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Research Article

Reduced-Order Algorithm for Eigenvalue Assignment of Singularly Perturbed Linear Systems

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In this paper, we present an algorithm for eigenvalue assignment of linear singularly perturbed systems in terms of reduced-order slow and fast subproblem matrices. No similar algorithm exists in the literature. First, we present an algorithm for the recursive solution of the singularly perturbed algebraic Sylvester equation used for eigenvalue assignment. Due to the presence of a small singular perturbation parameter that indicates separation of the system variables into slow and fast, the corresponding algebraic Sylvester equation is numerically ill-conditioned. The proposed method for the recursive reduced-order solution of the algebraic Sylvester equations removes ill-conditioning and iteratively obtains the solution in terms of four reduced-order numerically well-conditioned algebraic Sylvester equations corresponding to slow and fast variables. The convergence rate of the proposed algorithm is $O(\varepsilon)$, where ε is a small positive singular perturbation parameter.

1. Introduction

The classical method for numerical solution of the Sylvester algebraic equation dates back to reference [1]. Solving the Sylvester algebraic equation numerically is not a simple task [2, 3]. Namely, *it was stated in* [2, 3] *that the algorithm of* [1] *cannot produce a highly accurate solution.* Another method for solving the large-scale Sylvester equations is introduced in [4]. In [4], the authors have shown that researchers have developed some methods for the solution of large-scale Sylvester equations [5–7]. On the other hand, several research studies for solving the Sylvester algebraic equation have gained attention in engineering problem [8–10]. In the image-processing problem, the presentation developed in [8] has shown that the image fusion method applied to a large-scale image facilitates reduction of computational complexity based on the explicit solution of large-scale Sylvester equations. Furthermore, many problems of control theory such as regulator problem [9] and particle swarm theory [10] lead to a Sylvester equation.

The aim of our developed algorithm is to solve a large-scale Sylvester equation in order to overcome the numerical ill-conditioning problem of singularly perturbed systems presented in [11]. This leads to reduced-order regular algebraic Sylvester equations [12], combined with the techniques presented in [13, 14] which solves the eigenvalue assignment problem for singularly perturbed linear systems.

The general Sylvester equation is defined as

$$TA + MT + N = 0. \tag{1}$$

Its unique solution T exists under the assumption that matrices A and $-M$ have no eigenvalues in common [13].

Assumption 1. $\lambda(A) \neq \lambda(-M) \neq -\lambda(M)$.

Without loss of generality, we will consider the Sylvester equation encountered in the control system design of linear systems:

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{2}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control input vector, $y(t) \in R^p$ is the vector of system measurements, and A, B, and C are constant matrices. Forms of the Sylvester algebraic equations that appear in the observer and controller designs are given by

$$\begin{aligned} T_o A + M T_o - \overline{K} C &= 0, \quad \overline{K} = T_o K, \\ A T_c + T_c N - B \overline{F} &= 0, \quad \overline{F} = F T_c, \end{aligned} \tag{3}$$

where K stands for the observer feedback gain and F is the system feedback gain. These Sylvester equations were studied in [15]. The system-observer configuration has slow and fast modes since the observer must be much faster than the system, and hence, it represents implicitly a singularly perturbed system. Note that the main difficulty in the numerical solution of algebraic Sylvester equation (3) will come from the fact that the system to be studied has a singularly perturbed structure and not from the system-observer implicit singularly perturbed structure since the controller and the observer eigenvalue assignment problems are done independently using the separation principle. The main difficulty comes from the fact that since the system has slow and fast modes, then the observer must contain very fast modes, which leads to numerical ill-conditioning.

1.1. Problem Statement. In this section, we study the Sylvester algebraic equation corresponding to singularly perturbed systems defined by [11] (Chapter 2):

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) + B_1 u(t), x_1(t_0) = x_{10}, \\ \varepsilon \dot{x}_2(t) &= A_3 x_1(t) + A_4 x_2(t) + B_2 u(t), x_2(t_0) = x_{20}, \\ y(t) &= C_1 x_1(t) + C_2 x_2(t), \end{aligned} \tag{4}$$

where $x_1(t) \in R^{n_1}$ and $x_2(t) \in R^{n_2}$, $n_1 + n_2 = n$, are, respectively, slow and fast state variables and ε is a small positive singular perturbation parameter. Equation (4) can be obtained from (2) assuming that (2) has eigenvalues clustered into two groups: slow ones closer to the imaginary axis and fast ones farther from the imaginary axis (singularly perturbed structure). In such a case, a similarity transformation converts (2) into (4). The following is the standard assumption used in theory of singular perturbation [11] (Chapter 2).

Assumption 2. The matrix A_4 is nonsingular.

We study, without loss of generality, a variant of observer design algebraic Sylvester equation (3) given by

$$\begin{aligned} TA - A_{\text{des}} T &= \overline{K} C, \\ \overline{K} &= TK, \\ K &= \begin{bmatrix} K_1 \\ \frac{1}{\varepsilon} K_2 \end{bmatrix}. \end{aligned} \tag{5}$$

The matrix A_{des} is the matrix with the observer-desired closed-loop eigenvalues. The standard observer design assumptions are needed [13].

Assumption 3. .The pair (A, C) is observable, and the pair $(A_{\text{des}}, \overline{K})$ is controllable.

The general existence condition given in Assumption 1 and specialized to (5) leads to the following assumption.

Assumption 4. $\lambda(A) \cap \lambda(A_{\text{des}}) = \varnothing$.

Having found an invertible solution of (5), then the observer gain is given by $K = T^{-1}\overline{K}$. Note that Assumption 3 for single-input single-output systems is both sufficient and necessary condition for the existence of an invertible solution of (5). For multi-input multi-output systems, it is only a necessary condition [13], so that a repetitive design algorithm has to be performed until an invertible solution T is obtained (see Section 5). The matrices in (4) and (5) are partitioned as

$$\begin{aligned} A &= \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\varepsilon} A_3 & \frac{1}{\varepsilon} A_4 \end{bmatrix}, \\ A_{\text{des}} &= \begin{bmatrix} A_s & 0 \\ 0 & \frac{1}{\varepsilon} A_f \end{bmatrix}, \\ C &= [\, C_1 \;\; C_2 \,], \end{aligned} \tag{6}$$

where A_{des} contains the desired observer closed-loop eigenvalues, that is, $\lambda(A_{\text{des}}) = \lambda(A - KC)$. Note that they are placed far to the left in the complex plane to make the observer asymptotically stable and much faster than the closed-loop system. We have found that the following scaling is appropriate for the solution matrix T:

$$T = \begin{bmatrix} T_1 & \varepsilon T_2 \\ \varepsilon T_3 & \varepsilon T_4 \end{bmatrix}, \tag{7}$$

which is consistent with the structures of matrices defined in (5) and (6). Namely, the right-hand side of (5) is

$$\begin{aligned} \overline{K} C &= \begin{bmatrix} T_1 K_1 C_1 + T_2 K_2 C_1 & T_1 K_1 C_2 + T_2 K_2 C_2 \\ \varepsilon T_3 K_1 C_1 + T_4 K_2 C_1 & \varepsilon T_3 K_1 C_2 + T_4 K_2 C_2 \end{bmatrix} \\ &= -\begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} = \begin{bmatrix} O(1) & O(1) \\ O(1) & O(1) \end{bmatrix}. \end{aligned} \tag{8}$$

With the scaling chosen in (7), the left-hand side terms of (5), that is, TA and $A_{\text{des}}T$, are also both $O(1)$. Due to the structure of matrices A and A_{des}, the singularly perturbed algebraic Sylvester equation defined in (5) is numerically ill-conditioned. To overcome numerical ill-conditioning, we propose a new recursive algorithm for solving (5) in terms of reduced-order well-defined algebraic Sylvester equations. The dual version of (5) used for the system controller design is given by

$$AT_c - T_c A_{\text{des}}^c = B\overline{F},$$
$$\overline{F} = FT_c$$
$$B = \begin{bmatrix} B_1 \\ \frac{1}{\varepsilon}B_2 \end{bmatrix}, \tag{9}$$
$$A_{\text{des}}^c = \begin{bmatrix} A_{sc} & 0 \\ 0 & \frac{1}{\varepsilon}A_{fc} \end{bmatrix}.$$

Multiplying matrices in (9), we can get four algebraic equations for the partitioned matrix T_c as functions of the small parameter ε. It can be found from these equations [16, 17] that the structure of T_c is given by

$$T_c = \begin{bmatrix} T_{1c} & T_{2c} \\ T_{3c} & \frac{1}{\varepsilon}T_{4c} \end{bmatrix}. \tag{10}$$

Algebraic Sylvester equations (9)-(10) will be solved numerically in terms of reduced-order numerically well-conditioned algebraic Sylvester equations under the standard controller design assumptions [13].

Assumption 5. The pair (A, B) is controllable and the pair $(A_{\text{des}}^c, \overline{F})$ is observable.

Moreover, the existence of a unique solution of (9) requires the assumption dual to Assumption 4.

Assumption 6. $\lambda(A) \cap \lambda(A_{\text{des}}^c) = \varnothing$.

1.2. Parallel Algorithm for the Observer Sylvester Equation. The partitioned form of the Sylvester equation given in (5) subject to (6)–(8) is given by

$$\begin{aligned} T_1A_1 + T_2A_3 - A_sT_1 + Q_1 &= 0, \\ T_1A_2 + T_2A_4 - \epsilon A_sT_2 + Q_2 &= 0, \\ \varepsilon T_3A_1 + T_4A_3 - A_fT_3 + Q_3 &= 0, \\ \varepsilon T_3A_2 + T_4A_4 - A_fT_4 + Q_4 &= 0. \end{aligned} \tag{11}$$

It can be seen from (11) that this system has two independent sets of linear algebraic equations: the first one for T_1 and T_2 and the second one for T_3 and T_4.

Setting $\varepsilon = 0$ in (11), the algebraic equations for zeroth-order approximations of solutions $T_1^{(0)}$, $T_2^{(0)}$, $T_3^{(0)}$, and $T_4^{(0)}$ are obtained as

$$T_1^{(0)}A_1 + T_2^{(0)}A_3 - A_sT_1^{(0)} + Q_1 = 0, \tag{12}$$

$$T_1^{(0)}A_2 + T_2^{(0)}A_4 + Q_2 = 0, \tag{13}$$

$$T_4^{(0)}A_3 - A_fT_3^{(0)} + Q_3 = 0, \tag{14}$$

$$T_4^{(0)}A_4 - A_fT_4^{(0)} + Q_4 = 0. \tag{15}$$

These equations can be solved independently as follows. Unique solution $T_4^{(0)}$ can be obtained from Sylvester equation (15) under the following assumption.

Assumption 7. $\lambda(A_4) \neq \lambda(A_f)$.

Since A_f, defined in (6), is chosen by the designer as an asymptotically stable matrix, this assumption is easily satisfied. Having obtained $T_4^{(0)}$, from (13) and (14), we can obtain $T_3^{(0)}$ and $T_2^{(0)}$ independently as

$$T_3^{(0)} = A_f^{-1}\left(T_4^{(0)}A_3 + Q_3\right), \tag{16}$$

$$T_2^{(0)} = -\left(Q_2 + T_1^{(0)}A_2\right)A_4^{-1}. \tag{17}$$

Substituting (17) into (12) results in

$$T_1^{(0)}A_0 - A_sT_1^{(0)} + Q_0 = 0, \tag{18}$$

$$\begin{aligned} A_0 &= A_1 - A_2A_4^{-1}A_3, \\ Q_0 &= Q_1 - Q_2A_4^{-1}A_3. \end{aligned} \tag{19}$$

The unique solution $T_1^{(0)}$ of algebraic Sylvester equation (18) exists under the following assumption.

Assumption 8. $\lambda(A_0) \neq \lambda(A_s)$.

The solutions $T_1^{(0)}, T_2^{(0)}, T_3^{(0)}$, and $T_4^{(0)}$ are $O(\varepsilon)$ close to the exact solutions, that is,

$$\begin{aligned} T_1 &= T_1^{(0)} + \varepsilon E_1, \quad T_2 = T_2^{(0)} + \varepsilon E_2, \\ T_3 &= T_3^{(0)} + \varepsilon E_3, \quad T_4 = T_4^{(0)} + \varepsilon E_4. \end{aligned} \tag{20}$$

Now, we show that the values for $E_i, i = 1, 2, 3, 4$, can be obtained by running iterations on independent linear reduced-order algebraic equations. Subtracting (12)–(15) from (11) and using (20), we obtain the error equations (after some algebra) in the following form:

$$E_1A_1 - A_sE_1 = -E_2A_3, \tag{21}$$

$$E_2A_4 - \varepsilon A_sE_2 = -E_1A_2 + A_sT_2^{(0)}, \tag{22}$$

$$\varepsilon E_3A_1 - A_fE_3 = -T_3^{(0)}A_1 - E_4A_3, \tag{23}$$

$$E_4A_4 - A_fE_4 = -T_3^{(0)}A_2 - \varepsilon E_3A_2. \tag{24}$$

The error equations can be solved iteratively using the fixed-point algorithm, in which the cross-coupling terms multiplied by ε are delayed by one iteration. This idea has been used in several algorithms that involve small parameters.

Algorithm 1.

$$\begin{aligned}
E_1^{(i+1)}A_0 - A_sE_1^{(i+1)} &= -A_sT_2^{(0)}A_4^{-1}A_3 - \varepsilon A_sE_2^{(i)}A_4^{-1}A_3,\\
E_2^{(i+1)}A_4 - \varepsilon A_sE_2^{(i+1)} &= A_sT_2^{(0)} - E_1^{(i+1)}A_2,\\
\varepsilon E_3^{(i+1)}A_1 - A_fE_3^{(i+1)} &= -T_3^{(0)}A_1 - E_4^{(i)}A_3,\\
E_4^{(i+1)}A_4 - A_fE_4^{(i+1)} &= -T_3^{(0)}A_2 - \varepsilon E_3^{(i)}A_2,\\
E_2^{(0)} &= 0,\\
E_3^{(0)} &= A_f^{-1}\left(T_3^{(0)}A_1 + E_4^{(0)}A_3\right),\\
E_4^{(0)}A_4 - A_fE_4^{(0)} + T_3^{(0)}A_2 &= 0.
\end{aligned} \tag{25}$$

We first solve (22) as

$$E_2^{(i+1)} = \left[A_sT_2^{(0)} - E_1^{(i+1)}A_2 + \varepsilon A_sE_2^{(i)}\right]A_4^{-1}. \tag{26}$$

Substituting (26) into (21) gives

$$E_1^{(i+1)}A_0 - A_sE_1^{(i+1)} = -A_sT_2^{(0)}A_4^{-1}A_3 - \varepsilon A_sE_2^{(i)}A_4^{-1}A_3. \tag{27}$$

Equations (26) and (27) have nice forms since the quantity E_2 is multiplied by a small parameter ε. Similarly, equations for E_3 and E_4 can be iteratively solved as

$$\begin{aligned}
\varepsilon E_3^{(i+1)}A_1 - A_fE_3^{(i+1)} &= -T_3^{(0)}A_1 - E_4^{(i)}A_3,\\
E_4^{(i+1)}A_4 - A_fE_4^{(i+1)} &= -T_3^{(0)}A_2 - \varepsilon E_3^{(i)}A_2.
\end{aligned} \tag{28}$$

The following theorem presents the main feature of Algorithm 1.

Theorem 1. *Under Assumptions 7 and 8, Algorithm 1 converges to the exact solution E with the rate of convergence of $O(\varepsilon)$. The convergence is obtained for sufficiently small values of ε that makes the radius of convergence smaller than 1 in each iteration leading to a contraction mapping. Hence, after i iterations, the solution T is obtained with the accuracy of $O(\varepsilon^i)$, that is,*

$$T_j^{(i)} = T_j^{(0)} + \varepsilon E_j^{(i)} + O\left(\varepsilon^i\right), \quad j = 1,2,3,4;\ i = 1,2,\ldots. \tag{29}$$

Proof of Theorem 1.

For $i = 0$, (27) implies

$$\begin{aligned}
E_1^{(1)}A_0 - A_sE_1^{(1)} &= -A_sT_2^{(0)}A_4^{-1}A_3 - \varepsilon A_sE_2^{(0)}A_4^{-1}A_3\\
&= -A_sT_2^{(0)}A_4^{-1}A_3.
\end{aligned} \tag{30}$$

Note that $E_2^{(0)} = 0$. For $i = 1$, (27) produces

$$E_1^{(2)}A_0 - A_sE_1^{(2)} = -A_sT_2^{(0)}A_4^{-1}A_3 - \varepsilon A_sE_2^{(1)}A_4^{-1}A_3. \tag{31}$$

Subtracting (30) from (31), we have

$$\left(E_1^{(2)} - E_1^{(1)}\right)A_0 - A_s\left(E_1^{(2)} - E_1^{(1)}\right) = -\varepsilon A_sE_2^{(1)}A_4^{-1}A_3 = O(\varepsilon). \tag{32}$$

At this point, we conclude that

$$\left\|E_1^{(2)} - E_1^{(1)}\right\| = O(\varepsilon). \tag{33}$$

In a similar way, we can write the relationship between $E_1^{(3)}$ and $E_1^{(2)}$ as

$$\begin{aligned}
&\left(E_1^{(3)} - E_1^{(2)}\right)A_0 - A_s\left(E_1^{(3)} - E_1^{(2)}\right)\\
&\quad = \varepsilon A_sE_2^{(1)}A_4^{-1}A_3 - \varepsilon A_sE_2^{(2)}A_4^{-1}A_3 = O(\varepsilon),
\end{aligned} \tag{34}$$

which implies that

$$\left\|E_1^{(3)} - E_1^{(2)}\right\| = O(\varepsilon). \tag{35}$$

Continuing the same procedure, we obtain

$$\left\|E_1^{(i+1)} - E_1^{(i)}\right\| = O(\varepsilon). \tag{36}$$

Now, we work with E_2 using (26). For $i = 0$, we have

$$E_2^{(1)} = \left[A_sT_2^{(0)} - E_1^{(1)}A_2 + \varepsilon A_sE_2^{(0)}\right]A_4^{-1} \Rightarrow E_2^{(1)} = O(1). \tag{37}$$

For $i = 1$,

$$E_2^{(2)} = \left[A_sT_2^{(0)} - E_1^{(2)}A_2 + \varepsilon A_sE_2^{(1)}\right]A_4^{-1} \tag{38}$$

Using $E_2^{(0)} = 0$ and the result in (33), we get

$$\begin{aligned}
\left\|E_2^{(2)} - E_2^{(1)}\right\| &= \left\|E_1^{(2)} - E_1^{(1)}\right\|A_2A_4^{-1} + O(\varepsilon)\\
&\Rightarrow \left\|E_2^{(2)} - E_2^{(1)}\right\| = O(\varepsilon).
\end{aligned} \tag{39}$$

Considering (26) for $i = 2$ and using (39), we obtain

$$\left\|E_2^{(3)} - E_2^{(2)}\right\| = O(\varepsilon). \tag{40}$$

If we keep repeating this process, we conclude that

$$\left\|E_2^{(i+1)} - E_2^{(i)}\right\| = O(\varepsilon). \tag{41}$$

In addition, we have the following relationships. Subtracting (21)-(22) from (25) for $i = 0$ produces

$$\begin{aligned}
\left(E_1^{(1)} - E_1\right)A_0 - A_s\left(E_1^{(1)} - E_1\right) &= -\varepsilon A_s\left(E_2^{(0)} - E_2\right)A_4^{-1}A_3\\
&= O(\varepsilon) \Rightarrow \left\|E_1^{(1)} - E_1\right\| = O(\varepsilon),\\
\left(E_2^{(1)} - E_2\right)A_4 - \varepsilon A_s\left(E_2^{(1)} - E_2\right) &= -\left(E_1^{(1)} - E_1\right)A_2 = O(\varepsilon)\\
&\Rightarrow \left\|E_2^{(1)} - E_2\right\| = O(\varepsilon).
\end{aligned} \tag{42}$$

For $i = 1$, (21)-(22) and (25) produce

$$\begin{aligned}
\left(E_1^{(2)} - E_1\right)A_0 - A_s\left(E_1^{(2)} - E_1\right) &= -\varepsilon A_s\left(E_2^{(1)} - E_2\right)A_4^{-1}A_3\\
&= O\left(\varepsilon^2\right) \Rightarrow \left\|E_1^{(2)} - E_1\right\| = O\left(\varepsilon^2\right),\\
\left(E_2^{(2)} - E_2\right)A_4 - \varepsilon A_s\left(E_2^{(2)} - E_2\right) &= -\left(E_1^{(2)} - E_1\right)A_2 = O\left(\varepsilon^2\right)\\
&\Rightarrow \left\|E_2^{(2)} - E_2\right\| = O\left(\varepsilon^2\right).
\end{aligned} \tag{43}$$

Continuing the same procedure, we have

$$\left\|E_1^{(i)} - E_1\right\| = O\left(\varepsilon^i\right), \quad i = 1, 2, 3, \ldots, \tag{44}$$

$$\left\|E_2^{(i)} - E_2\right\| = O\left(\varepsilon^i\right), \quad i = 1, 2, 3, \ldots. \tag{45}$$

Similar procedures applied to (23)-(24) produces

$$\begin{aligned} \left\|E_4^{(i+1)} - E_4^{(i)}\right\| &= O(\varepsilon), \\ \left\|E_4^{(i)} - E_4\right\| &= O\left(\varepsilon^i\right), \\ i &= 1, 2, 3, \ldots, \end{aligned} \tag{46}$$

$$\begin{aligned} \left\|E_3^{(i+1)} - E_3^{(i)}\right\| &= O(\varepsilon), \\ \left\|E_3^{(i)} - E_3\right\| &= O\left(\varepsilon^i\right), \\ i &= 1, 2, 3, \ldots. \end{aligned} \tag{47}$$

Results established in (44)–(47) can be summarized in

$$\left\|E_j^{(i)} - E_j\right\| = O\left(\varepsilon^i\right), \quad j = 1, 2, 3, 4;\ i = 1, 2, \ldots, \tag{48}$$

which completes the proof of the stated theorem. □

2. Parallel Algorithm for Controller Sylvester Equation

The controller design algebraic Sylvester equation defined in (9)-(10) can be partitioned as

$$\begin{aligned} A_1T_{1c} + A_2T_{3c} - T_{1c}A_{sc} + Q_{1c} &= 0, \\ \varepsilon A_1T_{2c} + A_2T_{4c} - T_{2c}A_{fc} + Q_{2c} &= 0, \\ A_3T_{1c} + A_4T_{3c} - \varepsilon T_{3c}A_{sc} + Q_{3c} &= 0, \\ \varepsilon A_3T_{2c} + A_4T_{4c} - T_{4c}A_{fc} + Q_{4c} &= 0, \end{aligned} \tag{49}$$

$$Q_c = B\overline{F} = BFT_c = -\begin{bmatrix} Q_{1c} & \frac{1}{\varepsilon}Q_{2c} \\ \frac{1}{\varepsilon}Q_{3c} & \frac{1}{\varepsilon^2}Q_{4c} \end{bmatrix} = \begin{bmatrix} O(1) & O\left(\frac{1}{\varepsilon}\right) \\ O\left(\frac{1}{\varepsilon}\right) & O\left(\frac{1}{\varepsilon^2}\right) \end{bmatrix}. \tag{50}$$

Setting $\varepsilon = 0$ in (49), the zeroth-order approximations $T_{1c}^{(0)}, T_{2c}^{(0)}, T_{3c}^{(0)}$, and $T_{4c}^{(0)}$ can be obtained as follows:

$$A_1T_{1c}^{(0)} + A_2T_{3c}^{(0)} - T_{1c}^{(0)}A_{sc} + Q_{1c} = 0, \tag{51}$$

$$A_2T_{4c}^{(0)} - T_{2c}^{(0)}A_{fc} + Q_{2c} = 0, \tag{52}$$

$$A_3T_{1c}^{(0)} + A_4T_{3c}^{(0)} + Q_{3c} = 0, \tag{53}$$

$$A_4T_{4c}^{(0)} - T_{4c}^{(0)}A_{fc} + Q_{4c} = 0. \tag{54}$$

The unique solution $T_{4c}^{(0)}$ can be found from algebraic Sylvester equation (54) under the following assumption, which is easily satisfied since A_{fc} is an arbitrary stable closed-loop matrix.

Assumption 9. $\lambda(A_4) \neq \lambda(A_{fc})$.

From (52)-(53), we can obtain $T_{2c}^{(0)}$ and $T_{3c}^{(0)}$ as

$$T_{2c}^{(0)} = \left(A_2T_{4c}^{(0)} + Q_{2c}\right)A_{fc}^{-1}, \tag{55}$$

$$T_{3c}^{(0)} = -A_4^{-1}\left(A_3T_{1c}^{(0)} + Q_{3c}\right). \tag{56}$$

Substituting (56) into (51) results in

$$A_0T_{1c}^{(0)} - T_{1c}^{(0)}A_{sc} + Q_{0c} = 0, \tag{57}$$

$$\begin{aligned} A_0 &= A_1 - A_2A_4^{-1}A_3, \\ Q_{0c} &= Q_{1c} - A_2A_4^{-1}Q_{3c}. \end{aligned} \tag{58}$$

The solution $T_1^{(0)}$ can be obtained by solving algebraic Sylvester equation (57) under the following assumption.

Assumption 10. $\lambda(A_0) \neq \lambda(A_{sc})$.

Since A_{sc} is chosen by the designer, this assumption is easily satisfied. We define the approximation errors as

$$\begin{aligned} T_{1c} &= T_{1c}^{(0)} + \varepsilon E_{1c}, \quad T_{2c} = T_{2c}^{(0)} + \varepsilon E_{2c}, \\ T_{3c} &= T_{3c}^{(0)} + \varepsilon E_{3c}, \quad T_{4c} = T_{4c}^{(0)} + \varepsilon E_{4c}. \end{aligned} \tag{59}$$

Subtracting (51)–(54) from (49) and using (59), we obtain the error equations in the following form:

$$\begin{aligned} A_1E_{1c} - E_{1c}A_{sc} &= -A_2E_{3c}, \\ E_{2c}A_{fc} - \varepsilon A_1E_{2c} &= -A_1T_{2c}^{(0)} + A_2E_{4c}, \\ A_4E_{3c} - \varepsilon E_{3c}A_{sc} &= -A_3E_{1c} - T_{3c}^{(0)}A_{sc}, \\ E_{4c}A_{fc} - A_4E_{4c} &= A_3T_{2c}^{(0)} - \varepsilon A_3E_{2c}. \end{aligned} \tag{60}$$

These error equations can be solved using the fixed-point algorithm, dual to Algorithm 1, as follows.

Algorithm 2.

$$\begin{aligned} A_0E_{1c}^{(i+1)} - E_{1c}^{(i+1)}A_{sc} &= -\varepsilon A_2A_4^{-1}E_{3c}^{(i)}A_{sc} - A_2A_4^{-1}T_{3c}^{(0)}A_{sc}, \\ A_4E_{3c}^{(i+1)} - \varepsilon E_{3c}^{(i+1)}A_{sc} &= -A_3E_{1c}^{(i)} + T_{3c}^{(0)}A_{sc}, \\ E_{2c}^{(i+1)}A_{fc} - \varepsilon A_1E_{2c}^{(i+1)} &= A_1T_{2c}^{(0)} + A_2E_{4c}^{(i)}, \\ E_{4c}^{(i+1)}A_{fc} - A_4E_{4c}^{(i+1)} &= A_3T_{2c}^{(0)} + \varepsilon A_3E_{2c}^{(i)}, \\ E_{3c}^{(0)} &= 0, \\ E_{2c}^{(0)} &= 0, \\ A_0E_{1c}^{(0)} - E_{1c}^{(0)}A_{sc} + A_2A_4T_{3c}^{(0)}A_{sc} &= 0. \end{aligned} \tag{61}$$

The convergence proof of Algorithm 2 can be done via the dual arguments used in Algorithm 1. Similarly, we can state the corresponding theorem dual to Theorem 1.

Theorem 2. *Under Assumptions 5, 9, and 10, Algorithm 2 converges for sufficiently small values of ε with the rate of $O(\varepsilon)$ to the sought solution T_j, $j = 1, 2, 3, 4$; that is, after i iterations, we have*

$$T_{jc}^{(i)} = T_{jc}^{(0)} + \varepsilon E_{jc}^{(i)} + O\left(\varepsilon^{i}\right), \quad j = 1,2,3,4;\ i = 1,2,3,\ldots. \tag{62}$$

The proof of Theorem 2 parallels the one of Theorem 1.

3. Observer and Controller Designs

The general design of an observer and a controller using the Sylvester approach is presented in [13]. We will exploit two-time scale property so that the design is done in terms of reduced-order problems. The observer design procedure for the system defined in (2) has the following steps [13].

$$\dot{z}(t) = A_{\text{des}} z(t) + TBu(t) + \bar{K} y(t), \hat{x}(t) = T^{-1} z(t). \tag{63}$$

The state feedback controller for the system defined in (2) can be obtained using the steps.

Comment 1. according to our experience, we need only one repetition to obtain invertible T_c. According to [13], the pair (A, B) controllable and the pair $(A_{\text{des}}, \overline{F})$ observable are both the necessary and sufficient condition for invertibility of T_c in the case of single-input single-output systems. For multiple-input multiple-output systems, this condition is only sufficient.

The feedback system is given by [13]

$$\begin{aligned} \dot{x}(t) &= (A - BF)x(t), \\ \lambda(A - BF) &= \lambda_{\text{desired}}^{\text{system}}. \end{aligned} \tag{64}$$

Comment 2. well-conditioning with respect to matrix inversion of matrices T and T_c can be established by using the formula for matrix inversion of partitioned matrices [13]. For matrix T defined in (7), we obtain

$$T^{-1} = \begin{bmatrix} T_1 & \varepsilon T_2 \\ \varepsilon T_3 & \varepsilon T_4 \end{bmatrix}^{-1} = \begin{bmatrix} T_1^{-1} + O(\varepsilon) & O\left(\varepsilon^2\right) \\ O\left(\varepsilon^2\right) & \dfrac{1}{\varepsilon}(I + O(\varepsilon))^{-1} \end{bmatrix}. \tag{65}$$

Hence, the T matrix is well conditioned with respect to matrix inversion if matrix T_1 is well conditioned with respect to matrix inversion. Similarly, for matrix inversion of T_c, the matrix T_{4c} must be well conditioned with respect to the matrix inversion.

Comment 3. systems (63) and (64) have the structure of linear singularly perturbed systems.

4. Simulation Results

Consider a 4th–order system with the matrices A, B, and C taken from [11] (Chapter 3, 1999) as follows:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0.4000 & 0 & 0 \\ 0 & 0 & 0.3450 & 0 \\ 0 & -5.2400 & -4.6500 & 2.6200 \\ 0 & 0 & 0 & -10.0000 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \end{aligned} \tag{66}$$

The pair (A, C) is observable, and we can proceed with the observer design algorithm. The designer decides to place observer eigenvalues at the desired location by choosing matrices A_s and A_f. In the following, we will design a controller with the desired closed-loop eigenvalues placed at $\{-0.2, -0.3, -7, -8\}$. Note that (A, B) is controllable.

4.1. Observer Design Algorithm 1. We choose the observer eigenvalues such that it is roughly ten times faster than the closed-loop system. Consequently, we choose A_{des} as

$$A_{\text{des}} = \begin{bmatrix} A_s & 0 \\ 0 & \dfrac{1}{\varepsilon} A_f \end{bmatrix} = \begin{bmatrix} -50 & 0 & 0 & 0 \\ 0 & -60 & 0 & 0 \\ 0 & 0 & -500 & 0 \\ 0 & 0 & 0 & -600 \end{bmatrix}. \tag{67}$$

We choose $\overline{K}$ as

$$\overline{K} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \\ 4 & 6 \end{bmatrix}, \tag{68}$$

so that $(A_{\text{des}}, \overline{K})$ is controllable, as required in Step 2 of Algorithm 1.

The matrix Q defined in (8) is given as

$$Q = -\overline{K}C = \begin{bmatrix} -1 & 0 & -3 & 0 \\ -2 & 0 & -4 & 0 \\ -3 & 0 & -5 & 0 \\ -4 & 0 & -6 & 0 \end{bmatrix}. \tag{69}$$

The zeroth-order approximations $T_1^{(0)}, T_2^{(0)}, T_3^{(0)}$, and $T_4^{(0)}$ are obtained as

$$T_1^{(0)} = \begin{bmatrix} 0.0200000 & -0.0683039 \\ 0.0333333 & -0.0758390 \end{bmatrix},$$
$$T_2^{(0)} = \begin{bmatrix} -6.5022900 & -1.7035999 \\ -8.6584182 & -2.2685055 \end{bmatrix},$$
$$T_3^{(0)} = \begin{bmatrix} 0.0600000 & 0.001057 \\ 0.0666666 & 0.0008801 \end{bmatrix},$$
$$T_4^{(0)} = \begin{bmatrix} 0.1009387 & -0.0005397 \\ 0.1007810 & -0.0004475 \end{bmatrix}. \tag{70}$$

Performing iterations, we obtain the sixteen decimal digits accuracy after $i = 50$:

$$E_1^{(50)} = \begin{bmatrix} 0 & 0.75071350 \\ 0 & 0.81924891 \end{bmatrix},$$
$$E_2^{(50)} = \begin{bmatrix} 71.63296737 & 16.60304059 \\ 93.80712798 & 22.30657343 \end{bmatrix},$$
$$E_3^{(50)} = 10^{-3} \times \begin{bmatrix} 0 & -0.4800737 \\ 0 & -0.4444867 \end{bmatrix},$$
$$E_4^{(50)} = 10^{-5} \times \begin{bmatrix} -0.7033239 & 0.0037606 \\ -0.4842840 & 0.0021505 \end{bmatrix}. \tag{71}$$

The corresponding iterative solution $\widehat{T}$ and the exact solution T (obtained by using the MATLAB function lyap to solve the full-order Sylvester equation) are given by

$$\widehat{T} = \begin{bmatrix} 0.0200000 & 0.0067673 & 0.0661006 & -0.0043295 \\ 0.0333333 & 0.0060858 & 0.0722294 & -0.0037848 \\ 0.0060000 & 0.0001009 & 0.0100938 & -0.0000539 \\ 0.0066666 & 0.0000835 & 0.0100780 & -0.0000447 \end{bmatrix} = T. \tag{72}$$

The difference $\|T - \widehat{T}\|$ is

$$\|T - \widehat{T}\| = 1.882474211376878 \times 10^{-16}. \tag{73}$$

The solution T is invertible in the first run of Algorithm 1 (see Section 5). The corresponding observer gain $K = (\widehat{T})^{-1}\overline{K}$ is

$$K = 10^5 \times \begin{bmatrix} 0.0112290 & 0.0101385 \\ -0.7819285 & -0.9259744 \\ -0.0023409 & 0.0007244 \\ -1.2083683 & -1.3963744 \end{bmatrix}. \tag{74}$$

Checking the corresponding observer closed loop eigenvalues, we have

$$\lambda(A - KC) = \begin{bmatrix} -49.9999999999540 \\ -60.0000000000346 \\ -500.000000001148 \\ -599.9999999999814 \end{bmatrix}, \tag{75}$$

which with the accuracy of $O(10^{-12})$ is close to the chosen desired eigenvalues of the matrix A_{des}. The general Sylvester equation for the observed case is given as in (63).

4.2. Result of Comparison Algorithm 1 with Algorithm 432. To compare our proposed Algorithm 1 against the existing design algorithm, we implement the Algorithm 432 presented in [1]. Applying the Schur transformation to equation (5), we have

$$A'_{\text{des}} T' + T' A' = Q', \tag{76}$$

where

$$\begin{aligned} A'_{\text{des}} &= U^T(-A_{\text{des}})U, \\ T' &= U^T T V, \\ A' &= V^T A V, \\ Q' &= U^T Q V, \end{aligned} \tag{77}$$

where U and V are the Schur transformation in order to construct a lower triangular form for A_{des} and an upper triangular form for A given in (5) and used in [1]. The formula for the solution T in (5), based on the relationship in (77), is given in

$$T = U^{-T} T' V^{-1}. \tag{78}$$

Using the MATLAB Schur function and the similarity transformation given in (77), we have

$$U = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$
$$V = \begin{bmatrix} 1.0000000 & 0 & 0 & 0 \\ 0 & 0.6273910 & 0.7787043 & 0 \\ 0 & -0.7787043 & 0.6273910 & 0 \\ 0 & 0 & 0 & 1.0000000 \end{bmatrix},$$
$$A_{\text{des}}' = \begin{bmatrix} -600 & 0 & 0 & 0 \\ 0 & -500 & 0 & 0 \\ 0 & 0 & -60 & 0 \\ 0 & 0 & 0 & -50 \end{bmatrix},$$
$$A' = \begin{bmatrix} 0 & 0.2509564 & 0.3114817 & 0 \\ 0 & -0.4282066 & 5.5850000 & -2.0402054 \\ 0 & 0 & -4.2217933 & 1.6437644 \\ 0 & 0 & 0 & -10.0000000 \end{bmatrix},$$
$$Q' = \begin{bmatrix} -4.0000000 & 4.6722262 & -3.7643461 & 0 \\ -3.0000000 & 3.8935218 & -3.1369551 & 0 \\ -2.0000000 & 3.1148175 & -2.5095640 & 0 \\ -1.0000000 & 2.3361131 & -1.8821730 & 0 \end{bmatrix}. \tag{79}$$

Partitioning (76) results in

Step 1: choose A_{des} in (6) such that Assumption 4 is satisfied.
Step 2: guess $\overline{K}$ in (5) and (7) such that Assumption 3 is satisfied.
Step 3: solve $TA - A_{\text{des}}T = \overline{K}C$ using Algorithm 1.
Step 4: if T^{-1} does not exist, go back to Step 2 and guess another $\overline{K}$ and repeat the process until T^{-1} exists. The observer structure for $z(t)$ is given as in [13]:

Algorithm 1: (Observer design).

$$\begin{bmatrix} A'_{\text{des}_{11}} & 0 \\ A'_{\text{des}_{21}} & A'_{\text{des}_{22}} \end{bmatrix} \begin{bmatrix} T'_{11} & T'_{12} \\ T'_{21} & T'_{22} \end{bmatrix} + \begin{bmatrix} T'_{11} & T'_{12} \\ T'_{21} & T'_{22} \end{bmatrix} \begin{bmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{bmatrix} = \begin{bmatrix} Q'_{11} & Q'_{12} \\ Q'_{21} & Q'_{22} \end{bmatrix}, \tag{80}$$

where dimensions of $A'_{\text{des}_{11}}$, $A'_{\text{des}_{22}}$, A'_{11}, A'_{22}, T'_{11}, T'_{22}, and Q'_{11}, Q_{22}' are, respectively, $A'_{\text{des}_{11}} \in R^{n_1}$, $A'_{\text{des}_{22}} \in R^{n_2}$, $A'_{11} \in R^{n_1}$, $A'_{22} \in R^{n_2}$, $T'_{11} \in R^{n_1}$, $T'_{22} \in R^{n_2}$, and $Q'_{11} \in R^{n_1}$, $Q'_{22} \in R^{n_2}$ with the corresponding dimensions of $A'_{\text{des}_{12}}$, $A'_{\text{des}_{21}}$, A'_{12}, A'_{21}, T'_{12}, T'_{21}, and Q'_{12}, Q'_{21}.

From equation (80) and the condition $A'_{\text{des}_{21}} = 0$, we have

$$A'_{\text{des}_{11}} T'_{11} + T'_{11} A'_{11} = Q'_{11}, \tag{81}$$

$$A'_{\text{des}_{22}} T'_{21} + T'_{21} A'_{11} = Q'_{21}, \tag{82}$$

$$A'_{\text{des}_{22}} T'_{22} + T'_{22} A'_{22} = Q'_{22} - T'_{21} A'_{12}, \tag{83}$$

$$A'_{\text{des}_{11}} T'_{12} + T'_{12} A'_{22} = Q'_{12} - T'_{11} A_{12}'. \tag{84}$$

Solving reduced-order Sylvester equations (81)–(84), we have

$$\begin{aligned}
A'_{\text{des}_{11}} &= \begin{bmatrix} -600 & 0 \\ 0 & -500 \end{bmatrix}, \\
A'_{\text{des}_{22}} &= \begin{bmatrix} -60 & 0 \\ 0 & -50 \end{bmatrix}, \\
A'_{11} &= \begin{bmatrix} 0 & 0.2509564 \\ 0 & -0.4282066 \end{bmatrix}, \\
A'_{12} &= \begin{bmatrix} 0.3114817 & 0 \\ 5.5850000 & -2.0402054 \end{bmatrix}, \\
A'_{22} &= \begin{bmatrix} -4.2217933 & 1.6437644 \\ 0 & -10.0000000 \end{bmatrix}, \\
Q'_{11} &= \begin{bmatrix} -4.0000000 & 4.6722262 \\ -3.0000000 & 3.8935218 \end{bmatrix}, \\
Q'_{12} &= \begin{bmatrix} -3.7643461 & 0 \\ -3.1369551 & 0 \end{bmatrix}, \\
Q'_{21} &= \begin{bmatrix} -2.0000000 & 3.1148175 \\ -1.0000000 & 2.3361131 \end{bmatrix}, \\
Q'_{22} &= \begin{bmatrix} -2.5095640 & 0 \\ -1.8821730 & 0 \end{bmatrix}.
\end{aligned} \tag{85}$$

The corresponding solution of T' is given by

$$T' = \begin{bmatrix} T'_{11} & T'_{12} \\ T'_{21} & T'_{22} \end{bmatrix} = \begin{bmatrix} 0.0066666 & -0.0077787 & 0.0061616 & 0.0000426 \\ 0.0060000 & -0.0077773 & 0.0061389 & 0.0000508 \\ 0.0333333 & -0.0514073 & 0.0347675 & 0.0023147 \\ 0.0200000 & -0.0462259 & 0.0300659 & 0.0023955 \end{bmatrix}. \tag{86}$$

Original solution T using the inverse transformation of (78) is given by

$$T = \begin{bmatrix} 0.0200000 & -0.0055892 & 0.0548595 & 0.0023955 \\ 0.0333333 & -0.0051788 & 0.0618439 & 0.0023147 \\ 0.0060000 & -0.0000990 & 0.0099077 & 0.0000508 \\ 0.0066666 & -0.0000822 & 0.0099230 & 0.0000426 \end{bmatrix}. \tag{87}$$

The difference between T and $\widehat{T}$ of our proposed Algorithm 1 is

$$\|\widehat{T} - T\| = 0.0244206. \tag{88}$$

Since they are different, we conclude that our proposed Algorithm 1 has a better accuracy regarding the solution of the general Sylvester equation.

Algorithm 2(Controller design). Similarly, we design a controller for the same system using the algorithm for solving the controller algebraic Sylvester equation from Section 4. We choose

$$A^c_{\text{des}} = \begin{bmatrix} A^c_s & 0 \\ 0 & \frac{1}{\varepsilon} A^c_f \end{bmatrix} = \begin{bmatrix} -0.2 & 0 & 0 & 0 \\ 0 & -0.3 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -8 \end{bmatrix}. \tag{89}$$

The zeroth-order approximations $T^{(0)}_{1c}, T^{(0)}_{2c}, T^{(0)}_{3c}$, and $T^{(0)}_{4c}$ are obtained as

Step 1: choose A_{des}^{c} in (9) such that Assumption 6 is satisfied.
Step 2: guess $\overline{F}$ in (9) and (10) such that Assumption 5 is satisfied.
Step 3: solve $AT_c - TA_{\text{des}}^{c} = B\overline{F}$ using Algorithm 2.
Step 4: if T_c^{-1} exists, then $F = \overline{F}T_c^{-1}$, otherwise go back to Step 2, guess another $\overline{F}$, and repeat the process.

Algorithm 2: (Controller design).

$$\begin{aligned} T_{1c}^{(0)} &= \begin{bmatrix} -10.2973342 & -20.4370155 \\ 5.1486671 & 15.3277616 \end{bmatrix}, \\ T_{2c}^{(0)} &= \begin{bmatrix} 0 & 0 \\ 1.4652887 & 1.5177425 \end{bmatrix}, \\ T_{3c}^{(0)} &= \begin{bmatrix} -2.9847345 & -13.3284883 \\ 5.0000000 & 7.0000000 \end{bmatrix}, \\ T_{4c}^{(0)} &= \begin{bmatrix} -2.9730496 & -3.5194029 \\ 2.6666666 & 4.5000000 \end{bmatrix}. \end{aligned} \tag{90}$$

Using the proposed algorithm, we obtain after $i = 50$:

$$\begin{aligned} E_{1c}^{(50)} &= \begin{bmatrix} 2.4779252 & 31.3924384 \\ -1.2389626 & -23.5443288 \end{bmatrix}, \\ E_{2c}^{(50)} &= \begin{bmatrix} -0.7544016 & -0.7109162 \\ -1.4508588 & -0.9591003 \end{bmatrix}, \\ E_{3c}^{(50)} &= \begin{bmatrix} 0.7182392 & 20.4733294 \\ 1.0204081 & 2.1649484 \end{bmatrix}, \\ E_{4c}^{(50)} &= \begin{bmatrix} 2.9437715 & 2.2240006 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{91}$$

The iterative solution $\widehat{T}^c$ and the exact solution T^c are given by

$$\widehat{T}^c = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = T^c, \tag{92}$$

where

$$\begin{aligned} a &= \begin{bmatrix} -10.0495417 & -17.2977716 \\ 5.0247708 & 12.9733287 \end{bmatrix}, \\ b &= \begin{bmatrix} -0.0754401 & -0.0710916 \\ 1.3202028 & 1.4218325 \end{bmatrix}, \\ c &= \begin{bmatrix} -2.9129106 & -11.2811554 \\ 5.1020408 & 7.2164948 \end{bmatrix}, \\ d &= \begin{bmatrix} -26.7867249 & -32.9700291 \\ 26.6666666 & 45.0000000 \end{bmatrix}. \end{aligned} \tag{93}$$

Their difference is

$$\left\| T^c - \widehat{T}^c \right\| = 1.761132598848588 \times 10^{-14}. \tag{94}$$

The solution T^c is invertible in the first run of Algorithm 2 (see Section 5). The controller gain $F = \overline{F}(\widehat{T}^c)^{-1}$ is given as

$$F = \begin{bmatrix} 0.92930633 & 1.02725633 & 0.43128625 & 0.08500000 \end{bmatrix}. \tag{95}$$

Checking $\lambda(A - BF)$, we have

$$\lambda(A - BF) = \begin{bmatrix} -0.199999999999998 \\ -0.300000000000001 \\ -7.000000000000027 \\ -7.999999999999988 \end{bmatrix}, \tag{96}$$

which has produced the desired eigenvalues with the accuracy of $O(10^{-13})$.

5. Conclusions

It was shown that the numerically ill-conditioned Sylvester algebraic equation for singularly perturbed systems can be decomposed into four lower-order well-conditioned Sylvester equations. The recursive fixed-point-type methods were utilized in order to obtain numerical solutions for such lower-order algebraic Sylvester equations. The corresponding observer and controller design algorithms for assignment of observer and controller closed-loop eigenvalues in terms of reduced-order slow and fast subproblems were presented.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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