Research Article

Nonlinear Vibration of the Blade with Variable Thickness

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In this paper, the nonlinear dynamic responses of the blade with variable thickness are investigated by simulating it as a rotating pretwisted cantilever conical shell with variable thickness. The governing equations of motion are derived based on the von Kármán nonlinear relationship, Hamilton's principle, and the first-order shear deformation theory. Galerkin's method is employed to transform the partial differential governing equations of motion to a set of nonlinear ordinary differential equations. Then, some important numerical results are presented in terms of significant input parameters.

1. Introduction

Rotating blades can be found in different engineering problems as the key component of aircraft engines. Certain important design considerations require a thorough understanding of the structural dynamic characteristics of rotating blades. Extensive uses of pretwisted cantilever conical shells can be found in different engineering disciplines. The dynamic behaviors of the shell are significant for the analysis of the whole structure of blades.

The vibrations of different kinds of shells have been studied. Qinkai and Fulei [1] analyzed the instability of a rotating truncated conical shell subjected to periodic axial loads. Malekzadeh and Heydarpour [2] investigated the free vibration behaviors of rotating FG truncated conical shells with different edge conditions. Goldberg et al. [3] and Yang [4] analyzed the vibration of the conical shell by numerical integration. Srinivasan and Krishnan [5] presented an analysis of the dynamic behaviors for the conical panel using an integral equation technique. Liew et al. [6] and McGee and Chu [7] have investigated the vibration characteristics of pretwisted conical shells by the Ritz method while Lee et al. [8] have analyzed the vibration characteristics of twisted cantilevered conical composite shells by using the finite element method. Naidu and Sinha [9] studied the nonlinear free vibration of laminated composite shells in hygrothermal environment incorporating the Green–Lagrange-type

nonlinear strains. Kumari and Sinha [10] demonstrated the hygrothermal effects on the behavior of composite T-joints using a modified thick shell element. Ghosh [11] analyzed the hygrothermal effects on the initiation and propagation of damage in composite shells due to low-velocity impacts. Bandyopadhyay et al. [12] investigated the hygrothermal effects on the free vibration characteristics of delaminated carbon-epoxy composite pretwisted rotating conical shells. Hwu et al. [13] presented a modified FSDT model for obtaining closed-form solutions of natural frequencies for certain particular problems of sandwich plates and shells. Garg et al. [14] presented the closed form solutions based on the HSDT to study the free vibration characteristics of doubly curved sandwich shells. Rahmani et al. [15] proposed a sandwich plate theory that employs the FSDT for composite face-sheets and the HSDT for flexible cores to study the free vibration characteristics of sandwich cylindrical shells. Kumar et al. [16] employed an efficient FE model based on the higher-order zig-zag theory to solve the free vibration problem of multilayered composites and sandwich shallow shells. The study of free vibration characteristics of isotropic and multilayered composite shallow conical shells with pretwists have been carried out by Lim et al. [17] and Lee et al. [8] using the FSDT. Sofyev [18] investigated the free vibration characteristics of laminated orthotropic conical shells using the modified FSDT. The vibration and buckling of functionally graded material (FGM) conical shells using

various theories has been reported by Sofyev [19]. Wilkins et al. [20] investigated the symmetrical and unsymmetrical vibration modes of sandwich conical shells with various boundary conditions. Bardell et al. [21] developed an FE model to study the vibration characteristics of sandwich conical panels and carried out an experiment to verify the model. Nasihatgozar and Khalili [22] analyzed the vibration and buckling of sandwich truncated conical shells using the differential quadrature method. Sofyev and Osmancebioglu [23] studied the free vibration behavior of sandwich truncated conical shells containing FGM coatings using the FSDT. Deb Singha et al. [24] studied the influence of elevated temperature and moisture absorption on the free vibration behavior of rotating pretwisted sandwich conical shells consisting of two composite face sheets and a synthetic foam core. The stress, deformation, and snap-through conditions of thin, axisymmetric, shallow bimetallic shells were investigated by Jakomin and Kosel [25]. da Silva et al. [26] studied the influence of Young's modulus, shell thickness, and geometrical imperfection uncertainties on the parametric instability loads of simply supported axially excited cylindrical shells. Li et al. [27] had a research on the free vibration of the stiffened cylindrical shell with general boundary conditions.

The vibration of the cantilever blade with variable thickness has been less studied because of the mathematical complexity in describing the geometry and nonlinear dynamic behaviors. Thermoelastic behavior of a functionally graded blade with variable thickness subjected to the mechanical and thermal loadings has been investigated by Mirzaei et al. [28, 29].

In this paper, the nonlinear dynamic responses of the blade with variable thickness are investigated by simulating it as a rotating pretwisted cantilever conical shell with variable thickness. The governing equations of motion are derived based on the von Kármán nonlinear relationship, Hamilton's principle, and the first-order shear deformation theory. Galerkin's method is employed to transform the partial differential governing equations of motion to a set of nonlinear ordinary differential equations. Then, the effects of the varying rotating speed, external force, and external moment on the dynamic behavior of the blade are studied.

2. Theoretical Formulations

2.1. Description of Blade. As shown in Figure 1, the blade is regarded as a pretwisted thin-walled rotating cantilever conical shell with varying rotating speed and the thickness. The coordinate system $(x, \theta, \text{ and } z)$ is located in the middle surface of the pretwisted thin-walled cantilever conical shell. The coordinate components $(u, v, \text{ and } w)$ represent the displacements of an arbitrary point in the $x, \theta, \text{ and } z$ directions, respectively. The geometrical sizes of the pretwisted thin-walled cantilever conical shell are represented by the bottom inner radius r_{root} , subtended angle α , presetting angle β , pretwisted angle Φ , semivertex angle ψ , length L , and varying thickness $h = h_0(x)g(\theta)$, respectively. Therefore, we have the radius at any point along the length:

$$R = r_{\text{root}} - x \tan \psi. \quad (1)$$

Other geometric parameters of the pretwisted thin-walled cantilever conical shell are given as

$$h_0(x) = \max\{h(x)\},$$

$$g(\theta) = \frac{f(\theta)}{a},$$

$$a = \max\{f(\theta)\}, \quad (2a)$$

$$f(\theta) = \left(\sin \frac{\alpha}{2} - \sin \theta\right)^{(3/2)} \left(\sin \frac{\alpha}{2} - \sin \theta\right)^{(1/2)}, \quad (2b)$$

$$\theta \in \left[-\frac{\alpha}{2}, \frac{\alpha}{2}\right].$$

The pretwisted blade, which is installed on a rigid hub with the radius r_0 , rotates at the speed $\Omega(t) = \Omega_c + \Omega_v \cos \omega_r t$ about its polar axis [30, 31]. For the reason that the distribution of the load on the blade varies during one rotation, the load on the rotating blade is periodic and can be described as the force $F = F_0 + F_1 \cos \omega_r t$ and moment $M = M_0 + M_1 \cos \omega_r t$ on the upper surface of the pretwisted thin-walled cantilever conical shell. Both of the force F and the moment M include a mean and a simple harmonic component [32].

2.2. Displacement Field and Geometric Equation.

According to the first-order shear deformation theory, the displacement field $(u, v, \text{ and } w)$ of the pretwisted thin-walled cantilever conical shell is written as

$$u(x, \theta, z, t) = u_0(x, \theta, t) + z\varphi_x(x, \theta, t), \quad (3a)$$

$$v(x, \theta, z, t) = v_0(x, \theta, t) + z\varphi_\theta(x, \theta, t), \quad (3b)$$

$$w(x, \theta, z, t) = w_0(x, \theta, t), \quad (3c)$$

where (u_0, v_0, w_0) are, respectively, the midplane displacements in the direction of x, θ , and z for the pretwisted thin-walled cantilever conical shell and φ_x and φ_θ represent the midplane rotations of the transverse normal about the θ and x axes, respectively.

The strain components $(\varepsilon_x, \varepsilon_\theta, \gamma_{x\theta})$ at an arbitrary point of the pretwisted thin-walled cantilever conical shell are related to the middle surface strains $(\varepsilon_x^{(0)}, \varepsilon_\theta^{(0)}, \gamma_{x\theta}^{(0)})$ and to the changes in the curvature and the torsion of the middle surface $(\varepsilon_x^{(1)}, \varepsilon_\theta^{(1)}, \gamma_{x\theta}^{(1)})$. Using von Karman geometric strain-displacement relations, the expressions [33] can be obtained as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \gamma_{x\theta} \end{bmatrix} = \begin{bmatrix} \varepsilon_x^{(0)} \\ \varepsilon_\theta^{(0)} \\ \gamma_{x\theta}^{(0)} \end{bmatrix} + z \begin{bmatrix} \varepsilon_x^{(1)} \\ \varepsilon_\theta^{(1)} \\ \gamma_{x\theta}^{(1)} \end{bmatrix}, \quad (4)$$

where

$$\begin{aligned}
\begin{bmatrix} \varepsilon_x^{(0)} \\ \varepsilon_\theta^{(0)} \\ \gamma_{x\theta}^{(0)} \end{bmatrix} &= \begin{bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \\ \frac{1}{R \cos \psi} \frac{\partial v_0}{\partial \theta} + \frac{1}{R} w_0 + \frac{1}{R} u_0 \tan \psi + \frac{1}{2} \frac{1}{R^2 \cos^2 \psi} \left(\frac{\partial w_0}{\partial \theta} \right)^2 \\ \frac{1}{R \cos \psi} \frac{\partial u_0}{\partial \theta} - \frac{1}{R} v_0 \tan \psi + \frac{\partial v_0}{\partial x} + \frac{1}{R \cos \psi} \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial \theta} \end{bmatrix}, \\
\begin{bmatrix} \varepsilon_x^{(1)} \\ \varepsilon_\theta^{(1)} \\ \gamma_{x\theta}^{(1)} \end{bmatrix} &= \begin{bmatrix} \frac{\partial \varphi_x}{\partial x} \\ \frac{1}{R \cos \psi} \frac{\partial \varphi_\theta}{\partial \theta} + \frac{1}{R} \varphi_x \tan \psi \\ \frac{1}{R \cos \psi} \frac{\partial \varphi_x}{\partial \theta} - \frac{1}{R} \varphi_\theta \tan \psi + \frac{\partial \varphi_\theta}{\partial x} \end{bmatrix}, \\
\begin{bmatrix} \gamma_{\theta z} \\ \gamma_{xz} \end{bmatrix} &= \begin{bmatrix} \varphi_\theta + \frac{1}{R \cos \psi} \frac{\partial w_0}{\partial \theta} - \frac{1}{R} v_0 \\ \frac{\partial w_0}{\partial x} + \varphi_x \end{bmatrix}.
\end{aligned} \tag{5}$$

2.3. Constitutive Relation and Internal Force Relation. The stress-strain relations of the conical shell are expressed as

$$\begin{bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_{x\theta} \\ \sigma_{\theta z} \\ \sigma_{xz} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & & & \\ & Q_{12} & Q_{22} & & \\ & & & Q_{66} & \\ & & & & Q_{44} \\ & & & & & Q_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \gamma_{x\theta} \\ \gamma_{\theta z} \\ \gamma_{xz} \end{bmatrix}, \tag{6}$$

where Q_{mn} ($m, n = 1, 2, 4, 5, 6$) are the stiffness coefficient,

$$\begin{aligned}
Q_{11} &= Q_{22} = \frac{E}{1 - \nu^2}, \\
Q_{12} &= \frac{E\nu}{1 - \nu^2}, \\
Q_{44} &= Q_{55} = Q_{66} = \frac{E}{2(1 + \nu)},
\end{aligned} \tag{7}$$

where E denotes Young's modulus and ν is Poisson's ratio.

2.4. Governing Equations of Motion. The Hamilton principle

$$\int_0^t (\delta u_1 + \delta u_2 + \delta w - \delta K) dt = 0 \tag{8}$$

is employed to derive the governing equations of motion for the pretwisted thin-walled cantilever conical shell with variable thickness, where K , u_1 , u_2 , and W denote the kinetic energy, the strain energy, the centrifugal force potential

energy, and the virtual work of external forces, respectively, t represents time, and δ is the variation operator [34].

According to research given in Tong et al. [35] and König's theorem, the energy relationship between the pretwisted thin-walled rotating cantilever conical shell with variable thickness and that with constant thickness is described as

$$\frac{K}{K_{h_0}} = \frac{(1/2)mr_c^2\Omega^2 + o((1/2)mr_c^2\Omega^2)}{(1/2)m_{h_0}r_{h_0c}^2\Omega^2 + o((1/2)m_{h_0}r_{h_0c}^2\Omega^2)} \tag{9a}$$

$$\approx \frac{m}{m_{h_0}} = \frac{1}{\alpha} \int_{-(\alpha/2)}^{(\alpha/2)} f(\theta) d\theta = K_K,$$

$$\frac{u_2}{u_{2h_0}} = \frac{-(1/2)mr_c^2\Omega^2}{-(1/2)m_{h_0}r_{h_0c}^2\Omega^2} \tag{9b}$$

$$\approx \frac{m}{m_{h_0}} = \frac{1}{\alpha} \int_{-(\alpha/2)}^{(\alpha/2)} f(\theta) d\theta = K_{u_2},$$

$$\begin{aligned}
\frac{U_1}{U_{1h_0}} &= K_{U_1}, \\
\frac{W}{W_{h_0}} &= K_W,
\end{aligned} \tag{9c}$$

where K_{h_0} , u_{1h_0} , u_{2h_0} , and w_{h_0} are, respectively, the kinetic energy, the strain energy, the centrifugal force potential energy, and the virtual work of the conical shell with the constant thickness h_0 and K_K , K_{u_1} , K_{u_2} , and K_w are the constant coefficients.

FIGURE 1: The model of presetting and pretwisted cantilever conical shell with varying thickness.

The inertias of the pretwisted thin-walled cantilever conical shell I_i are calculated by

$$I_i = \int_{-(h_0/2)}^{(h_0/2)} \rho z^i dz, \quad i = 0, 1, 2, 3, 4. \quad (10)$$

The force and moment resultants are given as

$$\begin{bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \\ A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^{(0)} \\ \varepsilon_\theta^{(0)} \\ \gamma_{x\theta}^{(0)} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \\ B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^{(1)} \\ \varepsilon_\theta^{(1)} \\ \gamma_{x\theta}^{(1)} \end{bmatrix}, \quad (11a)$$

$$\begin{bmatrix} M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \\ B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^{(0)} \\ \varepsilon_\theta^{(0)} \\ \gamma_{x\theta}^{(0)} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \\ D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^{(1)} \\ \varepsilon_\theta^{(1)} \\ \gamma_{x\theta}^{(1)} \end{bmatrix}, \quad (11b)$$

$$\begin{bmatrix} Q_x \\ Q_\theta \end{bmatrix} = \begin{bmatrix} A_{44} \\ A_{55} \end{bmatrix} \begin{bmatrix} \gamma_{xz} \\ \gamma_{\theta z} \end{bmatrix}, \quad (11c)$$

where A_{ij} , B_{ij} , and D_{ij} are, respectively, called the extensional, the bending-extensional coupling, and the bending stiffness terms,

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-(h_0/2)}^{(h_0/2)} Q_{ij}(1, z, z^2) dz, \quad i, j = 1, 2, 6, \quad (12a)$$

$$A_{ij} = \int_{-(h_0/2)}^{(h_0/2)} Q_{ij} dz, \quad i, j = 4, 5. \quad (12b)$$

With one end clamped and others free, the boundary conditions of the pretwisted thin-walled rotating cantilever conical shell are written as

$$u_0 = v_0 = w_0 = \varphi_x = \varphi_\theta = 0, \quad x = 0, \quad (13a)$$

$$\begin{aligned} N_{xx} = N_{\theta\theta} = N_{x\theta} = M_{xx} = M_{\theta\theta} = M_{x\theta} = Q_x \\ = Q_\theta = 0, \quad x = L, \end{aligned} \quad (13b)$$

$$\begin{aligned} N_{xx} = N_{\theta\theta} = N_{x\theta} = M_{xx} = M_{\theta\theta} = M_{x\theta} \\ = Q_x = Q_\theta = 0, \quad \theta = -\frac{\pi}{2} \text{ and } \frac{\pi}{2}. \end{aligned} \quad (13c)$$

In general, the first two-mode truncation is accurate enough to analyze the dynamics response as the lower frequencies often play the most important role in the nonlinear dynamics systems [30, 31]. Hence, we focus on the first two modes. The approximate mode functions are

$$u_0(x, \theta, t) = u_1(t) \sin\left(\frac{\pi}{L}x\right) \cos(3\theta - 3x) + u_2(t) \sin\left(\frac{3\pi}{L}x\right) \cos(\theta - x), \quad (14a)$$

$$\begin{aligned} v_0(x, \theta, t) = v_1(t) \sin\left(\frac{\pi}{L}x\right) \sin(3\theta - 3x) \\ + v_2(t) \sin\left(\frac{3\pi}{L}x\right) \sin(\theta - x), \end{aligned} \quad (14b)$$

$$\begin{aligned} w_0(x, \theta, t) = w_1(t) \sin\left(\frac{\pi}{L}x\right) \cos(3\theta - 3x) \\ + w_2(t) \sin\left(\frac{3\pi}{L}x\right) \cos(\theta - x), \end{aligned} \quad (14c)$$

$$\begin{aligned}\varphi_x(x, \theta, t) &= \varphi_{x1}(t) \sin\left(\frac{\pi}{L}x\right) \cos(3\theta - 3x) \\ &+ \varphi_{x2}(t) \sin\left(\frac{3\pi}{L}x\right) \cos(\theta - x),\end{aligned}\quad (14d)$$

$$\begin{aligned}\varphi_\theta(x, \theta, t) &= \varphi_{\theta1}(t) \sin\left(\frac{\pi}{L}x\right) \sin(3\theta - 3x) \\ &+ \varphi_{\theta2}(t) \sin\left(\frac{3\pi}{L}x\right) \sin(\theta - x),\end{aligned}\quad (14e)$$

$$F_1 = F_{11} \sin\left(\frac{\pi}{L}x\right) \cos(3\theta - 3x) + F_{22} \sin\left(\frac{3\pi}{L}x\right) \cos(\theta - x), \quad (14f)$$

$$M_1 = M_{11} \sin\left(\frac{\pi}{L}x\right) \sin(3\theta - 3x) + M_{22} \sin\left(\frac{3\pi}{L}x\right) \sin(\theta - x). \quad (14g)$$

Using the equations above, the nonlinear equations of motion in terms of generalized displacements can be obtained. All the inertia terms of u_0 , v_0 , and φ_x in nonlinear equations of motion can be ignored since their influences are small compared to the inertia terms of w_0 and φ_θ [34]. Then, we can derive the displacements u_0 , v_0 , and φ_x with respect to the displacement w_0 and φ_θ . In the following analysis, we express u_0 , v_0 , and φ_x in terms of w_1 , w_2 , $\varphi_{\theta1}$, and $\varphi_{\theta2}$. Finally the nondimensional nonlinear system is obtained:

$$\begin{aligned}\ddot{w}_1 + \mu_1 \dot{w}_1 + \omega_1^2 w_1 + \alpha_1 w_1 \cos \omega_r t \\ + \alpha_2 w_1 (\cos \omega_r t)^2 + \alpha_3 w_1^2 + \alpha_4 w_2^2 + \alpha_5 w_1 w_2 \\ + \alpha_6 w_1^2 w_2 + \alpha_7 w_1^3 + \alpha_8 w_2^3 + \alpha_9 \varphi_{\theta1} \\ + \alpha_{10} \varphi_{\theta1} \cos \omega_r t + \alpha_{11} w_1 \varphi_{\theta2} + \alpha_{12} \varphi_{\theta1} (\cos \omega_r t)^2 \\ + \alpha_{13} w_1 \varphi_{\theta1} + \alpha_{14} w_2 \varphi_{\theta1} + \alpha_{15} w_2 \varphi_{\theta2} \\ = \alpha_{16} + f_1 \cos \omega_r t,\end{aligned}\quad (15a)$$

$$\begin{aligned}\ddot{w}_2 + \mu_2 \dot{w}_2 + \omega_2^2 w_2 + \beta_1 w_2 \cos \omega_r t \\ + \beta_2 w_2 (\cos \omega_r t)^2 + \beta_3 w_1^2 + \beta_4 w_2^2 + \beta_5 w_1 w_2 \\ + \beta_6 w_1^2 w_2 + \beta_7 w_1^3 + \beta_8 w_2^3 + \beta_9 \varphi_{\theta2} \\ + \beta_{10} \varphi_{\theta2} \cos \omega_r t + \beta_{11} w_1 \varphi_{\theta2} \\ + \beta_{12} \varphi_{\theta2} (\cos \omega_r t)^2 + \beta_{13} w_1 \varphi_{\theta1} + \beta_{14} w_2 \varphi_{\theta1} \\ + \beta_{15} w_2 \varphi_{\theta2} \\ = \beta_{16} + f_2 \cos \omega_r t,\end{aligned}\quad (15b)$$

$$\begin{aligned}\varphi_{\theta1} + \mu_3 \dot{\varphi}_{\theta1} + \omega_3^2 \varphi_{\theta1} + \gamma_1 \varphi_{\theta1} \cos \omega_r t \\ + \gamma_2 \varphi_{\theta1} (\cos \omega_r t)^2 + \gamma_3 w_1 + \gamma_4 w_1 \cos \omega_r t \\ + \gamma_5 w_1 (\cos \omega_r t)^2 + \gamma_6 w_1^2 + \gamma_7 w_1 w_2 + \gamma_8 w_2^2 \\ = \gamma_9 + m_1 \cos \omega_r t,\end{aligned}\quad (15c)$$

$$\begin{aligned}\varphi_{\theta2} + \mu_4 \dot{\varphi}_{\theta2} + \omega_4^2 \varphi_{\theta2} + \eta_1 \varphi_{\theta2} \cos \omega_r t \\ + \eta_2 \varphi_{\theta2} (\cos \omega_r t)^2 + \eta_3 w_1 + \eta_4 w_1 \cos \omega_r t \\ + \eta_5 w_1 (\cos \omega_r t)^2 + \eta_6 w_1^2 + \eta_7 w_1 w_2 \\ + \eta_8 w_2^2 \\ = \eta_9 + m_2 \cos \omega_r t.\end{aligned}\quad (15d)$$

3. Numerical Simulation

The nonlinear dynamic responses of the rotating cantilever conical shell under different rotating velocities and different excitations are studied in this part through some numerical simulations. To analyze the chaotic and periodic motions in the system, the phase portraits, time history diagrams, three-dimensional phase portraits, and power spectrum densities (PSD) are obtained.

3.1. Validation. Based on the kinetic energy and the potential energy of the blade deduced in the previous section, the Chebyshev polynomial and Ritz method are used to calculate the frequency of the model, which are compared with the frequency obtained by the finite element method.

The Chebyshev polynomial series $P_i(\chi)$ and $P_j(\chi)$ ($i = 1, 2, \dots; J = 1, 2, \dots$) are complete and orthogonal sets. The polynomial terms are taken as $i \times J = 10 \times 10$, where I and J are the truncation orders of the Chebyshev polynomial series.

The finite element analysis type is a modal analysis with the Block Lanczos method (using Ansys 15.0). The unit type is SOLID 95 with 20 nodes, and we adopt an intellectual finite element mesh division with four-stage precision for the mesh division.

3.1.1. The Comparative Analysis of Frequencies Using the Chebyshev–Ritz Method, the Finite Element Method, and a Modal Experiment. Completely free edge conditions are selected because they are the most exacting proof of any proposed theoretical solution scheme. And this work would provide a valuable reference datum for the subsequent theoretical study

To validate the accuracy of the frequencies worked out by the Chebyshev–Ritz method, we compare frequencies analytically calculated, with the frequencies obtained by the finite element method, the results from Ye et al. [36] and the datum from an experiment by Bardell et al. [21]. C-R represents the frequencies solved by the Chebyshev–Ritz method, FEM represents the frequencies obtained by the finite element method, and EX represents the frequencies by experiment.

Parameters of the blade are selected as

TABLE 1: Frequency comparison.

		1 st			2nd	
	C-R	FEM	EX	C-R	FEM	EX
Bardell et al. [21]		4.31	4.5		8.58	8.9
Ye et al. [36]	4.27			8.20		
Present	4.26	4.33		8.21	8.59	

TABLE 2: Frequency comparison.

		1 st		2nd	
	C-R	FEM	C-R	FEM	
0 rpm	1187.3	1243.5	1779.2	1935.1	
500 rpm	1199.7	1252.2	1790.6	1945.4

FIGURE 2: Continued.

FIGURE 2: The motion of the blade with varying rotating speeds when $\Omega_v = 0.17 \times 10^2$ (r/min). The time history of the first-order mode of (a) w_1 and (b) w_2 ; the phase portrait (c) on plane $(w_1, \dot{w}_1)$, (d) on plane $(w_2, \dot{w}_2)$, and (e) in the three-dimensional space $(w_1, \dot{w}_1, w_2)$; and (f) the PSD of w_1 .

FIGURE 3: Continued.

FIGURE 3: The motion of the blade with varying rotating speeds when $\Omega_v = 1.09 \times 10^2$ (r/min). The time history of the first-order mode of (a) w_1 and (b) w_2 ; the phase portrait (c) on plane $(w_1, \dot{w}_1)$, (d) on plane $(w_2, \dot{w}_2)$, and (e) in the three-dimensional space $(w_1, \dot{w}_1, w_2)$; and (f) the PSD of w_1 .

FIGURE 4: Continued.

FIGURE 4: The motion of the blade with varying rotating speeds when $\Omega_v = 1.8 \times 10^2$ (r/min). The time history of the first-order mode of (a) w_1 and (b) w_2 ; the phase portrait (c) on plane $(w_1, \dot{w}_1)$, (d) on plane $(w_2, \dot{w}_2)$, and (e) in the three-dimensional space $(w_1, \dot{w}_1, w_2)$; (f) the PSD of w_1 .

FIGURE 5: Continued.

FIGURE 5: The motion of the blade with changing forces when $F_1 = 1.1 \times 10^6$ (N/m²). The time history of the first-order mode of (a) w_1 and (b) w_2 ; the phase portrait (c) on plane $(w_1, \dot{w}_1)$, (d) on plane $(w_2, \dot{w}_2)$, and (e) in the three-dimensional space $(w_1, \dot{w}_1, w_2)$; (f) the PSD of w_1 .

FIGURE 6: Continued.

FIGURE 6: The motion of the blade with changing forces when $F_1 = 3.9 \times 10^6$ (N/m²). The time history of the first-order mode of (a) w_1 and (b) w_2 ; the phase portrait (c) on plane $(w_1, \dot{w}_1)$, (d) on plane $(w_2, \dot{w}_2)$, and (e) in the three-dimensional space $(w_1, \dot{w}_1, w_2)$; (f) the PSD of w_1 .

FIGURE 7: Continued.

FIGURE 7: The motion of the blade with changing forces when $F_1 = 4.7 \times 10^6$ (N/m²). The time history of the first-order mode of (a) w_1 and (b) w_2 ; the phase portrait (c) on plane $(w_1, \dot{w}_1)$, (d) on plane $(w_2, \dot{w}_2)$, and (e) in the three-dimensional space $(w_1, \dot{w}_1, w_2)$; (f) the PSD of w_1 .

FIGURE 8: Continued.

FIGURE 8: The motion of the blade with different moments when $M_1 = 0.89 \times 10^6$ (N/m). The time history of the first-order mode of (a) ϕ_{θ_1} and (b) ϕ_{θ_2} ; the phase portrait (c) on plane $(\phi_{\theta_1}, \dot{\phi}_{\theta_1})$, (d) on plane $(\phi_{\theta_2}, \dot{\phi}_{\theta_2})$, and (e) in the space $(\phi_{\theta_1}, \dot{\phi}_{\theta_1}, \phi_{\theta_2})$; (f) the PSD of ϕ_{θ_1} .

FIGURE 9: Continued.

FIGURE 9: The motion of the blade with different moments when $M_1 = 3.2 \times 10^6$ (N/m). The time history of the first-order mode of (a) $\phi_{\theta 1}$ and (b) $\phi_{\theta 2}$; the phase portrait (c) on plane $(\phi_{\theta 1}, \dot{\phi}_{\theta 1})$, (d) on plane $(\phi_{\theta 2}, \dot{\phi}_{\theta 2})$, and (e) in the space $(\phi_{\theta 1}, \dot{\phi}_{\theta 1}, \phi_{\theta 2})$; (f) the PSD of $\phi_{\theta 1}$.

FIGURE 10: Continued.

FIGURE 10: The motion of the blade with different moments when $M_1 = 4.0 \times 10^6$ (N/m). The time history of the first-order mode of (a) $\varphi_{\theta 1}$ and (b) $\varphi_{\theta 2}$; the phase portrait (c) on plane $(\varphi_{\theta 1}, \dot{\varphi}_{\theta 1})$, (d) on plane $(\varphi_{\theta 2}, \dot{\varphi}_{\theta 2})$, and (e) in the space $(\varphi_{\theta 1}, \dot{\varphi}_{\theta 1}, \varphi_{\theta 2})$; (f) the PSD of $\varphi_{\theta 1}$.

$$\begin{aligned}
 L &= 1.12 \text{ m}, & \rho &= 4450 (\text{kg/m}^3) \\
 r_{\text{root}} &= 0.16 \text{ m}, & \kappa &= 300 (\text{Ns/m}), \\
 \psi &= 0.15 \pi, & \gamma &= 300 (\text{Ns/m}) \\
 \nu &= 0.33, & F_0 &= 1.3 \times 10^6 (\text{N/m}^2), \\
 h_0 &= 0.002 \text{ m}, & M_0 &= 1.1 \times 10^6 (\text{N/m}), \\
 g(\theta) &= 1, & K_{u_1} &= 0.59, K_w \\
 E &= 70 \text{ GPa}, & & \\
 \alpha &= \pi, & & \\
 \rho &= 2700 (\text{kg/m}^3). & &
 \end{aligned} \tag{16}$$

Tables 1 and 2 show the frequencies of different orders obtained by the methods above. The model is reasonable because the frequencies calculated are in good agreement.

3.1.2. Comparison with the Other Work. To validate the model proposed, we compare frequencies analytically calculated with frequencies obtained by the finite element method. Parameters of the blade are selected as

$$\begin{aligned}
 L &= 0.15 \text{ m}, \\
 r_{\text{root}} &= 0.05 \text{ m}, \\
 \Psi &= \frac{\pi}{6}, \\
 \nu &= 0.33, \\
 E &= 102.04 \text{ GPa}, \\
 h_0 &= 0.03 \text{ m}, \\
 g(\theta) &= 1, \\
 \alpha &= \frac{\pi}{9}, \\
 \beta &= \frac{\pi}{6},
 \end{aligned}$$

3.2. Effect of Varying Velocities. In what follows, the parameters and the initial conditions are chosen as

$$\begin{aligned}
 L &= 0.21 \text{ m}, \\
 r_{\text{root}} &= 0.04 \text{ m}, \\
 \psi &= \frac{\pi}{6}, \\
 \nu &= 0.33, \\
 E &= 102.04 \text{ GPa}, \\
 h_0 &= 0.05 \text{ m}, \\
 \omega_r &= 100, \\
 \gamma &= 300 (\text{Ns/m}), \\
 \kappa &= 300 (\text{Ns/m}), \\
 K_{u_1} &= 0.59 \\
 \alpha &= \frac{\pi}{9},
 \end{aligned}$$

$$\begin{aligned}
\beta &= \frac{\pi}{6}, \\
\rho &= 4450 \text{ (kg/m}^3\text{)}, \\
K_w &= 0.62, \\
F_0 &= 1.5 \times 10^6 \text{ (N/m}^2\text{)}, \\
M_0 &= 1.2 \times 10^6 \text{ (N/m)}, \\
\Omega_c &= 2000 \text{ (r/min)}.
\end{aligned} \tag{18}$$

The chaotic and periodic responses can be identified by several conventional criteria. The waveform, phase portraits, and the power spectrum are utilized to verify the existence of the chaotic and periodic motion of the blade.

Under the different rotating speeds $\Omega_v = 0.17, 1.09$, and 1.8×10^2 (r/min), Figures 2–4 show the period-1 motion, multiperiod motion, and chaotic motion of the cantilever conical shell, respectively. It is observed that the rotating speed Ω_v makes great difference on the nonlinear dynamic responses of the cantilever shell.

Comparing Figures 2–4, one finds that both of the maximum amplitudes of w_1 and w_2 are increasing with the increasing rotating speed. Namely, the vibration mode changes as the increase of the rotating speed.

3.3. Effect of Changing Forces. The parameters of the conical shell and the initial conditions are chosen as

$$\begin{aligned}
L &= 0.19 \text{ m}, \\
r_{\text{root}} &= 0.03 \text{ m}, \\
\alpha &= \frac{\pi}{9}, \\
\beta &= \frac{\pi}{6}, \\
\omega_r &= 97, \\
\rho &= 4450 \text{ (kg/m}^3\text{)}, \\
\kappa &= 300 \text{ (Ns/m)}, \\
\psi &= \frac{\pi}{6}, \\
\nu &= 0.33, \\
h_0 &= 0.033 \text{ m}, \\
K_{u_1} &= 0.49, \\
E &= 102.04 \text{ GPa}, \\
\gamma &= 300 \text{ (Ns/m)}, \\
K_w &= 0.57, \\
F_0 &= 1.66 \times 10^6 \text{ (N/m}^2\text{)}, \\
\Omega_c &= 1600 \text{ (r/min)}, \\
M_0 &= 1.04 \times 10^6 \text{ (N/m)}.
\end{aligned} \tag{19}$$

Nonlinear oscillation of the system is investigated by choosing the rotating speed f_1 as the controlling parameter. Figures 5–7 present the period-1 motion, multiperiod motion, and chaotic motion of the rotating conical shell with the different force $f_1 = 1.1, 3.9$ and 4.7×10^6 (N/m²), respectively. It can be shown from figures that the amplitude of the first-order mode is larger than that of the second-order mode. The amplitudes of w_1 and w_2 , the distribution of the PSD, and phase portraits are very sensitive to the value of the force f_1 .

3.4. Effect of Different Moments. The parameters and the initial conditions are chosen as

$$\begin{aligned}
L &= 0.27 \text{ m}, \\
r_{\text{root}} &= 0.04 \text{ m}, \\
\psi &= \frac{\pi}{6}, \\
\alpha &= \frac{\pi}{9}, \\
E &= 102.04 \text{ GPa}, \\
\nu &= 0.33, \\
h_0 &= 0.03 \text{ m}, \\
\rho &= 4450 \text{ (kg/m}^3\text{)}, \\
\kappa &= 300 \text{ (Ns/m)}, \\
\omega_r &= 90, \\
\gamma &= 300 \text{ (Ns/m)}, \\
F_0 &= 1.4 \times 10^6 \text{ (N/m}^2\text{)}, \\
M_0 &= 1.1 \times 10^6 \text{ (N/m)}, \\
\Omega_c &= 1570 \text{ (r/min)}.
\end{aligned} \tag{20}$$

$K_{u_1} = 0.58$ and $K_w = 0.6$. When other parameters and the initial condition do not vary, we only change m_1 to detect the influence of the moment on vibration of the rotating cantilever blade. As the rotating speed and the force shown in the figures above, the moment m_1 also plays an important role for the nonlinear dynamic responses of the cantilever shell. Figures 8–10 contrast the period-1 motion, multiperiod motion, and chaotic motion in the different value of external moment $m_1 = 0.89, 3.2$ and 4.0×10^6 (N/m), respectively. These figures indicate that the amplitude, the phase portraits, and the power spectrum densities vary from the different value of the external moment. The amplitude of the rotating cantilever conical shell increases with the increase of the moment.

4. Conclusions

In order to describe the nonlinear dynamic responses of the blade with variable thickness, this paper presents a rotating pretwisted cantilever conical shell with variable thickness to investigate the effects of the varying rotating speed, external force, and external moment on the dynamic behaviors. Based on the von Kármán nonlinear relationship, Hamilton's principle, and the first-order shear deformation theory, the nonlinear governing equations of motion for the rotating cantilever blade with variable thickness are established. Galerkin's method is utilized to discretize the partial differential governing equations of motion as a nonlinear ordinary differential equation of motion. Numerical results show that the rotating velocity, external force, and moment have significant influences on nonlinear dynamic behaviors of the cantilever conical shell. It can be concluded as follows:

- (1) The numerical results demonstrate that nonlinear dynamic motions of the blade are sensitive to the rotating speed. The increase of the rotating speed could not only change the distribution of the PSD and the phase portraits but also initiate the amplitude increase. With the increasing rotating speed, the periodic motion, and chaotic motion of the blade which appears, the motion law of the rotating blade performs as follows: period-1 motion $\rightarrow$ multiperiod motion $\rightarrow$ chaotic motion. Occurrence of the chaotic motion means that the rotating cantilever blade may perform large amplitude nonlinear vibration, which leads to the damage of the system. We can control the responses of the system from the chaotic motions to the periodic motions by changing the rotating speed.
- (2) The phase portrait and power spectrum are conducted to demonstrate that periodic motions and chaotic motions occur in nonlinear vibrations of the rotating blade under certain conditions. The increase of forces and moments could not only change the distribution of the PSD and the phase portraits but also initiate the amplitude increase. With the increasing external excitation, the motion law of the rotating blade performs as follows: period-1 motion $\rightarrow$ multiperiod motion $\rightarrow$ chaotic motion. Since chaotic motion is the large amplitude nonlinear vibration, we can adjust the external force and external moment to control the responses of the system in order to avoid the appearance of the chaotic motion.

Study on nonlinear behavior of the blade is of great significance. Our results are useful to improve the stability of the rotating blade.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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