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Research Article

Optimal Investment Policy for Insurers under the Constant Elasticity of Variance Model with a Correlated Random Risk Process

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This paper investigates the optimal portfolio choice problem for a large insurer with negative exponential utility over terminal wealth under the constant elasticity of variance (CEV) model. The surplus process is assumed to follow a diffusion approximation model with the Brownian motion in which is correlated with that driving the price of the risky asset. We first derive the corresponding Hamilton–Jacobi–Bellman (HJB) equation and then obtain explicit solutions to the value function as well as the optimal control by applying a variable change technique and the Feynman–Kac formula. Finally, we discuss the economic implications of the optimal policy.

1. Introduction

Since the seminal work of Browne [1], there is a growing literature investigating the dynamic portfolio choice problems for insurers under the stochastic optimal control framework. However, Browne [1] assumes that the risky asset's price is driven by geometric Brownian motions (GBMs), which implies that the expected instantaneous return and volatility of the risky asset are constant and deterministic. To be more empirical, nowadays, there has been a series of works analyzing the insurer's portfolio optimization problems with different variants of stochastic market settings, such as the stochastic interest rate model (e.g., Guan and Liang [2]), stochastic return model (e.g., Li et al. [3]), and stochastic volatility model (e.g., Li et al. [4] and Gu et al. [5]).

As a special stochastic volatility model, the constant elasticity of variance (CEV) model is widely used in finance theory and practice. The CEV model is a generalization of the GBM of which the variance elasticity parameter equals to zero and has been successfully employed in the option pricing literature to model the empirical observed pattern of stock prices with heavy tail (e.g., Schroder [6], Boyle and Tian [7], Davydov and Linetsky [8], and Park and Kim [9]). Moreover, the CEV model helps explain volatility smiles (Cox and Ross [10] and Cox [11]). Jones [12] further suggested that compared to Heston's stochastic volatility model, the equity index return data are better represented by a stochastic variance model in the CEV class.

Recently, due to the empirical advantage and mathematical tractability, considerable research efforts have been devoted to considering optimal investment problems under the CEV model. The most commonly selected objectives include maximizing expected utility and mean-variance criterion. For the objective of maximizing expected utility, the traditional portfolio selection problems (e.g., Zhao and Rong [13], Bakkaloglu et al. [14], and Josa-Fombellida et al. [15]), the optimal investment problem for pension plans with different classes of hyperbolic absolute risk aversion (HARA) utility functions (e.g., Xiao et al. [16], Gao [17, 18], and Jung and Kim [19]), and the optimal investment and reinsurance problem (e.g., Gu et al. [5], Gu et al. [20], Lin and Li [21], Li et al. [22], Zheng et al. [23], K. Wu and W. Wu [24], Chunxiang et al. [25], and Wang et al. [26]) are extensively investigated under the CEV model. For the objective of mean-variance criterion, Basak and Chabakauri [27] derived the closed-form optimal time-consistent investment policy for a self-financing portfolio, and lately, Shen et al. [28] discussed the corresponding precommitment solution; Li et al. [29] and Zhao et al. [30] studied the time-consistent reinsurance-investment strategy for an insurance portfolio. Li et al. [31] investigated the time-consistent investment strategy for a defined contribution (DC) pension plan. Zhang and Chen [32] also explored the asset-liability management (ALM) problem.

Even though the optimal investment problem for an insurer is more fundamental than the optimal investment-reinsurance problem, it is still worthy to be considered under the CEV model. In practice, because the reinsurance service is not cheap, large insurers with adequate risk tolerance may prefer an investment-only policy to an investment-reinsurance policy. Moreover, most of the aforementioned works specialize the assumptions to that the uncertainty source in the insurer's surplus process is perfectly uncorrelated with that in the risky asset's price process described by the CEV model, which implies that the insurance market is independent of the financial market. As a result, the optimal investment strategy derived is independent of the insurer's surplus model (see Gu et al. [20], Lin and Li [21], Gu et al. [5], and Chunxiang et al. [25]). But, some literature goes to another extreme case by assuming that the two processes are perfectly correlated and obtain the explicit solution only in the case of special variance elasticity parameters (see Wang et al. [26]); meanwhile, they show that the diffusion part of the surplus has an effect on the optimal investment policy. Yuan and Lai [33] also adopted similar assumptions in Wang et al. [26] to study the optimal investment strategy of a family with a random household expenditure, and they only derive the approximate numerical solutions.

However, in reality, besides the idiosyncratic risk, the insurer's surplus process and the risky asset's price process are affected by the systematic risk, which leads to a dependence between the two uncertainty sources. To the best of our knowledge, Browne [1] considered the correlation between the risk of the insurer's surplus process and that of the risky asset's price process under the GBMs in the investment problem of an insurer, while there has been no literature focusing on the similar problems under the CEV or other stochastic market models so far.

In this paper, we focus on the correlation that occurs between Brownian motions in the insurer's surplus process and those in the risky asset's price process, which represents the common uncertainty between the insurance and financial markets. By extending the price model of the risky asset to the CEV model, we reconsider the negative exponential utility maximizing problem of Browne [1]. The surplus process is described by the diffusion approximation model, and particularly, the Brownian motion driving price process of the risky asset is correlated with that driving the surplus process. By the stochastic optimal control theory, we first establish a three-dimensional Hamilton–Jacobi–Bellman (HJB) equation for the optimization problem and then simplify it into two parabolic partial differential equations (pdes) via a variable change technique. By the Feynman–Kac formula, we solve the two pdes and obtain the explicit expressions of value function as well as the optimal investment strategy. Finally, we compare the result with that of Browne [1] and Gu et al. [5], respectively, which are special cases of our model.

Due to the consideration of the correlation, the corresponding HJB equation becomes more difficult to solve. Specifically, the Legendre dual transformation technique introduced by Xiao et al. [16] and then heavily used in many of the aforementioned works (see Gao [17, 18] and Jung and Kim [19]) to reduce the HJB equation into a linear pde cannot be directly adapted to models with general correlation. For example, following the way, Wang et al. [26] and Yuan and Lai [33] assumed that the correlation equals to ± 1 to remove the nonlinear parts of the HJB equation. In this paper, instead of that method, we directly conjecture the functional form of the value function and by which the HJB equation can be directly simplified into two parabolic pdes. Moreover, we relax the perfect correlation restrictions. So, obtaining the explicit solution of the investment strategy in the case of imperfectly correlated uncertainty sources is a technique contribution of our paper.

Moreover, another contribution of our paper is that many interesting implications are obtained after introducing the correlation. We find that the optimal investment strategy can be separated into four independent components: the myopic, dynamic, static, and delta hedging demands. The myopic demands, also known as the Kelly criterion, are to optimize over the next instant; the dynamical hedging demands are to hedge against the fluctuations of the instantaneous volatility; the static hedging demands are to hedge against the hedgeable risk of the surplus; and the delta hedging demands are to hedge the fluctuations risk of the static hedged portfolio. In particular, both static and delta demands vanish, and our results reduce to that of Gu et al. [5] if the correlation vanishes. Asymptotic analysis further shows that, as the variance elasticity parameters approach to zero, the dynamic and delta hedging demands both vanish, and the results are equivalent to those of Browne [1].

The remainder of the article is organized as follows. In Section 2, the insurer's optimal investment problem is formulated. In Section 3, the explicit solution of the value function as well as the optimal investment policy are derived. In Section 4, the economic interpretation of the optimal investment policy is provided. Section 5 concludes this paper. The details to derive the optimal investment strategy are postponed to Appendix.

2. Problem Formulation

In this section, we will give some basic assumptions and then formulate the insurer's portfolio choice problem.

We consider a continuous-time Markovian economy with a fixed and finite time horizon $[t, T]$. Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{t\le s\le T}, \mathbb{P})$ satisfying the usual conditions, where $\mathcal{F}_s = \sigma((Z_1(s) \times Z_2(s)); t \le s \le T)$ is the information available until time s and $Z_1(s), Z_2(s)$ are two independent standard winner processes under measure $\mathbb{P}$. In that follows, we also assume that all stochastic processes and random variables are adapted to $\{\mathcal{F}_s\}_{t\le s\le T}$, and their moments introduced are well defined, without explicitly stating the regular conditions. Meanwhile, we also assume that trading takes place continuously over time, no transaction costs or taxes are involved in the trading, and there is no difference between lending and borrowing rate.

Without loss of generality, we assume the insurer can invest in a risk-free asset (the bank account) and a risky asset (stock). The price of the bond is given by

$$\frac{dB(s)}{B(s)} = r ds, \quad B(t) = 1, \tag{1}$$

where $r > 0$ is the risk-free rate. The stock price, S, follows the CEV model:

$$\frac{dS(s)}{S(s)} = \mu ds + \sigma S(s)^{\alpha} dZ_1(s), \quad S(t) = S > 0, \tag{2}$$

where $\mu > r$ is the stock mean return; σ is a positive constant; 2α is the variance elasticity parameter; and $\sigma S(s)^{\alpha}$ is the instantaneous volatility. We also assume that $\alpha \geq 0$ as in Gu et al. [5], Li et al. [31], and Emanuel and Macbeth [34] for that the price may reach 0 for some negative α, which is also supported by the empirical estimations in Jones [12]. Since $S(s)$ is stochastic, the instantaneous volatility is also stochastic so that the insurer faces time-varying investment opportunities.

As in Browne [1], without investment, the wealth of the insurer follows a Brownian motion with drift, i.e., the diffusion limit of the classic Cramer-Lundberg risk model

$$dR(s) = \mu_m ds + \sigma_m \left(\rho dZ_1(s) + \sqrt{1-\rho^2}\, dZ_2(s)\right), \quad R(t) = 0, \tag{3}$$

where μ_m, σ_m, and ρ are all constants. As pointed out in Promislow and Young [35], when the parameters satisfy that (μ_m/σ_m) is big enough (at least 3), one will want to use this model in actuarial practice. We note that, under this setup, the market is incomplete as trading in the risky assets and bond cannot perfectly hedge against the surplus risk. However, in the special cases of perfect correlation between the stock return and risk process, $\rho = \pm 1$, dynamic market completeness is obtained.

A large insurer with adequate risk tolerance in this economy is endowed at time t with an initial wealth of W. The insurer chooses an investment policy w_s, where $w_s W(s)$ denotes the total money amount invested in the stock at time s and the remaining $1 - w_s$ portion of the wealth is invested in the risk-free asset. Thus, under policy w_s and with the initial wealth W, the wealth process $W(s)$ becomes

$$\begin{aligned} dW(s) &= \frac{dB(s)\left((1-w_s)W(s)\right)}{B(s)} + \frac{w_s W(s) dS(s)}{S(s)} + dR(s) \\ &= \left(\mu_m - r w_s W(s) + r W(s) + \mu w_s W(s)\right) ds \\ &\quad + \left(\rho\sigma_m + \sigma w_s W(s) S(s)^{\alpha}\right) dZ_1(s) \\ &\quad + \sqrt{1-\rho^2}\,\sigma_m dZ_2(s). \end{aligned} \tag{4}$$

A control policy is said to be admissible if $\forall s \in [t, T], w_s$ is $\mathcal{F}_s$ progressively measurable, (4) has a unique strong solution, and $E_t\left[\int_t^T (w_s W(s) S(s)^{\alpha})^2 ds\right] < \infty$. Denote by $\mathcal{A}$ the collection of all admissible policies. Given the initial wealth $W(t) = W$ and spot price $S(t) = S$, the insurer aims to maximize the expected utility over terminal wealth $W(T)$, i.e.,

$$\max_{w_s \in \mathcal{A}} \mathbb{E}\left[U(W_T) \,\middle|\, W(t) = W, S(t) = S\right], \tag{5}$$

where $U(\cdot)$ is utility function satisfying $U\prime(W) > 0, U''(W) < 0$. In this paper, we specialize our setting that the insurer is guided by a constant absolute risk aversion (CARA) preference

$$U(W) = \frac{-e^{-\gamma W}}{\gamma}, \tag{6}$$

where $\gamma = -(U''(W)/U'(W)) > 0$ is the absolute risk aversion coefficient. Note that this function plays a vital role in actuarial mathematics and insurance practice for that it is the unique utility function under the principle of "zero utility" giving a fair premium that is independent of the level of reserves of insurers.

Remark 1. If $\alpha = 0$, the CEV model turns into the geometric Brownian motion process; then, our problem is equivalent to the utility maximizing problem in Browne [1].

Remark 2. If $\rho = 0$, our problem degenerates into the optimal investment-only problem in Gu et al. [5].

3. Problem Solution

In this section, we first provide the general framework for optimization problems (4) and (5) by using the classical tools of stochastic optimal control and then try to simplify the corresponding Hamilton–Jacobi–Bellman (HJB) equation into two parabolic pdes via a variable change technique. Finally, we solve the two parabolic pdes by the Feynman–Kac formula and obtain the value function as well as the optimal investment policy.

3.1. General Framework. For portfolio choice problem (5) with dynamic budget constrain (4), since the wealth process $W(s)$ contains two state variables $S(s), W(s)$, we can write the value function as

$$J(t, W, S) = \max_{w \in \mathcal{A}} \mathbb{E}\left[U(W_T) \,\middle|\, W(t) = W, S(t) = S\right]. \tag{7}$$

According to the classical dynamic programming principle, $J(t, W, S)$ satisfies the following HJB equation:

$$0 = \max_{w\in\mathscr{A}} \mathscr{A}^w J(t, W, S), \tag{8a}$$

$$\begin{aligned}\mathscr{A}^w f(t, W, S) = {} & f_W\left(\mu_m - r(w-1)W + \mu w W\right) \\ & + f_{WS}\left(\rho\sigma\sigma_m S^{\alpha+1} + \sigma^2 w W S^{2\alpha+1}\right) \\ & + f_{WW}\left(\frac{\sigma_m^2}{2} + \rho\sigma w W \sigma_m S^{\alpha} + \frac{1}{2}\sigma^2 w^2 W^2 S^{2\alpha}\right) \\ & + \frac{1}{2}\sigma^2 f_{SS} S^{2\alpha+2} + \mu S f_S + f_t,\end{aligned} \tag{8b}$$

$$J(T, W, S) = U(W), \quad \forall S, \tag{8c}$$

where $\mathscr{A}^w f(t, W, S)$ is the infinitesimal generator for w controlled stochastic process (4). Assuming there exists a sufficiently smooth solution $J(t, W, S)$ to the HJB equation, differentiating (8a)–(8c) with respect to w gives the first-order condition:

$$w^* = \frac{J_W(r-\mu)S^{-2\alpha}}{\sigma^2 W J_{WW}} - \frac{S J_{WS}}{W J_{WW}} - \frac{\rho\sigma_m S^{-\alpha}}{\sigma W}, \tag{9}$$

while the second-order condition is $\sigma^2 W^2 S^{2\alpha} J_{SS} \le 0$, which, in other words, says that $J(t, W, S)$ must be concave as a function of W. Inserting first-order condition (9) into (8a)–(8c) yields after some routine manipulations the following two-dimensional nonlinear parabolic pde:

$$\begin{aligned}0 = {} & J_W\left(\frac{S J_{WS}(r-\mu)}{J_{WW}} + \mu_m + \frac{\rho\sigma_m(r-\mu)S^{-\alpha}}{\sigma} + rW\right) \\ & - \frac{1}{2}\left(\rho^2 - 1\right) J_{WW}\sigma_m^2 \\ & - \frac{J_W^2(r-\mu)^2 S^{-2\alpha}}{2\sigma^2 J_{WW}} + \frac{1}{2}\sigma^2 J_{SS} S^{2\alpha+2} - \frac{\sigma^2 J_{WS}^2 S^{2\alpha+2}}{2 J_{WW}} \\ & + \mu S J_S + J_t,\end{aligned} \tag{10}$$

with boundary condition (8c).

Here, we notice that the optimal portfolio choice problem has been transformed into a nonlinear pde. The goal now is to construct an explicit solution to (10) and then incorporate the solution in (9) and obtain the optimal investment policy. However, as it stands, it is difficult to solve (10). Therefore, we shall first reduce the dimension and remove the nonlinearity of HJB equation (10).

3.2. Simplifying the Nonlinear PDE. Inspired by Browne [1], Gu et al. [5], and Gao [18], we try to conjecture a solution to (10) taking the following functional form:

$$J(t, W, S) = -\frac{\exp(g(t,S))\exp(-\gamma\exp(r(T-t))(h(t,S)+W))}{\gamma}. \tag{11}$$

Then, we have

$$\left(\frac{J_S}{J}\right) = g_S - \gamma h_S e^{r(T-t)}, \tag{12a}$$

$$\left(\frac{J_{SS}}{J}\right) = -\gamma e^{r(T-t)}\left(2 g_S h_S + h_{SS}\right) + g_S^2 + g_{SS} + \gamma^2 h_S^2 e^{2r(T-t)}, \tag{12b}$$

$$\left(\frac{J_W}{J}\right) = \gamma\left(-e^{r(T-t)}\right), \tag{12c}$$

$$\left(\frac{J_{WS}}{J}\right) = \gamma e^{r(T-t)}\left(\gamma h_S e^{r(T-t)} - g_S\right), \tag{12d}$$

$$\left(\frac{J_{WW}}{J}\right) = \gamma^2 e^{2r(T-t)}, \tag{12e}$$

$$\left(\frac{J_t}{J}\right) = g_t + \gamma e^{r(T-t)}\left(r(h+W) - h_t\right). \tag{12f}$$

Incorporating these partial derivatives in (10) and after some simplifications, we obtain

$$\begin{aligned}0 = {} & rSg_S + \frac{1}{2}\sigma^2 g_{SS} S^{2\alpha+2} + g_t - \frac{(r-\mu)^2 S^{-2\alpha}}{2\sigma^2} \\ & - \gamma e^{r(T-t)}\left(rSh_S - hr + \frac{1}{2}\sigma^2 h_{SS} S^{2\alpha+2} + h_t\right. \\ & \left. + \mu_m + \frac{\rho\sigma_m(r-\mu)S^{-\alpha}}{\sigma} + \frac{1}{2}\gamma\left(\rho^2-1\right)\sigma_m^2 e^{r(T-t)}\right).\end{aligned} \tag{13}$$

We can decompose (13) into two independent linear parabolic pdes (14) and (16), and also by (6), we have boundary conditions (15) and (17):

$$0 = rSg_S + \frac{1}{2}\sigma^2 g_{SS} S^{2\alpha+2} + g_t - \frac{(r-\mu)^2 S^{-2\alpha}}{2\sigma^2}, \tag{14}$$

$$0 = g(T, S), \quad \forall S, \tag{15}$$

$$\begin{aligned}0 = {} & rSh_S - hr + \frac{1}{2}\sigma^2 h_{SS} S^{2\alpha+2} + h_t + \mu_m \\ & + \frac{\rho\sigma_m(r-\mu)S^{-\alpha}}{\sigma} + \frac{1}{2}\gamma\left(\rho^2-1\right)\sigma_m^2 e^{r(T-t)},\end{aligned} \tag{16}$$

$$0 = h(T, S), \quad \forall S. \tag{17}$$

Here, we have simplified the nonlinear pde to two linear parabolic pdes; the problem now is to solve (14) and (16) and replace the solutions in (11) and (9) so as to find the optimal investment policy.

3.3. Explicit Solution for the Insurer's Optimal Investment Problem. Since the problem for $\alpha = 0$ has been investigated in Browne [1], we, here, mainly focus on the solutions for $\alpha > 0$. By the techniques of stochastic analysis, we can obtain explicit solutions to (14) and (16), and we have the following theorem.

Theorem 1. *For portfolio choice problem (5), a solution to HJB equation (8a)–(8c) with terminal condition (10) is given by $J(t,W,S)$, and the corresponding optimal investment policy is given by w^* in feedback form if $\alpha>0$, where*

$$J(t,W,S) = -\frac{\exp(g(t,S))\exp(-\gamma\exp(r(T-t))(h(t,S)+W))}{\gamma}, \tag{18}$$

$$\begin{aligned} w^* &= \frac{S^{-2\alpha}e^{r(t-T)}(\mu-r-2\alpha\sigma^2 f_1(t))}{\gamma\sigma^2 W} - \frac{\rho\sigma_m S^{-\alpha}}{\sigma W} - \frac{Sh_S}{W} \\ &= \frac{S^{-2\alpha}e^{r(t-T)}(\mu-r-2\alpha\sigma^2 f_1(t))}{\gamma\sigma^2 W} - \frac{\rho\sigma_m S^{-\alpha}}{\sigma W} \\ &\quad - \frac{\rho\sigma_m S(r-\mu)\left(\int_t^T e^{-r(s-t)}Dl(s,S)\mathrm{d}s\right)}{\sigma W}, \end{aligned} \tag{19}$$

$$\begin{aligned} g(t,S) &= f_1(t)S^{-2\alpha} + f_0(t), \\ f_1(t) &= \frac{(r-\mu)^2(e^{-2\alpha r(T-t)}-1)}{4\alpha r\sigma^2}, \\ f_0(t) &= \frac{(2\alpha+1)(r-\mu)^2((2\alpha r(t-T)+1)-e^{-2\alpha r(t-T)})}{8\alpha r^2}, \\ h(t,S) &= \frac{\rho\sigma_m(r-\mu)}{\sigma}\int_t^T e^{-r(s-t)}l(s,S)\mathrm{d}s + \frac{\mu_m(1-e^{-r(t-T)})}{r} - \frac{\gamma(\rho^2-1)\sigma_m^2\sinh(r(t-T))}{2r}, \\ l(s,S) &= \frac{\Gamma((3/2)+(1/2\alpha))e^{\alpha(-r)(s-t)}{}_1F_1(-(1/2);\,(2\alpha+1/2\alpha);\,-S^{-2\alpha}M(s))}{\Gamma(1+(1/2\alpha))\sqrt{M(s)}}, \\ Dl(s,S) &= -\frac{\alpha\Gamma((3/2)+(1/2\alpha))\sqrt{M(s)}S^{-2\alpha-1}e^{\alpha r(t-s)}{}_1F_1((1/2);2+(1/2\alpha);-S^{-2\alpha}M(s))}{\Gamma(2+(1/2\alpha))}, \\ M(s) &= \frac{r}{\alpha\sigma^2(e^{2\alpha r(s-t)}-1)}. \end{aligned} \tag{20}$$

Here, $\Gamma(\cdot)$ is the gamma function and ${}_1F_1(a;b;z)$ denotes the confluent hypergeometric function, which is defined as follows:

$${}_1F_1(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}\exp(zt)\mathrm{d}t. \tag{21}$$

Proof. The solution for $g(t,S), h(t,S)$ can be seen from Appendix. Inserting (18) into (9) yields (19). □

Remark 3. Obviously, Theorem 3 of Gu et al. [5] is the special case of our results as $\rho=0$. Furthermore, let $\sigma_m=0$, and the optimal investment policy also reduces to the results of Gao [17, 18].

Remark 4. According to Theorem 1, we find that the dollar amount invested in the risky asset, w^*W, is independent of the insurer's current wealth W, which results from the property of CARA utility function and is consistent with the results of Gu et al. [5], Gao [17, 18], and Gu et al. [20].

Remark 5. $h(t,S)$ is the indifference pricing of the surplus, which is an analog of equation (80) in Browne [1]. In general, indifference pricing depends on the instantaneous volatility of the risky assets. Consequently, the indifference pricing under the CEV model must be function of S unless $\alpha=0$. However, as $\alpha\longrightarrow 0^+$, $h(t,S)$ would reduce to equation (80) in Browne [1], which will be shown later in (34).

4. Discussion

In this section, we discuss the economic interpretation of the optimal investment policy by decomposing it in different parts.

4.1. Economic Interpretation of the Optimal Investment Policy. It is easy to write the optimal investment policy w^* as the following structure:

$$w^* = w_{\text{myopic}}(t) + w_{\text{dhedge}}(t) + w_{\text{shedge}}(t) + w_{\text{deltahedge}}(t), \tag{22}$$

where

$$
\begin{aligned}
w_{\text{myopic}}(t) &= \frac{(\mu - r)S^{-2\alpha}e^{-r(T-t)}}{\gamma\sigma^2 W},\\
w_{\text{dhedge}}(t) &= -\frac{2\alpha f_1(t)S^{-2\alpha}e^{r(t-T)}}{\gamma W},\\
w_{\text{shedge}}(t) &= -\frac{\rho\sigma_m S^{-\alpha}}{\sigma W},\\
w_{\text{deltahedge}}(t) &= -\frac{\rho\sigma_m S^{-\alpha}}{\sigma W}.
\end{aligned}
\tag{23}
$$

To gain a detailed insight on the optimal investment policy, we first consider the results for special model parameters.

Corollary 1. *If $\mu_m = 0$ and $\sigma_m = 0$, then $h(t, S) = 0$, $w_{shedge}(t) = 0$, and $w_{deltahedge}(t) = 0$, and the value function $J(t, W, S)$ is given by*

$$
J(t, W, S) = -\frac{\exp(g(t, S))\exp(-\gamma\exp(r(T - t))W)}{\gamma},
\tag{24}
$$

and the corresponding investment policy reduces to

$$
w^* = w_{\text{myopic}}(t) + w_{\text{dhedge}}(t).
\tag{25}
$$

Note that if $\mu_m = 0$ and $\sigma_m = 0$, the surplus process of the insurer vanishes, and the portfolio is self-financing. If the insurer invests all the wealth in the risk-free asset, the expected utility at time T would be $J(t, W, S) = -(e^{-\gamma W e^{r(T-t)}}/\gamma)$. However, by taking an optimal investment policy on both risk-free asset and risky asset, the value function is modified by a factor $e^{g(t,S)}$. Since $g(t, S) < 0, \forall t < T, S > 0$, the insurer's welfare is always improved by investment in the risky asset.

The optimal investment policy given by (25) consists of the myopic demands and the dynamic hedging demands. The myopic demands, $w_{\text{myopic}}(t)$, also known as the Kelly criterion, would be the investment policy for an insurer who optimizes over the next instant, not accounting for her future investment. The dynamical hedging demands, $w_{\text{dhedge}}(t)$, arise due to the need to hedge against the fluctuations in the investment opportunities because of the stochastic volatility.

Different from the strategy in Gu et al. [5], besides the myopic and dynamic hedging parts, there exist another two terms in the optimal investment strategy in our model. We denote $w_{\text{shedge}}(t)$ and $-(Sh_S/W)$ as static hedging demands and delta hedging demands, respectively. To investigate the role of static hedging demands, suppose that the insurer adopts a policy by investing $w_{\text{shedge}}W = -(\rho\sigma_m S(s)^{-\alpha}/\sigma)$ dollar amount of money in the risky asset at time s; then, the dynamics of the wealth, $W_c(s)$, would be

$$
\begin{aligned}
\mathrm{d}W_c(s) &= -\frac{\rho\sigma_m S(s)^{-\alpha}}{\sigma}(\mathrm{d}S(s)/S(s)) + \mathrm{d}R(s) + r\frac{\rho\sigma_m S(s)^{-\alpha}}{\sigma}\mathrm{d}s\\
&= \left(\mu_m - \frac{(\mu - r)\rho\sigma_m S(s)^{-\alpha}}{\sigma}\right)\mathrm{d}s + \sqrt{1-\rho^2}\sigma_m \mathrm{d}Z_2(s).
\end{aligned}
\tag{26}
$$

Note that (26) contains only $\mathrm{d}Z_2(s)$, the unhedgeable part of risk of the surplus, while the hedgeable risk $\mathrm{d}Z_1(s)$ vanishes, which implies that static hedging demands w_{shedge} perfectly hedge the hedgeable risk of the surplus. For special case of complete market, i.e., $\rho = \pm 1$, the unhedgeable risk also vanishes.

However, unless $\alpha = 0$, the wealth of the insurer in (26) is still stochastic as the drift term contains $S(s)$, which results in the delta hedging demands. The delta hedge demands can also be understood in the viewpoint of derivative pricing. Recall $h(t, S)$ is the indifference pricing of the surplus; by Ito's lemma, the dynamics for the portfolio $(h(t, S), -Sh_S(t, S))$ become

$$
\begin{aligned}
&\mathrm{d}h(s, S(s)) - S(s)h_{SS}(s, S(s))(\mathrm{d}S(s) - r)\\
&= \mathrm{d}s\left(rS(s)h_{SS} + \frac{1}{2}\sigma^2 S(s)^{2\alpha+2}h_{SS}(s, S(s)) + h_t(s, S(s))\right),
\end{aligned}
\tag{27}
$$

which implies that the portfolio is free of risk. Therefore, we call $-(Sh_S/W)$ the delta hedge demands.

Corollary 2. *If $\rho\sigma_m = 0$, investment policy is given by*

$$
w^* = w_{\text{myopic}}(t) + w_{\text{dhedge}}(t).
\tag{28}
$$

On the one hand, if $\rho = 0$, i.e., the risk of the financial market is perfectly uncorrelated with that of the surplus, taking positions in the risky asset does not help to reduce the surplus risk so that the static hedging demands vanish. Moreover, the surplus is independent of risk asset states, which leads to the delta hedge demands vanishing as well. On the other hand, if $\sigma_m = 0$, the surplus is equivalent to an annuity contract with continuous-time payoff μ_m, and $h(t, S) = (\mu_m(1 - e^{-r(T-t)})/r)$ is exactly its present value. Both static and delta hedge demands vanish for that the surplus and its indifference pricing are deterministic, and there is no need to hedging.

In addition to the parameters of the surplus, the variance elasticity parameter, α, is also worthy of discussing. Obviously, the optimal investment policy given by Theorem 1 is not well defined at $\alpha = 0$; instead, we are interested in its asymptotic behavior as $\alpha \longrightarrow 0^+$ and obtain the following corollary.

Corollary 3.

$$
\lim_{\alpha \longrightarrow 0^+} w^* = \frac{(\mu - r)e^{r(t-T)}}{\gamma\sigma^2 W} - \frac{\rho\sigma_m}{\sigma W}.
\tag{29}
$$

Proof. As $\alpha \longrightarrow 0^+$, we have

$$
\begin{aligned}
\lim_{\alpha \longrightarrow 0^+} f_1(t) &= \frac{(r - \mu)^2(t - T)}{2\sigma^2},\\
\lim_{\alpha \longrightarrow 0^+} f_0(t) &= 0.
\end{aligned}
\tag{30}
$$

Henceforth,

Table 1: Four parts of the optimal investment policy versus the parameters of the surplus and elasticity variance.

	Myopic	Dynamic	Static	Delta	Related works
$\alpha = 0, \rho\sigma_m = 0$	√				Special cases of Browne [1]
$\alpha \neq 0, \rho\sigma_m = 0$	√	√			Gu et al. [5] and Gao [17, 18]
$\alpha = 0, \rho\sigma_m \neq 0$	√		√		Browne [1]
$\alpha \neq 0, \rho\sigma_m \neq 0$	√	√	√	√	This paper

$$\lim_{\alpha \longrightarrow 0^+} g(t,S) = \frac{(r-\mu)^2 (t-T)}{2\sigma^2}. \tag{31}$$

As

$$\begin{aligned} -\frac{r(\coth(\alpha r\tau)-1)}{2\alpha\sigma^2} &= -\frac{1}{2\alpha^2(\sigma^2\tau)} + \frac{r}{2\alpha\sigma^2} - \frac{r^2\tau}{6\sigma^2} \\ &\quad + \frac{\alpha^2 r^4 \tau^3}{90\sigma^2} + O(\alpha^4), \\ {}_1F_1(a;b;z) &= \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}(1+O(1/z)) \\ &\quad + \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b}(1+O(1/z)), \\ &\quad \text{if } (|z| \longrightarrow \infty), \end{aligned} \tag{32}$$

we have

$$\begin{aligned} \lim_{\alpha \longrightarrow 0^+} l(s,S) &= 1, \\ \lim_{\alpha \longrightarrow 0^+} Dl(s,S) &= 0. \end{aligned} \tag{33}$$

Thus,

$$\begin{aligned} \lim_{\alpha \longrightarrow 0^+} h(t,S) = &-\frac{\gamma(\rho^2-1)\sigma_m^2 \sinh(r(t-T))}{2r} \\ &- \frac{\rho\sigma_m(r-\mu)(e^{r(t-T)}-1)}{r\sigma} \\ &- \frac{\mu_m(e^{r(t-T)}-1)}{r}, \end{aligned} \tag{34}$$

and the corresponding investment policy is (29). □

Remark 6. The CEV model with $\alpha = 0$ reduces to the GMBs, and our results in (29) is the same as those in Theorem 5 of Browne [1], which implies that the optimal investment policy is right continuous at $\alpha = 0$.

Note that if $\alpha = 0$, the investment opportunities are constant so that the dynamical hedging demands disappear. Meanwhile, the dynamics of $W_c(s)$ given by (29) become to

$$dW_c(s) = \left(\mu_m - \frac{(\mu - r)\rho\sigma_m}{\sigma}\right) ds + \sqrt{1-\rho^2}\sigma_m dZ_2(s), \tag{35}$$

which implies that the static hedged portfolio is independent of the market states S. As a result, the delta hedge demands also vanish.

In total, we summarize the results of the above corollaries in Table 1.

5. Conclusion

We had investigated the optimal portfolio choice problem for a large insurer with negative exponential utility over terminal wealth. In particular, we applied the constant elasticity of variance (CEV) model to describe the price dynamics of the risky asset and allowed the Brownian motion driving the price process which was correlated with the Brownian motion driving the surplus process. By adopting a stochastic control approach, variable change technique, and the Feynman–Kac formula, we had obtained the explicit form expressions for the value function as well as the optimal investment policy. We had decomposed the optimal investment policy into four independent parts and discussed the effects of each component.

In future research concerning the optimal portfolio choice under the CEV model, it would be very interesting to extend our analysis to the case of more sophisticated surplus model, such as the classic Cramér–Lundberg risk models, jump-diffusion models, perturbed compound Poisson risk model (e.g., Peng et al. [36] and Yu et al. [37]), absolute ruin insurance risk model (e.g., Yu et al. [38]), and Lévy process (e.g., Huang et al. [39] and Zhang et al. [40]). Besides the negative exponential utility, optimal investment problems in terms of other utilities or even other objectives such as minimizing the ruin probability and the mean-variance criterion are also worthy of being investigated. Moreover, other stochastic optimal control or actuarial problems with the geometric Brownian motion settings (e.g., Yu et al. [41, 42]) can also be extended to the CEV model.

Appendix

We give some technical lemmas that are used in the proof of the main results in the paper.

Lemma 1. *If $S(s)$ follows the constant elasticity of variance process*

$$\frac{dS(s)}{S(s)} = r ds + \sigma S(s)^{\alpha} dZ_1(s), \quad S(t) = S > 0, \tag{A.1}$$

with $\alpha \geq 0$, then $S(s)^{-2\alpha}$ follows the Cox–Ingersoll–Ross (CIR) model and admits a unique strong solution.

Proof. By Ito's lemma, we have

$$\begin{aligned}\mathrm{d}S(s)^{-2\alpha} &= \alpha\left((2\alpha+1)\sigma^2 - 2rS(s)^{-2\alpha}\right)\mathrm{d}s - 2\alpha\sigma S(s)^{-\alpha}\mathrm{d}Z_1(s)\\ &= \kappa\left(\lambda - S(s)^{-2\alpha}\right)\mathrm{d}s + \theta\sqrt{S(s)^{-2\alpha}}\mathrm{d}Z_1(s),\end{aligned} \tag{A.2}$$

where $\kappa = 2\alpha r, \lambda = ((2\alpha+1)\sigma^2/2r), \theta = -2\alpha\sigma$. Feller's square root condition $2\kappa\lambda - \theta^2 = 2\alpha\sigma^2 \geq 0$ is satisfied for all $\alpha \geq 0$; henceforth, $S(s)^{-2\alpha}$ admits a unique strong solution and is positive for all $s \in [t, T]$. □

Lemma 2. *If $X(s)$ is described by the Cox–Ingersoll–Ross (CIR) process*

$$\mathrm{d}X(s) = \kappa(\lambda - X(s))\mathrm{d}s + \theta\sqrt{X(s)}\mathrm{d}Z_1(s), \quad X(t) = X, \tag{A.3}$$

then

$$\mathbb{E}\left[X(s)^p \mid X(t) = X\right] = \frac{e^{\kappa(-p)\tau}\left(2\kappa/\theta^2\left(e^{\kappa\tau}-1\right)\right)^{-p}\Gamma\left(p + \left(2\kappa\lambda/\theta^2\right)\right){}_1F_1\left(-p; \left(2\kappa\lambda/\theta^2\right); -\left(2\kappa/\theta^2\right)\left(X/\left(e^{\kappa\tau}-1\right)\right)\right)}{\Gamma\left(2\kappa\lambda/\theta^2\right)}, \tag{A.4}$$

for all $p > 0$, where $\tau = s - t > 0, \Gamma(\cdot)$ is the gamma function and ${}_1F_1(a; b; z)$ denotes the confluent hypergeometric function.

Proof. See Section 3 of Dereich et al. [43]. □

Proof of Theorem 1. □

Proof. We will first solve $g(t, S)$. By the Feynman–Kac formula (cf. Theorem 1 of Appendix E in Duffie [44]), $g(t, S)$ can be expressed as a conditional expectation:

$$g(t, S) = -\widetilde{\mathbb{E}}\left[\int_t^T \left(\frac{(r-\mu)^2\widetilde{S}(s)^{-2\alpha}}{2\sigma^2}\right)\mathrm{d}s \middle| \widetilde{S}(t) = S\right], \quad \forall t < T, \tag{A.5}$$

where $\widetilde{\mathbb{E}}[\widetilde{S}(t) = S]$ denotes the expectation under a new probability measure $\widetilde{\mathbb{P}}$, and $\widetilde{S}(s)$ follows dynamics:

$$\frac{\mathrm{d}\widetilde{S}(s)}{\widetilde{S}(s)} = r\mathrm{d}s + \sigma\widetilde{S}(s)^{\alpha}\mathrm{d}\widetilde{Z}(s), \quad \widetilde{S}(t) = S. \tag{A.6}$$

Here, $\widetilde{Z}(s)$ is a $\widetilde{\mathbb{P}}$-measure standard Brownian motion. By Lemma 1, $\widetilde{S}(s)^{-2\alpha}$ follows the Cox–Ingersoll–Ross model, and by Lemma 2, the conditional expectation is given by

$$\begin{aligned}\widetilde{\mathbb{E}}_t\left[\widetilde{S}(s)^{-2\alpha}\right] &= \widetilde{\mathbb{E}}[\widetilde{X}(s) \mid \widetilde{X}(t) = S]\\ &= \frac{(2\alpha+1)\sigma^2}{2r} + e^{-2\alpha r(s-t)}\left(S^{-2\alpha} - \frac{(2\alpha+1)\sigma^2}{2r}\right).\end{aligned} \tag{A.7}$$

Inserting (A.7) into (A.5) and after some manipulation yield

$$g(t, S) = f_1(t)S^{-2\alpha} + f_0(t), \tag{A.8}$$

where

$$\begin{aligned}f_1(t) &= \frac{(r-\mu)^2\left(e^{-2\alpha r(T-t)} - 1\right)}{4\alpha r\sigma^2},\\ f_0(t) &= \frac{(2\alpha+1)(r-\mu)^2\left((2\alpha r(t-T)+1) - e^{-2\alpha r(T-t)}\right)}{8\alpha r^2}.\end{aligned} \tag{A.9}$$

Note that since $\alpha > 0$, we have $f_1(t) < 0, f_0(t) < 0, \forall t < T$. Henceforth, $g(t, S) < 0, \forall S > 0, t < T$.

For (18), by the Feynman–Kac theorem, $h(t, S)$ can also be expressed as a conditional expectation:

$$\begin{aligned}h(t, S) &= \widetilde{\mathbb{E}}\left[\int_t^T e^{-r(s-t)}\left(\frac{\rho\sigma_m(r-\mu)\widetilde{S}(s)^{-\alpha}}{\sigma}\right.\right.\\ &\qquad \left.\left. + \mu_m - \frac{1}{2}\gamma\left(1-\rho^2\right)\sigma_m^2 e^{r(T-s)}\right)\mathrm{d}s \mid \widetilde{S}(t) = S\right]\\ &= \int_t^T e^{-r(s-t)}\widetilde{\mathbb{E}}\left[\left(\frac{\rho\sigma_m(r-\mu)\widetilde{S}(s)^{-\alpha}}{\sigma}\right) \mid \widetilde{S}(t) = S\right]\mathrm{d}s + A(t)\\ &= \frac{\rho\sigma_m(r-\mu)}{\sigma}\int_t^T e^{-r(s-t)}l(s, S)\mathrm{d}s + A(t),\end{aligned} \tag{A.10}$$

where

$$\begin{aligned}A(t) &= \frac{\mu_m\left(1 - e^{-r(T-t)}\right)}{r} - \frac{\gamma\left(\rho^2 - 1\right)\sigma_m^2\sinh(r(t-T))}{2r},\\ l(s, S) &= \widetilde{\mathbb{E}}\left[\widetilde{S}(s)^{-\alpha} \middle| \widetilde{S}(t) = S\right].\end{aligned} \tag{A.11}$$

Also, together by Lemmas 1 and 2, we have

$$\begin{aligned}l(s, S) &= \widetilde{\mathbb{E}}\left[\widetilde{X}(s)^{(1/2)} \middle| \widetilde{X}(t) = S^{-2\alpha}\right]\\ &= \frac{\Gamma((3/2) + (1/2\alpha))e^{\alpha(-r)\tau}{}_1F_1\left(-(1/2); (2\alpha + 1/2\alpha); -S^{-2\alpha}M(\tau)\right)}{\Gamma(1 + (1/2\alpha))\sqrt{M(\tau)}},\end{aligned} \tag{A.12}$$

where $\tau = s - t, M(\tau) = \left(r/\alpha\sigma^2\left(e^{2\alpha r\tau} - 1\right)\right)$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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