Research Article

Copious Closed Forms of Solutions for the Fractional Nonlinear Longitudinal Strain Wave Equation in Microstructured Solids

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A computational scheme is employed to investigate various types of the solution of the fractional nonlinear longitudinal strain wave equation. The novelty and advantage of the proposed method are illustrated by applying this model. A new fractional definition is used to convert the fractional formula of these equations into integer-order ordinary differential equations. Soliton, rational functions, the trigonometric function, the hyperbolic function, and many other explicit wave solutions are obtained.

1. Introduction

Fractional nonlinear evolution equation is one of the noticeable branches of science, particularly in recent years. Fractional calculus has a great profound physical background where it is able to formulate many various phenomena in distinct fields such as physics, mechanical engineering, economics, chemistry, signal processing, food supplement, applied mathematics, quasichaotic dynamical systems, hydrodynamics, system identification, statistics, finance, fluid mechanics, solid-state biology, dynamical systems with chaotic dynamical behavior, optical fibers, electric control theory, and economics and diffusion problems. The mathematical modeling of these phenomena will contain a fractional derivative which provides a great explanation of the nonlocal property of these models since it depends on both historical and current states of the problem in contrast with the classical calculus which depends on the current state only. Based on the importance of this kind of calculus, many definitions have been being derived such as conformable fractional derivative, fractional Riemann–Liouville derivatives, Caputo, and Caputo–Fabrizio definition [1–17]. These definitions have been being

employed to convert the fractional nonlinear partial differential equations to nonlinear integer-order ordinary differential equation and then the computational and numerical schemes can be applied to get various types of solutions for these models and the examples of these schemes [18–30].

Recently, the mK method is formulated and applied to distinct physical models such as the complex Ginzburg–Landau model, the (2 + 1)-dimensional KD equation and KdV equation, fractional (N + 1) sinh-Gordon, biological population, equal width, modified equal width, and Duffing equations [31–40].

This method depends on a new auxiliary equation, which is equal to the Riccati equation [41]. The auxiliary equation of the mK method is given by

$$\mathcal{E}'(\varphi) = \frac{1}{\ln(\mathcal{Q})} [\delta \mathcal{Q}^{\mathcal{E}(\varphi)} + \varrho \mathcal{Q}^{-\mathcal{E}(\varphi)} + X], \quad (1)$$

where δ , ϱ , χ , and $\mathcal{Q}$ are the arbitrary constants. The Riccati equation is given by

$$\mathcal{R}'(\varphi) = \mathcal{A}_0 + \mathcal{A}_1 \mathcal{R}(\varphi) + \mathcal{A}_2 \mathcal{R}^2(\varphi), \quad (2)$$

where $\mathcal{A}_0, \mathcal{A}_1$, and $\mathcal{A}_2$ are the arbitrary constants. So, equations (1) and (2) are equal when $[\mathcal{E}(\varphi) = \mathcal{R}(\varphi), \chi = A_1, \varrho = A_0, \delta = A_2]$. Using this technique, it leads to the equalling of the mK auxiliary equation with many other analytical methods, but the mK method can obtain more solutions than most of them. This equivalence shows superiority, power, and productivity of the mK method.

In this context, the mK method is employed to construct new formulas of solutions for the fractional nonlinear longitudinal strain wave equation which is given in [42–48]:

$$\mathcal{D}_{tt}^{2\vartheta} \mathcal{M} + \mathcal{M}_{xx} - \gamma [\mathcal{L}_1 (\mathcal{M}^2)_{xx} - \mathcal{L}_2 \mathcal{M}_{xxxx} + \mathcal{L}_3 \mathcal{D}_{tt}^{2\vartheta} (\mathcal{M}_{xx})] = 0, \quad (3)$$

where $[\gamma, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3]$ are the arbitrary constants.

This model is considered as one of the fundamental models in the microstructure of a material that is used to determine the elasticity, which is caused by the dissipation/energy input in the content. This model contains nonlinear and dissipation terms which are the officials of construct the kink and shock waves. This shows the permanent form in medium points to a possible presence of dispersion or dissipation. The following definition of the $\mathcal{ABR}$ fractional operator [49–52] is applied to equation (3).

Definition 1. It is given in [17] that

$${}^{\mathcal{ABR}}\mathcal{D}_{a+}^{\alpha} \mathcal{F}(t) = \frac{\mathcal{B}(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t \mathcal{F}(x) \mathcal{G}_{\alpha} \left(\frac{-\alpha(t-x)^{\alpha}}{1-\alpha} \right) dx, \quad (4)$$

where $\mathcal{G}_{\alpha}$ is the Mittag–Leffler function defined by the following formula:

$$\mathcal{G}_{\alpha} \left(\frac{-\alpha(t-x)^{\alpha}}{1-\alpha} \right) = \sum_{n=0}^{\infty} \frac{(-\alpha/1-\alpha)^n (t-x)^{\alpha n}}{\Gamma(\alpha n + 1)}, \quad (5)$$

and $\mathcal{B}(\alpha)$ is a normalisation function. Thus,

$${}^{\mathcal{ABR}}\mathcal{D}_{a+}^{\alpha} \mathcal{F}(x) = \frac{\mathcal{B}(\alpha)}{1-\alpha} \sum_{n=0}^{\infty} \left(\frac{-\alpha}{1-\alpha} \right)^n {}^{RL}\mathcal{J}_a^{\alpha n} \mathcal{F}(x), \quad (6)$$

leads to

$$\mathcal{M}(x, t) = \mathcal{M}(\varphi),$$

$$\varphi = x + \frac{c(1-\alpha)t^{-\alpha n}}{B(\alpha) \sum_{n=0}^{\infty} (-\alpha/1-\alpha)^n \Gamma(1-\alpha n)}, \quad (7)$$

where c is the arbitrary constant. This wave transformation converts equation (3) to ODE. Integrating the obtained ODEs twice with zero constant of the integration gives

$$\mathcal{M} - \alpha \mathcal{M}^2 + \lambda \mathcal{M}'' = 0, \quad (8)$$

where $[\alpha = \gamma \mathcal{L}_1 / c^2 - 1, \lambda = (\gamma(\mathcal{L}_2 - c^2 \mathcal{L}_3)) / c^2 - 1]$. Calculating the homogeneous balance value in equation (8) yields $n = 2$. Thus, both equations have same general formula of solution and it is given according to the mK method by

$$\begin{aligned} \mathcal{M}(\varphi) &= \sum_{i=1}^n a_i \mathcal{Q}^{i\mathcal{E}(\varphi)} + \sum_{i=1}^n b_i \mathcal{Q}^{-i\mathcal{E}(\varphi)} + a_0 \\ &= a_1 \mathcal{Q}^{\mathcal{E}(\varphi)} + a_2 \mathcal{Q}^{2\mathcal{E}(\varphi)} + a_0 + b_2 \mathcal{Q}^{-2\mathcal{E}(\varphi)} + b_1 \mathcal{Q}^{-\mathcal{E}(\varphi)}, \end{aligned} \quad (9)$$

where a_0, a_1, a_2, b_1 , and b_2 are the arbitrary constants.

The order for the rest of this article is shown as follows: Section 2 applies the mK method to the nonlinear fractional strain wave equation. Section 3 discusses the obtained computational results and explains the comparison between them and that obtained in previous work. Moreover, it shows the comparison between the obtained numerical results. Section 4 gives the conclusion of the whole research.

2. Abundant Wave Solutions of the Fractional Strain Wave Equation

Applying the mK method with its auxiliary equation and the suggested general solutions for the fractional strain wave equation leads to a system of algebraic equations. Using Mathematica 11.2 to find the values of the parameters in this system leads to the following.

2.1. Family I.

$$\left[a_0 \longrightarrow \frac{b_2 \left(-\sqrt{(\chi^2 - 4\delta\varrho)^2} + 8\delta\varrho + \chi^2 \right)}{12\varrho^2}, a_1 \longrightarrow 0, a_2 \longrightarrow 0, b_1 \longrightarrow \frac{b_2 \chi}{\varrho}, \alpha \longrightarrow -\frac{6\varrho^2}{b_2 \sqrt{(4\delta\varrho - \chi^2)^2}}, \right. \\ \left. \lambda \longrightarrow -\frac{1}{\sqrt{(\chi^2 - 4\delta\varrho)^2}}, \text{ where } (\varrho \neq 0, \chi^2 \neq 4\delta\varrho) \right]. \quad (10)$$

Consequently, the closed forms of solutions for the fractional strain waves model are given as follows:

When $[\chi^2 - 4\delta\varrho < 0 \ \& \ \delta \neq 0]$,

$$\begin{aligned}\mathcal{M}_1(x, t) &= \frac{b_2}{12} \left[\frac{48\delta^2}{(\chi - \sqrt{4\delta\varrho - \chi^2} \tan((1/2)\sqrt{4\delta\varrho - \chi^2} (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n))))^2} \right. \\ &\quad - \frac{24\delta\chi}{\chi\varrho - \varrho\sqrt{4\delta\varrho - \chi^2} \tan((1/2)\sqrt{4\delta\varrho - \chi^2} (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n))))} \\ &\quad \left. + \frac{-\sqrt{(\chi^2 - 4\delta\varrho)^2 + 8\delta\varrho + \chi^2}}{\varrho^2} \right], \\ \mathcal{M}_2(x, t) &= \frac{b_2}{12} \left[\frac{48\delta^2}{(\chi - \sqrt{4\delta\varrho - \chi^2} \cot((1/2)\sqrt{4\delta\varrho - \chi^2} (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n))))^2} \right. \\ &\quad - \frac{24\delta\chi}{\chi\varrho - \varrho\sqrt{4\delta\varrho - \chi^2} \cot((1/2)\sqrt{4\delta\varrho - \chi^2} (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n))))} \\ &\quad \left. + \frac{-\sqrt{(\chi^2 - 4\delta\varrho)^2 + 8\delta\varrho + \chi^2}}{\varrho^2} \right].\end{aligned}\tag{11}$$

When $[\chi^2 - 4\delta\varrho > 0 \ \& \ \delta \neq 0]$,

$$\begin{aligned}\mathcal{M}_3(x, t) &= \frac{b_2}{12} \left[\frac{48\delta^2}{(\chi - \sqrt{4\delta\varrho - \chi^2} \tanh((1/2)\sqrt{\chi^2 - 4\delta\varrho} (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) + \chi)^2} \right. \\ &\quad - \frac{24\delta\chi}{\chi\varrho - \varrho\sqrt{4\delta\varrho - \chi^2} \tanh((1/2)\sqrt{\chi^2 - 4\delta\varrho} (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) + \chi\varrho} \\ &\quad \left. + \frac{-\sqrt{(\chi^2 - 4\delta\varrho)^2 + 8\delta\varrho + \chi^2}}{\varrho^2} \right], \\ \mathcal{M}_4(x, t) &= \frac{b_2}{12} \left[\frac{48\delta^2}{(\chi - \sqrt{4\delta\varrho - \chi^2} \coth((1/2)\sqrt{\chi^2 - 4\delta\varrho} (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) + \chi)^2} \right. \\ &\quad - \frac{24\delta\chi}{\chi\varrho - \varrho\sqrt{4\delta\varrho - \chi^2} \coth((1/2)\sqrt{\chi^2 - 4\delta\varrho} (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) + \chi\varrho + \left(-\sqrt{(\chi^2 - 4\delta\varrho)^2 + 8\delta\varrho + \chi^2/\varrho^2}\right)} \\ &\quad \left. \right].\end{aligned}\tag{12}$$

When $[\delta\varrho > 0 \ \& \ \varrho \neq 0 \ \& \ \delta \neq 0 \ \& \ \chi = 0]$,

$$\begin{aligned}\mathcal{M}_5(x, t) &= \frac{b_2 \left(\delta\varrho (3 \cot^2(\sqrt{\delta\varrho} (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) + 2) - \sqrt{\delta^2\varrho^2} \right)}{3\varrho^2}, \\ \mathcal{M}_6(x, t) &= \frac{b_2 \left(\delta\varrho (3 \tan^2(\sqrt{\delta\varrho} (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) + 2) - \sqrt{\delta^2\varrho^2} \right)}{3\varrho^2}.\end{aligned}\tag{13}$$

When $[\delta \varrho < 0 \ \& \ \varrho \neq 0 \ \& \ \delta \neq 0 \ \& \ \chi = 0]$,

$$\begin{aligned}\mathcal{M}_7(x, t) &= -\frac{b_2 \left(3\delta \varrho \operatorname{csch}^2(\sqrt{-\delta \varrho} (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) + \sqrt{\delta^2 \varrho^2 + \delta \varrho} \right)}{3\varrho^2}, \\ \mathcal{M}_8(x, t) &= -\frac{b_2 \left(\delta \varrho \left(3 \tanh^2(\sqrt{-\delta \varrho} (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) - 2 \right) + \sqrt{\delta^2 \varrho^2} \right)}{3\varrho^2}.\end{aligned}\quad (14)$$

When $[\chi = 0 \ \& \ \varrho = -\delta]$,

$$\begin{aligned}\mathcal{M}_9(x, t) &= \frac{b_2}{3} \left[\frac{12}{(\exp(2\varrho(x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) + 1)^2} \right. \\ &\quad \left. + 6 \tanh\left(\varrho\left(x - \frac{(\alpha - 1)ct^{-2\alpha}}{B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)}\right)\right) - \frac{\sqrt{\varrho^4}}{\varrho^2} - 5 \right] a.\end{aligned}\quad (15)$$

When $[\chi = (\varrho/2) = \kappa \ \& \ \delta = 0]$,

$$\begin{aligned}\mathcal{M}_{10}(x, t) &= b_2 \left[\frac{1}{(\exp(\kappa(x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) - 2)^2} \right. \\ &\quad \left. + \frac{1}{2 \exp(\kappa(x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) - 4} - \frac{\sqrt{\kappa^4}}{48\kappa^2} + \frac{1}{48} \right].\end{aligned}\quad (16)$$

When $[\chi = \delta = 0 \ \& \ \varrho \neq 0]$,

$$\mathcal{M}_{11}(x, t) = \frac{b_2}{\varrho^2 (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))^2}.\quad (17)$$

When $[\chi = 0 \ \& \ \varrho = \delta]$,

$$\mathcal{M}_{12}(x, t) = \frac{1}{3} b_2 \left(3 \cot^2\left(-\frac{(\alpha - 1)c\varrho t^{-2\alpha}}{B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)} + C + x\varrho\right) - \frac{\sqrt{\varrho^4}}{\varrho^2} + 2 \right).\quad (18)$$

When $[\delta = 0 \ \& \ \chi \neq 0 \ \& \ \varrho \neq 0]$,

$$\begin{aligned}\mathcal{M}_{13}(x, t) &= \frac{b_2}{12} \left[\frac{\chi^2 (1 - (12\varrho/\varrho - \chi \exp(\chi(x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))))) - \sqrt{\chi^4}}{\varrho^2} \right. \\ &\quad \left. + \frac{12}{(\exp(\chi(x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) - (\varrho/\chi))^2} \right].\end{aligned}\quad (19)$$

2.2. Family II.

$$\begin{aligned}\left[a_0 \longrightarrow \frac{\sqrt{(\chi^2 - 4\delta\varrho)^2 - 8\delta\varrho - \chi^2}}{2\alpha\sqrt{(\chi^2 - 4\delta\varrho)^2}}, a_1 \longrightarrow -\frac{6\delta\chi}{\alpha\sqrt{(\chi^2 - 4\delta\varrho)^2}}, a_2 \longrightarrow -\frac{6\delta^2}{\alpha\sqrt{(4\delta\varrho - \chi^2)^2}}, b_1 \longrightarrow 0, b_2 \longrightarrow 0, \right. \\ \left. \lambda \longrightarrow -\frac{1}{\sqrt{(\chi^2 - 4\delta\varrho)^2}}, \text{ where } (\delta \neq 0, \alpha \neq 0, 4\delta\varrho \neq \chi^2) \right].\end{aligned}\quad (20)$$

Consequently, the closed forms of solutions for the fractional strain waves model are given as follows:

When $[\chi^2 - 4\delta\varrho < 0 \text{ \& } \delta \neq 0]$,

$$\begin{aligned}\mathcal{M}_{14}(x, t) &= \left[3(\chi^2 - 4\delta\varrho) \tan^2 \left(\frac{1}{2} \sqrt{4\delta\varrho - \chi^2} \left(x - \frac{(\alpha - 1)ct^{-2\alpha}}{B(\alpha) \sum_{n=0}^{\infty} \frac{(-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)}{(-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)}} \right) \right) + \sqrt{(\chi^2 - 4\delta\varrho)^2 - 8\delta\varrho + 2\chi^2} \right] \\ &\quad \times \left(\frac{1}{2\alpha \sqrt{(\chi^2 - 4\delta\varrho)^2}} \right), \\ \mathcal{M}_{15}(x, t) &= \left[3(\chi^2 - 4\delta\varrho) \cot^2 \left(\frac{1}{2} \sqrt{4\delta\varrho - \chi^2} \left(x - \frac{(\alpha - 1)ct^{-2\alpha}}{B(\alpha) \sum_{n=0}^{\infty} \frac{(-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)}{(-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)}} \right) \right) + \sqrt{(\chi^2 - 4\delta\varrho)^2 - 8\delta\varrho + 2\chi^2} \right] \\ &\quad \times \left(\frac{1}{2\alpha \sqrt{(\chi^2 - 4\delta\varrho)^2}} \right).\end{aligned}\tag{21}$$

When $[\chi^2 - 4\delta\varrho > 0 \text{ \& } \delta \neq 0]$,

$$\begin{aligned}\mathcal{M}_{16}(x, t) &= \left[3(\chi^2 - 4\delta\varrho) \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{\chi^2 - 4\delta\varrho} \left(x - \frac{(\alpha - 1)ct^{-2\alpha}}{B(\alpha) \sum_{n=0}^{\infty} \frac{(-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)}{(-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)}} \right) \right) + \sqrt{(\chi^2 - 4\delta\varrho)^2 + 4\delta\varrho - \chi^2} \right] \\ &\quad \times \left(\frac{1}{2\alpha \sqrt{(\chi^2 - 4\delta\varrho)^2}} \right), \\ \mathcal{M}_{17}(x, t) &= \left[3(4\delta\varrho - \chi^2) \coth^2 \left(\frac{1}{2} \sqrt{\chi^2 - 4\delta\varrho} \left(x - \frac{(\alpha - 1)ct^{-2\alpha}}{B(\alpha) \sum_{n=0}^{\infty} \frac{(-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)}{(-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)}} \right) \right) + \sqrt{(\chi^2 - 4\delta\varrho)^2 - 8\delta\varrho + 2\chi^2} \right] \\ &\quad \times \left(\frac{1}{2\alpha \sqrt{(\chi^2 - 4\delta\varrho)^2}} \right).\end{aligned}\tag{22}$$

When $\delta\varrho > 0 \text{ \& } \varrho \neq 0 \text{ \& } \delta \neq 0 \text{ \& } \chi = 0$,

$$\begin{aligned}\mathcal{M}_{18}(x, t) &= \frac{1}{2\alpha \sqrt{\delta^2 \varrho^2}} \left[-3\delta\varrho \sec^2 \left(\sqrt{\delta\varrho} \left(x - \frac{(\alpha - 1)ct^{-2\alpha}}{B(\alpha) \sum_{n=0}^{\infty} \frac{(-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)}{(-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)}} \right) \right) + \sqrt{\delta^2 \varrho^2 + \delta\varrho} \right], \\ \mathcal{M}_{19}(x, t) &= \frac{1}{2\alpha \sqrt{\delta^2 \varrho^2}} \left[-3\delta\varrho \csc^2 \left(\sqrt{\delta\varrho} \left(x - \frac{(\alpha - 1)ct^{-2\alpha}}{B(\alpha) \sum_{n=0}^{\infty} \frac{(-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)}{(-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)}} \right) \right) + \sqrt{\delta^2 \varrho^2 + \delta\varrho} \right].\end{aligned}\tag{23}$$

When $[\delta \varrho < 0 \ \& \ \varrho \neq 0 \ \& \ \delta \neq 0 \ \& \ \chi = 0]$,

$$\begin{aligned} \mathcal{M}_{20}(x, t) &= \frac{1}{2\alpha} \left[\frac{\delta \varrho (3 \tanh^2(\sqrt{-\delta \varrho} (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) - 2)}{\sqrt{\delta^2 \varrho^2}} + 1 \right], \\ \mathcal{M}_{21}(x, t) &= \frac{1}{2\alpha \sqrt{\delta^2 \varrho^2}} \left[3\delta \varrho \operatorname{csch}^2 \left(\sqrt{-\delta \varrho} \left(x - \frac{(\alpha - 1)ct^{-2\alpha}}{B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)} \right) \right) + \sqrt{\delta^2 \varrho^2} + \delta \varrho \right]. \end{aligned} \quad (24)$$

When $[\chi = 0 \ \& \ \varrho = -\delta]$,

$$\mathcal{M}_{22}(x, t) = \frac{1}{2\alpha \sqrt{\varrho^4}} \left[-3\varrho^2 \coth^2 \left(\varrho \left(x - \frac{(\alpha - 1)ct^{-2\alpha}}{B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)} \right) \right) + \sqrt{\varrho^4} + 2\varrho^2 \right]. \quad (25)$$

When $[\chi = \delta = \kappa \ \& \ \varrho = 0]$,

$$\mathcal{M}_{23}(x, t) = \frac{1}{2\alpha} \left[1 - \frac{\sqrt{\kappa^4} (3 \operatorname{csch}^2((1/2)\kappa (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) + 1)}{\kappa^2} \right]. \quad (26)$$

When $\varrho = 0 \ \& \ \chi \neq 0 \ \& \ \delta \neq 0$,

$$\begin{aligned} \mathcal{M}_{24}(x, t) &= \frac{-1}{2\alpha} \left[\frac{\sqrt{\chi^4}}{\chi^2} \left(\frac{48}{(\delta \exp(\chi (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) - 2)} \right. \right. \\ &\quad \left. \left. + \frac{24}{\delta \exp(\chi (x - ((\alpha - 1)ct^{-2\alpha}/B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)))) - 2} + 1 \right) - 1 \right]. \end{aligned} \quad (27)$$

When $[\chi = 0 \ \& \ \varrho = \delta]$,

$$\mathcal{M}_{25}(x, t) = \frac{1}{2\alpha \sqrt{\varrho^4}} \left[-3\varrho^2 \sec^2 \left(-\frac{(\alpha - 1)c\varrho t^{-2\alpha}}{B(\alpha) \sum_{n=0}^{\infty} (-(\alpha/1 - \alpha))^n \Gamma(1 - \alpha n)} + C + x\varrho \right) + \sqrt{\varrho^4} + \varrho^2 \right]. \quad (28)$$

3. Results and Discussion

This section is divided into two main parts. The first part shows studying the obtained computational solutions for the fractional suggested model, while the second part presents a comparison between them and other obtained results in previous work.

(1) The solutions obtained in this paper:

- (i) In this research paper, the fractional nonlinear longitudinal strain wave equation is investigated by the employment of the mK method and a new fractional definition ($\mathcal{ABR}$). Abundant explicit closed forms of solutions are obtained (twenty five solutions).

(2) The previously obtained solutions in previous work:

- (i) In [53], two analytical methods were applied to three different models involving our two

investigated models. However, they used two schemes but a very few special solutions were obtained.

- (ii) The two analytical schemes in [53] are just a special case of the mK method when $[\mathcal{Q}^{\mathcal{M}(\varphi)} = (G'/G), \varrho = -\mu, \chi = -\lambda, \delta = 1]$.
- (iii) Equation (22) is equal to equation (3.9) in [53] when $[e_0 = -12\delta(\mu + d(d - \lambda)), -3(\lambda^2 - 4\mu) = \delta\lambda\varrho]$.
- (iv) All other solutions obtained in this paper are considered new solutions when compared with those obtained in [53].

4. Conclusion

In our research paper, we solved the flaws and disadvantages of the (G'/G) -expansion methods that are used in [53] by M. Ali Akbar et al., and as shown in the previous section, it is just a particular case of our applied method in this research paper. Moreover, a new definition of fractional derivative is

used, successfully converting the fractional from our abovementioned models to integer-order ordinary differential equations. Abundance new solutions for both the models were obtained.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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