho_p}{\rho_s} \frac{l_p}{l} \right)^2.
$$

(45)

where $k_{31} = \varepsilon_{31}^2 / (\varepsilon_{33} Y_p)$ is the electromechanical coupling factor of the piezoelectric film, $k_{31}$ is the electromechanical coupling coefficient of piezoelectric material, $s$ is the material and structure influence factor of the piezoelectric oscillator (hereinafter referred to as the influence factor), and its size is related to the modulus ratio $Y_p/Y_s$, density ratio $\rho_p/\rho_s$, length ratio $l_p/l$, and thickness ratio $h_p/h_s$ of the piezoelectric film and the substrate.

For most piezoelectric materials, the general range of variation $k_{31}$ is 0.05–0.4, where piezoelectric ceramics are generally 0.3–0.4, and piezoelectric polymers are usually 0.12. Comparatively speaking, piezoelectric ceramics have stronger electromechanical coupling effect. The following is a study of the relationship between the influence factor $s$ and

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Substrate</th>
<th>Piezoelectric thin film</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_s$(cm)</td>
<td>14.5</td>
<td>10.0</td>
</tr>
<tr>
<td>$B$(cm)</td>
<td>3.9</td>
<td>3.9</td>
</tr>
<tr>
<td>$H$(cm)</td>
<td>1.03</td>
<td>0.50</td>
</tr>
<tr>
<td>$Y$(GPa)</td>
<td>113</td>
<td>2.5</td>
</tr>
<tr>
<td>$\rho$(kg/m$^3$)</td>
<td>8800</td>
<td>1760</td>
</tr>
<tr>
<td>Piezoelectric strain constant (pc/N)</td>
<td>—</td>
<td>11</td>
</tr>
<tr>
<td>Relative permittivity</td>
<td>—</td>
<td>9.5</td>
</tr>
</tbody>
</table>
the modulus ratio $Y_p/Y_s$, density ratio $\rho_p/\rho_s$, length ratio $l_p/l$, and thickness ratio $h_p/h_s$ of the piezoelectric thin film and the substrate and the relationship between the maximum influence factor $s_{\text{max}}$ and the corresponding optimum length ratio $(l_p/l)_{\text{opt}}$, thickness ratio $(h_p/h_s)_{\text{opt}}$, and modulus ratio and density ratio under the conditions of comprehensive optimization of various parameters. In the study, the first-order mode is taken as the research object.

4.1. Modulus Ratio and Density Ratio. Figure 5 shows the length ratio $l_p/l = 1$ and thickness ratio $h_p/h_s = 1$, and the influence factor $s$ varies with the modulus ratio and density ratio of the piezoelectric film and the substrate. It can be seen from the figure that the influence factor increases with the increase of modulus ratio and changes very slightly with the change of density ratio. When the modulus ratio changes from 0.12 to 0.56, the maximum influence factor can be obtained by adjusting the density ratio under this modulus ratio condition, which increases nearly four times. Among the influence factors of material parameters, modulus density ratio plays a dominant role.

4.2. Length Ratio and Thickness Ratio. Figure 6 shows the dependence of the influence factor $s$ on the length ratio $(l_p/l)$
and thickness ratio \((h_p/\rho_s)\) of the piezoelectric film to the substrate when the modulus ratio \(Y_p/Y_s = 2.5\) and density ratio \(\rho_p/\rho_s = 1.55\). It can be seen from the figure that the influence factor increases first and then decreases with the increase of length ratio and thickness ratio. When \(l_p/l = 0.68\) and \(h_p/h_s = 0.38\), the influence factor has the maximum value \(s = 0.68\). With the increase of thickness ratio, the length ratio of the maximum influencing factor under the condition of thickness ratio increases gradually, and the increasing speed is faster initially and then slows down gradually. With the increase of length ratio, the thickness ratio of the maximum influencing factor under the condition of length ratio increases gradually, and the increase rate is slow initially. When \(l_p/l > 0.55\), the increase rate is slightly faster.

### 4.3. Comprehensive Effect of Parameters

Figure 7 shows the relationship between the maximum influence factor \(s_{\text{max}}\) obtained by optimizing the length and thickness ratio and the modulus ratio and density ratio of the piezoelectric thin film and the substrate.

Figure 8 shows the optimal length ratio \((l_p/l)_{\text{opt}}\) for modulus ratio and density ratio.

Figure 9 shows the relationship between the optimum thickness ratio \((h_p/h_s)_{\text{opt}}\) and modulus ratio and density ratio. On the whole, the influence factor of modulus ratio is more significant, and the influence of density ratio is weaker.

Figure 8 shows the optimal length ratio \((l_p/l)_{\text{opt}}\), even if the influence factor reaches the maximum length ratio, with the change of modulus ratio and density ratio. It can be seen from the figure that the optimum length ratio increases with the increase of modulus ratio and decreases with the increase of density ratio. The magnitude of increase or decrease is very limited, and the overall range of change is between 0.55 and 0.75. Figure 9 shows the optimum thickness ratio \((h_p/h_s)_{\text{opt}}\), even if the influence factor reaches the maximum thickness ratio, with the change of modulus ratio and density ratio. It can be seen from the figure that the optimum thickness ratio decreases with the increase of modulus ratio, the relationship between the optimum thickness ratio and density ratio is more complex, and there is a nonsingular corresponding relationship under most conditions. Generally speaking, the effect of modulus density ratio on the optimum thickness ratio is more significant.

### 5. Conclusions

Based on the Euler–Bernoulli beam theory, the paper presented the structural vibration equation and the circuit balance equation of the piezoelectric vibrator with external pure resistance. The analytical expressions of the vibration response and the voltage at both ends of the external...
resistance are obtained by employing the modal method under the condition of the near natural frequency bearing simple harmonic vibration, and the model is verified by experiments. The following conclusions are drawn:

(1) Theoretical analysis shows that when the circuit impedance is matched, the electromechanical conversion efficiency of the piezoelectric vibrator is the highest, and the electromechanical conversion efficiency of the piezoelectric vibrator after resistance optimization is determined by the ratio of the electric damping caused by electromechanical coupling to the total damping of the vibrator.

(2) The electric coupling coefficient of the piezoelectric vibrator is composed of the electromechanical coupling factor of the piezoelectric film and the material and structure influence factor of the piezoelectric vibrator.

(3) With the increase of the ratio of piezoelectric to substrate modulus, the influence factor increases. The density ratio of piezoelectric to substrate has little effect on the influencing factors. With the increase of piezoelectric to substrate length ratio and thickness ratio, the influence factor increases first and then decreases. There is an optimal length ratio and thickness ratio to maximize the influence factor.

The research results can be adapted to guide the design of the piezoelectric vibrator.

Data Availability
As our paper is based on government funded projects, we need to follow the policies of its institutions and funders. According to the agreements and requirements signed between us and relevant government institutions, we cannot share relevant data and materials with the outside world. If we need to publish them, we must obtain their authorization.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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