Research Article

Kinematic Accuracy Method of Mechanisms Based on Tolerance Theories

Li Zhang,1,2 Hong Nie,1,2 and Xiaohui Wei,1,2

1Key Laboratory of Fundamental Science for National Defense Advanced Design Technology of Flight Vehicle, Nanjing University of Aeronautics and Astronautics, Nanjing 210001, China
2State Key Laboratory of Mechanics and Control of Mechanical Structures, Nanjing University of Aeronautics and Astronautics, Nanjing 210001, China

Correspondence should be addressed to Hong Nie; hnie@nuaa.edu.cn and Xiaohui Wei; wei_xiaohui@nuaa.edu.cn

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Traditional tolerance analysis is mostly restricted to static analysis. However, tolerances of different components also affect the movement accuracy in a mechanism. In this paper, the idea of kinematic tolerance analysis is advanced. In the interest of achieving movement precision considering tolerance, a kinematic Jacobian model is established on the basis of a traditional dimensional chain and an original Jacobian model. The tolerances of functional element (FE) pairs are expressed as small-displacement screws. In addition, joint clearances resulting from tolerance design also influence the kinematic accuracy, and they are modeled by FE pairs. Two examples are presented to illustrate the rationality and the validity of the kinematic tolerance model. The results of the two examples are shown, and the discussion is presented. A physical model of the 2D example is also built up in 3DCS software. Based on the discussion, a comparison between the statistical and physical models is carried out, and the merits and demerits of both are listed.

1. Introduction

For an assembly, component tolerance reflects the actual relationships between mating parts. Conventionally, tolerance analysis is used to estimate the accumulation of the assembly dimensions. As a consequence, tolerance analysis is mostly restricted to static tolerance analysis in an assembly. However, the tolerances of different parts also affect the accuracy of movement in an assembly. With this need for tolerance analysis, the effects of component tolerances on the kinematic accuracy of mechanisms require study.

Tolerance analysis includes three steps: the tolerance representation, propagation model, and analysis method [1–13]. In the early years, the propagation model and analysis method were often discussed together. The work of this paper mainly focuses on the topic of tolerance propagation.

In terms of tolerance propagation and analysis, the traditional approach is a dimensional chain. The main methods that have been developed in recent years are the matrix model [14, 15], vector loop model [16–19], SOV model [20, 21], and Jacobian model [22, 23]. Concerning the traditional method, a tolerance stack-up function is established on the basis of the relevant dimensional chain. The first-order differential value of the corresponding dimension is taken as the deviation. This method is mainly used in static tolerance analysis and has been applied in many cases [24, 25]. Utilizing homogeneous matrix transformation, a matrix model transfers the displacements from local reference frames to a global reference frame. The matrix propagation and representation models are used together to perform the tolerance analysis. A vector loop model uses vectors to represent the dimensions in an assembly [17, 19]. The vectors are arranged in chains or loops representing those dimensions that stack together to determine the resultant assembly dimensions [14]. In an SOV model, the dimensional deviations that represent the influences of tooling and part errors (including part-to-fixture, part-to-part, and interstation interactions) are considered as product and process factors; the whole assembly process is modeled as a state-space model. The accumulations of dimensional
deviations are represented by state transfer functions. The SOV model is suitable for complex assemblies. In recent decades, increasingly more attention has been directed to the development of three-dimensional (3D) tolerance analysis, and a Jacobian model is advanced with the tide. Laperrière and Lafond adopted virtual joints in robotics to explore a tolerance transformation process, and the Jacobian model was obtained by first associating a coordinate frame to every virtual joint [23, 26]. In this model, functional element (FE) pairs and functional requirement (FR) are used to represent the relationship between parts. Based on this innovation, many other studies ensued [27–33]. In addition to the mainstream tolerance analysis methods, new ideas are also burgeoning [18, 34, 35]. Wang et al. [36] invented an assembly dimensional model based on the shortest path. The tolerance propagation function can be acquired through the shortest path, as described in their theory. Based on equivalence of the deviation source, Zhao et al. [18] proposed a combined deviation accumulation method. Although the angles of view are new, the core propagation approaches are mentioned above. Among the propagation models introduced in recent years, the Jacobian model is the most suitable one for the kinematic tolerance analysis.

Regarding tolerance representation, different models exist in the relevant literature. Requicha introduced the mathematical definition of tolerance semantics and proposed a solid offset approach initially [37]. Salomons et al. specified tolerances based on technologically and topologically related surfaces (TTRS) [38, 40]. Desrochers and Riviere presented a homogeneous matrix approach coupled with the notion of constraints for the representation of tolerance zones [15]. T-Maps were set up as well as a hypothetical volume of points that correspond to all possible locations and variations of a segment of a plane that can arise from tolerances on size, form, and orientation [41]. The skin model is a basic concept within GeoSpelling and the ISO standards. Skin model shapes [42–47] are skin model representatives that comprise various kinds of geometric deviations. With this method, geometric deviations are represented by discretizing features into points and measuring the distances between the points of the nominal model and the skin model shape. The main advantage of this method is to model geometric tolerance more practically. To apply geometric tolerances in the Jacobian model, Polini and Corrado [48–50] presented an approach and, whereafter, also integrated manufacturing signature and operation conditions into the Jacobian model based on the skin model shapes. In recent years, there has been an increasing amount of literature on small-displacement screw models [51–55]. A small-displacement screw model was first discussed by Bourdet and Clement [56]. As variations of a surface and its features from the nominal position can be represented by a screw, a small-displacement model was gradually developed for tolerance analysis in the following years.

Most of the tolerance studies are limited to static tolerance analysis. However, for a mechanism, the tolerance of every part in an assembly also affects the kinematic performance, e.g., the accuracy of movement. Walter et al. [57] used “the integrated tolerance analysis in motion” approach to obtain the effects of manufacturing-caused and operation-depending deviations on a system’s FKCs. The propagation function is derived from a dimensional chain, while in the three-dimensional space, the ideal propagation method for kinematic tolerance analysis is the Jacobian matrix; meanwhile, the study concerns more about the operation-depending effect on the tolerance, not the tolerance influence on the kinematic accuracy. Utilizing the skin model shapes, Schleich et al. [58, 59] constituted a framework for the deviation analysis of contact and mobility in an assembly. The skin model shapes are a tolerance representation method, and their study mainly focuses on the geometric modeling of the contact in motion. The mobility is involved; however, the propagation method is also a dimensional chain, and the application is limited to some extent. Zhou et al. [60] proposed a kinematic accuracy method based on DP-SDT theory. Although the influences of motional displacements, force direction, and vibration can be calculated using the method, the propagation function is complicated.

The purpose of this paper is to propose a novel kinematic accuracy method for mechanisms based on tolerance analysis. First, joint clearances resulting from the fit tolerances are represented as tolerances by small-displacement screws. Then, the kinematic Jacobian model is established by combining an original tolerance Jacobian model and a traditional dimensional chain. Finally, the kinematic movement error can be obtained through the analysis of the kinematic tolerance Jacobian model. These ideas will generate important insights into tolerance design and kinematic accuracy; a small constituent part of the establishment method can also be used independently and integrated with other theories (e.g., robust theory) so that several kinds of practical problems in engineering can be solved [61].

The outline of this work is as follows: Section 2 reviews related theory. Section 3 discusses the details of the proposed method. Section 4 presents a case study to verify the feasibility and efficiency of the new method, and a physical model is also given to prove the legitimacy of the results in this section. Meanwhile, a practical mechanism in engineering is also chosen as an example in Section 5, and the discussion of the results is given. Section 6 summarizes the work and points out the future study.

2. Related Theory

2.1. Jacobian Model. A Jacobian model adopting kinematic theory has been advanced in robotics. Several basic concepts of an original Jacobian model are listed in Table 1 [61, 62].

In an assembly, it is essential to establish the tolerance chain first and then define FE pairs in the established tolerance chain.

2.2. Deviation of a Functional Element Pair from Its Nominal Value (Based on Dimensional Chain Theory). A tolerance chain is a representation of how a functional requirement $x$ depends on a known set of functional dimensions $d_i$ [61, 63]. Each dimension has a deviation from its nominal value: $\Delta d_i = d_i - \bar{d}_i$, and, as a consequence, a deviation will occur on the functional requirement: $\Delta x = x - \bar{x}$. 
Table 1: Basic concepts of a Jacobian model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Functional element (FE)</td>
<td>Points, curves, or surfaces that belong to parts in the assembly. An FE can be real, e.g., the plane surface of a block, or constructed, e.g., the axis of a cylinder.</td>
</tr>
<tr>
<td>Functional requirement (FR)</td>
<td>An important condition to be satisfied between two FEs on different parts, e.g., a fitting condition.</td>
</tr>
<tr>
<td>Kinematic pair</td>
<td>If two FEs are on different parts and there is physical contact between them, then they constitute a kinematic pair.</td>
</tr>
<tr>
<td>Internal pair</td>
<td>If two FEs are on the same part and both of them participate in a contact relation with some other parts, then they form an internal pair.</td>
</tr>
<tr>
<td>Functional element pair (FE pair)</td>
<td>Two FEs on different parts or on the same part, including kinematic pairs and internal pairs.</td>
</tr>
<tr>
<td>Virtual joints</td>
<td>For the purpose of tolerancing, some coordinate frames are associated with the tolerated FEs in an FE pair, assuming a set of virtual joints exist in each FE pair and can make the tolerated FEs &quot;move&quot; relative to the other FEs, in order to simulate manufacturing inaccuracies.</td>
</tr>
<tr>
<td>Global reference frame (GRF)</td>
<td>The frame in which the FR and global dimensional chain are established.</td>
</tr>
<tr>
<td>Local reference frame (LRF)</td>
<td>The frames in which FE pairs are established.</td>
</tr>
</tbody>
</table>

In the general case, \( x = f(d_1, d_2, \ldots, d_n) \) is nonlinear and possibly unknown, and the above assumption \( \Delta x \) is difficult to calculate. However, the equation can be linearized by a first-order Taylor approximation (as the deviations on the dimensions are small compared to nominal values, higher-order terms of a Taylor series can be omitted):

\[
x + \Delta x = f(d_1, d_2, \ldots, d_n) + \left[ \frac{\partial f}{\partial d_1} \right] \Delta d_1 + \cdots + \left[ \frac{\partial f}{\partial d_n} \right] \Delta d_n
\]

(1)

Then,

\[
\Delta x = \sum_{i=1}^{n} \frac{\partial f}{\partial d_i} \Delta d_i
\]

(2)

FR and FE pairs are both established between two FEs. Even the length of an FE pair can be seen as a FR of its elements, letting \( x = f(d_1, d_2, \ldots, d_n) \) be the length of an FE pair, then.

Then,

\[
\Delta x = \sum_{i=1}^{n} \frac{\partial f}{\partial d_i} \Delta d_i
\]

(3)

where \( \Delta x \) is the small deviation of the length of the functional element pair and \( \Delta d_i \) is its deviation from the nominal value of \( d_i \) (in tolerancing, it can be seen as a designed tolerance of dimension \( d \)).

For an FE pair in Cartesian coordinates, the projections of lengths along three axes are

\[
x = f_x(d_1, d_2, \ldots, d_n),
\]

\[
y = f_y(d_1, d_2, \ldots, d_n),
\]

\[
z = f_z(d_1, d_2, \ldots, d_n),
\]

(4)

and the respective angles about three axes are

\[
\theta_x = f_{\theta x}(d_1, d_2, \ldots, d_n), \quad \theta_y = f_{\theta y}(d_1, d_2, \ldots, d_n), \quad \theta_z = f_{\theta z}(d_1, d_2, \ldots, d_n).
\]

(5)

respectively, where \( \Delta x, \Delta y, \Delta z, \Delta \theta_x, \Delta \theta_y, \text { and } \Delta \theta_z \) are, respectively, small deviations along and about the three Cartesian axes.

3. Jacobian Model for Kinematic Tolerance Analysis (Kinematic Jacobian Model)

In this article, the generic forms of small-displacement screws are used to model the tolerances associated with features and gaps in an assembly.

3.1. Tolerance Modeling Using the Small-Displacement Screw.

Let \( \Delta = [\delta_1, \delta_2, \delta_3; \theta_1, \theta_2, \theta_3] \) be a small-displacement screw model, where \( [\delta_1, \delta_2, \delta_3] \) is a small translational displacement and \( [\theta_1, \theta_2, \theta_3] \) is a small rotational displacement.

Concerning the existence of tolerances, when mechanisms move, FE pairs move, the lengths of FE pairs change, and the small-displacement screw models change as well. We let \( \eta \) be the input variable of a mechanism; then, the projections of lengths and angles, respectively, along and about three axes are
\[ x = f_x(d, \eta), \quad y = f_y(d, \eta), \quad z = f_z(d, \eta), \]
\[ \theta_x = f_{\theta x}(d, \eta), \quad \theta_y = f_{\theta y}(d, \eta), \quad \theta_z = f_{\theta z}(d, \eta), \]
and then,
\[ \Delta x = \sum_{i=1}^{n} \frac{\partial f_x(d, \eta)}{\partial d_i} \Delta d_i, \]
\[ \Delta y = \sum_{i=1}^{n} \frac{\partial f_y(d, \eta)}{\partial d_i} \Delta d_i, \]
\[ \Delta z = \sum_{i=1}^{n} \frac{\partial f_z(d, \eta)}{\partial d_i} \Delta d_i, \]
\[ \Delta \theta_x = \sum_{i=1}^{n} \frac{\partial f_{\theta x}(d, \eta)}{\partial d_i} \Delta d_i, \]
\[ \Delta \theta_y = \sum_{i=1}^{n} \frac{\partial f_{\theta y}(d, \eta)}{\partial d_i} \Delta d_i, \]
\[ \Delta \theta_z = \sum_{i=1}^{n} \frac{\partial f_{\theta z}(d, \eta)}{\partial d_i} \Delta d_i, \]
respectively, where \( \Delta x, \Delta y, \Delta z, \Delta \theta_x, \Delta \theta_y, \) and \( \Delta \theta_z \) are, respectively, small deviations that are functions of the input variables \( \eta \) and \( \Delta d_i \). As a consequence, the small-displacement screw can be expressed as

\[
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\theta_1 \\
\theta_2 \\
\theta_3 \\
\Delta x \\
\Delta y \\
\Delta z \\
\Delta \theta_x \\
\Delta \theta_y \\
\Delta \theta_z
\end{bmatrix} = \begin{bmatrix}
\sum_{i=1}^{n} \frac{\partial f_x(d, \eta)}{\partial d_i} \\
\sum_{i=1}^{n} \frac{\partial f_y(d, \eta)}{\partial d_i} \\
\sum_{i=1}^{n} \frac{\partial f_z(d, \eta)}{\partial d_i} \\
\sum_{i=1}^{n} \frac{\partial f_{\theta x}(d, \eta)}{\partial d_i} \\
\sum_{i=1}^{n} \frac{\partial f_{\theta y}(d, \eta)}{\partial d_i} \\
\sum_{i=1}^{n} \frac{\partial f_{\theta z}(d, \eta)}{\partial d_i}
\end{bmatrix} \Delta d_i
\]

\[ \Delta \text{FE}_i = \begin{bmatrix}
\delta_{i1} \\
\delta_{i2} \\
\delta_{i3} \\
\theta_{i1} \\
\theta_{i2} \\
\theta_{i3} \\
\Delta x_{i1} \\
\Delta y_{i1} \\
\Delta z_{i1} \\
\Delta \theta_{x_{i1}} \\
\Delta \theta_{y_{i1}} \\
\Delta \theta_{z_{i1}}
\end{bmatrix} = \begin{bmatrix}
\sum_{i=1}^{n} \frac{\partial f_{x_{i1}}(d, \eta)}{\partial d_i} \\
\sum_{i=1}^{n} \frac{\partial f_{y_{i1}}(d, \eta)}{\partial d_i} \\
\sum_{i=1}^{n} \frac{\partial f_{z_{i1}}(d, \eta)}{\partial d_i} \\
\sum_{i=1}^{n} \frac{\partial f_{\theta x_{i1}}(d, \eta)}{\partial d_i} \\
\sum_{i=1}^{n} \frac{\partial f_{\theta y_{i1}}(d, \eta)}{\partial d_i} \\
\sum_{i=1}^{n} \frac{\partial f_{\theta z_{i1}}(d, \eta)}{\partial d_i}
\end{bmatrix} \Delta d_i
\]

Here, the clearance in an assembly is seen as an FE pair of the tolerance chain.

3.2. Jacobian Matrix. For the Jacobian model, FE pairs are used to represent the dimensions and variations in an assembly. The representations of virtual joints and coordinate frames in an FE pair are shown in Figure 1.

The transformation matrix can be deduced to be
The Jacobian matrix introduced previously is written as
\[ J = \left[ \begin{array}{ccc}
J_{11} & J_{12} & \cdots & J_{1n} \\
J_{21} & J_{22} & \cdots & J_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
J_{m1} & J_{m2} & \cdots & J_{mn}
\end{array} \right]_{m \times n} \]
where \( J_{ij} \) is the Jacobian matrix associated with the FE of the \( i \)th FE pair (internal or kinematic) to which the tolerances are applied, with \( i = 1 \sim n \).

For small rotational virtual joints, the \( i \)th column of the Jacobian matrix \( J_i \) is computed as
\[ J_i = \left[ \begin{array}{c}
\omega_{0i} \\
\omega_{1i} \\
\vdots \\
\omega_{ni}
\end{array} \right] = \frac{\left[ \begin{array}{c}
\omega_{0i} \\
\omega_{1i} \\
\vdots \\
\omega_{ni}
\end{array} \right]}{Z_0^{i-1}} \times \left[ \begin{array}{c}
\omega_{n+1} \\
\omega_{n+2} \\
\vdots \\
\omega_{2n}
\end{array} \right], \]
where \( Z_0^{i-1} \) is the third column of \( T_0^{i-1} \) and \( d_0^{i-1} \) is the last column of \( T_0^{i-1} \).

For a mechanism containing \( n \) FE pairs, the Jacobian model for kinematic tolerance analysis in the GRF can be expressed as

\[
\delta_1 \\
\delta_2 \\
\delta_3 \\
\theta_1 \\
\theta_2 \\
\theta_3_{FE}
\]

This model is called the kinematic Jacobian model, where
\[
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\theta_1 \\
\theta_2 \\
\theta_3_{FE}
\end{bmatrix}_{FR}
\] is the small displacement of the FR in the GRF. The Jacobian matrix \( J \) and small-displacement screws of FE pairs (\( \Delta FE_i \)) change with input variable \( \eta \) in the mechanism. The clearance between parts is seen as an FE pair and modeled as a small-displacement screw.

3.3. Steps to Establish a Kinematic Jacobian Model in Kinematic Tolerance Analysis. The basic steps of the kinematic Jacobian model for tolerance analysis are as follows:

(1) Identify FE pairs and define the LRF for each FE and the virtual joints

(2) Create the tolerance chain in the mechanism, and obtain small-displacement screws for FE pairs; then, establish the Jacobian model for kinematic tolerance analysis

(3) Compute the Jacobian model using a statistical method, worst-case method, or Monte Carlo method

(4) Analyze the results

4. Case Study 1 for the Kinematic Tolerance Analysis

In Section 4, a crank-slider mechanism is taken as an example to analyze the tolerance effect. In addition to the results obtained using the kinematic Jacobian model introduced above, a software simulation is also performed and is discussed in Section 4. Figure 2 shows the vector representation of a classic crank-slider mechanism with a clearance \( r \) in the revolving joint resulting from the tolerance design. B and C are centers of the hole and shaft,
respectively. The distance between A and D is seen as a functional requirement. The frame established at point A is seen as the GRF, while the other frames are taken as LRFs. The effect of small angle change is omitted in this case.

As \( r_e \) is the clearance resulting from the fit tolerances predesigned, considering the contact theories stated by Johnson, the following two assumptions are provided before the analysis:

1. There is no deformation caused by the contacts in joints
2. Contacts between the holes and shafts are supposed to be line contacts

Based on the above two basic assumptions, the angle between \( r_e \) and the horizontal line can be concluded to be equal to the angle between \( r_2 \) and the horizontal line. The parameters of the crank-slider are listed in Table 2.

In this case, the Monte Carlo method is used in the calculation; therefore, the tolerance of the connecting rod obeys norm distribution, as shown in Table 2. The functional pairs of the crank-slider are listed in Table 3. It is worth noting that the \( z \) directions of small-displacement screws are different from the original virtual joints; therefore, for functional pairs, small displacements in the \( z \) directions are negative.

In Table 4, \((\bar{d}_0^n - \bar{d}_0^{n-1}) = f_i(d, \eta)\), which changes with the movement of the kinematic chain. In this case, \( \eta = \theta \), and \((\bar{d}_0^n - \bar{d}_0^{n-1}) \) is actually the coordinates of each functional pair in the GRF.

### 4.1. Kinematic Tolerance Analysis Using the Jacobian Model

To establish small-displacement screws for the three FE pairs, we have

\[
\Delta F E_1 = \begin{bmatrix}
\delta_{11} & \Delta r_1 \sin \theta \\
\delta_{12} & 0 \\
\delta_{13} & \Delta r_1 \cos \theta \\
\theta_{11} & 0 \\
\theta_{12} & 0 \\
\theta_{13} & 0
\end{bmatrix},
\Delta F E_2 = \begin{bmatrix}
\delta_{21} & -\Delta r_e \sin \gamma \\
\delta_{22} & 0 \\
\delta_{23} & \Delta r_e \cos \gamma \\
\theta_{21} & 0 \\
\theta_{22} & 0 \\
\theta_{23} & 0
\end{bmatrix},
\Delta F E_3 = \begin{bmatrix}
\delta_{31} & -\Delta r_3 \sin \gamma \\
\delta_{32} & 0 \\
\delta_{33} & \Delta r_3 \cos \gamma \\
\theta_{31} & 0 \\
\theta_{32} & 0 \\
\theta_{33} & 0
\end{bmatrix}.
\]

To establish the Jacobian matrix for tolerance analysis, we have

\[
J^T = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
x_2 + x_3 & 0 & 0 & 1 & 0 & 0 \\
z_1 & 0 & 1 & 0 & 0 & 0 \\
x_3 & 0 & 0 & 1 & 0 & 0 \\
z_2 & 0 & 1 & 0 & 0 & 0 \\
x_3 & 0 & 0 & 1 & 0 & 0 \\
z_2 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}.
\]

The overall Jacobian model is

\[
J = \begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\theta_1 \\
\theta_2 \\
\theta_3
\end{bmatrix}_{\text{FR}} = J \cdot \begin{bmatrix}
\Delta F E_1 \\\n\Delta F E_2 \\
\Delta F E_3
\end{bmatrix}.
\]

In this case, \( \delta_3 \) is the kinematic tolerance change in the \( x \) direction of the FR; calculating the Jacobian model, the stack-up function of \( \delta_3 \) is formalized as

\[
\delta_3 = 1 \cdot \delta_{13} - z_1 \cdot \theta_{13} + 1 \cdot \delta_{23} - z_2 \cdot \theta_{23} + 1 \cdot \delta_{13}
\]

\[
\begin{align*}
&= 1 \cdot (\Delta r_1 \cos \theta - z_1 \cdot \theta_{13}) + 1 \cdot (\Delta r_e \cos \gamma - z_2 \cdot \theta_{23}) + 1 \cdot (\Delta r_3 \cos \gamma - z_3 \cdot \theta_{33}) \\
&+ 1(\Delta r_2 \cos \gamma) = \Delta r_1 \cos \theta + \Delta r_e \cos \gamma + \Delta r_3 \cos \gamma.
\end{align*}
\]

In the GRF, \( (r_2 + r_e) \cdot \sin \gamma + r_1 \cdot \sin \theta = 0 \); therefore, \( \sin \gamma = (-r_1 \cdot \sin \theta/(r_2 + r_e)) \).

Letting \( \theta \) to change from 0 to 2\( \pi \), calculations are taken every 15°. For the tolerances of the crank-slider, the Monte Carlo method is used here, and the test number is chosen as 100,000. The histograms for each angle are shown in Figures 3–8.

PD represents the probability density of deviation with every \( \delta \), while the small possible deviations of the functional requirements are denoted as offset values in Figures 3–8. Meanwhile, mean offset values are plotted in Figure 9.

### 4.2. Kinematic Tolerance Analysis Using the Physical Model

As tolerance analysis can be processed with 3DCS software, here a physical model has been established, and numerical simulation has also been completed (Figure 10). In 3DCS, the fits are endowed with tolerances, and the analysis is carried out. Basic parameters are the same as analysis using the kinematic Jacobian model. The mean values are listed in Figure 11.
4.3. Discussion and Comparison. When the kinematic Jacobian model is used as the stack-up function, Figures 3–8 show the probability densities of deviations for functional requirements at every angle. Meanwhile, the mean values of all offset values are drawn in Figure 9, from which it can be seen that the offset values of the functional requirements are strongly dependent on the input variable of the crank-slider; that is, they change as the input variable changes. The figure reaches the peak when \( \theta \) is near \( \pi \cdot i / 2 (i = 2k + 1, k = 1, 2, \ldots, n) \).

Table 3: Screw models of FE pairs.

<table>
<thead>
<tr>
<th>Functional pairs</th>
<th>Parameters of each functional pair</th>
</tr>
</thead>
<tbody>
<tr>
<td>FE pair 1 (A and B)</td>
<td>( x_1 = r_1 \cdot \cos \theta ) ( y_1 = 0 ) ( z_1 = -r_1 \cdot \sin \theta )</td>
</tr>
<tr>
<td>FE pair 2 (B and C)</td>
<td>( x_2 = r_2 \cdot \cos \gamma ) ( y_2 = 0 ) ( z_2 = -r_2 \cdot \sin \gamma )</td>
</tr>
<tr>
<td>FE pair 3 (C and D)</td>
<td>( x_3 = r_3 \cdot \cos \delta ) ( y_3 = 0 ) ( z_3 = -r_3 \cdot \sin \delta )</td>
</tr>
<tr>
<td>FR (A and D)</td>
<td>Unknown and must be calculated</td>
</tr>
</tbody>
</table>

Figure 10 illustrates the results obtained when the crank-slider is analyzed using 3DCS software. In general, the results are influenced by the input variable of the crank-slider as well, and the figure also reaches the peak when \( \theta \) is near \( \pi \cdot i / 2 \pm 1 / 2 \) \((i = 2k + 1, k = 1, 2, \ldots, n)\).

In Figure 12, the results obtained from the kinematic Jacobian model and the results obtained by 3DCS software are both plotted. From the comparison in Figure 12, it can be concluded that the trends of two curves are almost the same. However, for the curve obtained by 3DCS, the peak of the curve is brought forward or postponed compared with the curve of the kinematic Jacobian model results. The reason is that the tiny unalterable tolerance setting in software will inevitably affect the tolerance results.

It can be seen from the above results that the kinematic Jacobian model obtains comparatively stable results and clearly shows the trend of error change. The results obtained with the kinematic Jacobian model are conservative, however, and the basic error-change tendency is the same as that found via the software analysis.
Figure 3: Probability density of deviation ($\theta \in \pi/12, \pi/6, \pi/4, \pi/3$). (a) $\theta = \pi/12$. (b) $\theta = 2 * \pi/12$. (c) $\theta = 3 * \pi/12$. (d) $\theta = 4 * \pi/12$.

Figure 4: Probability density of deviation ($\theta = 5\pi/12, \pi/2, 7\pi/12, 2\pi/3$). (a) $\theta = 5 * \pi/12$. (b) $\theta = 6 * \pi/12$. (c) $\theta = 7 * \pi/12$. (d) $\theta = 8 * \pi/12$. 
Figure 5: Probability density of deviation ($\theta = 9\pi/12, 5\pi/6, 11\pi/12, \pi$). (a) $\theta = 9\pi/12$. (b) $\theta = 10\pi/12$. (c) $\theta = 11\pi/12$. (d) $\theta = 12\pi/12$.

Figure 6: Probability density of deviation ($\theta = 13\pi/12, 7\pi/6, 5\pi/4, 4\pi/3$). (a) $\theta = 13\pi/12$. (b) $\theta = 14\pi/12$. (c) $\theta = 15\pi/12$. (d) $\theta = 16\pi/12$. 
Figure 7: Probability density of deviation ($\theta = 17\pi/12, 3\pi/2, 19\pi/12, 10\pi/6$). (a) $\theta = 17 \pi/12$. (b) $\theta = 18 \pi/12$. (c) $\theta = 19 \pi/12$. (d) $\theta = 20 \pi/12$.

Figure 8: Probability density of deviation ($\theta = 7\pi/4, 11\pi/6, 23\pi/12, 2\pi$). (a) $\theta = 21 \pi/12$. (b) $\theta = 22 \pi/12$. (c) $\theta = 23 \pi/12$. (d) $\theta = 24 \pi/12$. 
The kinematic Jacobian model is comparatively simple in the example, provided in this paper. This coincides with expectation. Major trends are clearly observed using the kinematic Jacobian model, implying that it can show the trends of a mechanism more distinctly. As the basic trends of the statistical model and physical model are almost identical,
the kinematic Jacobian model can be used as the tolerance stack-up function in kinematic tolerance analysis, and its validity for this purpose is proved.

5. Case Study 2 for the Kinematic Tolerance Analysis

In this section, a three-dimensional landing gear retraction mechanism is taken as an example to illustrate the theory introduced. The overall structure and representation of the landing gear retraction system are shown in Figures 13 and 14. The mechanism is composed of five components: AB is the upper connecting joint, BC is the upper brace, CD is the lower brace, DE is the lower connecting joint, and GF is the main strut. A and G are localized. The main strut rotates around GH. When the landing gear mechanism retracts or extends, the movement of the wheel is not stable. It closely relates to the tolerances of components, and the kinematic tolerance analysis is performed next. The landing gear retraction mechanism holds the same assumptions as Section 4.

The parameters of the landing gear retraction mechanism are listed in Table 5. On the ground of the experience and the mechanical engineering handbook, the tolerances of the components and fits are chosen as medium level. The Monte Carlo method is used in the calculation, and the tolerances of the components obey norm distribution, as shown in Table 5.

The FE pairs of the landing gear retraction mechanism are listed in Table 6. The $d_\theta = d_\theta^{l-1}$s of FE pairs are listed in Table 7.

To establish small-displacement screws for eight FE pairs, we have

\[
\Delta FE_i = \begin{bmatrix}
\delta_{i1} \\
\delta_{i2} \\
\delta_{i3} \\
\phi_{i1} \\
\phi_{i2} \\
\phi_{i3}
\end{bmatrix} = \begin{bmatrix}
\Delta AB \cdot \cos \theta_{i1} \\
\Delta AB \cdot \cos \theta_{i1} \\
\Delta AB \cdot \cos \theta_{i1} \\
\Delta \phi \\
\Delta \phi \\
\Delta \phi
\end{bmatrix}
\]

\[
\Delta FE_2 = \begin{bmatrix}
\delta_{21} \\
\delta_{22} \\
\delta_{23} \\
\phi_{21} \\
\phi_{22} \\
\phi_{23}
\end{bmatrix} = \begin{bmatrix}
\Delta r_B \cdot \cos \theta_{i1} \\
\Delta r_B \cdot \cos \theta_{i1} \\
\Delta r_B \cdot \cos \theta_{i1} \\
\Delta \phi \\
\Delta \phi \\
\Delta \phi
\end{bmatrix}
\]

\[
\Delta FE_3 = \begin{bmatrix}
\delta_{31} \\
\delta_{32} \\
\delta_{33} \\
\phi_{31} \\
\phi_{32} \\
\phi_{33}
\end{bmatrix} = \begin{bmatrix}
\Delta BC \cdot \cos \theta_{i2} \\
\Delta BC \cdot \cos \theta_{i2} \\
\Delta BC \cdot \cos \theta_{i2} \\
\Delta \phi \\
\Delta \phi \\
\Delta \phi
\end{bmatrix}
\]

\[
\Delta FE_4 = \begin{bmatrix}
\delta_{41} \\
\delta_{42} \\
\delta_{43} \\
\phi_{41} \\
\phi_{42} \\
\phi_{43}
\end{bmatrix} = \begin{bmatrix}
\Delta CD \cdot \cos \theta_{i3} \\
\Delta CD \cdot \cos \theta_{i3} \\
\Delta CD \cdot \cos \theta_{i3} \\
\Delta \phi \\
\Delta \phi \\
\Delta \phi
\end{bmatrix}
\]

\[
\Delta FE_5 = \begin{bmatrix}
\delta_{51} \\
\delta_{52} \\
\delta_{53} \\
\phi_{51} \\
\phi_{52} \\
\phi_{53}
\end{bmatrix} = \begin{bmatrix}
\Delta D \cdot \cos \theta_{i3} \\
\Delta D \cdot \cos \theta_{i3} \\
\Delta D \cdot \cos \theta_{i3} \\
\Delta \phi \\
\Delta \phi \\
\Delta \phi
\end{bmatrix}
\]

\[
\Delta FE_6 = \begin{bmatrix}
\delta_{61} \\
\delta_{62} \\
\delta_{63} \\
\phi_{61} \\
\phi_{62} \\
\phi_{63}
\end{bmatrix} = \begin{bmatrix}
\Delta DE \cdot \cos \theta_{i4} \\
\Delta DE \cdot \cos \theta_{i4} \\
\Delta DE \cdot \cos \theta_{i4} \\
\Delta \phi \\
\Delta \phi \\
\Delta \phi
\end{bmatrix}
\]

\[
\Delta FE_7 = \begin{bmatrix}
\delta_{71} \\
\delta_{72} \\
\delta_{73} \\
\phi_{71} \\
\phi_{72} \\
\phi_{73}
\end{bmatrix} = \begin{bmatrix}
\Delta DE \cdot \cos \theta_{i4} \\
\Delta DE \cdot \cos \theta_{i4} \\
\Delta DE \cdot \cos \theta_{i4} \\
\Delta \phi \\
\Delta \phi \\
\Delta \phi
\end{bmatrix}
\]

\[
\Delta FE_8 = \begin{bmatrix}
\delta_{81} \\
\delta_{82} \\
\delta_{83} \\
\phi_{81} \\
\phi_{82} \\
\phi_{83}
\end{bmatrix} = \begin{bmatrix}
\Delta DE \cdot \cos \theta_{i4} \\
\Delta DE \cdot \cos \theta_{i4} \\
\Delta DE \cdot \cos \theta_{i4} \\
\Delta \phi \\
\Delta \phi \\
\Delta \phi
\end{bmatrix}
\]
Figure 13: The landing gear retraction mechanism.

Figure 14: The representation of the landing gear mechanism.
### Table 5: Basic parameters of the landing gear.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB (mm)</td>
<td>Length</td>
<td>159</td>
</tr>
<tr>
<td>BC (mm)</td>
<td>Length</td>
<td>757</td>
</tr>
<tr>
<td>CD (mm)</td>
<td>Length</td>
<td>1031.4</td>
</tr>
<tr>
<td>DD' (mm)</td>
<td>Length</td>
<td>126</td>
</tr>
<tr>
<td>D'E (mm)</td>
<td>Length</td>
<td>159</td>
</tr>
<tr>
<td>GF (mm)</td>
<td>Length</td>
<td>2980</td>
</tr>
<tr>
<td>$\theta_x$</td>
<td>Input variable</td>
<td>$\sim N(0, 0.083^2)$</td>
</tr>
<tr>
<td>$\theta_y$</td>
<td>Tolerance</td>
<td>$\Delta_{BC}$~N (0, 0.2^2)</td>
</tr>
<tr>
<td>$\theta_z$</td>
<td>Tolerance</td>
<td>$\Delta_{CD}$~N (0, 0.133^2)</td>
</tr>
<tr>
<td>$\Delta$AB</td>
<td>Tolerance</td>
<td>$\Delta_{DE}$~N (0, 0.083^2)</td>
</tr>
<tr>
<td>$\Delta$BC</td>
<td>Tolerance</td>
<td>$\Delta_{GF}$~N (0, 0.333^2)</td>
</tr>
<tr>
<td>$\Delta$CD</td>
<td>Tolerance</td>
<td>$\Delta_{GR}$~N (0.0345, 0.00375^2)</td>
</tr>
<tr>
<td>$\Delta$DE</td>
<td>Tolerance</td>
<td>$\Delta_{GR}$~N (0.0345, 0.00375^2)</td>
</tr>
<tr>
<td>$\Delta$GF</td>
<td>Tolerance</td>
<td>$\Delta_{GR}$~N (0.0345, 0.00375^2)</td>
</tr>
<tr>
<td>$\Delta\theta$</td>
<td>Angle tolerance</td>
<td>$\Delta\theta$~N (0, 0.00015^2)</td>
</tr>
</tbody>
</table>

### Table 6: Screw models of FE pairs.

<table>
<thead>
<tr>
<th>Functional pairs</th>
<th>Parameters of each functional pair</th>
</tr>
</thead>
<tbody>
<tr>
<td>FE pair 1 (A and B')</td>
<td>$\delta_{i1} = \Delta_{AB} \cdot \cos \theta_{z1}, \delta_{i2} = \Delta_{AB} \cdot \cos \theta_{z2}, \delta_{i3} = \Delta_{AB} \cdot \cos \theta_{z3}$</td>
</tr>
<tr>
<td>FE pair 2 (B' and B)</td>
<td>$\delta_{i1} = \Delta_{BC} \cdot \cos \theta_{z1}, \delta_{i2} = \Delta_{BC} \cdot \cos \theta_{z2}, \delta_{i3} = \Delta_{BC} \cdot \cos \theta_{z3}$</td>
</tr>
<tr>
<td>FE pair 3 (B and C')</td>
<td>$\delta_{i1} = \Delta_{CD} \cdot \cos \theta_{z1}, \delta_{i2} = \Delta_{CD} \cdot \cos \theta_{z2}, \delta_{i3} = \Delta_{CD} \cdot \cos \theta_{z3}$</td>
</tr>
<tr>
<td>FE pair 4 (C' and C)</td>
<td>$\delta_{i1} = \Delta_{DE} \cdot \cos \theta_{z1}, \delta_{i2} = \Delta_{DE} \cdot \cos \theta_{z2}, \delta_{i3} = \Delta_{DE} \cdot \cos \theta_{z3}$</td>
</tr>
<tr>
<td>FE pair 5 (C and D')</td>
<td>$\delta_{i1} = \Delta_{GF} \cdot \cos \theta_{z1}, \delta_{i2} = \Delta_{GF} \cdot \cos \theta_{z2}, \delta_{i3} = \Delta_{GF} \cdot \cos \theta_{z3}$</td>
</tr>
<tr>
<td>FE pair 6 (D' and D)</td>
<td>$\delta_{i1} = \Delta_{GR} \cdot \cos \theta_{z1}, \delta_{i2} = \Delta_{GR} \cdot \cos \theta_{z2}, \delta_{i3} = \Delta_{GR} \cdot \cos \theta_{z3}$</td>
</tr>
<tr>
<td>FE pair 7 (D and E)</td>
<td>$\delta_{i1} = \Delta_{GR} \cdot \cos \theta_{z1}, \delta_{i2} = \Delta_{GR} \cdot \cos \theta_{z2}, \delta_{i3} = \Delta_{GR} \cdot \cos \theta_{z3}$</td>
</tr>
<tr>
<td>FE pair 8 (G and F)</td>
<td>$\delta_{i1} = \Delta_{GR} \cdot \cos \theta_{z1}, \delta_{i2} = \Delta_{GR} \cdot \cos \theta_{z2}, \delta_{i3} = \Delta_{GR} \cdot \cos \theta_{z3}$</td>
</tr>
<tr>
<td>FR (A and F)</td>
<td>$\delta_{i1} = \Delta_{GR} \cdot \cos \theta_{z1}, \delta_{i2} = \Delta_{GR} \cdot \cos \theta_{z2}, \delta_{i3} = \Delta_{GR} \cdot \cos \theta_{z3}$</td>
</tr>
</tbody>
</table>

### Table 7: $\frac{d_0 - d_{i-1}}{d_0 - d_i}$ of FE pairs in the GRF.

<table>
<thead>
<tr>
<th>Functional pairs</th>
<th>$\frac{d_0 - d_{i-1}}{d_0 - d_i}$ of each functional pair</th>
</tr>
</thead>
<tbody>
<tr>
<td>FE pair 1 (A and B)</td>
<td>$x_i' = x_1 - x_5, y_i' = -y_3, z_i' = -z_5$</td>
</tr>
<tr>
<td>FE pair 2 (B' and B)</td>
<td>$x_i' = x_1 - x_5, y_i' = -y_3, z_i' = -z_5$</td>
</tr>
<tr>
<td>FE pair 3 (B and C')</td>
<td>$x_i' = x_2 - x_5, y_i' = y_2 - y_3, z_i' = z_2 - z_3$</td>
</tr>
<tr>
<td>FE pair 4 (C' and C)</td>
<td>$x_i' = x_2 - x_5, y_i' = y_2 - y_3, z_i' = z_2 - z_3$</td>
</tr>
<tr>
<td>FE pair 5 (C and D')</td>
<td>$x_i' = x_2 - x_5, y_i' = y_2 - y_3, z_i' = z_2 - z_3$</td>
</tr>
<tr>
<td>FE pair 6 (D' and D)</td>
<td>$x_i' = x_2 - x_5, y_i' = y_2 - y_3, z_i' = z_2 - z_3$</td>
</tr>
<tr>
<td>FE pair 7 (D and E)</td>
<td>$x_i' = x_4 - x_5, y_i' = y_4 - y_5, z_i' = z_4 - z_5$</td>
</tr>
<tr>
<td>FE pair 8 (G and F)</td>
<td>$x_i' = 0, y_i' = 0, z_i' = 0$</td>
</tr>
</tbody>
</table>
For FE pairs 1 and 2, they have the same \( \frac{d_n}{d_{n-1}} \); this situation holds for FE pairs 3 and 4 and FE pairs 5 and 6. Therefore, the Jacobian matrix can be shortened. To establish this situation holds for FE pairs 3 and 4 and FE pairs 5 and 6.

The overall Jacobian model is

\[
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\phi_1 \\
\phi_2 \\
\phi_3 \\
\end{bmatrix} = J \begin{bmatrix}
\text{FE}_1 \\
\text{FE}_2 \\
\text{FE}_3 \\
\text{FE}_4 \\
\text{FE}_5 \\
\text{FE}_6 \\
\end{bmatrix}
\]

(23)

In this case, \( \delta_1, \delta_2, \) and \( \delta_3 \) are the kinematic tolerance changes in the three directions of FR. Calculating the Jacobian model, the stack-up function of three tolerance changes can be formalized as

\[
\delta_1 = (\Delta_{AB} + \Delta r_B) \cdot \cos \theta_{x_1} + (x_3 - x_1) \cdot \Delta y_{23} + (\Delta \beta_{22} + (x_3 - x_1)) \cdot \cos \theta_{x_2} + (y_2 - y_3) \cdot \Delta \beta_{42} + (x_3 - x_1) \cdot \Delta y_{43} + (\Delta \beta_6 + \Delta \beta_{62}) + \Delta \beta_{63} + \Delta \beta_{64} + \Delta \beta_{65} \cdot \cos \theta_{z_5},
\]

\[
\delta_2 = (\Delta_{AB} + \Delta r_B) \cdot \cos \theta_{y_1} + (x_1 - x_2) \cdot \Delta \alpha_{21} + z_5 \cdot \Delta \beta_{22} + (\Delta \beta_{42} + (z_5 - z_5)) \cdot \Delta \beta_{42} + (\Delta \beta_{62} + \Delta \beta_{64}) + \Delta \beta_{65} + \Delta \beta_{66} \cdot \cos \theta_{z_5},
\]

\[
\delta_3 = (\Delta_{AB} + \Delta r_B) \cdot \cos \theta_{z_1} + y_5 \cdot \Delta \alpha_{21} + (\Delta \beta_{22} + (\Delta \beta_{42} + (\Delta \beta_{44} + \Delta \beta_{46})) \cdot \cos \theta_{y_2} + (y_5 - y_3) \cdot \Delta \beta_{23} + (\Delta \beta_{43} + \Delta \beta_{44}) + \Delta \beta_{45} + \Delta \beta_{46} \cdot \cos \theta_{z_5},
\]

(24)

Letting \( \theta_{z_5} \) change from \(-89^\circ\) to \(-10^\circ\), calculations are taken every \(0.3^\circ\). For the tolerances of the landing gear retraction mechanism, the Monte Carlo method is used here, and the test number is chosen as 100,000. Mean offset values are plotted in Figures 15–17.

Figures 15–17 display the mean offset values for \( \delta_1, \delta_2, \) and \( \delta_3 \). They change as the input variable \( \theta_{z_5} \) changes. Although small fluctuations exist throughout the process, the overall trend can be observed.

In Figure 15, mean offset values for \( \delta_1 \) increase from first to last as the input variable \( \theta_{z_5} \) increases. However, the absolute values for \( \delta_1 \) decrease as \( \theta_{z_5} \) increases, they drop from 0.04 to 0, and then increase to 0.01. In the beginning of the curve, the fluctuations are comparatively larger.

As shown in Figure 16, when the input variable \( \theta_{z_5} \) is within the range of \(-90^\circ\) to \(-70^\circ\), mean offset values have no significant change for \( \delta_2 \). When \( \theta_{z_5} \) is \(-70^\circ\), they begin to increase gradually from 0.035 to 0.07. From the beginning to the end, the fluctuations have no apparent change.

Looking at Figure 17, at the very beginning of the curve, no significant change can be observed. Then, the mean values for \( \delta_3 \) increase gradually and continually. Meanwhile, the absolute values for \( \delta_3 \) decrease first and then increase. It can be seen from the curve that the fluctuations are comparatively smaller when the input variable \( \theta_{z_5} \) is within the range of \(-90^\circ\) to \(-50^\circ\).

Similarly, major trends are also easily observed using the kinematic Jacobian model in this three-dimensional case, implying that this model can show the trends of the mechanism more distinctly.
6. Conclusion

In this article, the idea of kinematic tolerance analysis was advanced, and a kinematic Jacobian model was built up. Two study cases are given as examples to expand traditional static tolerance analysis to kinematic tolerance analysis. Major contributions and comments are summarized as follows:

1. By combining an original tolerance Jacobian model and a traditional dimensional chain, a kinematic Jacobian model is established. The kinematic movement error resulting from the tolerance design can be obtained through the analysis of the kinematic Jacobian model.

2. An elementary crank-slider was used to demonstrate the proposed methodology. Meanwhile, a physical model was also built up, and simulations were performed. Using the final results, a comparison between the statistical model and physical model was made. The comparison indicates the rationality, intrinsic logic, and its validity.

3. A three-dimensional landing gear retraction mechanism is also taken as an example to illustrate the theory introduced. Two examples simultaneously show that the Jacobian model can exhibit the inner regularity between the input variable and the movement accuracy more easily.

4. In general, the proposed kinematic accuracy method is plain and distinct from the computational point of view. Utilizing this method, joint clearances resulting from the tolerance design and tolerance of components are modeled by small-displacement screws; their effects on the final movement error can be estimated. This method can strongly enhance the efficiency of kinematic tolerance analysis.

5. This work presents the key idea and the first step of expanding traditional static tolerance analysis to kinematic tolerance analysis. Future work aims to further modify the existing Jacobian model and make it more suitable to consider variables (such as geometrical tolerance and angle tolerance) in the kinematic analysis.

Data Availability

The data supporting the conclusions of this study are included within the article, and these are also available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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