Research Article

Fractional Ostrowski Type Inequalities via Generalized Mittag–Leffler Function

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Received 23 April 2020; Revised 12 May 2020; Accepted 13 May 2020; Published 4 June 2020

Guest Editor: Praveen Agarwal

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If we study the theory of fractional differential equations then we notice the Mittag–Leffler function is very helpful in this theory. On the contrary, Ostrowski inequality is also very useful in numerical computations and error analysis of numerical quadrature rules. In this paper, Ostrowski inequalities with the help of generalized Mittag–Leffler function are established. In addition, bounds of fractional Hadamard inequalities are given as straightforward consequences of these inequalities.

1. Introduction

Exponential function plays a vital role in the theory of integer order differential equations. The symbol $E_\alpha(z)$ is well known as the Mittag–Leffler function and it is a generalization of exponential function. It occurs in the solutions of fractional differential equations such as exponential function which exists in the solutions of differential equations. Due to its importance, Mittag–Leffler function is generalized by many mathematicians: For example, Wiman [1], Prabhakar [2], Shukla and Prajapati [3], Salim [4], Salim and Faraj [5], and Rahman et al. [6]. Mittag–Leffler function is also used in the formation of fractional integral operators. These fractional integral operators provide generalizations of fractional differential equations and modeling of dynamic systems. Fractional integral operators also play a vital role in the advancement of classical mathematical inequalities. For example, Hadamard inequality, Ostrowski inequality, Gruss inequality, and many others have been presented for fractional integral and derivative operators, see [7–16]. The aim of this paper is to study well-known Ostrowski inequality for an integral operator which is directly associated with many fractional integral operators defined in near past.

Recently, in [7], Andrić et al. defined the extended generalized Mittag–Leffler function $E_{\mu,\sigma,l}^{\gamma,\delta,k,c}(\cdot; p)$ as follows.

Definition 1. Let $\mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, and $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Then, the extended generalized Mittag–Leffler function $E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(t; p)$ is defined as

$$E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(t; p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{nd}}, \quad (1)$$

where β_p is the generalized beta function defined as $\beta_p(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{-(p/t(1-t))} dt$ and $(c)_{nk}$ is the Pochhammer symbol given by $(c)_{nk} = (\Gamma(c + nk))/\Gamma(c)$.

The corresponding left- and right-sided generalized fractional integrals $\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c}$ and $\epsilon_{\mu,\alpha,l,\omega,b^-}^{\gamma,\delta,k,c}$ are defined as follows.

Definition 2 (see [7]). Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Let $\psi_1 \in L_1[a, b]$ and $x \in [a, b]$. Then, the generalized fractional integrals $\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} \psi_1$ and $\epsilon_{\mu,\alpha,l,\omega,b^-}^{\gamma,\delta,k,c} \psi_1$ are defined as

$$\left({}^c_{\mu, \alpha, l, \omega, a^+} \psi_1 \right)(x; p) = \int_a^x (x-t)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(x-t)^\mu; p) \psi_1(t) dt, \quad (2)$$

$$\left({}^c_{\mu, \alpha, l, \omega, b^-} \psi_1 \right)(x; p) = \int_x^b (t-x)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(t-x)^\mu; p) \psi_1(t) dt. \quad (3)$$

Recently, Farid defined a unified integral operator in [17] (also see [18]). This unifies several kinds of fractional and conformable integrals in a compact formula and is given as follows.

Definition 3. Let $\psi_1, \psi_2: [a, b] \rightarrow \mathbb{R}$, $0 < a < b$, be the functions such that ψ_1 be positive and $\psi_1 \in L_1[a, b]$, and ψ_2 be differentiable and strictly increasing. Also, let ϕ/x be an increasing function on $[a, \infty)$ and $\alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$, $p, \mu, \delta \geq 0$, and $0 < k \leq \delta + \mu$. Then, for $x \in [a, b]$ the left and right integral operators are defined by

$$\left({}^c_{\mu, \alpha, l, \omega, a^+} \psi_1 \right)(x; p) = \int_a^x (\psi_2(x) - \psi_2(t))^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(x) - \psi_2(t))^\mu; p) \times \psi_1(t) \psi_2'(t) dt, \quad (6)$$

$$\left({}^c_{\mu, \alpha, l, \omega, b^-} \psi_1 \right)(x; p) = \int_x^b (\psi_2(t) - \psi_2(x))^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(t) - \psi_2(x))^\mu; p) \times \psi_1(t) \psi_2'(t) dt. \quad (7)$$

It can be noted that

$$\left({}^c_{\mu, \alpha, l, \omega, a^+} 1 \right)(x; p) = (\psi_2(x) - \psi_2(t))^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(x) - \psi_2(t))^\mu; p), \quad (8)$$

$$\left({}^c_{\mu, \alpha, l, \omega, b^-} 1 \right)(x; p) = (\psi_2(t) - \psi_2(x))^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(t) - \psi_2(x))^\mu; p). \quad (9)$$

In the following, we state the Ostrowski inequality which is proved by Ostrowski [19] in 1938.

Theorem 1. Let $\psi_1: I \rightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a mapping differentiable in I° , the interior of I and $a, b \in I^\circ$, $a < b$. If $|\psi_1'(t)| \leq M$ for all $t \in [a, b]$, then for $x \in [a, b]$ we have

$$\left| \psi_1(x) - \frac{1}{b-a} \int_a^b \psi_1(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - ((a+b)/2))^2}{(b-a)^2} \right] (b-a)M. \quad (10)$$

The Ostrowski inequality has been studied by many researchers to obtain its refinements, generalizations, and extensions. Also, their applications are analyzed for establishing the bounds of relations among special means and for

$$\left({}^c_{\mu, \alpha, l, \omega, a^+} \psi_1 \right)(x, \omega; p) = \int_a^x K_x^\gamma(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi) \psi_1(y) d(\psi_2(y)), \quad (4)$$

$$\left({}^c_{\mu, \alpha, l, \omega, b^-} \psi_1 \right)(x, \omega; p) = \int_x^b K_y^\gamma(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi) \psi_1(y) d(\psi_2(y)), \quad (5)$$

where $K_x^\gamma(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g; \phi) = ((\phi(\psi_2(x) - \psi_2(y)))/(\psi_2(x) - \psi_2(y))) E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(x) - \psi_2(y))^\mu; p)$.

The following definition can be deduced from Definition 3 (see [16]).

Definition 4. Let $\psi_1, \psi_2: [a, b] \rightarrow \mathbb{R}$, $0 < a < b$, be the functions such that ψ_1 be positive and $\psi_1 \in L_1[a, b]$, and ψ_2 be differentiable and strictly increasing and $\alpha, l, \gamma, c \in \mathbb{R}_+, c > \gamma$, $p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$. Then, for $x \in [a, b]$ the left and right integral operators are defined by

estimations of numerical quadrature rules. For recent developments of Ostrowski inequality, we refer the reader to [8, 9, 11, 20–26] and references therein.

In Section 2, fractional version of Ostrowski inequalities with the help of Mittag–Leffler function has been established. The presented results may be useful in the study of fractional integral operators and their applications. Also, the error bounds of fractional Hadamard inequalities are presented in Section 3.

2. Main Results

First, we establish the following lemma for extended generalized Mittag–Leffler function.

Lemma 1. If $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0, \delta > 0$ and $0 < k < \delta + \Re(\mu)$, then

$$\begin{aligned} & \left(\frac{d}{dt} \right) \left[(\psi_2(t))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega \psi_2(t)^\mu; p) \right] \\ &= \psi_2'(t) \psi_2(t)^{\alpha-2} E_{\mu,\alpha-1,l}^{\gamma,\delta,k,c} (\omega \psi_2(t)^\mu; p). \end{aligned} \quad (11)$$

Proof. We have

$$\begin{aligned} \left(\frac{d}{dt} \right) \left[(\psi_2(t))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega \psi_2(t)^\mu; p) \right] &= \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{\omega^n (\mu n + \alpha - 1) (\psi_2(t))^{\mu n + \alpha - 2} \psi_2'(t)}{(l)_{n\delta}} \\ &= \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha - 1)} \frac{\omega^n (\psi_2(t))^{\mu n + \alpha - 2} \psi_2'(t)}{(l)_{n\delta}}. \end{aligned} \quad (12)$$

After simple computation, one can obtain (11).

Next, we give the generalized fractional Ostrowski type inequality containing extended generalized Mittag–Leffler function. $\square$

Theorem 2. Let $\psi_1: I \rightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a mapping differentiable in I° , the interior of I and $a, b \in I^\circ$,

$a < b$. If ψ_1 is an integrable function, $|\psi_1'(\psi_2(t))| \leq M$ for all $t \in [a, b]$ and $\psi_2: [a, b] \rightarrow \mathbb{R}$ be an increasing and positive function on $(a, b]$, having continuous derivative ψ_2' on (a, b) , then for $\alpha, \beta \geq 1$, the following inequality for fractional integrals (6) and (7) holds:

$$\begin{aligned} & \left| \psi_1(\psi_2(x)) ((\psi_2(b) - \psi_2(x))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c} (\omega (\psi_2(b) - \psi_2(x))^\mu; p) + (\psi_2(x) - \psi_2(a))^{\alpha-1} \right. \\ & \quad \times E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega (\psi_2(x) - \psi_2(a))^\mu; p) - \left(\left(\psi_2 Y_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(x; p) + \left(\psi_2 Y_{\mu,\beta-1,l,\omega,b^-}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(x; p) \right) \Big| \\ & \leq M \left((\psi_2(x) - \psi_2(a))^\alpha \times E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega (\psi_2(x) - \psi_2(a))^\mu; p) + (\psi_2(b) - \psi_2(x))^\beta E_{\mu,\beta,l}^{\gamma,\delta,k,c} (\omega (\psi_2(b) - \psi_2(x))^\mu; p) \right. \\ & \quad \left. - \left(\left(\psi_2 Y_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} \right)(x; p) + \left(\psi_2 Y_{\mu,\beta-1,l,\omega,b^-}^{\gamma,\delta,k,c} \right)(x; p) \right) \right). \end{aligned} \quad (13)$$

Proof. Let $x \in [a, b]$, $t \in [a, x]$, and $\alpha \geq 1$. Then, the following inequality holds for the monotonically increasing function ψ_2 and the Mittag–Leffler function (1):

$$(\psi_2(x) - \psi_2(t))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega (\psi_2(x) - \psi_2(t))^\mu; p) \psi_2'(t) \leq (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega (\psi_2(x) - \psi_2(a))^\mu; p) \psi_2'(t). \quad (14)$$

From (14) and given condition of boundedness of ψ_1' , one can have the following integral inequalities:

$$\begin{aligned} & \int_a^x (M - \psi_1'(\psi_2(t))) (\psi_2(x) - \psi_2(t))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega (\psi_2(x) - \psi_2(t))^\mu; p) \psi_2'(t) dt \\ & \leq (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega (\psi_2(x) - \psi_2(a))^\mu; p) \int_a^x (M - \psi_1'(\psi_2(t))) \psi_2'(t) dt, \end{aligned} \quad (15)$$

$$\begin{aligned} & \int_a^x (M + \psi_1'(\psi_2(t))) (\psi_2(x) - \psi_2(t))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega (\psi_2(x) - \psi_2(t))^\mu; p) \psi_2'(t) dt \\ & \leq (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega (\psi_2(x) - \psi_2(a))^\mu; p) \int_a^x (M + \psi_1'(\psi_2(t))) \psi_2'(t) dt. \end{aligned} \quad (16)$$

First, we consider inequality (15) as follows:

$$\begin{aligned}
 & M \int_a^x (\psi_2(x) - \psi_2(t))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(t))^\mu; p) \psi_2'(t) dt - \int_a^x (\psi_2(x) - \psi_2(t))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(t))^\mu; p) \\
 & \quad \cdot \psi_1'(\psi_2(t)) \psi_2'(t) dt \\
 & \leq (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) \int_a^x (M - \psi_1'(\psi_2(t))) \psi_2'(t) dt.
 \end{aligned} \tag{17}$$

Therefore, (17) takes the following form after integrating by parts and using derivative property (11) and a simple computation:

$$\begin{aligned}
 & (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) \psi_1(\psi_2(x)) \\
 & \quad - \left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(x; p) \leq M \left((\psi_2(x) - \psi_2(a))^\alpha \right. \\
 & \quad \left. E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) - \left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} 1 \right)(x; p) \right).
 \end{aligned} \tag{18}$$

Similarly, adopting the same pattern from (16), one can obtain

$$\begin{aligned}
 & (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) \psi_1(\psi_2(x)) \\
 & \quad - \left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(x; p) \geq -M (\psi_2(x) - \psi_2(a))^\alpha \\
 & \quad E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) - \left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} 1 \right)(x; p).
 \end{aligned} \tag{19}$$

From (18) and (19), the following inequality is obtained:

$$\begin{aligned}
 & \left| (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) \psi_1(\psi_2(x)) - \left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(x; p) \right| \\
 & \leq M \left((\psi_2(x) - \psi_2(a))^\alpha E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) - \left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} 1 \right)(x; p) \right).
 \end{aligned} \tag{20}$$

Now, on the contrary, we let $x \in [a, b]$, $t \in [x, b]$, and $\beta \geq 1$. Then, the following inequality holds for Mittag-Leffler function:

$$(\psi_2(t) - \psi_2(x))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(t) - \psi_2(x))^\mu; p) \psi_2'(t) \leq (\psi_2(b) - \psi_2(x))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) \psi_2'(t). \tag{21}$$

From (21) and the condition of boundedness of ψ_1' , one can have the following integral inequalities:

$$\begin{aligned}
 & \int_x^b (M - \psi_1'(\psi_2(t))) (\psi_2(t) - \psi_2(x))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(t) - \psi_2(x))^\mu; p) \psi_2'(t) dt \\
 & \leq (\psi_2(b) - \psi_2(x))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) \int_x^b (M - \psi_1'(\psi_2(t))) \psi_2'(t) dt,
 \end{aligned} \tag{22}$$

$$\begin{aligned}
 & \int_x^b (M + \psi_1'(\psi_2(t))) (\psi_2(t) - \psi_2(x))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(t) - \psi_2(x))^\mu; p) \psi_2'(t) dt \\
 & \leq (\psi_2(b) - \psi_2(x))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) \int_x^b (M + \psi_1'(\psi_2(t))) \psi_2'(t) dt.
 \end{aligned} \tag{23}$$

Following the same procedure as we did for (15) and (16), one can obtain from (22) and (23) the following modulus inequality:

$$\begin{aligned} & \left| (\psi_2(b) - \psi_2(x))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) \psi_1(\psi_2(x)) - \left(\psi_2 Y_{\mu,\beta-1,l,\omega,b^-}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(x; p) \right| \\ & \leq M \left((\psi_2(b) - \psi_2(x))^\beta E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) - \left(\psi_2 Y_{\mu,\beta-1,l,\omega,b^-}^{\gamma,\delta,k,c} 1 \right)(x; p) \right). \end{aligned} \quad (24)$$

Inequalities (20) and (24) give (13) which is the required inequality.

In the following, we give direct consequences of above theorem. $\square$

Corollary 1. If we put $\alpha = \beta$ in (13), then we get the following fractional integral inequality:

$$\begin{aligned} & \left| \psi_1(\psi_2(x)) (\psi_2(b) - \psi_2(x))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) + (\psi_2(x) - \psi_2(a))^{\alpha-1} \times E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) \right. \\ & \quad \left. - \left(\left(\psi_2 Y_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(x; p) + \left(\psi_2 Y_{\mu,\alpha-1,l,\omega,b^-}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(x; p) \right) \right| \\ & \leq M \left((\psi_2(x) - \psi_2(a))^\alpha \times E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) + (\psi_2(b) - \psi_2(x))^\alpha E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) \right. \\ & \quad \left. - \left(\left(\psi_2 Y_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} 1 \right)(x; p) + \left(\psi_2 Y_{\mu,\alpha-1,l,\omega,b^-}^{\gamma,\delta,k,c} 1 \right)(x; p) \right) \right). \end{aligned} \quad (25)$$

Remark 1

- (i) If we put $\psi_2(x) = x$ in (13), then we obtain Theorem 5 in [9]
- (ii) If we put $\omega = p = 0$ and $\psi_2(x) = x$ in (13), then we obtain Theorem 1 in [6]
- (iii) If we put $\alpha = \beta = 1$, $\psi_2(x) = x$, and $\omega = p = 0$ in (13), then we obtain Ostrowski inequality (10)
- (iv) If we put $\psi_2(x) = x$ in (25), then we obtain Corollary 1 in [9]

The next result is a general form of fractional Ostrowski inequality containing generalized Mittag-Leffler function.

Theorem 3. Let $\psi_1: I \rightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a mapping differentiable in I° , the interior of I and $a, b \in I^\circ$, $a < b$. If ψ_1 is integrable function and $m < \psi_1'(\psi_2(t)) \leq M$ for all $t \in [a, b]$ and $\psi_2: [a, b] \rightarrow \mathbb{R}$ be an increasing and positive function on $(a, b]$, having continuous derivative ψ_2' on (a, b) , then, for $\alpha, \beta \geq 1$, the following inequalities for fractional integrals (6) and (7) hold:

$$\begin{aligned} & \psi_1(\psi_2(x)) (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) - (\psi_2(b) - \psi_2(x))^{\beta-1} \\ & \quad \times E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) - \left(\psi_2 Y_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(x; p) \\ & \quad - \left(\psi_2 Y_{\mu,\beta-1,l,\omega,b^-}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(x; p) \leq M (\psi_2(x) - \psi_2(a))^\alpha \\ & \quad \times E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) - \left(\psi_2 Y_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} 1 \right)(x; p) - m (\psi_2(b) - \psi_2(x))^\beta \\ & \quad \times E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) - \left(\psi_2 Y_{\mu,\beta-1,l,\omega,b^-}^{\gamma,\delta,k,c} 1 \right)(x; p), \end{aligned} \quad (26)$$

$$\begin{aligned} & \psi_1(\psi_2(x)) (\psi_2(b) - \psi_2(x))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) - (\psi_2(x) - \psi_2(a))^{\alpha-1} \\ & \quad E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) - \left(\left(\psi_2 Y_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(x; p) - \left(\psi_2 Y_{\mu,\beta-1,l,\omega,b^-}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(x; p) \right) \\ & \leq M \left((\psi_2(b) - \psi_2(x))^\beta \times E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) - \left(\psi_2 Y_{\mu,\beta-1,l,\omega,b^-}^{\gamma,\delta,k,c} 1 \right)(x; p) \right) \\ & \quad - m \left((\psi_2(x) - \psi_2(a))^\alpha E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) - \left(\psi_2 Y_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} 1 \right)(x; p) \right). \end{aligned} \quad (27)$$

Proof. The proof is similar to the proof of Theorem 2, just after comparing conditions on derivative of ψ_1 , so we left it for the reader.

Some comments on the abovementioned result are given as follows. $\square$

Remark 2

- (i) If we put $\omega = p = 0$ and $\psi_2(x) = x$ in (26) and (27), then we obtain Theorem 1 in [6]
- (ii) If we put $m = -M$ in Theorem 3, then with some rearrangements we obtain Theorem 2

- (iii) If we put $\psi_2(x) = x$ in (26) and (27), then we obtain Theorem 6 in [9]

In the following, we have established a result related to fractional Ostrowski inequality containing generalized Mittag-Leffler function.

Theorem 4. Let $\psi_1: I \rightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a mapping differentiable in I° , the interior of I and $a, b \in I^\circ$, $a < b$. If ψ_1 is integrable function, $|\psi_1'(\psi_2(t))| \leq M$ for all $t \in [a, b]$ and $\psi_2: [a, b] \rightarrow \mathbb{R}$ be an increasing and positive function on $(a, b]$, then, for $\alpha, \beta \geq 1$, the following inequality for fractional integrals (6) and (7) holds:

$$\begin{aligned} & \left| \psi_1(\psi_2(b))(\psi_2(b) - \psi_2(x))^{\beta-1} E_{\mu, \beta, l}^{\gamma, \delta, k, c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) + \psi_1(\psi_2(a)) \right. \\ & \cdot (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) - \left(\left(\psi_2 Y_{\mu, \beta-1, l, \omega, x^+}^{\gamma, \delta, k, c} \psi_1 \circ \psi_2 \right)(b; p) - \left(\psi_2 Y_{\mu, \alpha-1, l, \omega, x^-}^{\gamma, \delta, k, c} \psi_1 \circ \psi_2 \right)(a; p) \right) \Big| \\ & \leq M \left((\psi_2(a) - \psi_2(x))^\alpha \times E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) + (\psi_2(b) - \psi_2(x))^\beta E_{\mu, \beta, l}^{\gamma, \delta, k, c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) \right. \\ & \quad \left. - \left(\left(\psi_2 Y_{\mu, \alpha-1, l, \omega, x^-}^{\gamma, \delta, k, c} \right)(a; p) + \left(\psi_2 Y_{\mu, \beta-1, l, \omega, x^+}^{\gamma, \delta, k, c} \right)(b; p) \right) \right). \end{aligned} \quad (28)$$

Proof. Let $x \in [a, b]$, $t \in [a, x]$, and $\alpha \geq 1$. Then, the following inequality holds true for Mittag-Leffler function:

$$\begin{aligned} & (\psi_2(t) - \psi_2(a))^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(t) - \psi_2(a))^\mu; p) \psi_2'(t) \\ & \leq (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) \psi_2'(t). \end{aligned} \quad (29)$$

From (29) and given condition of boundedness on ψ_1' , one can have the following integral inequalities:

$$\begin{aligned} & \int_a^x (M - \psi_1'(\psi_2(t))) (\psi_2(t) - \psi_2(a))^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(t) - \psi_2(a))^\mu; p) \psi_2'(t) dt \\ & \leq (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) \int_a^x (M - \psi_1'(\psi_2(t))) \psi_2'(t) dt, \end{aligned} \quad (30)$$

$$\begin{aligned} & \int_a^x (M + \psi_1'(\psi_2(t))) (\psi_2(t) - \psi_2(a))^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(t) - \psi_2(a))^\mu; p) \psi_2'(t) dt \\ & \leq (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) \int_a^x (M + \psi_1'(\psi_2(t))) \psi_2'(t) dt. \end{aligned} \quad (31)$$

First, we consider inequality (30) as follows:

$$\begin{aligned} & M \int_a^x (\psi_2(t) - \psi_2(a))^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(t) - \psi_2(a))^\mu; p) \psi_2'(t) dt \\ & - \int_a^x (\psi_2(t) - \psi_2(a))^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(t) - \psi_2(a))^\mu; p) \psi_2'(t) \psi_1'(\psi_2(t)) dt \\ & \leq (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) \int_a^x (M - \psi_1'(\psi_2(t))) \psi_2'(t) dt. \end{aligned} \quad (32)$$

Therefore, (32) takes the following form after integrating by parts and using derivative property (11) and a simple computation

$$\begin{aligned} & (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) \psi_1(\psi_2(a)) - \left(\psi_2 Y_{\mu,\alpha-1,l,\omega,x^-}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(a; p) \\ & \leq M \left((\psi_2(x) - \psi_2(a))^\alpha \times E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) - \left(\psi_2 Y_{\mu,\alpha-1,l,\omega,x^-}^{\gamma,\delta,k,c} 1 \right)(a; p) \right). \end{aligned} \quad (33)$$

Similarly, adopting the same pattern from (31), one can obtain

$$\begin{aligned} & (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) \psi_1(\psi_2(a)) - \left(\psi_2 Y_{\mu,\alpha-1,l,\omega,x^-}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(a; p) \\ & \geq -M \left((\psi_2(x) - \psi_2(a))^\alpha \times E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) - \left(\psi_2 Y_{\mu,\alpha-1,l,\omega,x^-}^{\gamma,\delta,k,c} 1 \right)(a; p) \right). \end{aligned} \quad (34)$$

From (33) and (34), the following inequality is obtained:

$$\begin{aligned} & \left| (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\psi_2(x) - \psi_2(a))^\mu; p) \psi_1(\psi_2(a)) - \left(\psi_2 Y_{\mu,\alpha-1,l,\omega,x^-}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(a; p) \right| \\ & \leq M \left((\psi_2(x) - \psi_2(a))^\alpha \times E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega((\psi_2(x) - \psi_2(a))^\alpha)^\mu; p) - \left(\psi_2 Y_{\mu,\alpha-1,l,\omega,x^-}^{\gamma,\delta,k,c} 1 \right)(a; p) \right). \end{aligned} \quad (35)$$

Now, on the contrary, we let $x \in [a, b]$, $t \in [x, b]$, and $\beta \geq 1$. Then, the following inequality holds for Mittag-Leffler function:

$$\begin{aligned} & (\psi_2(b) - \psi_2(t))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(t))^\mu; p) \psi_2'(t) \leq (\psi_2(b) - \psi_2(x))^{\beta-1} \\ & \times E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) \psi_2'(t). \end{aligned} \quad (36)$$

From (36) and given condition of boundedness of ψ_1' , one can have the following integral inequalities:

$$\begin{aligned} & \int_x^b (M - \psi_1'(\psi_2(t))) (\psi_2(b) - \psi_2(t))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(t))^\mu; p) \psi_2'(t) dt \\ & \leq (\psi_2(b) - \psi_2(x))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) \int_x^b (M - \psi_1'(\psi_2(t))) \psi_2'(t) dt, \end{aligned} \quad (37)$$

$$\begin{aligned} & \int_x^b (M + \psi_1'(\psi_2(t))) (\psi_2(b) - \psi_2(t))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(t))^\mu; p) \psi_2'(t) dt \\ & \leq (\psi_2(b) - \psi_2(x))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c}(\omega(\psi_2(b) - \psi_2(x))^\mu; p) \int_x^b (M + \psi_1'(\psi_2(t))) \psi_2'(t) dt. \end{aligned} \quad (38)$$

Following the same procedure as we did for (30) and (31), one can obtain from (37) and (38) the following modulus inequality:

$$\begin{aligned} & \left| (\psi_2(b) - \psi_2(x))^{\beta-1} E_{\mu,\beta,l}^{\gamma,\delta,k,c} (\omega(\psi_2(b) - \psi_2(x))^\mu; p) \psi_1(\psi_2(b)) - \left(\psi_2 \Upsilon_{\mu,\beta-1,l,\omega,x^+}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(b; p) \right| \\ & \leq M \left((\psi_2(b) - \psi_2(x))^\beta \times E_{\mu,\beta,l}^{\gamma,\delta,k,c} (\omega(\psi_2(b) - \psi_2(x))^\mu; p) - \left(\psi_2 \Upsilon_{\mu,\beta-1,l,\omega,x^+}^{\gamma,\delta,k,c} 1 \right)(b; p) \right). \end{aligned} \quad (39)$$

Inequalities (35) and (39) give (28) which is required inequality.

Some direct consequences of the above theorem are given below. $\square$

Corollary 2. If we put $\alpha = \beta$ in (28), then we get the following fractional integral inequality:

$$\begin{aligned} & \left| \psi_1(\psi_2(b)) (\psi_2(b) - \psi_2(x))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega(\psi_2(b) - \psi_2(x))^\mu; p) + \psi_1(\psi_2(a)) \right. \\ & \quad \cdot (\psi_2(x) - \psi_2(a))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega(\psi_2(x) - \psi_2(a))^\mu; p) - \left(\left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,x^+}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(b; p) + \left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,x^-}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(a; p) \right) \Big| \\ & \leq M \left((\psi_2(x) - \psi_2(a))^\alpha \times E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega(\psi_2(x) - \psi_2(a))^\mu; p) + (\psi_2(b) - \psi_2(x))^\alpha E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega(\psi_2(b) - \psi_2(x))^\mu; p) \right. \\ & \quad \left. - \left(\left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,x^-}^{\gamma,\delta,k,c} 1 \right)(a; p) + \left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,x^+}^{\gamma,\delta,k,c} 1 \right)(b; p) \right) \right). \end{aligned} \quad (40)$$

Remark 3

- (i) If we put $\psi_2(x) = x$ in (28), then we obtain Theorem 5 in [9]
- (ii) If we put $\omega = p = 0$ and $\psi_2(x) = x$ in (28), then we obtain Theorem 1 in [6]
- (iii) If we put $\psi_2(x) = x$ in (40), then we obtain Corollary 2 in [9]

3. Applications

In this section, we just describe some applications of Theorem 4 and leave such applications of other results for the reader. By applying Theorem 4 at end points of the interval $[a, b]$ and adding the resulting inequalities, one obtains the error bounds of compact form of the fractional Hadamard inequality.

Theorem 5. Under the assumptions of Theorem 4, the following estimation of Hadamard inequality can be obtained:

$$\begin{aligned} & \left| \frac{(\psi_1(\psi_2(b)) + \psi_1(\psi_2(a)))}{2} \left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} 1 \right)(b; p) - \frac{1}{2} \left(\left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,b^-}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(a; p) + \left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(b; p) \right) \right| \\ & \leq \frac{M}{2} (\psi_2(b) - \psi_2(a))^\alpha E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega(\psi_2(b) - \psi_2(a))^\mu; p) - \frac{M}{2} \left(\left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,b^-}^{\gamma,\delta,k,c} 1 \right)(a; p) \right. \\ & \quad \left. + \left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} 1 \right)(b; p) \right). \end{aligned} \quad (41)$$

Proof. By putting $x = a$, $\alpha = \beta$, and $x = b$ in (40) then adding the resulting inequalities, we obtain

$$\begin{aligned} & \left| (\psi_1(\psi_2(b)) + \psi_1(\psi_2(a))) (\psi_2(b) - \psi_2(a))^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega(\psi_2(b) - \psi_2(a))^\mu; p) \right. \\ & \quad \left. - \left(\left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,b^-}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(a; p) + \left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} \psi_1 \circ \psi_2 \right)(b; p) \right) \right| \\ & \leq M (\psi_2(b) - \psi_2(a))^\alpha E_{\mu,\alpha,l}^{\gamma,\delta,k,c} (\omega(\psi_2(b) - \psi_2(a))^\mu; p) - \left(\left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,b^-}^{\gamma,\delta,k,c} 1 \right)(a; p) + \left(\psi_2 \Upsilon_{\mu,\alpha-1,l,\omega,a^+}^{\gamma,\delta,k,c} 1 \right)(b; p) \right). \end{aligned} \quad (42)$$

Multiplying both sides of the above inequality by $1/2$ and using (8) and (9), inequality (41) can be obtained. $\square$

Remark 4. If in (41) α is replaced by $\alpha + 1$ and ω by $\omega^\dagger = \omega / ((g(b) - g(a))^\mu)$, then we get an error bound of the Hadamard inequality given in Theorem 1 in [20].

$$\begin{aligned} & \left| \psi_1(\psi_2(b)) \left(\psi_2 Y_{\mu, \alpha-1, l, \omega, b^-}^{\gamma, \delta, k, c} \right) \left(\frac{a+b}{2}; p \right) + \psi_1(\psi_2(a)) \left(\psi_2 Y_{\mu, \alpha-1, l, \omega, a^+}^{\gamma, \delta, k, c} \right) \left(\frac{a+b}{2}; p \right) \right. \\ & \quad \left. - \left(\left(\psi_2 Y_{\mu, \alpha-1, l, \omega, (a+b/2)^+}^{\gamma, \delta, k, c} \right) \psi_1 \circ \psi_2 \right) (b; p) + \left(\psi_2 Y_{\mu, \alpha-1, l, \omega, (a+b/2)^-}^{\gamma, \delta, k, c} \right) \psi_1 \circ \psi_2 (a; p) \right| \\ & \leq M \left(\left(\psi_2 \left(\frac{a+b}{2} \right) - \psi_2(a) \right)^\alpha E_{\mu, \alpha, l}^{\gamma, \delta, k, c} \left(\omega \left(\psi_2 \left(\frac{a+b}{2} \right) - \psi_2(a) \right)^\mu; p \right) \right. \\ & \quad \left. + \left(\psi_2(b) - \psi_2 \left(\frac{a+b}{2} \right) \right)^\alpha E_{\mu, \alpha, l}^{\gamma, \delta, k, c} \left(\omega \left(\psi_2(b) - \psi_2 \left(\frac{a+b}{2} \right) \right)^\mu; p \right) \right. \\ & \quad \left. - \left(\left(\psi_2 Y_{\mu, \alpha-1, l, \omega, (a+b/2)^-}^{\gamma, \delta, k, c} \right) (a; p) + \left(\psi_2 Y_{\mu, \alpha-1, l, \omega, (a+b/2)^+}^{\gamma, \delta, k, c} \right) (b; p) \right) \right). \end{aligned} \quad (43)$$

Proof. By putting $x = (a+b)/2$ for $\alpha = \beta$ in (40), (43) can be obtained. $\square$

4. Concluding Remarks

We have established generalized fractional integral inequalities of Ostrowski type. By applying boundedness of a differentiable function and using properties of an extended generalized Mittag-Leffler function different generalized versions of Ostrowski type inequalities are analyzed. Also, some deductions from results of this paper are connected with already published results. Furthermore, all the results can be calculated for fractional integral operators defined in [2, 3, 5, 6, 27], and we left it for the reader.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors do not have any conflicts of interest.

Authors' Contributions

All authors have equal contributions.

Acknowledgments

This research work was supported by the Higher Education Commission of Pakistan under NRP 2016, Project no. 5421.

Theorem 6. Under the assumptions of Theorem 4, the following inequality can be obtained:

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