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Research Article

Some New Refinements of Hermite–Hadamard-Type Inequalities Involving ψ_k-Riemann–Liouville Fractional Integrals and Applications

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The main objective of this article is to establish some new fractional refinements of Hermite–Hadamard-type inequalities essentially using new ψ_k-Riemann–Liouville fractional integrals, where $k > 0$. Using this new fractional integral, we also derive two new fractional integral identities. Applications of the obtained results are also discussed.

1. Introduction and Preliminaries

Let $f: I = [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function; then,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

The above inequality is known as Hermite–Hadamard's inequality [1–5]. This inequality provides us a necessary and sufficient condition for a function to be convex. It can be considered as one of the most extensively studied results pertaining to convexity. Since the appearance of this result in the literature, it gained popularity, and many new generalizations for this classical result have been obtained. This can be attributed to its applications in various other fields such as in numerical analysis and in mathematical statistics. For more details on generalizations of convexity, Hermite–Hadamard-like inequalities, and its applications, see [6–14].

Fractional calculus is a calculus in which we study about the integrals and derivatives of any arbitrary real or complex order. The history of fractional calculus is not very much old, but in the short span of time, it experienced a rapid development. Recently, the generalizations [15–25], extensions [26–32], and applications [33–46] for fractional calculus have been made by many researchers. The Riemann–Liouville fractional integrals are defined as follows.

Definition 1 (see [47]). Let $f \in L_1[a, b]$. Then, Riemann–Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad x > a, \tag{2}$$

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \quad x < b, \tag{3}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1}\mathrm{d}x, \tag{4}$$

is the well-known gamma function.

Sarikaya et al. [10] elegantly utilized this concept in establishing fractional analogue of Hermite–Hadamard's inequality. This idea motivated other researchers, and consequently, many new generalizations of Hermite–Hadamard's inequality have been obtained using the concept of Riemann–Liouville fractional integrals.

Sarikaya and Karaca [12] introduced k-analogue of Riemann–Liouville fractional integrals and discussed some of its basic properties. They defined this concept in the following way: to be more precise, let f be piecewise continuous on $I^* = (0,\infty)$ and integrable on any finite subinterval of $I = [0,\infty]$. Then, for $t > 0$, we consider k-Riemann–Liouville fractional integral of f of order α as

$$ {}_kJ_a^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{(\alpha/k)-1} f(t)\mathrm{d}t, \quad x > a, k > 0. \tag{5}$$

If $k \longrightarrow 1$, then k-Riemann–Liouville fractional integrals reduce to classical the Riemann–Liouville fractional integral. It is worth to mention here that the concept of the k-Riemann–Liouville fractional integral is a significant generalization of Riemann–Liouville fractional integrals; as for $k \neq 1$, the properties of k-Riemann–Liouville fractional integrals are quite different from the classical Riemann–Liouville fractional integrals.

Another important generalization of Riemann–Liouville fractional integrals is ψ_k-Riemann–Liouville fractional integrals.

Definition 2 (see [6]). Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also, let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ on (a,b). Then, the left- and right-sided ψ-Riemann–Liouville fractional integrals of a function f with respect to another function ψ on $[a,b]$ are defined as

$$\begin{aligned} I_{a^+}^{\alpha;\psi} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} f(t)\mathrm{d}t, \\ I_{b^-}^{\alpha;\psi} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{\alpha-1} f(t)\mathrm{d}t, \end{aligned} \tag{6}$$

respectively; $\Gamma(\cdot)$ is the gamma function.

For some recent research works, see [48].

Recently, Liu et al. [14] obtained some interesting results pertaining to Hermite–Hadamard's inequality involving ψ_k-Riemann–Liouville fractional integrals. Motivated by the research work of Liu et al. [14], we obtain some new refinements of fractional Hermite–Hadamard's inequality essentially using ψ_k-Riemann–Liouville fractional integrals. We also discuss applications of the obtained results to means. We show that our results represent significant generalization of some previous results.

2. Hermite–Hadamard's Inequality

In this section, we derive a new refinement of Hermite–Hadamard's inequality via the ψ_k-Riemann–Liouville fractional integral.

Definition 3. Let $k > 0$, (a,b) be a finite interval of the real line $\mathbb{R}$, and $\alpha > 0$. Also, let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ on (a,b). Then, the left- and right-sided ψ_k-Riemann–Liouville fractional integrals of a function f with respect to another function ψ on $[a,b]$ are defined as

$$\begin{aligned} {}_kI_{a^+}^{\alpha;\psi} f(x) &= \frac{1}{k\Gamma_k(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{(\alpha/k)-1} f(t)\mathrm{d}t, \\ {}_kI_{b^-}^{\alpha;\psi} f(x) &= \frac{1}{k\Gamma_k(\alpha)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{(\alpha/k)-1} f(t)\mathrm{d}t, \end{aligned} \tag{7}$$

respectively;

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-(t^k/k)}\mathrm{d}t, \quad \mathfrak{R}(x) > 0, \tag{8}$$

is the k-analogue of gamma function.

The k-analogues of beta function and incomplete beta function are, respectively, defined as

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{(x/k)-1}(1-t)^{(x/k)-1}\mathrm{d}t, \tag{9}$$

$$B_k(z;x,y) = \frac{1}{k}\int_0^z t^{(x/k)-1}(1-t)^{(b/y)-1}\mathrm{d}t. \tag{10}$$

We now derive the main result of this section.

Theorem 1. *Let $0 \le e < f$ and g: $[e,f] \longrightarrow \mathbb{R}$ be a positive function and $g \in L_1[e,f]$. Also, suppose that g is a convex function on $[e,f]$, $\psi(x)$ is an increasing and positive monotone function on $(e,f]$, having a continuous derivative $\psi'(x)$ on (e,f), and $\alpha \in (0,1)$. Then, for $k > 0$, the following k-fractional integral inequalities hold:*

$$\begin{aligned} g\left(\frac{e+f}{2}\right) &\le \frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI_{\psi^{-1}(e)^+}^{\alpha;\psi}(g\circ\psi)\left(\psi^{-1}(f)\right)\right. \\ &\quad \left. + {}_kI_{\psi^{-1}(f)^-}^{\alpha;\psi}(g\circ\psi)\left(\psi^{-1}(e)\right)\right] \le \frac{g(e)+g(f)}{2}. \end{aligned} \tag{11}$$

Proof. Using the convexity of g, we have

$$2g\left(\frac{e+f}{2}\right) \le g(tc+(1-t)f) + g((1-t)e+td). \tag{12}$$

Multiplying both sides by $t^{(\alpha/k)-1}$ and then integrating with respect to t on $[0,1]$, we have

$$\begin{aligned} \frac{2k}{\alpha} g\left(\frac{e+f}{2}\right) &\le \int_0^1 t^{(\alpha/k)-1} g(tc+(1-t)f)\mathrm{d}t \\ &\quad + \int_0^1 t^{(\alpha/k)-1} g((1-t)e+td)\mathrm{d}t. \end{aligned} \tag{13}$$

Now, making the substitution $t = (\psi(v) - f/e - f)$, $s = (\psi(v) - e/f - e)$, we have

$$\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]$$
$$=\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\frac{1}{k\Gamma_k(\alpha)}\left[\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)}(g\circ\psi)(v)\psi'(v)\mathrm{d}v\right.$$
$$\left.+\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(\psi(v)-e)^{(\alpha/k)}(g\circ\psi)(v)\psi'(v)\mathrm{d}v\right] \tag{14}$$
$$=\frac{\alpha}{2k}\left[\int_0^1 t^{(\alpha/k)-1}g(tc+(1-t)f)\mathrm{d}t+\int_0^1 t^{(\alpha/k)-1}g((1-t)e+td)\mathrm{d}t\right]$$
$$\geq g\left(\frac{e+f}{2}\right).$$

Also, using the convexity property of g, we have

$$g(tc+(1-t)f)+g((1-t)e+td)\leq g(e)+g(f). \tag{15}$$

Multiplying both sides by $t^{(\alpha/k)-1}$ and then integrating it with respect to t on $[0,1]$, we obtain

$$\int_0^1 t^{(\alpha/k)-1}g(tc+(1-t)f)\mathrm{d}t+\int_0^1 t^{(\alpha/k)-1}g((1-t)e+td)\mathrm{d}t$$
$$\leq\frac{k}{\alpha}[g(e)+g(f)]. \tag{16}$$

This implies

$$\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.$$
$$\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]\leq\frac{g(e)+g(f)}{2}. \tag{17}$$

The proof is completed. □

3. Some More Fractional Inequalities of Hermite–Hadamard Type

We now derive two new fractional integral identities involving ψ_k-Riemann–Liouville fractional integrals. These results will serve as auxiliary results for obtaining our next results.

Lemma 1. *Let $e<f$ and g: $[e,f]\longrightarrow\mathbb{R}$ be a differentiable mapping on (e,f). Also, suppose that $g'\in L[e,f]$, $\psi(x)$ is an increasing and positive monotone function on $(e,f]$, having a continuous derivative $\psi'(x)$ on (e,f), and $\alpha\in(0,1)$. Then, for $k>0$, the following identity holds:*

$$\frac{g(e)+g(f)}{2}-\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.$$
$$\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]$$
$$=\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}\left[(\psi(v)-e)^{(\alpha/k)}\right.$$
$$\left.-(f-\psi(v))^{(\alpha/k)}\right](g'\circ\psi)(v)\psi'(v)\mathrm{d}v. \tag{18}$$

Proof. Consider $J_1=(\Gamma_k(\alpha+k)/2(f-e)^{(\alpha/k)})_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)(\psi^{-1}(f))$ and $J_2=(\Gamma_k(\alpha+k)/2(f-e)^{(\alpha/k)})_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)(\psi^{-1}(e))$.

Now,

$$J_1=\frac{\alpha}{2k(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)-1}(g\circ\psi)(v)\psi'(v)\mathrm{d}v$$
$$=-\frac{1}{2k(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(g\circ\psi)(v)\mathrm{d}(f-\psi(v))^{(\alpha/k)} \tag{19}$$
$$=\frac{g(e)}{2}+\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)}(g'\circ\psi)(v)\psi'(v)\mathrm{d}v.$$

Similarly,

$$\begin{aligned} J_2 &= \frac{\alpha}{2k(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(\psi(v)-e)^{(\alpha/k)-1} \\ &\quad\cdot (g\circ\psi)(v)\psi'(v)\mathrm{d}v \\ &= \frac{1}{2k(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(g\circ\psi)(v)\mathrm{d} \\ &\quad\cdot(\psi(v)-e)^{(\alpha/k)} \\ &= \frac{g(f)}{2}-\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)} \\ &\quad\cdot(\psi(v)-e)^{(\alpha/k)}(g'\circ\psi)(v)\psi'(v)\mathrm{d}v. \end{aligned} \tag{20}$$

It follows that

$$\begin{aligned} \frac{g(e)+g(f)}{2}-(J_1+J_2) &= \frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)} \\ &\quad\cdot\left[(\psi(v)-e)^{(\alpha/k)}-(f-\psi(v))^{(\alpha/k)}\right] \\ &\quad\cdot(g'\circ\psi)(v)\psi'(v)\mathrm{d}v. \end{aligned} \tag{21}$$

□

Example 1. Let $c=2, d=3, \alpha=(1/2), k=2, g(x)=x^2, \psi(x)=x$. Then, all the assumptions in Lemma 1 are satisfied. Observe that $(g(c)+g(d)/2)=(13/2)$.

$$\begin{aligned} &\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right)\right. \\ &\quad\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right] \\ &=\frac{\Gamma_{(2)}(1/2)}{2}\left[\frac{1}{\Gamma_{(2)}(1/2)}\int_2^3 v^2(3-v)^{-(3/4)}\mathrm{d}v\right. \\ &\quad\left.+\frac{1}{\Gamma_{(2)}(1/2)}\int_2^3 v^2(v-2)^{-(3/4)}\mathrm{d}v\right]=\frac{577}{90}. \end{aligned} \tag{22}$$

This implies

$$\begin{aligned} &\frac{g(c)+g(d)}{2}-\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right)\right. \\ &\quad\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right]=\frac{4}{45}. \end{aligned} \tag{23}$$

Also,

$$\begin{aligned} &\frac{1}{2(d-c)^{(\alpha/k)}}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}\left[(\psi(v)-c)^{(\alpha/k)}-(d-\psi(v))^{(\alpha/k)}\right] \\ &\quad\cdot(g'\circ\psi)(v)\psi'(v)\mathrm{d}v \\ &\quad=\int_2^3 v(v-2)^{(1/4)}\mathrm{d}v-\int_2^3 v(3-v)^{(1/4)}\mathrm{d}v=\frac{4}{45}. \end{aligned} \tag{24}$$

Example 2. Let $c=2, d=3, \alpha=(1/2), k=(1/2), g(x)=x^2, \psi(x)=x$. Then, all the assumptions in Lemma 1 are satisfied. Observe that $(g(c)+g(d)/2)=(13/2)$.

$$\begin{aligned} &\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right)+{}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right] \\ &=\frac{\Gamma_{(1/2)}(1/2)}{2}\left[\frac{1}{\Gamma_{(1/2)}(1/2)}\int_2^3 v^2\mathrm{d}v+\frac{1}{\Gamma_{(1/2)}(1/2)}\int_2^3 v^2\mathrm{d}v\right]=\frac{19}{3}. \end{aligned} \tag{25}$$

This implies

$$\begin{aligned} &\frac{g(c)+g(d)}{2}-\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right)\right. \\ &\quad\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right]=\frac{1}{6}. \end{aligned} \tag{26}$$

Also,

$$\begin{aligned} &\frac{1}{2(d-c)^{(\alpha/k)}}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}\left[(\psi(v)-c)^{(\alpha/k)}-(d-\psi(v))^{(\alpha/k)}\right] \\ &\quad\cdot(g'\circ\psi)(v)\psi'(v)\mathrm{d}v \\ &\quad=\int_2^3 v(v-2)\mathrm{d}v-\int_2^3 v(3-v)\mathrm{d}v=\frac{1}{6}. \end{aligned} \tag{27}$$

Lemma 2. *Let $e<f$ and g: $[e,f]\longrightarrow\mathbb{R}$ be a differentiable mapping on (e,f). Also, suppose that $g'\in L[e,f]$, $\psi(x)$ is an increasing and positive monotone function on $(e,f]$, having a continuous derivative $\psi'(x)$ on (e,f), and $\alpha\in(0,1)$. Then, for $k>0$, the following identity holds:*

$$\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]-g\left(\frac{e+f}{2}\right)$$
$$=\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}h\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v+\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}\left[(\psi(v)-e)^{(\alpha/k)}-(f-\psi(v))^{(\alpha/k)}\right]\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v, \quad (28)$$

where

$$h=\begin{cases}\dfrac{1}{2}, & \text{for } \psi^{-1}\left(\dfrac{e+f}{2}\right)\le v\le\psi^{-1}(f),\\[2ex] -\dfrac{1}{2}, & \text{for } \psi^{-1}(e)\le v\le\psi^{-1}\left(\dfrac{e+f}{2}\right).\end{cases} \quad (29)$$

Proof. Suppose

$$I_1=-\frac{1}{2}\int_{\psi^{-1}(e)}^{\psi^{-1}(e+f/2)}\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v=-\frac{1}{2}g\left(\frac{e+f}{2}\right)+\frac{g(e)}{2},$$
$$I_2=\frac{1}{2}\int_{\psi^{-1}(e+f/2)}^{\psi^{-1}(f)}\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v=-\frac{1}{2}g\left(\frac{e+f}{2}\right)+\frac{g(f)}{2},$$
$$I_3=\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)}\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v$$
$$=-\frac{g(e)}{2}+\frac{\alpha}{2k(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(f-\psi(v))^{(\alpha/k)-1}(g\circ\psi)(v)\psi'(v)\mathrm{d}v$$
$$=-\frac{g(e)}{2}+\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right), \quad (30)$$

$$I_4=-\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(\psi(v)-e)^{(\alpha/k)}\left(g'\circ\psi\right)(v)\psi'(v)\mathrm{d}v$$
$$=-\frac{g(f)}{2}+\frac{\alpha}{2k(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}(\psi(v)-e)^{(\alpha/k)-1}(g\circ\psi)(v)\psi'(v)\mathrm{d}v \quad (31)$$
$$=-\frac{g(f)}{2}+\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right).$$

Summing I_1, I_2, I_3, and I_4, we get the required result. □

Example 3. Let $c=2, d=3, \alpha=(1/2), k=2, g(x)=x^2, \psi(x)=x$. Then, all the assumptions in Lemma 2 are satisfied. Note that $g(c+d/2)=(25/4)$.

$$\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right)+{}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right]$$
$$=\frac{\Gamma_{(1/2)}(1/2)}{8}\left[\frac{1}{\Gamma_{(1/2)}(1/2)}\int_2^3 v^2(3-v)^{-(3/4)}\mathrm{d}v+\frac{1}{\Gamma_{(1/2)}(1/2)}\int_2^3 v^2(v-2)^{-(3/4)}\mathrm{d}v\right]=\frac{577}{90}. \quad (32)$$

This implies

$$\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right)\right.$$
$$\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right]-g\left(\frac{c+d}{2}\right)=\frac{29}{180}. \tag{33}$$

Also,

$$\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}h\left(g'\circ\psi\right)(\nu)\psi'(\nu)\mathrm{d}\nu=\frac{1}{4}, \tag{34}$$

where h is defined in Lemma 2.

$$\frac{1}{2(d-c)^{(\alpha/k)}}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}\left[(d-\psi(\nu))^{(\alpha/k)}-(\psi(\nu)-c)^{(\alpha/k)}\right]$$
$$\cdot\left(g'\circ\psi\right)(\nu)\psi'(\nu)\mathrm{d}\nu$$
$$=\int_2^3\nu(\nu-2)^{(1/4)}\mathrm{d}\nu-\int_2^3\nu(3-\nu)^{(1/4)}\mathrm{d}\nu=-\frac{4}{45}. \tag{35}$$

This implies

$$\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}h\left(g'\circ\psi\right)(\nu)\psi'(\nu)\mathrm{d}\nu+\frac{1}{2(d-c)^{(\alpha/k)}}$$
$$\cdot\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}\left[(d-\psi(\nu))^{(\alpha/k)}\right.$$
$$\left.-(\psi(\nu)-c)^{(\alpha/k)}\right]\left(g'\circ\psi\right)(\nu)\psi'(\nu)\mathrm{d}\nu=\frac{29}{180}. \tag{36}$$

Example 4. Let $c=2,d=3,\alpha=(1/2),k=(1/2),g(x)=x^2,\psi(x)=x$. Then, all the assumptions in Lemma 2 are satisfied. Note that $g(c+d/2)=(25/4)$.

$$\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right)+{}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right]$$
$$=\frac{\Gamma_{(1/2)}(1/2)}{2}\left[\frac{1}{\Gamma_{(1/2)}(1/2)}\int_2^3\nu^2\mathrm{d}\nu+\frac{1}{\Gamma_{(2)}(1/2)}\int_2^3\nu^2\mathrm{d}\nu\right]=\frac{19}{3}. \tag{37}$$

This implies

$$\frac{\Gamma_k(\alpha+k)}{2(d-c)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(c)^+}(g\circ\psi)\left(\psi^{-1}(d)\right)\right.$$
$$\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(d)^-}(g\circ\psi)\left(\psi^{-1}(c)\right)\right]-g\left(\frac{c+d}{2}\right)=\frac{1}{12}. \tag{38}$$

Also,

$$\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}h\left(g'\circ\psi\right)(\nu)\psi'(\nu)\mathrm{d}\nu=\frac{1}{4}, \tag{39}$$

where h is defined in Lemma 2.

$$\frac{1}{2(d-c)^{(\alpha/k)}}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}\left[(d-\psi(\nu))^{(\alpha/k)}\right.$$
$$\left.-(\psi(\nu)-c)^{(\alpha/k)}\right]\left(g'\circ\psi\right)(\nu)\psi'(\nu)\mathrm{d}\nu$$
$$=\int_2^3\nu(\nu-2)\mathrm{d}\nu-\int_2^3\nu(3-\nu)\mathrm{d}\nu=-\frac{1}{6}. \tag{40}$$

This implies

$$\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}h\left(g\prime\circ\psi\right)(\nu)\psi'(\nu)\mathrm{d}\nu+\frac{1}{2(d-c)^{(\alpha/k)}}\int_{\psi^{-1}(c)}^{\psi^{-1}(d)}$$
$$\cdot\left[(d-\psi(\nu))^{(\alpha/k)}-(\psi(\nu)-c)^{(\alpha/k)}\right]$$
$$\cdot\left(g\prime\circ\psi\right)(\nu)\psi'(\nu)\mathrm{d}\nu=\frac{1}{12}. \tag{41}$$

Before proceeding to next results, let us recall the definition of s-convex function of Breckner type.

Definition 4 (see [49]). A function $g\colon [0,\infty)\longrightarrow[0,\infty)$ is said to be s-convex function of Breckner type if

$$g((1-t)x+ty)\leq(1-t)^s g(x)+t^s g(y),$$
$$\forall x,y\in[0,\infty),t\in[0,1],s\in(0,1]. \tag{42}$$

Theorem 2. *Let $e<f$ and $g\colon [e,f]\longrightarrow\mathbb{R}$ be a differentiable mapping on (e,f). Also, suppose that $|g'|$ is Breckner type of s-convex on $[e,f]$, $\psi(x)$ is an increasing and positive monotone function on $(e,f]$, having a continuous derivative $\psi'(x)$ on (e,f), and $\alpha\in(0,1)$. Then, for $k>0$, the following inequality holds:*

$$\left|\frac{g(e)+g(f)}{2}-\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.\right.$$
$$\left.\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]\right|$$
$$\leq\frac{f-e}{2}\left[L_1\left|g'(e)\right|+L_2\left|g'(f)\right|\right], \tag{43}$$

where

$$L_1:=2k\mathrm{B}_k\left(\frac{1}{2};1+s,\frac{k+\alpha}{k}\right)+\frac{k\left(1-2^{-(ks+\alpha/k)}\right)}{k+ks+\alpha}-\mathrm{B}_k\left(1+s,\frac{k+\alpha}{k}\right), \tag{44}$$

$$L_2:=\frac{k\left(1-2^{-(ks+\alpha/k)}\right)}{k+ks+\alpha}-2k\mathrm{B}_k\left(\frac{1}{2};\frac{k+\alpha}{k},1+s\right)-\mathrm{B}_k\left(\frac{k+\alpha}{k},1+s\right), \tag{45}$$

respectively.

Proof. Using Lemma 1 and the fact that $|g'|$ is Breckner type of s-convex function, we have

$$\begin{aligned}
&\left|\frac{g(e)+g(f)}{2}-\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]\right|\\
&\leq\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}\left|(\psi(v)-e)^{(\alpha/k)}-(f-\psi(v))^{(\alpha/k)}\right|\left|(g\prime\circ\psi)(v)\right|\left|\psi'(v)\right|\mathrm{d}v\\
&=\frac{f-e}{2}\int_0^1\left|(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right|\left|g'(tc+(1-t)f)\right|\mathrm{d}t\\
&\leq\frac{f-e}{2}\int_0^1\left|(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right|\left[t^s\left|g'(e)\right|+(1-t)^s\left|g'(f)\right|\right]\mathrm{d}t\\
&=\frac{f-e}{2}\left[\left|g'(e)\right|\int_0^{(1/2)}t^s\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]\mathrm{d}t+\left|g'(f)\right|\int_0^{(1/2)}(1-t)^s\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]\mathrm{d}t\right.\\
&\left.+\left|g'(e)\right|\int_{(1/2)}^1t^s\left[t^{(\alpha/k)}-(1-t)^{(\alpha/k)}\right]\mathrm{d}t+\left|g'(f)\right|\int_{(1/2)}^1(1-t)^s\left[t^{(\alpha/k)}-(1-t)^{(\alpha/k)}\right]\mathrm{d}t\right]\\
&=\frac{f-e}{2}\left[L_1\left|g'(e)\right|+L_2\left|g'(f)\right|\right],
\end{aligned}\tag{46}$$

where

$$\begin{aligned}
L_1 &:= H_1+H_3=\int_0^{(1/2)}t^s\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]\mathrm{d}t+\int_{(1/2)}^1t^s\left[t^{(\alpha/k)}-(1-t)^{(\alpha/k)}\right]\mathrm{d}t\\
&=2k\mathrm{B}_k\left(\frac{1}{2};1+s,\frac{k+\alpha}{k}\right)+\frac{k\left(1-2^{-(ks+\alpha/k)}\right)}{k+ks+\alpha}-\mathrm{B}_k\left(1+s,\frac{k+\alpha}{k}\right),
\end{aligned}\tag{47}$$

$$\begin{aligned}
L_2 &:= H_2+H_4=\int_0^{(1/2)}(1-t)^s\left[(1-t)^{(\alpha/k)}-t^{(\alpha/k)}\right]\mathrm{d}t+\int_{(1/2)}^1(1-t)^s\left[t^{(\alpha/k)}-(1-t)^{(\alpha/k)}\right]\mathrm{d}t\\
&=\frac{k\left(1-2^{-(ks+\alpha/k)}\right)}{k+ks+\alpha}-2k\mathrm{B}_k\left(\frac{1}{2};\frac{k+\alpha}{k},1+s\right)-\mathrm{B}_k\left(\frac{k+\alpha}{k},1+s\right).
\end{aligned}\tag{48}$$

This completes the proof. □

Theorem 3. *Let $g\colon [e,f]\longrightarrow\mathbb{R}$ be a differentiable function on (e,f) with $e<f$. Also, suppose that $|g'|$ is Breckner type of s-convex function. If $\psi(x)$ is an increasing and positive monotone function on $(e,f]$, having a continuous derivative $\psi'(x)$ on (e,f) and $\alpha\in(0,1)$, then for $k>0$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.\right.\\
&\left.\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)\left(\psi^{-1}(e)\right)\right]-g\left(\frac{e+f}{2}\right)\right|\\
&\leq\frac{|g(f)-g(e)|}{2}+\frac{f-e}{2}\left[L_1\left|g'(e)\right|+L_2\left|g'(f)\right|\right],
\end{aligned}\tag{49}$$

where L_1 and L_2 are given by (44) and (45), respectively.

Proof. Using Lemma 2, the property of modulus, and the given hypothesis of the theorem, we have

$$\begin{aligned}
&\left|\frac{\Gamma_k(\alpha+k)}{2(f-e)^{(\alpha/k)}}\left[{}_kI^{\alpha;\psi}_{\psi^{-1}(e)^+}(g\circ\psi)\left(\psi^{-1}(f)\right)\right.\right.\\
&\left.\left.+{}_kI^{\alpha;\psi}_{\psi^{-1}(f)^-}(g\circ\psi)(\pi)\right]-g\left(\frac{e+f}{2}\right)\right|\\
&\leq\left|\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}h(g'\circ\psi)(v)\psi'(v)\mathrm{d}v\right|\\
&+\left|\frac{1}{2(f-e)^{(\alpha/k)}}\int_{\psi^{-1}(e)}^{\psi^{-1}(f)}\left[(f-\psi(v))^{(\alpha/k)}\right.\right.\\
&\left.\left.-(\psi(v)-e)^{(\alpha/k)}\right](g\prime\circ\psi)(v)\psi'(v)\mathrm{d}v\right|\\
&=I_1+I_2.
\end{aligned}\tag{50}$$

Using substitution $t = (\psi(v) - e/f - e)$ and the fact that $|g'|$ is Breckner type of s-convex function, we have

$$I_1 \leq \frac{f-e}{2}\left[L_1\left|g'(e)\right| + L_2\left|g'(f)\right|\right], \tag{51}$$

where L_1 and L_2 are given by (44) and (45), respectively. And

$$I_2 = \frac{|g(f) - g(e)|}{2}. \tag{52}$$

This completes the proof. □

4. Applications

In this section, we discuss some applications of Theorem 2 to means by considering a particular example of s-convexity. First of all, we recall some previously known concepts related to means [50].

For arbitrary real numbers α, β, $\alpha \neq \beta$, we define the following:

(1) Arithmetic mean:

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in \mathbb{R}; \tag{53}$$

(2) Logarithmic mean:

$$\overline{L}(\alpha,\beta) = \frac{\beta-\alpha}{\ln|\beta| - \ln|\alpha|}, \quad \alpha,\beta \in \mathbb{R}\backslash\{0\}; \tag{54}$$

(3) Generalized log-mean:

$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{(1/n)}, \quad n \in \mathbb{N}, n \geq 1, \alpha,\beta \in \mathbb{R}, \alpha < \beta. \tag{55}$$

We now give the main results of this section.

Proposition 1. *Let $e, f \in \mathbb{R}^+$ with $e < f$; then,*

$$\left|A\left(e^s, f^s\right) - L_s^s(e,f)\right| \leq \frac{s(f-e)}{2}\left[W_1|e|^{s-1} + W_2|f|^{s-1}\right], \tag{56}$$

where

$$W_1 := 2\mathrm{B}\left(\frac{1}{2}; 1+s, 2\right) + \frac{1-2^{-1-s}}{2+s} - \mathrm{B}(1+s,2), \tag{57}$$

$$W_2 := \frac{1-2^{-1-s}}{2+s} - 2\mathrm{B}\left(\frac{1}{2}; 2, 1+s\right) - \mathrm{B}(2, 1+s), \tag{58}$$

respectively.

Proof. Applying Theorem 2 for $g(x) = x^s$, $\psi(x) = x$, and $\alpha = 1 = k$, we obtain the required result. □

Proposition 2. *Let $e, f \in \mathbb{R}^+$ with $e < f$; then,*

$$\left|A\left(e^s, f^s\right) - L_s^s(e,f)\right| \leq \frac{\left|f^s - g^s\right|}{2} + \frac{s(f-e)}{2}\left[W_1|e|^{s-1} + W_2|f|^{s-1}\right], \tag{59}$$

where W_1 and W_2 are given by (57) and (58), respectively.

Proof. Applying Theorem 3 for $g(x) = x^s$, $\psi(x) = x$, and $\alpha = 1 = k$, we obtain the required result. □

5. Conclusion

In this article, we obtain some new fractional estimates of Hermite–Hadamard's inequality essentially using a new k-analogue of ψ_k-fractional integrals. We derive two new fractional integral identities in the setting of k-fractional calculus. In order to check the validity of these identities, we discuss some particular examples. In the final section, we have discussed applications of Theorems 2 and 3 to means.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

[1] J. Hadamard, "Étude sur les propriétés des fonctions entières et en particulier dune fonction considérée par Riemann," *Journal de Mathématiques Pures et Appliquées*, vol. 58, pp. 171–215, 1893.

[2] M. Adil Khan, Y. Khurshid, T.-S. Du, and Y.-M. Chu, "Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals," *Journal of Function Spaces*, vol. 2018, Article ID 5357463, 12 pages, 2018.

[3] M. A. Latif, S. Rashid, S. S. Dragomir, and Y.-M. Chu, "Hermite-Hadamard type inequalities for co-ordinated convex and qausi-convex functions and their applications," *Journal of Inequalities and Applications*, vol. 2019, p. 33, 2019.

[4] A. Iqbal, M. Adil Khan, S. Ullah, and Y.-M. Chu, "Some new Hermite-Hadamard-type inequalities associated with conformable fractional integrals and their applications," *Journal of Function Spaces*, vol. 2020, Article ID 9845407, 18 pages, 2020.

[5] M. Adil Khan, N. Mohammad, E. R. Nwaeze, and Y.-M. Chu, "Quantum Hermite-Hadamard inequality by means of a

green function," *Advances in Difference Equations*, vol. 2020, Article ID 99, no. 1, p. 20, 2020.

[6] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.

[7] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, Melbourne, Australia, 2000, https://rgmia.org/monographs/hermite_hadamard.html.

[8] G. Cristescu and L. Lupşa, *Non-Connected Convexities and Applications*, Kluwer Academic Publishers, Dordrecht, Netherlands, 2002.

[9] C. P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications*, Springer, New York, USA, 2006.

[10] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013.

[11] M. A. Noor, K. I. Noor, M. U. Awan, and S. Khan, "Fractional Hermite-Hadamard inequalities for some new classes of Godunova-Levin functions," *Applied Mathematics & Information Sciences*, vol. 8, no. 6, pp. 2865–2872, 2014.

[12] M. Z. Sarikaya and A. Karaca, "On the k-Riemann-Liouville fractional integral and applications," *International Journal of Mathematics and Statistics*, vol. 1, no. 3, pp. 33–43, 2014.

[13] G. Cristescu, M. A. Noor, and M. U. Awan, "Bounds of the second degree cumulative frontier gaps of functions with generalized convexity," *Carpathian Journal of Mathematics*, vol. 31, no. 2, pp. 173–180, 2015.

[14] K. Liu, J.-R. Wang, and D. O'Regan, "On the Hermite-Hadamard type inequality for ψ-Riemann-Liouville fractional integrals via convex functions," *Journal of Inequalities and Applications*, vol. 2019, Article ID 27, no. 1, p. 10, 2019.

[15] S. Mubeen and G. M. Habibullah, "k-fractional integrals and application," *International Journal of Contemporary Mathematical Sciences*, vol. 7, no. 1–4, pp. 89–94, 2012.

[16] C. Huang and L. Liu, "Sharp function inequalities and boundness for Toeplitz type operator related to general fractional singular integral operator," *Publications de l'Institut Mathématique (Belgrade)*, vol. 92, no. 106, pp. 165–176, 2012.

[17] K. S. Nisar, G. Rahman, J. Choi, S. Mubeen, and M. Arshad, "Generalized hypergeometric k-functions via (k,s)-fractional calculus," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 4, pp. 1791–1800, 2017.

[18] G. Rahman, P. Agarwal, S. Mubeen, and M. Arshad, "Fractional integral operators involving extended Mittag-Leffler function as its kernel," *Boletín de la Sociedad Matemática Mexicana*, vol. 24, no. 2, pp. 381–392, 2018.

[19] A. Atangana, "Non validity of index law in fractional calculus: a fractional differential operator with Markovian and non-Markovian properties," *Physica A: Statistical Mechanics and Its Applications*, vol. 505, pp. 688–706, 2018.

[20] P. Agarwal, M. Chand, J. Choi, and G. Singh, "Certain fractional integrals and image formulas of generalized k-Bessel function," *Communications of the Korean Mathematical Society*, vol. 33, no. 2, pp. 423–436, 2018.

[21] M. Bohner, G. Rahman, S. Mubeen, and K. S. Nisar, "A further extension of the extended Riemann-Liouville fractional derivative operator," *Turkish Journal of Mathematics*, vol. 42, no. 5, pp. 2631–2642, 2018.

[22] F. Liu, L. Feng, V. Anh, and J. Li, "Unstructured-mesh Galerkin finite element method for the two-dimensional multi-term time-space fractional Bloch-Torrey equations on irregular convex domains," *Computers & Mathematics with Applications*, vol. 78, no. 5, pp. 1637–1650, 2019.

[23] S. Zhou and Y. Jiang, "Finite volume methods for N-dimensional time fractional Fokker-Planck equations," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 6, pp. 3167–3186, 2019.

[24] Y. Jiang and X. Xu, "A monotone finite volume method for time fractional Fokker-Planck equations," *Science China Mathematics*, vol. 62, no. 4, pp. 783–794, 2019.

[25] P. Agarwal and J. E. Restrepo, "An extension by means of ω-weighted classes of the generalized Riemann-Liouville k-fractional integral inequalities," *Journal of Mathematical Inequalities*, vol. 14, no. 1, pp. 35–46, 2020.

[26] W. Tan, F.-L. Jiang, C.-X. Huang, and L. Zhou, "Synchronization for a class of fractional-order hyperchaotic system and its application," *Journal of Applied Mathematics*, vol. 2012, Article ID 974639, 11 pages, 2012.

[27] D. Baleanu and B. Shiri, "Collocation methods for fractional differential equations involving non-singular kernel," *Chaos, Solitons & Fractals*, vol. 116, pp. 136–145, 2018.

[28] G. Singh, P. Agarwal, S. Araci, and M. Acikgoz, "Certain fractional calculus formulas involving extended generalized Mathieu series," *Advances in Difference Equations*, vol. 2018, Article ID 144, no. 1, p. 30, 2018.

[29] P. Agarwal, F. Qi, M. Chand, and G. Singh, "Some fractional differential equations involving generalized hypergeometric functions," *Journal of Applied Analysis*, vol. 25, no. 1, pp. 37–44, 2019.

[30] M. H. Heydari and A. Atangana, "A cardinal approach for nonlinear variable-order time fractional Schrödinger equation defined by Atangana-Baleanu-Caputo derivative," *Chaos, Solitons & Fractals*, vol. 128, pp. 339–348, 2019.

[31] D. Baleanu and G. C. Wu, "Some further results of the Laplace transform for variable-order fractional difference equations," *Fractional Calculus and Applied Analysis*, vol. 22, no. 6, pp. 1641–1654, 2019.

[32] D. Baleanu, S. Rezapour, and Z. Saberpour, "On fractional integro-differential inclusions via the extended fractional Caputo-Fabrizio derivation," *Boundary Value Problems*, vol. 2019, Article ID 79, no. 1, p. 17, 2019.

[33] S. X. Zhou, C.-X. Huang, H.-J. Hu, and L. Liu, "Inequality estimates for the boundedness of multilinear singular and fractional integral operators," *Journal of Inequalities and Applications*, vol. 2013, Article ID 303, 15 pages, 2013.

[34] E. Şen, M. Acikgoz, J. J. Seo, S. Araci, and K. Oruçoğlu, "Existence and uniqueness of positive solutions of boundary-value problems for fractional differential equations with p-Laplacian operator and identities on the some special polynomials," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 753171, 11 pages, 2013.

[35] J. Wu and Y.-C. Liu, "Uniqueness results and convergence of successive approximations for fractional differential equations," *Hacettepe Journal of Mathematics and Statistics*, vol. 42, no. 2, pp. 149–158, 2013.

[36] S. Araci, E. Şen, M. Acikgoz, and H.-M. Srivastava, "Existence and uniqueness of positive and nondecreasing solutions for a class of fractional boundary value problems involving the p-Laplacian operator," *Advances in Difference Equations*, vol. 2015, Article ID 40, no. 1, p. 12, 2015.

[37] S. Araci, E. Şen, M. Acikgoz, and K. Oruçoğlu, "Identities involving some new special polynomials arising from the applications of fractional calculus," *Applied Mathematics & Information Sciences*, vol. 9, no. 5, pp. 2657–2662, 2015.

[38] A. A. El-Sayed and P. Agarwal, “Numerical solution of multiterm variable-order fractional differential equations via shifted Legendre polynomials,” *Mathematical Methods in the Applied Sciences*, vol. 42, no. 11, pp. 3978–3991, 2019.

[39] V. F. Morales-Delgado, J. F. Gómez-Aguilar, K. M. Saad, M. A. Khan, and P. Agarwal, “Analytic solution for oxygen diffusion from capillary to tissues involving external force effects: a fractional calculus approach,” *Physica A: Statistical Mechanics and Its Applications*, vol. 523, pp. 48–65, 2019.

[40] S. Jain and P. Agarwal, “On new applications of fractional calculus,” *Boletim da Sociedade Paranaense de Matemática*, vol. 37, no. 3, pp. 113–118, 2019.

[41] H. Khalil, R. A. Khan, D. Baleanu, and M. M. Rashidi, “Some new operational matrices and its application to fractional order Poisson equations with integral type boundary constrains,” *Computers & Mathematics with Applications*, vol. 78, no. 6, pp. 1826–1837, 2019.

[42] K. S. Nisar, G. Rahman, and K. Mehrez, “Chebyshev type inequalities via generalized fractional conformable integrals,” *Journal of Inequalities and Applications*, vol. 2019, Article ID 245, no. 1, p. 9, 2019.

[43] P. A. Feulefack, J. D. Djida, and A. Atangana, “A new model of groundwater flow within an unconfined aquifer: application of Caputo-Fabrizio fractional derivative,” *Discrete and Continuous Dynamical Systems—Series B*, vol. 24, no. 7, pp. 3227–3247, 2019.

[44] A. Atangana and S. İ. Araz, “Analysis of a new partial integro-differential equation with mixed fractional operators,” *Chaos, Solitons & Fractals*, vol. 127, pp. 257–271, 2019.

[45] G. Rahman, K. S. Nisar, A. Ghaffar, and F. Qi, “Some inequalities of the Grüss type for conformable *k*-fractional integral operators,” *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales: Serie A. Matemáticas*, vol. 114, no. 1, Article ID 9, 2020.

[46] S. Shahmorad, M. H. Ostadzad, and D. Baleanu, “A Tau-like numerical method for solving fractional delay integro-differential equations,” *Applied Numerical Mathematics*, vol. 151, pp. 322–336, 2020.

[47] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, Netherlands, 2006.

[48] J. V. C. Sousa and E. C. Oliveira, “On the ψ-Hilfer fractional derivative,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 60, pp. 72–91, 2018.

[49] W. W. Breckner, “Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen,” *Publications de l’Institut Mathématique*, vol. 23, no. 37, pp. 13–20, 1978.

[50] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalities*, D. Reidel Publishing Co., Dordrecht, Netherlands, 1988.