Research Article

Compact Local Structure-Preserving Algorithms for the Nonlinear Schrödinger Equation with Wave Operator

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Received 7 November 2019; Accepted 6 January 2020; Published 28 January 2020

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Combining the compact method with the structure-preserving algorithm, we propose a compact local energy-preserving scheme and a compact local momentum-preserving scheme for the nonlinear Schrödinger equation with wave operator (NSEW). The convergence rates of both schemes are \(O(h^4 + \tau^2)\). The discrete local conservative properties of the presented schemes are derived theoretically. Numerical experiments are carried out to demonstrate the convergence order and local conservation laws of the developed algorithms.

1. Introduction

The nonlinear Schrödinger equation with wave operator (NSEW) is a very important model in mathematical physics with applications in a wide range, such as plasma physics, water waves, nonlinear optics, and bimolecular dynamics [1, 2]. In this paper, we consider the periodic initial-boundary value problem of the NSEW as

\[
\begin{align*}
\partial_t u - uu_{xx} + i \alpha \partial_x u + \beta |u|^2 u &= 0, \quad x \in (x_L, x_R), \quad 0 < t \leq T, \\
u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in [x_L, x_R], \\
u(x_L, t) &= \nu(x_R, t), \quad 0 \leq t \leq T,
\end{align*}
\]

where \(u(x, t)\) is a complex function, \(u_0(x)\) and \(u_1(x)\) are known complex functions, \(\alpha\) and \(\beta\) are real constants, and \(i^2 = -1\). Several numerical algorithms have been studied for solving the NSEW [Refs. [3–11] and references therein].

Recently, structure-preserving algorithms were proposed to solving the Hamiltonian systems [12–15] and applied to various PDEs, such as the nonlinear Schrödinger-type equation [16–19], wave equation [20], and KdV equation [21]. The important feature of the structure-preserving algorithm is that it can maintain certain invariant quantities and has the ability of long-term simulation. It should be pointed out that Wang et al. [22] presented the concept of the local structure-preserving algorithm for PDEs and then proposed several algorithms which preserved the multisymplectic conservation law and local energy and momentum conservation laws for the Klein–Gordon equation. For the next few years, the theory of the local structure-preserving algorithm was used successfully for solving the PDEs (Refs. [23–30] and references therein), and the main advantage of the method is that it can keep the local structures of PDEs independent of boundary conditions.

As is known to all, high accuracy [31] and conservation algorithm are two important aspects of the numerical solutions. However, there are few local structure-preserving algorithms with high-order approximation to equation (1) in the literature. In this paper, using the high precision of the compact algorithm, we construct two new schemes (i.e., compact local energy-preserving scheme and compact local momentum-preserving scheme) with fourth-order accuracy in the space for the NSEW.

The outline of this paper is as follows: In Section 2, some preliminary knowledge is given, such as divided grid point, operator definitions, and their properties. In Section 3, the local energy-preserving algorithm is proposed and the local energy conservation law is proved. In Section 4, the local momentum-
preserving algorithm is presented and the local momentum conservation law is analyzed. Numerical experiments are shown in Section 5. At last, we make some conclusions in Section 6.

2. Preliminary Knowledge

Firstly, let \( N \) and \( N_p \) be two positive integers; we then divide the space region \([x_L, x_R]\) and the time interval, respectively, into \( N \) parts and \( N_p \) parts. Thus, we introduce some notations: \( h = (x_R - x_L)/N, \tau = T/N, \) \( x_j = x_L + jh, j = 0, 1, \ldots, N, \) and \( t_n = nt, n = 0, 1, \ldots, N_p, \) where \( h \) is the spatial step span and \( \tau \) is the temporal length. The numerical solution and exact value of the function \( u(x, t) \) at the node \((x_j, t_n)\) are denoted by \( u^n_j \) and \( u(x_j, t_n) \), respectively.

Secondly, we define operators as follows:

\[
D_x u^n_j = \frac{u^n_{j+1} - u^n_j}{h}, \quad D_x u^n_j = \frac{u^n_{j+1} - u^n_j}{\tau}, \quad A_x u^n_j = \frac{u^n_{j+1} + u^n_j}{2}, \quad A_x u^n_j = \frac{u^n_{j+1} + u^n_j}{2},
\]

\[
A^2_x u^n_j = \frac{u^n_{j+2} + 2u^n_{j+1} + u^n_j}{4}, \quad A^2_x u^n_j = \frac{u^n_{j+2} + 2u^n_{j+1} + u^n_j}{4}, \quad \delta_x u^n_j = \frac{u^n_{j+1} + 4u^n_j + u^n_{j-1}}{6}, \quad \delta_x u^n_j = \frac{u^n_{j+1} + 4u^n_j + u^n_{j-1}}{6}, \quad (4)
\]

According to Taylor’s expansion, it is easy to get that

\[
D_x u^n_j = \delta_x \left( \frac{\partial u^n}{\partial x} \right)_j + O(h^4),
\]

\[
D^2_x u^n_{j-1} = \delta_x^2 \left( \frac{\partial^2 u^n}{\partial x^2} \right)_j + O(h^4).
\]

Then,

\[
\delta_x^{-1} D_x u^n_j = \left( \frac{\partial u^n}{\partial x} \right)_j + O(h^4),
\]

\[
\delta_x^{-1} D^2_x u^n_{j-1} = \left( \frac{\partial^2 u^n}{\partial x^2} \right)_j + O(h^4).
\]

By some simple computations, it is not difficult to obtain the following:

(i) Commutative law:

\[
D_x D_x = D_x D_x, \quad A D = A D,
\]

where \( A \) represents \( A_x \) or \( A \) and \( D \) represents \( D_x \) or \( D \).

(ii) Chain rule:

\[
D_x F(u^n) = D_x F(u^n) D_x u^n + O(\tau),
\]

\[
D_x F(u^n) = D_x F(u^n) D_x u^n + O(h).
\]

(iii) Discrete Leibniz rule:

\[
D_x (f \cdot g) = (\theta f_{j+1} + (1-\theta) f_j) \cdot D_x g_j + (1-\theta) g_{j+1} \cdot D_x f_j, \quad \forall 0 \leq \theta \leq 1,
\]

\[
D_x (f \cdot g)^n = \left( \theta f^n_{j+1} + (1-\theta) f^n_j \right) \cdot D_x g^n_j + (1-\theta) g^n_{j+1} \cdot D_x f^n_j, \quad \forall 0 \leq \theta \leq 1.
\]

In particular, we have the following two equalities for \( \theta = 1/2 \) and \( \theta = 1 \):

\[
\theta = \frac{1}{2}, \quad D_x (f \cdot g) = A_x f_j \cdot D_x g_j + D_x f_j \cdot A_x g_j,
\]

\[
\theta = 1, \quad D_x (f \cdot g) = f_{j+1} \cdot D_x g_j + D_x f_j \cdot g_j.
\]

Letting \( f = g = u \), we have

\[
D_x \left( \frac{1}{2} (u^n)^2 \right) = D_x u^n \cdot A_x u^n,
\]

\[
D_x \left( \frac{1}{2} (u^n \cdot u^{n-1}) \right) = u^n \cdot D_x (A_x u^{n-1}).
\]

Furthermore, we get

\[
D_x (\delta_x^{-1} f g)_{j+1} = \delta_x^{-1} f_{j+1} \cdot D_x g_j + D_x \delta_x^{-1} f_j \cdot g_j.
\]

For any \( u, v \in V_N = \{v = (v_0, v_1, \ldots, v_N)^T, v_0 = v_N\} \), we define the inner product and norms as

\[
(u, v) = h \sum_{j=0}^{N} u_j v_j,
\]

\[
\|u\|_\infty = \max_{0 \leq j \leq N} |u_j|.
\]

For constructing algorithms conveniently, taking \( u = p + i q \) in equation (1), where \( p \) and \( q \) are real-valued functions, we derive

\[
\begin{align*}
\rho_{nt} - p_{xx} - a_{qi} + b(\rho^2 + q^2) & = 0, \\
q_{nt} - q_{xx} + a_{pi} + b(\rho^2 + q^2) & = 0.
\end{align*}
\]

Furthermore, letting \( p_i, q_i, \xi, q_i = \eta, p_x = \omega, \) and \( q_x = \nu \), system (14) can be written as
\[
\begin{align*}
\xi_t - a \xi_q - \omega_x &= -\beta(p^2 + q^2)\rho, \\
\eta_t + \alpha \eta_p - \nu_x &= -\beta(p^2 + q^2)q, \\
\rho_t &= \xi, \quad q_t = \eta, \\
p_x &= \omega, \quad q_x = \nu, \quad \xi_x = \phi, \quad \eta_x = \phi.
\end{align*}
\]

For system (15), using \(\omega_x = \phi\) and \(\nu_x = \phi\), we obtain
\[
\begin{align*}
p_x &= \omega, \quad q_x = \nu, \quad \xi_x = \phi, \quad \eta_x = \phi, \\
\rho_t &= \xi, \quad q_t = \eta.
\end{align*}
\]

(16) \hspace{1cm} \hspace{1cm} (17)

\[
\begin{align*}
\xi_t - a \xi_q - \phi + \beta(p^2 + q^2)p &= 0, \\
\eta_t + \alpha \eta_p - \phi + \beta(p^2 + q^2)q &= 0.
\end{align*}
\]

(18) \hspace{1cm} \hspace{1cm} (19)

3. Compact Local Energy-Preserving Algorithm

Firstly, we consider the local energy conservation law for system (15). Multiplying the first line of equation (15) by \(p_t\), and the second line of equation (15) by \(q_t\), we derive
\[
\begin{align*}
\xi_t p_t - a \xi_q p_t - \omega_x p_t + \beta(p^2 + q^2)p p_t &= 0, \\
\eta_t q_t + \alpha \eta_p q_t - \nu_x q_t + \beta(p^2 + q^2)q q_t &= 0.
\end{align*}
\]

(20) \hspace{1cm} \hspace{1cm} (21)

Summing equations (20) and (21), we obtain
\[
\begin{align*}

\xi_t p_t + \eta_t q_t - \omega_x p_t - \nu_x q_t + \beta(p^2 + q^2)(p p_t + q q_t) &= 0.
\end{align*}
\]

(22)

Also by equations (16) and (17), we have
\[
\begin{align*}
\xi_t + \eta_t + \omega_x + \nu_x + \beta(p^2 + q^2)(p p_t + q q_t) - \omega_x \xi + \nu_x \eta &= 0.
\end{align*}
\]

(23)

Through further processing, we know that system (15) admits the local energy conservation law:
\[
\begin{align*}
\partial_t \left( \frac{1}{2} (\xi^2 + \eta^2 + \omega^2 + \nu^2) + \frac{\beta}{4} (p^2 + q^2) \right) &= \partial_x (\omega \xi + \nu \eta).
\end{align*}
\]

(24)

In equations (16)–(19), discretizing the space derivatives by using the compact leap-frog rule, the time derivatives by using the midpoint rule, and the nonlinear term with the discrete chain rule in the time direction, we obtain
\[
\begin{align*}
D_x p^n_j &= \delta_x w^n_{j+1}, \quad D_x w^n_{j+1} = \delta_x \phi^n_j, \quad D_x q^n_{j+1} = \delta_x v^n_{j+1}, \\
D_x v^n_{j+1} &= \delta_x \phi^n_j, \\
D_t p^n_j &= A_t \xi^n_j, \quad D_t q^n_j = A_t \eta^n_j, \\
D_t \xi^n_j - a D_t A_t q^n_{j-1} - \phi^n_j + \frac{\beta}{4} (A_t p^n_j)^2 + (A_t q^n_j)^2 &= 0.
\end{align*}
\]

(25) \hspace{1cm} \hspace{1cm} (26) \hspace{1cm} \hspace{1cm} (27)

\[
\begin{align*}
D_t A_t \xi^n_{j-1} + a D_t A_t q^n_{j-1} - \phi^n_j + \frac{\beta}{4} (A_t p^n_j)^2 + (A_t q^n_j)^2 &= 0.
\end{align*}
\]

(28)

From equations (25)–(28), eliminating \(\phi\) and \(\phi\), we have
\[
\begin{align*}
D_t A_t \xi^n_{j-1} &= a D_t A_t q^n_{j-1} - D_x \delta_x \omega^n_j + \frac{\beta}{4} (A_t p^n_j)^2 + (A_t q^n_j)^2, \\
+ (A_t q^n_j)^2 + (A_t q^n_{j-1})^2 + (A_t q^n_{j-1})^2 (A_t p^n_j + A_t p^n_{j-1}) &= 0.
\end{align*}
\]

(29)

Furthermore, eliminating \(\omega, \nu, \xi, \eta\), we get the following discrete scheme:
\[
\begin{align*}
D_t^2 p^n_{j-1} &= a D_t A_t q^n_{j-1} - D_x^2 \delta_x^2 p^n_{j-1} + \frac{\beta}{4} (A_t p^n_j)^2 + (A_t q^n_j)^2, \\
+ (A_t p^n_j)^2 + (A_t q^n_{j-1})^2 (A_t p^n_j + A_t p^n_{j-1}) &= 0, \\
D_t^2 q^n_{j-1} &= a D_t A_t p^n_{j-1} - D_x^2 \delta_x q^n_{j-1} + \frac{\beta}{4} (A_t p^n_j)^2 + (A_t q^n_j)^2, \\
+ (A_t q^n_j)^2 + (A_t q^n_{j-1})^2 (A_t q^n_j + A_t q^n_{j-1}) &= 0, \\
\end{align*}
\]

(30)

i.e.,
\[
\begin{align*}
D_t^2 u^n_{j-1} + \beta D_t A_t u^n_{j-1} - D_x^2 \delta_x u^n_{j-1} + \frac{\beta}{4} (A_t u^n_j)^2 + (A_t u^n_{j-1})^2 &= 0.
\end{align*}
\]

(31)

Lemma 1. Grid function \(f^n_j\) satisfies the following identical equation:
\[
\begin{align*}
f^n_{j+1} D_t f^n_{j+1} + f^n_{j+1} D_x f^n_{j+1} &= \frac{1}{2\tau} [hD_x(f^n_{j+1} - f^n_{j+1}) + \tau D_t(f^n_{j+1} - f^n_{j+1}) - hD_x(f^n_{j+1} - f^n_{j+1}) \tau D_t(f^n_{j+1} - f^n_{j+1})].
\end{align*}
\]

(32)

Proof. The left-hand term in equation (33) is equal to
\[f^n_{j+1}D_t A_t f^{n-1}_j + f^n_{j+1}D_j A_j f^{n-1}_j = \frac{1}{2\tau} \left( f^n_{j+1}f^{n+1}_{j+2} - f^n_{j+1}f^{n-1}_{j+2} + f^n_{j+1}f^{n+1}_j - f^n_{j+1}f^{n-1}_j \right) \]

\[= \frac{1}{2\tau} \left( f^n_{j+1}f^{n+1}_{j+2} - f^n_{j+1}f^{n-1}_{j+2} + (f^n_{j+1}f^{n+1}_j - f^n_{j+1}f^{n-1}_j) \right) \]

\[= \frac{1}{2\tau} \left[ (f^n_{j+1}f^{n+1}_{j+2} - f^n_{j+1}f^{n-1}_{j+2}) + (f^n_{j+1}f^{n+1}_j - f^n_{j+1}f^{n-1}_j) \right] \]

\[= \frac{1}{2\tau} \left[ (f^n_{j+1}f^{n+1}_{j+2} - f^n_{j+1}f^{n-1}_{j+2}) + (f^n_{j+1}f^{n+1}_j - f^n_{j+1}f^{n-1}_j) \right] \]

This completes the proof.

**Theorem 1.** Scheme (32) meets the discrete local energy conservation law:

\[\epsilon^n_j = D_t \left\{ \frac{1}{2} \left[ (A_t \xi_j^{n-1})^2 + (A_t \eta_j^{n-1})^2 \right] \right\} + \frac{1}{24} \left[ 10(\delta_x^{-1} \omega_j^{n-1} \delta_x \omega_j^{n+1}) + \delta_x^{-1} \omega_j^{n-1} \delta_x \omega_j^{n+1} + 10(\delta_x^{-1} \omega_j^{n-1} \omega_j^{n+1}) \right] \]

\[+ \delta_x^{-1} \nu_j^{n-1} \delta_x \nu_j^{n+1} + \delta_x^{-1} \nu_j^{n-1} \nu_j^{n+1} + \frac{\beta}{4} \left[ (A_t \xi_j^{n-1})^2 + (A_t \eta_j^{n-1})^2 \right]^2 \]

\[- D_x \left\{ \delta_x^{-1} \omega_j^{n-1} \delta_x \omega_j^{n+1} - \frac{h}{24\tau} \left( \delta_x^{-1} \omega_j^{n-1} \omega_j^{n+1} - \delta_x^{-1} \omega_j^{n-1} \omega_j^{n+1} \right) \right\} = 0. \]

That is to say, scheme (32) is a local energy-preserving algorithm.

**Proof.** Multiplying equation (29) by \(D_t A_t \nu_j^{n-1}\) and equation (30) by \(D_j A_j \nu_j^{n-1}\) and then adding them together, we get

\[D_t A_t \xi_j^{n-1} D_t A_t \nu_j^{n-1} - D_x \delta_x^{-1} \omega_j^{n-1} D_x A_t \nu_j^{n-1} + D_x A_t \delta_x^{n-1} D_x A_t \nu_j^{n-1} \]

\[- D_x \delta_x^{-1} \nu_j^{n-1} D_x A_t \nu_j^{n-1} + \frac{\beta}{4} \left[ (A_t \xi_j^{n-1})^2 + (A_t \eta_j^{n-1})^2 + (A_t \nu_j^{n-1})^2 + (A_t \nu_j^{n-1})^2 \right] \]

\[= \left[ (A_t \nu_j^{n-1})^2 (A_t \nu_j^{n-1})^2 + (A_t \nu_j^{n-1})^2 (A_t \nu_j^{n-1})^2 \right] = 0. \]

By the discrete Leibniz rule and equations (26)–(28), the first and third terms in the left side of (36) are

\[D_t A_t \xi_j^{n-1} D_t A_t \nu_j^{n-1} = D_t A_t \xi_j^{n-1} A_t A_t \xi_j^{n-1} = D_t \left\{ \frac{1}{2} (A_t \xi_j^{n-1})^2 \right\}, \]

\[D_t A_t \eta_j^{n-1} D_t A_t \nu_j^{n-1} = D_t A_t \eta_j^{n-1} A_t A_t \eta_j^{n-1} = D_t \left\{ \frac{1}{2} (A_t \eta_j^{n-1})^2 \right\}. \]

From Lemma 1, equation (25), and the second term in the left side of equation (36), we obtain
\[ D_x \delta_x^{-1} \omega^n_j D_x A_p^{-1} = D_x \left( D_x \omega^n_j D_x A_p^{-1} \right) - \delta_x^{-1} \omega^n_{j+1} D_x A_i D_x p^{-1} \]
\[ = D_x \left( D_x \omega^n_j D_x A_p^{-1} \right) - \delta_x^{-1} \omega^n_{j+1} D_x \delta_x \omega^n_{j+1} \]
\[ = D_x \left( D_x \omega^n_j D_x A_p^{-1} \right) - \frac{1}{12} \delta_x^{-1} \omega^n_{j+1} \left( D_x A_i \delta_x^{-1} \omega^n_{j+1} + 10D_x A_i \delta_x^{-1} \omega^n_{j+1} + D_x A_i \delta_x^{-1} \omega^n_{j+1} \right) \]
\[ = D_x \left( D_x \omega^n_j D_x A_p^{-1} \right) - \frac{1}{24} D_x \left[ 10 \left( \delta_x^{-1} \omega^n_{j+1} \delta_x^{n-1} \omega^n_{j+1} \right) \right] - \frac{1}{24} \left[ hD_x \left( \delta_x^{-1} \omega^n_{j+1} \delta_x^{n-1} \omega^n_{j+1} \right) \right] \]
\[ + \frac{1}{24} \left( \delta_x^{-1} \omega^n_{j+1} \delta_x^{n-1} \omega^n_{j+1} \right) \]
\[ = D_x \left[ \delta_x^{-1} \omega^n_j D_x A_i p^{-1} - \frac{h}{24} \left( \delta_x^{-1} \omega^n_j \delta_x^{n-1} \omega^n_{j+1} - \delta_x^{-1} \omega^n_j \delta_x^{n-1} \omega^n_{j+1} \right) \right] \]
\[ - D_x \left[ \frac{1}{24} \left( 10 \delta_x^{-1} \omega^n_{j+1} \delta_x^{n-1} \omega^n_{j+1} + \delta_x^{-1} \omega^n_{j+1} \delta_x^{n-1} \omega^n_{j+1} + \delta_x^{-1} \omega^n_{j+1} \delta_x^{n-1} \omega^n_{j+1} \right) \right]. \]

Similarly, the fourth term in the left side of equation (36) is equal to

\[ D_x \delta_x^{-1} \eta^n_j D_x q_j^{n-1} = D_x \left[ \delta_x^{-1} \eta^n_j D_x A_i q_j^{n-1} \right] \]
\[ - D_x \left[ \frac{1}{24} \left( 10 \delta_x^{-1} \eta^n_{j+1} \delta_x^{n-1} \eta^n_{j+1} + \delta_x^{-1} \eta^n_{j+1} \delta_x^{n-1} \eta^n_{j+1} + \delta_x^{-1} \eta^n_{j+1} \delta_x^{n-1} \eta^n_{j+1} \right) \right]. \]

The last term in the left side of equation (36) is

\[ \frac{\beta}{4} \left[ \left( A_i p_j^n \right)^2 + \left( A_i q_j^n \right)^2 \right] \]
\[ - \left[ \left( A_i p_j^{n-1} \right)^2 + \left( A_i q_j^{n-1} \right)^2 \right] \]
\[ - \frac{\beta}{4} \left[ \left( A_i p_j^n \right)^2 + \left( A_i q_j^n \right)^2 \right] \]
\[ - \left[ \left( A_i p_j^{n-1} \right)^2 + \left( A_i q_j^{n-1} \right)^2 \right] \]
\[ \times \frac{\delta_x^{-1} \omega^n_j \delta_x^{n-1} \omega^n_{j+1} - \delta_x^{-1} \omega^n_j \delta_x^{n-1} \omega^n_{j+1}}{\tau} \]
\[ = D_x \left[ \frac{\beta}{4} \left( \left( A_i p_j^{n-1} \right)^2 + \left( A_i q_j^{n-1} \right)^2 \right) \right]. \]

From equations (36)–(41), we complete the proof of Theorem 1. \[ \square \]

**4. Compact Local Momentum-Preserving Algorithm**

Now, we consider the local momentum conservation law for system (15). Multiplying the first line of equation (15) by \( p_x \), and the second line of equation (15) by \( q_x \), we have

\[ \xi_t p_x - \alpha q_x p_x - \omega_x p_x + \beta \left( p^2 + q^2 \right) pp_x = 0, \]
\[ \eta_t q_x + \alpha p_x q_x - \nu_x q_x + \beta \left( p^2 + q^2 \right) qq_x = 0. \]

Then adding equations (42) to (43), we get

\[ \xi_t p_x + \eta_t q_x - \alpha q_x p_x + \alpha p_x q_x - \omega_x p_x - \nu_x q_x + \beta \left( p^2 + q^2 \right) \cdot \left( pp_x + qq_x \right) = 0. \]

Additionally,
\[\partial_x (\xi p_x) = \xi q_x p_x + \xi q_{xt} = \xi q_x + \xi q_{x} = \xi q_x + \partial_x \left( \frac{1}{2} q_x^2 \right),\]

\[\partial_x (\eta q_x) = \eta q_x + \eta q_{xt} = \eta q_x + \eta q_x = \eta q_x + \partial_x \left( \frac{1}{2} q_x^2 \right),\]

\[p_x q_x - q_x p_x = \partial_x \left( \frac{1}{2} p_x q_x - \frac{1}{2} q_x p_x \right) + \frac{1}{2} p_x q_x - \frac{1}{2} p_x q_x - \frac{1}{2} q_x p_x + \frac{1}{2} q_x p_x\]

\[= \partial_x \left( \frac{1}{2} p_x q_x - \frac{1}{2} q_x p_x \right) + \left( \frac{1}{2} p_x q_x + \frac{1}{2} q_x p_x \right) - \left( \frac{1}{2} q_x p_x + \frac{1}{2} p_x q_x \right)\]

\[= \partial_x \left( \frac{1}{2} p_x q_x - \frac{1}{2} q_x p_x \right) + \partial_x \left( \frac{1}{2} q_x p_x - \frac{1}{2} p_x q_x \right)\]

\[\omega_x p_x + v_x q_x = \omega_x \omega + v_x \gamma = \frac{1}{2} \partial_x (\omega^2 + v^2),\]

\[\beta (p^2 + q^2) (p p_x + q q_x) = \partial_x \left[ \frac{\beta}{4} (p^2 + q^2)^2 \right].\]

Thus, system (15) possesses the following local momentum conservation law:

\[\partial_x \left( \frac{1}{2} (p q_t - q p_t) + \frac{1}{2} (\xi^2 + \eta^2 + \omega^2 + v^2) - \frac{\beta}{4} (p^2 + q^2)^2 \right)\]

\[= \partial_x \left( \xi p_x + \eta q_x + \frac{\alpha}{2} p q_x - \frac{\alpha}{2} q p_x \right).\]

(46)

In equations (16)–(19), applying the compact midpoint rule to space derivatives, the midpoint rule to non-linear term in the spatial direction, we obtain

\[D_x p_j = \delta_x A_x \omega_j, \quad D_x \omega_j = \delta_x A_x \phi_j, \quad D_x q_j = \delta_x A_x \gamma_j, \quad D_x \gamma_j = \delta_x A_x \phi_j.\]

\[D_x p_j = \delta_x A_x \omega_j, \quad D_x \omega_j = \delta_x A_x \phi_j, \quad D_x q_j = \delta_x A_x \gamma_j.\]

(47)

\[D_x A_x A_x^2 \phi_{j-1} - aD_x A_x A_x^2 \phi_j - D_x A_x^2 A_x \phi_j - aD_x A_x^2 \phi_j - \frac{1}{2} \beta \left( A_x^2 A_x \phi_{j-1} + A_x^2 A_x \phi_j + A_x^2 A_x \phi_{j+1} + (A_x^2 A_x \phi_{j-1})^2 + (A_x^2 A_x \phi_j)^2 + (A_x^2 A_x \phi_{j+1})^2 \right) = 0.\]

(48)

\[D_x A_x A_x^2 \phi_{j-1} - aD_x A_x A_x^2 \phi_j - D_x A_x^2 A_x \phi_j - aD_x A_x^2 \phi_j - \frac{1}{2} \beta \left( A_x^2 A_x \phi_{j-1} + A_x^2 A_x \phi_j + A_x^2 A_x \phi_{j+1} + (A_x^2 A_x \phi_{j-1})^2 + (A_x^2 A_x \phi_j)^2 + (A_x^2 A_x \phi_{j+1})^2 \right) = 0.\]

(49)

\[D_x A_x A_x^2 \phi_{j-1} - aD_x A_x A_x^2 \phi_j - D_x A_x^2 A_x \phi_j - aD_x A_x^2 \phi_j - \frac{1}{2} \beta \left( A_x^2 A_x \phi_{j-1} + A_x^2 A_x \phi_j + A_x^2 A_x \phi_{j+1} + (A_x^2 A_x \phi_{j-1})^2 + (A_x^2 A_x \phi_j)^2 + (A_x^2 A_x \phi_{j+1})^2 \right) = 0.\]

(50)

By equations (47)–(50), we have
i.e.,
\[ D_x^2 A_x^2 u_j^{n-1} + \text{i} \alpha D_x A_x A_x^2 u_j^{n-1} - D_x A_x^2 \delta_x^2 u_j^{n-1} \]
\[ + \beta \left( \left| A_x^2 A_x u_j^{n-1} \right|^2 + \left| A_x^2 A_x u_j^{n-1} \right|^2 \right) (A_x^2 A_x u_j^{n-1}) = 0. \]

(54)

Lemma 2. Grid function \( f_j^n \) satisfies the following identical equation:
\[ D_x f_j^{n-1} A_x f_{j+1}^{n-1} + D_x f_j^{n-1} A_x f_{j-1}^{n-1} \]
\[ = \frac{1}{2h} \left[ (f_{j+1}^{n-1} - f_j^{n-1})(f_{j+2}^{n-1} + f_j^{n-1}) + (f_{j-1}^{n-1} - f_j^{n-1})(f_{j-2}^{n-1} + f_j^{n-1}) \right] \]
\[ = \frac{1}{2h} (f_{j+1}^{n-1} f_{j+2}^{n-1} + f_{j+1}^{n-1} f_{j-1}^{n-1} - f_j^{n-1} f_{j+2}^{n-1} - f_j^{n-1} f_{j-1}^{n-1}) \]
\[ + f_j^{n-1} f_j^{n-1} + f_j^{n-1} f_j^{n-1} - f_j^{n-1} f_j^{n-1}) \]
\[ = \frac{1}{2} D_x (f_j^{n-1} f_{j+1}^{n-1} + f_j^{n-1} f_j^{n-1} + f_j^{n-1} f_{j-1}^{n-1} - f_{j+1}^{n-1} f_{j-1}^{n-1}). \]

This completes the proof.

Theorem 2. Scheme (54) possesses the discrete local momentum conservation law as

\[ m_j^n = D_x \left\{ \frac{\alpha}{2} (D_x A_x A_x p_j^{n-1} A_x^2 q_j^{n-1} - D_x A_x A_x q_j^{n-1} A_x^2 p_j^{n-1}) - \frac{1}{2} \left( A_x^2 A_x q_j^{n-1} \right)^2 + \left( A_x^2 A_x p_j^{n-1} \right)^2 \right\} \]
\[ - \frac{1}{24} \left( 11 (A_x^2 A_x \delta_x^1 \omega_j^{n-1})^2 + A_x^2 A_x \delta_x^3 \omega_j^{n-1} A_x^2 A_x \delta_x^1 \omega_j^{n-1} + A_x^2 A_x \delta_x^1 \omega_j^{n-1} A_x^2 A_x \delta_x^1 \omega_j^{n-1} \right) \]
\[ - 11 (A_x^2 A_x \delta_x^1 \omega_j^{n-1})^2 + A_x^2 A_x \delta_x^3 \omega_j^{n-1} + 11 (A_x^2 A_x \delta_x^1 \omega_j^{n-1})^2 + A_x^2 A_x \delta_x^1 \omega_j^{n-1} A_x^2 A_x \delta_x^1 \omega_j^{n-1} \]
\[ + A_x^2 A_x \delta_x^3 \omega_j^{n-1} A_x^2 A_x \delta_x^1 \omega_j^{n-1} - A_x^2 A_x \delta_x^3 \omega_j^{n-1} A_x^2 A_x \delta_x^1 \omega_j^{n-1} \]
\[ = \frac{\beta}{4} \left( A_x^2 A_x p_j^{n-1} \right)^2 \]
\[ + \left( A_x^2 A_x q_j^{n-1} \right)^2 \right\} + D_x \left\{ A_x^2 A_x p_j^{n-1} D_x A_x A_x p_j^{n-1} + A_x^2 A_x q_j^{n-1} D_x A_x A_x q_j^{n-1} \right\} \]
\[ + \frac{\alpha}{2} \left( A_x^2 A_x p_j^{n-1} D_x A_x A_x q_j^{n-1} - A_x^2 A_x q_j^{n-1} D_x A_x A_x p_j^{n-1} \right) \}
\[ = 0. \]

That is to say, scheme (54) is a local momentum-preserving algorithm.

Proof. Multiplying (51) by \( D_x A_x^2 A_x p_j^{n-1} \) and (52) by \( D_x A_x^2 A_x q_j^{n-1} \) and then adding them together, we derive
\[ D_t A_i A_x^2 \xi^n_j D_x A_i A_x^2 p_j^{n-1} - a D_t A_i A_x^2 q^n_j D_x A_i A_x p_j^{n-1} - D_x A_i^2 A_x \delta_x \omega_j^{n-1} D_x A_i^2 A_x p_j^{n-1} \]
\[ + D_t A_i A_x^2 \eta^n_j D_x A_i A_x q_j^{n-1} + a D_t A_i A_x^2 p_j^{n-1} D_x A_i^2 A_x q_j^{n-1} - D_x A_i^2 A_x \delta_x v_j^{n-1} D_x A_i^2 A_x q_j^{n-1} \]
\[ + \frac{\beta}{2} \left[ (A_i^2 A_x p_j^{n-1})^2 + (A_i^2 A_x q_j^{n-1})^2 + (A_i^2 A_x p_j^{n-1})^2 + (A_i^2 A_x q_j^{n-1})^2 \right] \]
\[ \times (A_i^2 A_x p_j^{n-1} D_x A_i^2 A_x p_j^{n-1} + A_i^2 A_x q_j^{n-1} D_x A_i^2 A_x q_j^{n-1}) = 0. \] (58)

According to discrete Leibniz rules, the first term in the left-hand side of (58) can be expressed as

\[ D_t A_i A_x^2 \xi^n_j D_x A_i A_x^2 p_j^{n-1} = D_i \left( A_i A_x^2 \xi^n_j D_x A_i A_x p_j^{n-1} \right) - A_i^2 A_x^2 \xi^n_j D_i D_x A_i A_x p_j^{n-1} \]
\[ = D_i \left( A_i A_x^2 \xi^n_j D_x A_i A_x p_j^{n-1} \right) - A_i^2 A_x^2 \xi^n_j D_i D_x A_i A_x p_j^{n-1} \] (59)

Similarly, from the fourth term in the left side of (58), we have

\[ D_t A_i A_x^2 \eta^n_j D_x A_i A_x q_j^{n-1} = D_i \left( A_i A_x^2 \eta^n_j D_x A_i A_x q_j^{n-1} \right) - D_x \left[ \frac{1}{2} (A_i^2 A_x \xi^n_j)^2 \right]. \] (60)

The second and fifth terms in the left side of (58) are equal to

\[ a D_t A_i A_x^2 p_j^{n-1} D_x A_i A_x q_j^{n-1} - a D_t A_i A_x^2 q_j^{n-1} D_x A_i A_x p_j^{n-1} \]
\[ = D_i \left( \frac{a}{2} A_i A_x^2 p_j^{n-1} D_x A_i A_x q_j^{n-1} - \frac{a}{2} A_i A_x^2 q_j^{n-1} D_x A_i A_x p_j^{n-1} \right) + \frac{a}{2} D_t A_i A_x^2 p_j^{n-1} D_x A_i A_x q_j^{n-1} \]
\[ - \frac{a}{2} A_i^2 A_x^2 q_j^{n-1} D_x D_t A_i A_x q_j^{n-1} - \frac{a}{2} D_t A_i A_x^2 q_j^{n-1} D_x A_i^2 A_x p_j^{n-1} + \frac{a}{2} A_i^2 A_x^2 p_j^{n-1} D_x D_t A_i A_x p_j^{n-1} \] (61)

\[ = D_i \left( \frac{a}{2} A_i A_x^2 p_j^{n-1} D_x A_i A_x q_j^{n-1} - \frac{a}{2} A_i A_x^2 q_j^{n-1} D_x A_i A_x p_j^{n-1} \right) + D_x \left( \frac{a}{2} D_t A_i A_x p_j^{n-1} A_i^2 A_x q_j^{n-1} \right) - \frac{a}{2} D_t A_i A_x q_j^{n-1} A_i^2 A_x p_j^{n-1}. \]
From the third term in the left side of (58) and Lemma 2, we can obtain

\[ D_x A^2_x \delta_x^{-1} \omega_j^{n-1} D_x A^2_x p_j^{n-1} = D_x A^2_x \delta_x^{-1} \omega_j^{n-1} A^2_x A^2_x \delta_x \omega_j^{n+1} \]

\[ = D_x A^2_x \delta_x^{-1} \omega_j^{n+1} A^2_x A^2_x \delta_x \omega_j^{n+1} \]

\[ = \frac{1}{12} \left( 10 D_x A^2_x \delta_x^{-1} \omega_j^{n+1} A^2_x A^2_x \delta_x \omega_j^{n+1} \right) \]

\[ + D_x A^2_x \delta_x^{-1} \omega_j^{n+1} A^2_x A^2_x \delta_x \omega_j^{n+1} \]

\[ = \frac{1}{24} \left[ 10 \left( A^2_x A^2_x \delta_x^{-1} \omega_j^{n+1} \right)^2 + A^2_x A^2_x \delta_x^{-1} \omega_j^{n+1} A^2_x A^2_x \delta_x \omega_j^{n+1} \right] \]

\[ = \frac{1}{24} \left[ 10 \left( A^2_x A^2_x \delta_x^{-1} \omega_j^{n-1} \right)^2 + A^2_x A^2_x \delta_x^{-1} \omega_j^{n-1} A^2_x A^2_x \delta_x \omega_j^{n-1} \right] \]

Similarly, the sixth term in the left side of (58) is equal to

\[ D_x A^2_x \delta_x^{-1} \omega_{j+1}^{n+1} D_x A^2_x q_j^{n+1} \]

\[ = \frac{1}{24} \left[ 11 \left( A^2_x A^2_x \delta_x^{-1} \omega_{j+1}^{n+1} \right)^2 + A^2_x A^2_x \delta_x^{-1} \omega_{j+1}^{n+1} A^2_x A^2_x \delta_x \omega_{j+1}^{n+1} \right] \]

\[ = \frac{1}{24} \left[ 11 \left( A^2_x A^2_x \delta_x^{-1} \omega_{j+1}^{n-1} \right)^2 + A^2_x A^2_x \delta_x^{-1} \omega_{j+1}^{n-1} A^2_x A^2_x \delta_x \omega_{j+1}^{n-1} \right] \]

From the last term in the left-hand side of (58), we have

\[ \frac{\beta}{2} \left( \left( A^2_x A^2_x \delta_x \omega_j^{n-1} \right)^2 + (A^2_x A^2_x \delta_x \omega_j^{n+1})^2 \right) \]

\[ \times \left( A^2_x A^2_x \delta_x \omega_j^{n+1} D_x A^2_x p_j^{n+1} + A^2_x A^2_x \delta_x \omega_j^{n+1} D_x A^2_x q_j^{n+1} \right) \]

\[ = \frac{\beta}{2} \left[ \left( A^2_x A^2_x \delta_x \omega_j^{n+1} \right)^2 + (A^2_x A^2_x \delta_x \omega_j^{n+1})^2 \right] \]

\[ \times \left( A^2_x A^2_x \delta_x \omega_j^{n+1} D_x A^2_x p_j^{n+1} + A^2_x A^2_x \delta_x \omega_j^{n+1} D_x A^2_x q_j^{n+1} \right) \]

\[ = \frac{\beta}{2} \left[ \left( A^2_x A^2_x \delta_x \omega_j^{n-1} \right)^2 + (A^2_x A^2_x \delta_x \omega_j^{n-1})^2 \right] \]

\[ \times \left( A^2_x A^2_x \delta_x \omega_j^{n-1} D_x A^2_x p_j^{n-1} + A^2_x A^2_x \delta_x \omega_j^{n-1} D_x A^2_x q_j^{n-1} \right) \]

\[ = D_x \left[ \frac{\beta}{4} \left( A^2_x A^2_x \delta_x \omega_j^{n-1} \right)^2 + (A^2_x A^2_x \delta_x \omega_j^{n-1})^2 \right] \]

Through equations (58)–(64), we complete the proof of Theorem 2. \( \square \)

5. Numerical Experiments

In this section, numerical experiments are designed to show the accuracies and conservation properties of the schemes which we have obtained above. Taking \( \alpha = \beta = 1 \), equation (1) has exact solution \( u(x, t) = e^{i(x+\tau)} \). In numerical calculations, we let \( x \in [-\pi, \pi] \).

In order to verify the convergence rates of the proposed schemes (32) and (54), we define \( \varepsilon = u^{N_j} - u(x_j, t_{N_j}) \) and \( \text{Re}(L_\rho) = \| \text{Re}(\varepsilon) \|_p, \text{Im}(L_\rho) = \| \text{Im}(\varepsilon) \|_p \), where \( \rho = \infty \) or \( \rho = 2 \). Tables 1–4 show the \( L_\infty \) and \( L_2 \) errors of the numerical solutions with respect to the exact ones for both schemes. In these tables, we can confirm that the two schemes have the accuracy of \( O(\tau^3 + h^4) \).

Next, we investigate the local conservation properties of the schemes (32) and (54). To compute the discrete local energy and local momentum at \( t_n = n\tau \), we define \( e^{\text{local}} = \max_{0 \leq j \leq N} |e_j^{n}| \) and \( m^{\text{local}} = \max_{0 \leq j \leq N} |m_j^{n}| \), where \( e_j^{n} \) and \( m_j^{n} \) can be calculated by (35) and (57), respectively. In our experiments, we take \( T = 100, h = \pi/128 \), and \( \tau = 0.02 \). Figures 1 and 2 show the numerical results for the local
Table 1: Temporal errors and convergence rates of scheme (32) at $T=1$ with $h = \pi/64$.

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<th>$\tau$</th>
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<tr>
<td>1/160</td>
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<td>5.86E-06</td>
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Table 2: Spatial errors and convergence rates of scheme (32) at $T=1$ with $\tau = 0.0005$.

<table>
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Table 3: Temporal errors and convergence rates of scheme (54) at $T=1$ with $h = \pi/64$.

<table>
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Table 4: Spatial errors and convergence rates of scheme (54) at $T=1$ with $\tau = 0.0005$.

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<td>4.02</td>
<td>4.02</td>
</tr>
</tbody>
</table>

Figure 1: Local energy $e_{\text{local}}$ of scheme (32).

Figure 2: Local momentum $m_{\text{local}}$ of scheme (54).
energy and local momentum errors using schemes (32) and (54), respectively. The figures indicate that the developed schemes can preserve the local energy and local momentum of the system very well over long-time simulations, which is consistent with the theoretical results of Theorem 1 and Theorem 2 in this paper.

6. Conclusions
In this paper, two new compact local structure-preserving algorithms are constructed for solving the NSEW. Local conservation laws of the proposed schemes are derived theoretically. Numerical results are shown to verify the accuracy, validity, and long-time numerical behavior of the schemes obtained in this work. Hence, the compact local structure-preserving method can be used for many Hamiltonian systems.

Data Availability
The data of the numerical experiment used to support the findings of this study are included within the article.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Acknowledgments
This work was partially supported by the Key Laboratory of Applied Mathematics of Fujian Province University (Putian University) (no. SX201802), the National Science Foundation of China (nos. 11701196 and 11701197), the Promotion Program for Young and Middle-Aged Teachers in Science and Technology Research of Huaqiao University (no. ZQN-YX502), and the Fundamental Research Funds for the Central Universities (no. ZQN-702).

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