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Research Article

Mixed H_2/H_∞ Control for Itô-type Stochastic Time-Delay Systems with Applications to Clothing Hanging Device

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This paper deals with the problem of mixed H_2/H_∞ control for Itô-type stochastic time-delay systems. First, the H_2/H_∞ control problem for stochastic time-delay systems is presented, which considers the mean square stability, H_2 control performance index, and the ability of disturbance attenuation of the closed-loop systems. Second, by choosing an appropriate Lyapunov–Krasoviskii functional and using matrix inequality technique, some sufficient conditions for the existence of state feedback H_2/H_∞ controller for stochastic time-delay systems are obtained in the form of linear matrix inequalities. Third, two convex optimization problems with linear matrix inequality constraints are formulated to design the optimal mixed H_2/H_∞ controller which minimizes the guaranteed cost of the closed-loop systems with known and unknown initial functions, and the corresponding algorithm is given to optimize H_2/H_∞ performance index. Finally, a numerical example is employed to show the effectiveness and feasibility of the proposed method.

1. Introduction

Over the past decades, there has been a rapid increase of interest in the study of stochastic systems due to the importance of stochastic models in science and engineering, such as finance systems [1] and power systems [2]. And a lot of excellent results have been obtained. For example, Zhu et al. [3] investigated the tracking control issue of stochastic systems subject to time-varying full state constraints and input saturation. In [4], the stability of a class of discrete-time stochastic nonlinear systems with external disturbances was considered. The finite-time tracking control of a class of stochastic quantized nonlinear systems was studied in [5]. Furthermore, since stochasticity and time delay are the main sources resulting in the complexities of systems in reality, considerable interests have been focused on a general model of stochastic time-delay systems. For example, the problem of guaranteed cost robust stable control was considered via state feedback for a class of uncertain stochastic systems with time-varying delay in [6]. In [7], the mean square exponential stability of neutral-type linear stochastic time-delay systems with three different delays by using the Lyapunov–Krasovskii functionals was studied. In [8], the finite-time dissipative control for stochastic interval systems with time delay and Markovian switching was investigated. Some other nice results can be referred to [9–17] and the references therein.

At present, H_∞ control has been receiving increased attention because it can suppress external interference, and many efforts have been devoted to extending the results for H_∞ control over the last few decades. For instance, Ma and Liu [18] investigated the finite-time H_∞ control problem for singular Markovian jump system with actuator fault through the sliding mode control approach. In [19], the problem of nonfragile observer-based H_∞ control for stochastic time-delay systems was considered. The problems of robust stabilization and robust H_∞ control with maximal decay rate were investigated for discrete-time stochastic systems with time-varying norm-bounded parameter uncertainties in [20]. Some other nice results can be referred to [21–26]. On the contrary, H_∞ control is an effective way to attenuate the disturbance, while H_2 control can guarantee quadratic performance cost. By combining H_2 control and H_∞ control theory, the mixed H_2/H_∞ control theory is obtained. Owing to the fact that the mixed H_2/H_∞ control can minimize a desired control performance and eliminate the effect of disturbance, it is more attractive than the sole H_∞ control in engineering practice. For example, Gao et al. [12] investigated the problem of H_2/H_∞ control for nonlinear stochastic systems with time-delay and state-dependent noise. H_2/H_∞ control problem of stochastic systems with random jumps was solved in [27]. Sathananthan et al. [28] studied guaranteed cost H_∞ control of linear stochastic Markovian switching systems. Although the problem of H_2/H_∞ control has been investigated, there are few literature studies on Itô-type stochastic time-delay systems.

Motivated by the abovementioned discussions, in this work, we aim to investigate the mixed H_2/H_∞ control for Itô-type stochastic time-delay systems. It is difficult to design state feedback H_2/H_∞ controller because of the complicated structure of the system. The main contributions of this paper are as follows. (i) The definition of H_2/H_∞ control for Itô-type stochastic time-delay systems is presented, which considers stability, H_2 control performance index, and H_∞ control performance index. (ii) The new sufficient conditions for the existence of state feedback H_2/H_∞ controller are provided in the form of linear matrix inequalities. (iii) An algorithm is given to optimize H_2/H_∞ performance index.

The organization of this paper is as follows. Section 2 is devoted to the problem statement, preliminaries, and lemmas. Section 3 provides the sufficient conditions for the existence of state feedback H_2/H_∞ controller for Itô-type stochastic time-delay systems. Section 4 gives an algorithm to solve the theorems. Section 5 presents a numerical example to demonstrate the effectiveness of the proposed method. Section 6 is our conclusions.

Notations: A' denotes the transpose of matrix A; $\mathrm{tr}(A)$ indicates the trace of matrix A; $A > 0\,(A \geq 0)$ indicates that A is a positive definite (positive semidefinite) matrix; $I_{n\times n}$ represents a n-dimensional identity matrix; $\mathscr{R}^n$ shows n-dimensional Euclidean space; E represents the mathematical expectation of random process; and the asterisk "*" in the matrix indicates symmetry term.

2. Preliminaries

Consider the following Itô-type stochastic time-delay system described by

$$\begin{cases} dx(t) = [A_{11}x(t) + A_{12}x(t-\tau) + B_{11}u(t) + B_{12}v(t)]dt + [A_{21}x(t) + A_{22}x(t-\tau) + B_{21}u(t)]dw(t), \\ z(t) = C_1x(t) + D_1u(t), \\ x(t) = \phi(t), \forall t \in [-\tau, 0], \end{cases} \tag{1}$$

where $x(t)$ is the state of the system, $u(t)$ is the control input, $z(t)$ is the control output, $\phi(t)$ is the initial state function, and $w(t)$ is a one-dimensional standard Wiener process defined on probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathrm{P})$. $\mathscr{F}_t$ stands for the smallest σ-algebra generated by $w(s)$, $0 \leq s \leq t$, i.e., $\mathscr{F}_t = \sigma\{w(s) \mid 0 \leq s \leq t\}$. $\tau > 0$ is the time delay. A_{11}, A_{12}, A_{21}, A_{22}, B_{11}, B_{12}, C_1, and D_1 are constant matrices with appropriate dimensions.

Next, a new definition of the mean square stability for system (1) is given.

Definition 1. System (1) ($u(t) \equiv 0$ and $v(t) \equiv 0$) is said to be mean square stable, if

$$\lim_{t\longrightarrow\infty} E\|x(t)\|^2 = 0. \tag{2}$$

Then, some lemmas for obtaining the main results are introduced.

Lemma 1 (see [29]). *Let $V(t,x) \in C^{1,2}(R_+, R^n)$ be a scalar function, and $V(t,x) > 0$, for the following stochastic system:*

$$dx(t) = a(x)dt + b(x)dw(t). \tag{3}$$

The Itô formula of $V(t,x)$ is given as follows:

$$dV(t,x) = LV(t,x)dt + \frac{\partial V'(t,x)}{\partial x} b(x)dw(t), \tag{4}$$

where

$$LV(t,x) = \frac{\partial V(t,x)}{\partial t} + \frac{\partial V'(t,x)}{\partial x} a(x) + \frac{1}{2} b'(x) \frac{\partial^2 V(t,x)}{\partial x^2} b(x). \tag{5}$$

Lemma 2 (see [30]). *For given $x \in R^n$, $y \in R^m$, $N \in R^{n\times m}$, and $\rho > 0$, then we have*

$$2x'Ny \leq \rho x'x + \frac{1}{\rho} y'N'Ny. \tag{6}$$

Lemma 3 (see [6]). *For some real matrices $N, M' = M$ and $R = R' > 0$, the following three conditions are equivalent:*

$$M + NR^{-1}N' < 0,$$
$$\begin{bmatrix} M & N \\ N' & -R \end{bmatrix} < 0, \tag{7}$$
$$\begin{bmatrix} M & -N \\ -N' & -R \end{bmatrix} < 0.$$

3. Mixed H_2/H_∞ Control for Stochastic Time-Delay Systems

In this section, a state feedback H_2/H_∞ controller will be designed.

We consider a state feedback controller for system (1) is

$$u(t) = Kx(t), \tag{8}$$

where K is the state feedback gain to be determined.

The closed-loop system can be obtained by substituting (2) into (1):

$$\begin{cases} dx(t) = [A_{11}x(t) + A_{12}x(t-\tau) + B_{11}Kx(t) + B_{12}v(t)]dt + [A_{21}x(t) + A_{22}x(t-\tau) + B_{21}Kx(t)]dw(t), \\ z(t) = C_1x(t) + D_1Kx(t), \\ x(t) = \phi(t), \quad \forall t \in [-\tau, 0]. \end{cases} \tag{9}$$

Associated with system (1), the cost function is provided as follows:

$$J_s(x(t), u(t)) = E\int_0^\infty (x'(t)Tx(t) + u'(t)Ru(t))\mathrm{d}t, \tag{10}$$

where $T = T' > 0$ and $R = R' > 0$ are the given positive scalars or given weighting matrices.

By substituting (2) into (4), we can obtain

$$J_s(x(t)) = E\int_0^\infty (x'(t)Tx(t) + x'(t)K'RKx(t))\mathrm{d}t. \tag{11}$$

Based on the above analysis, the problem of H_2/H_∞ control for stochastic time-delay systems is provided as follows.

Definition 2. For a given scalar $\gamma > 0$, if there exist a positive scalar J_s^* and a state feedback controller (2) such that

(i) The closed-loop system (3) is asymptotically stable in mean square sense.

(ii) H_2 cost function (5) satisfies $J_s(x(t)) \le J_s^*$ under the condition of $v(t) = 0$.

(iii) For any nonzero disturbance $v(t)$, the control output $z(t)$ satisfies the following inequality with zero initial condition:

$$E\int_0^\infty \|z(t)\|^2 \mathrm{d}s < \gamma^2 E\int_0^\infty \left\|v(t)^2\right\| \mathrm{d}s, \tag{12}$$

then (2) is said to be a state feedback H_2/H_∞ controller for system (1).

The sufficient conditions for the existence of the state feedback H_2/H_∞ controller (2) are given below. For this reason, an important lemma is first given.

Lemma 4. *For a given scalar $\gamma > 0$ and two symmetric positive definite matrices T and R, if there are two symmetric positive definite matrices P and Q such that*

$$\begin{bmatrix} \Gamma_{11} & PA_{12} + (A_{21} + B_{21}K)'PA_{22} & PB_{12} \\ * & -Q + A_{22}'PA_{22} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{13}$$

hold, where $\Gamma_{11} = Q + (A_{21} + B_{21}K)'P(A_{21} + B_{21}K) + 2(A_{11} + B_{11}K)'P + (C_1 + D_1K)'(C_1 + D_1K) + T + K'RK$, then (2) is a mixed H_2/H_∞ controller of system (3), and the corresponding guaranteed cost for system (3) is $J_s^* = E[x'(0)Px(0) + \int_{-\tau}^0 x'(s)Qx(s)ds]$.

Proof. The following proof is divided into three parts. First, it is proved that the closed-loop system (3) is mean square stable.

According to Lemma 3, condition (7) implies

$$\begin{bmatrix} \Gamma_{11} + \dfrac{1}{\gamma^2}PB_{12}B_{12}'P & PA_{12} + (A_{21} + B_{21}K)'PA_{22} \\ * & -Q + A_{22}'PA_{22} \end{bmatrix} < 0. \tag{14}$$

Due to $T > 0$, $R > 0$, and $\gamma > 0$, we can obtain $(C_1 + D_1K)'(C_1 + D_1K) > 0$, $(1/\gamma^2)PB_{12}B_{12}'P > 0$, and $K'RK > 0$; then, (8) implies

$$\begin{bmatrix} \Sigma_{11} & PA_{12} + (A_{21} + B_{21}K)'PA_{22} \\ * & -Q + A_{22}'PA_{22} \end{bmatrix} < 0, \tag{15}$$

where $\Sigma_{11} = Q + (A_{21} + B_{21}K)'P(A_{21} + B_{21}K) + 2(A_{11} + B_{11}K)'P$.

Let a quadratic function $V(x(t), t) = x'(t)Px(t) + \int_{-\tau}^0 x'(t+s)Qx(t+s)\mathrm{d}s$ and the differential generation operator of system (3) be $L_1V(x(t))$ with $v = 0$; then,

$$\begin{aligned}
&L_1V(x(t),t)\\
&= x'(t)Qx(t) - x'(t-\tau)Qx(t-\tau) + 2\left[x'(t)(A_{11}+B_{11}K)' + x'(t-\tau)A'_{12}\right]Px(t)\\
&\quad + \left[x'(t)(A_{21}+B_{21}K)' + x'(t-\tau)A'_{22}\right]P\left[(A_{21}+B_{21}K)x(t) + A_{22}x(t-\tau)\right]\\
&= x'(t)Qx(t) - x'(t-\tau)Qx(t-\tau) + 2\left[x'(t)(A_{11}+B_{11}K)' + x'(t-\tau)A'_{12}\right]Px(t)\\
&\quad + x'(t-\tau)A'_{22}PA_{22}x(t-\tau) + x'(t)(A_{21}+B_{21}K)'P(A_{21}+B_{21}K)x(t)\\
&\quad + 2x'(t-\tau)A'_{22}P(A_{21}+B_{21}K)x(t),
\end{aligned} \tag{16}$$

that is,

$$L_1V = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}' \begin{bmatrix} \Omega_{11} & PA_{12} + (A_{21}+B_{21}K)'PA_{22} \\ * & -Q + A'_{22}PA_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}, \tag{17}$$

where $\Omega_{11} = Q + (A_{11}+B_{11}K)'P + P(A_{11}+ B_{11}K) + (A_{21}+B_{21}K)'P(A_{21}+B_{21}K)$.

In the light of (9), we can derive that $L_1V(x(t),t) < 0$, that is, the closed-loop system (3) is asymptotically stable in mean square sense.

Secondly, we prove that the control output $z(t)$ satisfies H_∞ index for any nonzero disturbance $v(t)$ under zero initial condition.

According to (7), $T > 0$, and $K'RK > 0$, we can obtain

$$\begin{bmatrix} \Psi_{11} & PA_{12} + (A_{21}+B_{21}K)'PA_{22} & PB_{12} \\ * & -Q + A'_{22}PA_{22} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{18}$$

where $\Psi_{11} = Q + (A_{21}+B_{21}K)' P(A_{21}+B_{21}K) + 2(A_{11}+B_{11}K)'P + (C_1+D_1K)'(C_1+D_1K)$.

Notice that

$$\begin{aligned}
&E\int_0^\infty \left(z'z - \gamma^2 v'v\right)\mathrm{d}t = E\int_0^\infty\\
&\quad \left\{\left[x'(t)(C_1+D_1K)'(C_1+D_1K)x(t) - \gamma^2 v'v\right]\right.\\
&\quad \left.\mathrm{d}t + L_2V - L_2V\right\}\\
&\le E\int_0^\infty \left\{\left[x'(t)(C_1+D_1K)'(C_1+D_1K)x(t) - \gamma^2 v'v\right]\right.\\
&\quad \left.\mathrm{d}t + L_2V\right\},
\end{aligned} \tag{19}$$

where $L_2V(x(t))$ is the infinitesimal operator of system (3) for any nonzero disturbance $v(t)$, and

$$\begin{aligned}
&L_2V(x(t),t)\\
&= x'(t)Qx(t) - x'(t-\tau)Qx(t-\tau) + 2\\
&\left[(A_{11}+B_{11}K)x(t) + A_{12}x(t-\tau) + B_{12}v(t)\right]'Px(t)\\
&\quad + \left[(A_{21}+B_{21}K)x(t) + A_{22}x(t-\tau)\right]'P\left[(A_{21}+B_{21}K)x(t) + A_{22}x(t-\tau)\right]\\
&= x'(t)Qx(t) - x'(t-\tau)Qx(t-\tau)\\
&\quad + 2x'(t)(A_{11}+B_{11}K)'Px(t) + x'(t)(A_{21}+B_{21}K)'\\
&P(A_{21}+B_{21})x(t) + 2x'(t-\tau)A'_{12}Px(t) + 2v'(t)B'_{12}Px(t)\\
&\quad + 2x'(t)(A_{21}+B_{21}K)'PA_{22}x(t-\tau) + x'(t-\tau)A'_{22}PA_{22}x(t-\tau).
\end{aligned} \tag{20}$$

Then, we can see that

$$E\int_0^\infty \left(z'z - \gamma^2 v'v\right)\mathrm{d}t \le E\int_0^\infty \begin{bmatrix} x(t) \\ x(t-\tau) \\ v(t) \end{bmatrix}' \begin{bmatrix} \Psi_{11} & \Psi_{12} & PB_{12} \\ * & -Q + A'_{22}PA_{22} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \\ v(t) \end{bmatrix} \mathrm{d}t, \tag{21}$$

where $\Psi_{12} = PA_{12} + (A_{21} + B_{21}K)'PA_{22}$.

Based on (12), we can see $E\int_0^{\infty}(z'z - \gamma^2 v'v)dt \le 0$, that is, (12) implies that $E\int_0^{\infty}\|z(t)\|^2 dt < \gamma^2 E\int_0^{\infty}\|v(t)\|^2 dt$. Therefore, system (3) satisfies H_∞ index.

Thirdly, we prove that system (3) satisfies H_2 index under the condition of $v(t) = 0$.

Based on (13) and (14), $(C_1 + D_1K)'(C_1 + D_1K) > 0$, and $(1/\gamma^2)PB_{12}B_{12}'P > 0$, we obtain that

$$\begin{bmatrix} \Theta_{11} & PA_{12} + (A_{21} + B_{21}K)'PA_{22} \\ * & -Q + A_{22}'PA_{22} \end{bmatrix} < 0, \quad (22)$$

holds, where $\Theta_{11} = \Omega_{11} + T + K'RK$.

Due to

$$\begin{aligned} &E\int_0^{\infty}(x'(t)Tx(t) + u'(t)Ru(t))dt \\ &= E\int_0^{\infty}[(x'(t)(T + K'RK)x(t) + L_1V - L_1V]dt \\ &= E\int_0^{\infty}[x'(t)(T + K'RK)x(t) + L_1V]dt - E\int_0^{\infty}dV \\ &\quad = E\int_0^{\infty}[x'(t)(T + K'RK)x(t) + L_1V] \\ &\quad dt - EV(x(\infty),\infty) + EV(x(0),0) \\ &\quad = E\int_0^{\infty}\begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}'\begin{bmatrix}\Theta_{11} & \Psi_{12} \\ * & \Theta_{14}\end{bmatrix}\begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix} \\ &\quad dt + EV(x(0),0), \end{aligned} \quad (23)$$

where $\Theta_{14} = -Q + A_{22}'PA_{22}$.

In view of (22), we obtain

$$\begin{bmatrix} \Theta_{11} & \Psi_{12} \\ * & \Theta_{14} \end{bmatrix} < 0. \quad (24)$$

According to (23) and (24), we can see

$$\begin{aligned} J_s &= E\int_0^{\infty}(x'(t)Tx(t) + u'(t)Ru(t))dt \\ &< EV(x(0),0) \\ &= E\left[x'(0)Px(0) + \int_{-\tau}^{0}x'(s)Qx(s)ds\right] \\ &= J_s^*. \end{aligned} \quad (25)$$

The proof is completed here. □

In order to solve the complex problem to seek the solution caused by the nonlinear terms in Lemma 4, we give the following Lemma 5.

Lemma 5. *For a given scalar $\gamma > 0$ and two symmetric positive definite matrices $\widetilde{T}$ and $\widetilde{R}$, if there are two symmetric positive definite matrices $\widetilde{P}$ and $\widetilde{Q}$ and a matrix M such that*

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & A_{12}\widetilde{Q} & 0 & \widetilde{P} & 2\widetilde{P}A_{21}' & 2M'B_{21}' & B_{12} \\ * & \Xi_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\widetilde{Q} & \sqrt{3}\,\widetilde{Q}A_{22}' & 0 & 0 & 0 & 0 \\ * & * & * & -\widetilde{P} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\widetilde{Q} & 0 & 0 & 0 \\ * & * & * & * & * & -\widetilde{P} & 0 & 0 \\ * & * & * & * & * & * & -\widetilde{P} & 0 \\ * & * & * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \quad (26)$$

hold, where $\Xi_{11} = \widetilde{P}A_{11}' + A_{11}\widetilde{P} + M'B_{11}' + B_{11}M$, $\Xi_{12} = [\sqrt{3}\,\widetilde{P}C_1', \widetilde{P}, \sqrt{3}\,M'D_1', M']$, and $\Xi_{22} = diag\{-I, -\widetilde{T}, -I, -\widetilde{R}\}$; then, (8) is a mixed H_2/H_∞ controller of system (9), and the corresponding guaranteed cost for system (9) is $J_s^* = E[x'(0)\widetilde{P}^{-1}x(0) + \int_{-\tau}^{0}x'(s)\widetilde{Q}^{-1}x(s)ds]$. In this case, $K = M\widetilde{P}^{-1}$.

Proof. According to Lemma 2 and (13), if the following inequality

$$\begin{bmatrix} \Upsilon_{11} & PA_{12} & PB_{12} \\ * & -Q + 3A_{22}'PA_{22} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \quad (27)$$

hold, where $\Upsilon_{11} = Q + 4A_{21}'PA_{21} + 4K'B_{21}'PB_{21}K + 2A_{11}'P + 2K'B_{11}'P + 3C_1'C_1 + 3K'D_1'D_1K + T + K'RK$, then (13) holds.

Using $\operatorname{diag}\{P^{-1}, Q^{-1}, I\}$ to premultiply and postmultiply inequality (27), we have

$$\begin{bmatrix} \widetilde{\Upsilon}_{11} & A_{12}Q^{-1} & B_{12} \\ * & -Q^{-1} + 3Q^{-1}A_{22}'PA_{22}Q^{-1} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \quad (28)$$

hold, where $\widetilde{\Upsilon}_{11} = P^{-1}QP^{-1} + 4P^{-1}A_{21}'PA_{21}P^{-1} + 4P^{-1}K'B_{21}'PB_{21}KP^{-1} + 2P^{-1}A_{11}' + 2P^{-1}K'B_{11}' + 3P^{-1}C_1'C_1P^{-1} + 3P^{-1}K'D_1'D_1KP^{-1} + P^{-1}TP^{-1} + P^{-1}K'RKP^{-1}$. Let $\widetilde{P} = P^{-1}$, $M = K\widetilde{P}$, $\widetilde{Q} = Q^{-1}$, $\widetilde{R} = R^{-1}$, and $\widetilde{T} = T^{-1}$; by Lemma 3, we obtain (26) from (28).

Summarizing the process, the proof is completed. □

Next, in order to get the least upper bound for cost function among all the possible solutions to inequality (26), the convex optimization problem is provided as follows.

Theorem 1. *For system (9), if the following optimization problem*

$$\min_{\gamma>0,\alpha>0,W>0,\widetilde{P}>0,\widetilde{Q}>0,M} [\alpha + tr(W)] \quad (29)$$

subject to (26) and

$$\begin{bmatrix} -\alpha & x'(0) \\ x(0) & -\widetilde{P} \end{bmatrix} < 0, \quad (30)$$

$$\begin{bmatrix} -W & N' \\ N & -\widetilde{Q} \end{bmatrix} < 0, \quad (31)$$

has a solution α, W, $\widetilde{P}$, $\widetilde{Q}$, and M, then controller $u(t) = M\widetilde{P}^{-1}x(t)$ is an optimal state feedback H_2/H_∞ controller which ensures the minimization of guaranteed cost $J_s^* = E[x'(0)\widetilde{P}^{-1}x(0) + \int_{-\tau}^{0} x'(s)\widetilde{Q}^{-1}x(s)ds]$ for system (9), where $\int_{-\tau}^{0} x(s)x'(s)ds = NN'$.

Proof. From Lemma 5, the controller $u(t) = M\widetilde{P}^{-1}x(t)$ is a guaranteed cost control law of system (9). (30) is equivalent to $x'(0)\widetilde{P}^{-1}x(0) < \alpha$; (31) is equivalent to $N'\widetilde{Q}^{-1}N < W$.

Therefore, we can obtain

$$\begin{aligned} \int_{-\tau}^{0} x'(s)\widetilde{Q}x(s)ds &= \int_{-\tau}^{0} \operatorname{tr}\left(x'(s)\widetilde{Q}^{-1}x(s)\right)ds \\ &= \operatorname{tr}\left(NN'\widetilde{Q}^{-1}\right) = \operatorname{tr}\left(N'\widetilde{Q}^{-1}N\right) < \operatorname{tr}(W). \end{aligned} \quad (32)$$

Thus, we can obtain $J_s^* < \alpha + \operatorname{tr}(W)$.

Therefore, the minimization of $\alpha + \operatorname{tr}(W)$ implies the minimization of guaranteed cost for system (9).

The proof is completed here. □

Remark 1. It is an ideal case that the initial function is known. However, in general, the initial function of system (1) is not known, but the guaranteed cost depends on it. In order to avoid the dependence, we assume that the initial function is a white noise process with zero expectation function and unit covariance function.

When the initial function is not known, we have

$$\begin{aligned} J_s^* &= E\left[x'(0)\widetilde{P}^{-1}x(0) + \int_{-\tau}^{0} x'(s)\widetilde{Q}^{-1}x(s)ds\right] \\ &= E\left[\operatorname{tr}\left(x(0)x'(0)\widetilde{P}^{-1}\right)\right] + \int_{-\tau}^{0} E\left[\operatorname{tr}\left(x(s)x'(s)\widetilde{Q}^{-1}\right)\right]ds \\ &= \operatorname{tr}\left(E\left(x(0)x'(0)\widetilde{P}^{-1}\right)\right) + \int_{-\tau}^{0} \operatorname{tr}\left(E\left(x(s)x'(s)\widetilde{Q}^{-1}\right)\right)ds \\ &= \operatorname{tr}\left(\widetilde{P}^{-1}\right) + \tau \times \operatorname{tr}\left(\widetilde{Q}^{-1}\right). \end{aligned} \quad (33)$$

Therefore, we have the following optimization problem:

$$\min_{\gamma>0,W_1>0,W_2>0,\widetilde{P}>0,\widetilde{Q}>0,M} \left[\operatorname{tr}(W_1) + \tau \times \operatorname{tr}(W_2)\right], \quad (34)$$

which subjects to (26) and

$$\begin{bmatrix} W_1 & I \\ I & \widetilde{P} \end{bmatrix} > 0, \quad (35)$$

$$\begin{bmatrix} W_2 & I \\ I & \widetilde{Q} \end{bmatrix} > 0. \quad (36)$$

Theorem 2. *If there exist solution to (26), (34)–(36) then controller $u(t) = M\widetilde{P}^{-1}x(t)$ is an optimal state feedback H_2/H_∞ controller which ensures the minimization of guaranteed cost (18) for system (1).*

Proof. From Lemma 5, the controller $u(t) = M\widetilde{P}^{-1}x(t)$ is a H_2/H_∞ controller of system (9). We can see (34) is equivalent to $0 < \widetilde{P}^{-1} < W_1$ and (35) is equivalent to $0 < \widetilde{Q}^{-1} < W_2$ from Lemma 3. Therefore, the minimization of $tr(W_1) + \tau \times tr(W_2)$ implies the minimization of the guaranteed cost for system (1).

The proof is complete. □

4. Numerical Algorithms

In this section, an algorithm is presented in order to find the minimum value of $\alpha + \operatorname{tr}(W)$ in Theorem 1. The similar algorithm can also be applied to Theorem 2.

By analyzing (26), (30), (31) in Theorem 1, we find that if (26), (30), (31) have no feasible solutions when γ takes the initial value, then (26), (30), (31) will have no feasible solutions for all $\gamma > 0$. Next, we search for γ from the initial value that makes (26), (30), (31) have feasible solutions to optimize $\alpha + \mathrm{tr}(W)$ by using linear search algorithm. The specific algorithm is as follows.

5. Numerical Examples

The coefficient matrices of system (1) are given as follows:

$$\begin{aligned} A_{11} &= \begin{bmatrix} -30 & 10 \\ -10 & -20 \end{bmatrix}, \\ A_{12} &= \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix}, \\ A_{21} &= \begin{bmatrix} -2.7 & 0.8 \\ 0.9 & -1.6 \end{bmatrix}, \\ A_{22} &= \begin{bmatrix} -0.2 & -0.5 \\ -0.3 & -1.4 \end{bmatrix}, \\ B_{11} &= \begin{bmatrix} -4 \\ 3 \end{bmatrix}, \\ B_{12} &= \begin{bmatrix} 0.1 & -0.1 \\ 0.5 & 0.7 \end{bmatrix}, \\ B_{21} &= \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}, \\ D_1 &= \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \\ N &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \\ T &= \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}, \\ R &= 1, \\ \tau &= 1. \end{aligned} \tag{37}$$

First case: when the initial function is known and $x(0) = [\,1 \;\; 2\,]'$, $t \in [-1, 0]$. In order to find the minimum value of $\alpha + \mathrm{tr}(W)$, we obtain the relationship between $\alpha + \mathrm{tr}(W)$ and γ by Algorithm 1, which is shown in Figure 1.

As can be seen from Figure 1, $\alpha + \mathrm{tr}(W)$ decreases with the increase of γ, and $\min[\alpha + \mathrm{tr}(W)] = 38.4173$ when $\gamma = 0.4$, and $\min[\alpha + \mathrm{tr}(W)] = 29.7250$ when $\gamma = 1.98$.

Take $\gamma = 0.8$, according to Theorem 1, we obtain that

$$\begin{aligned} \widetilde{P} &= \begin{bmatrix} 0.8484 & 0.0215 \\ 0.0215 & 0.5896 \end{bmatrix}, \\ \widetilde{Q} &= \begin{bmatrix} 18.8352 & -4.2480 \\ -4.2480 & 1.0503 \end{bmatrix}, \\ M &= [\,0.0629 \;\; -0.0472\,], \\ W &= 23.0449, \\ \alpha &= 7.7981. \end{aligned} \tag{38}$$

Therefore, the optimal state feedback H_2/H_∞ controller is $u(t) = [0.0762 - 0.0828]x(t)$, and the guaranteed cost of closed-loop system is $J_s^* = 30.8430$.

Take external disturbance $v(t) = \sin(t)$, then we can obtain the curves of x_1 and x_2 and $E\|x(t)\|^2$ in Figure 2. From Figure 2, we can see that $E\|x(0)\|^2 = 5$ and $\lim_{t \to \infty} E\|x(t)\|^2 = 0$, that is, closed-loop system (9) is mean square stable.

Second case: when the initial function is a white noise process with zero expectation function and unit covariance function, in order to find the minimum value of $\mathrm{tr}(W_1) + \tau \times \mathrm{tr}(W_2)$, we obtain the relationship between $\mathrm{tr}(W_1) + \tau \times \mathrm{tr}(W_2)$ and γ by Algorithm 1, which is shown in Figure 3.

As can be seen from Figure 3, $\mathrm{tr}(W_1) + \tau \times \mathrm{tr}(W_2)$ decreases with the increase of γ, and $\min[\mathrm{tr}(W_1) + \tau \times \mathrm{tr}(W_2)] = 19.5450$ when $\gamma = 0.4$, $\min[\mathrm{tr}(W_1) + \tau \times \mathrm{tr}(W_2)] = 13.6648$ when $\gamma = 1.98$.

Take $\gamma = 0.8$, according to Theorem 2, we obtain that

$$\begin{aligned} \widetilde{P} &= \begin{bmatrix} 0.8520 & 0.0217 \\ 0.0217 & 0.5889 \end{bmatrix}, \\ \widetilde{Q} &= \begin{bmatrix} 20.5511 & -4.6765 \\ -4.6765 & 1.1567 \end{bmatrix}, \\ M &= [\,0.0629 \;\; -0.0472\,], \\ \mathrm{tr}(W_1) &= 2.8746, \\ \mathrm{tr}(W_2) &= 11.4205. \end{aligned} \tag{39}$$

Step 1: Given the values of τ.
Step 2: Using linear search algorithm, if a series of $\gamma_i\,(i = 1, \cdots, n)$ can be found to make inequalities (26), (30), (31) have feasible solutions, then turn to *Step 3*; otherwise, turn to *Step 7*.
Step 3: Let $i = 1$, then we take γ_i.
Step 4: Solve the following minimization problem:
$\min_{s.t.\,(20),(24),(25)} \alpha + \mathrm{tr}\,(W)$.
Step 5: Let $i = i + 1$, if $i + 1 > n$, then turn to *Step 6*; otherwise, let $\gamma_i = \gamma_{i+1}$, and turn to *Step 4*.
Step 6: There are solutions to this problem, printing data, and then stop.
Step 7: There is no solution to this problem and stop.

Algorithm 1: Linear search algorithm.

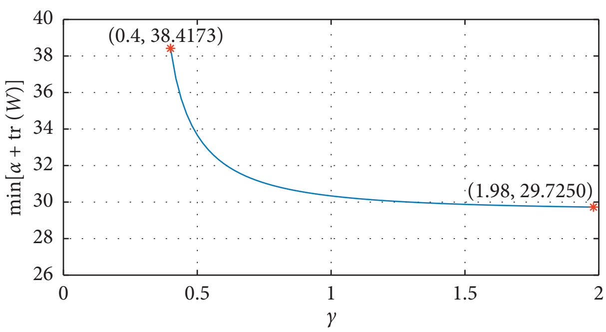

Figure 1: When $\gamma \in [0, 2]$, the minimum upper bound of $\alpha + \mathrm{tr}\,(W)$.

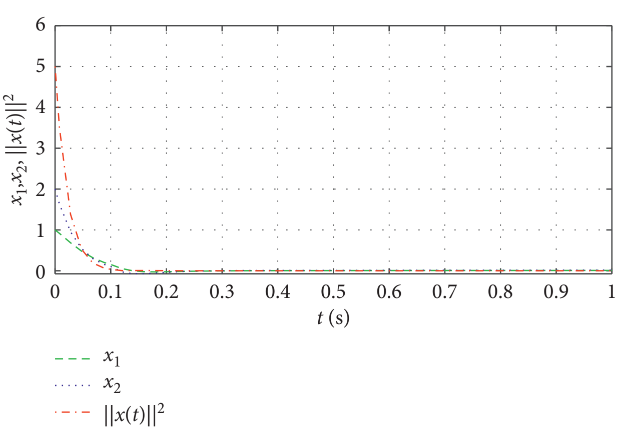

Figure 2: When $t \in [0, 1]$, the response for $E\|x(t)\|^2$.

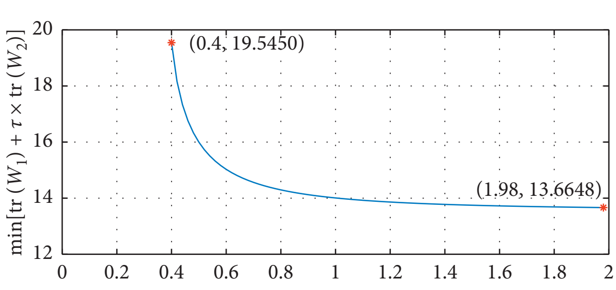

Figure 3: When $\gamma \in [0, 2]$, the minimum upper bound of $\mathrm{tr}\,(W_1) + \tau \times \mathrm{tr}\,(W_2)$.

Therefore, the optimal state feedback H_2/H_∞ controller is $u(t) = [0.0759 - 0.0829]x(t)$, and the guaranteed cost of closed-loop system is $J_s^* = 14.2951$.

6. Conclusion

In this paper, the mixed H_2/H_∞ control problem for Itô-type stochastic time-delay systems is presented, and the description of H_2/H_∞ control problem for stochastic time-delay systems is given. On the basis of matrix transformation and convex optimization method, state feedback H_2/H_∞ controller is obtained to make the system satisfy H_∞ performance index and H_2 performance index. Moreover, an algorithm is given to solve state feedback controller and optimize H_2/H_∞ performance index. Finally, a numerical example is used to show the feasibility of the results. In the future work, we will investigate mixed H_2/H_∞ control for the more complex systems, such as, stochastic Markov jump systems with time delay.

Data Availability

The data used to support the findings of this study are available from the corresponding upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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