Research Article

Positive Definite Advection Transport Algorithm for Conservation Law Equations on Nonuniform Irregular Grids

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1. Introduction

Numerical modeling of atmospheric phenomena often necessitates solving the advection equation for positive definite scalar functions. Owing to the large spatial variation range, a large gradient and strong discontinuity exist in the distribution of these scalar fields [1]. These two factors lead to oscillation and negative values and cause unacceptable dispersion errors in the solution of ordinary advection transport schemes. These problems typically destroy the inherent spatial distribution and conservation of scalar substances such as water vapor. In addition, the positive definite scheme [2] is also of great significance in the calculation of advection transport of atmospheric aerosols and pollutants.

Smolarkiewicz introduced a predictor-iterative-corrector scheme in upwind schemes and proposed the multidimensional positive definite advection transport algorithm (MPDATA) [3, 4] for approximating the advection terms in fluid equations. MPDATA is second-order accurate, positive definite, conservative, and computationally efficient. Since the first article in 1983 [3], the MPDATA has developed into a calculation method of multiple equations and global atmospheric model schemes [5]. However, its theoretical core is still a continuation of the ideas of 1983. Through numerical experiments, we found that the existing MPDATA for the nonuniform grid domains will be a loss of accuracy. In fact, for the nonuniform grid, many existing numerical schemes cannot achieve the design accuracy. In [6], Xie studies such precision problems. Numerical solutions on spheres are the key to developing global atmospheric models. In particular, uniform mesh distribution for spherical atmospheric dynamics simulation does not exist.

Therefore, we hope to improve the numerical precision of the MPDATA on the nonuniform grid. The new MPDATA has two differences from the former one. On the
one hand, it is the calculation of boundary derivative. Here, we use the multipoint Taylor expansion method to calculate the boundary derivative, which will be introduced in Section 2. On the other hand, the upwind format of the former MPDATA scheme is replaced with a suitable form for the nonuniform grid. In recent years, Kurganov et al. integrated over nonuniform polygonal control volumes of different shapes and derived a fully discrete central-upwind scheme to present a family of central-upwind schemes on general triangulations. Those schemes are simpler than their upwind counterparts, have a genuinely multidimensional structure, and can be used to solve problems on irregular domains [7, 8]. Based on the work of Kurganov et al., we will improve the MPDATA for the nonuniform grid.

In Section 2, we will analyze the accuracy of the former MPDATA numerical scheme and propose our numerical precision solution. Furthermore, numerical simulation is employed to verify the result that forms the conclusion of this study.

2. Numerical Methods

2.1. Analysis of the Previous MPDATA. The irregular meshes discussed in this paper all refer to nonuniform irregular meshes. In this section, we analyze the reasons why previous MPDATA methods are not applicable to irregular grids. The one-dimensional advection equation is

\[
\frac{\partial}{\partial t} \psi + \frac{\partial}{\partial x} (u \psi) = 0, \tag{1}
\]

where \( \psi \) is a nondiffusive scalar quantity and \( u \) is the one-dimensional velocity. The former MPDATA scheme is proposed by Smolarkiewicz and Margolin [3], which is

\[
\psi_{i+1}^* = \psi_i^n - F(\psi_i^n, \psi_{i+1}^n, u_{i+1/2}^n) - F(\psi_{i-1}^n, \psi_i^n, u_{i-1/2}^n), \tag{2}
\]

\[
\psi_{i+1}^* = \psi_i^n - F(\psi_i^n, \psi_{i+1}^*, \bar{u}_{i+1/2}) - F(\psi_{i-1}^*, \psi_i^n, \bar{u}_{i-1/2}), \tag{3}
\]

where

\[
F(\psi_i^n, \psi_{i+1}^n, u_{i+1/2}^n) = \left( |u_{i+1/2}^n| \right) \psi_i^n + \left( |u_{i+1/2}^n - |u_{i+1/2}^n| \right) \psi_{i+1}^n \right] \frac{\Delta t}{2 \Delta x}.
\]

Here, \( \psi_i^n \) is the value of \( \psi \) at the \( i \)th grid point for the \( n \)th time step. \( \bar{u}_{i+1/2} \) is the numerical approximation of \( u_d \) at \( x_{i+1/2} \), and it is

\[
\bar{u}_{i+1/2} = \frac{\left[ |u_{i+1/2}^n| \Delta x - |u_{i+1/2}^n| \right] \left( \psi_{i+1}^n - \psi_i^n \right) + \epsilon}{\left( \psi_{i+1}^n + \psi_i^n \right) \Delta x} \tag{5}
\]

where \( \epsilon \) is a small value (e.g., \( 10^{-15} \)) to ensure \( \bar{u} = 0 \) when \( \psi_i^n = \psi_i^* = 0 \).

For a regular grid, the numerical schemes (2) and (3) are an effective way to obtain positive definite solutions (see Section 3.1). However, neither (2) nor (3) are able to provide the second-order numerical accuracy solutions for irregular grids. There are two reasons for this problem. The first is the compatibility of numerical discretion of the “diffusion velocity” \( u_d \). The second is the compatibility of the first-order numerical scheme. To overcome the first problem, we have the following propositions.

**Proposition 1.** In (5), the numerical discretation of \( \partial \psi / \partial x |_{i+1/2} \) is

\[
\frac{\partial \psi}{\partial x |_{i+1/2}} = \frac{\psi_{i+1}^* - \psi_i^n}{\Delta x} + O(\Delta x), \tag{6}
\]

for regular grids, and discretation (6) is the first order. For an irregular grid, the discretation of (6) prevents step (3) from reaching first order. As a result, the MPDATA schemes (2) and (3) cannot reach the second order.

**Proof.** For the irregular grid, discretation (6) is

\[
\frac{\partial \psi}{\partial x |_{i+1/2}} = \frac{\psi_{i+1}^* - \psi_i^n}{\Delta x} + C_0, \tag{7}
\]

where \( C_0 \) is a constant. This enables us to obtain the relationship between the numerical approximation values and the exact values:

\[
\psi_i^n = \psi(x_i) + O(\Delta x), \tag{8}
\]

\[
\bar{u} = u(x_i) + C_0. \tag{9}
\]

Substituting the abovementioned equations (8) and (9) into (4), we have

\[
F(\psi_{i+1}^*, \psi_{i+1}^*, \bar{u}_{i+1/2}) = \left[ |\bar{u}(x_{i+1/2}) + |\bar{u}(x_{i+1/2})| \right] \psi(x_i)
\]

\[
+ \left( |\bar{u}(x_{i+1/2}) - |\bar{u}(x_{i+1/2})| \right) \psi(x_{i+1})
\]

\[
+ \left( |\bar{u}(x_{i+1/2})| \Delta x + \psi(x_{i+1}) \right) \frac{\Delta t}{2 \Delta x},
\]

\[
\psi(x_{i+1})C_2 + O\left( \Delta x^3 \right) \frac{\Delta t}{2 \Delta x} \tag{10}
\]

where \( C_1 \) and \( C_2 \) are constant. Using the same method for \( \partial \psi / \partial x |_{i-1/2} \), we have

\[
F(\psi_i^n, \psi_{i+1}^*, \bar{u}_{i+1/2}^*) = F(\psi_{i+1}^*, \psi_{i+1}^*, \bar{u}_{i+1/2}^*) = F(\psi(x_i), \psi(x_{i+1}), \bar{u}(x_{i+1/2})) - F(\psi(x_{i+1}), \psi(x_i), \bar{u}(x_{i-1/2})
\]

\[
+ \left[ |\bar{u}(x_{i+1/2}) - |\bar{u}(x_{i-1/2})| \Delta t + C_1 \psi(x_{i-1}) + C_2 \psi(x_i) \right]
\]

\[
+ C_1 \psi(x_{i+1}) + O(\Delta x^3), \tag{11}
\]

where \( \bar{u}(x_{i+1/2}) = u(x_{i+1/2}) + \epsilon \).
where $C'_1$, $C'_2$, and $C'_3$ are constant. When $\Delta x \rightarrow 0$, 
\[ C'_1 \psi(x_{i-1}) + C'_2 \psi(x_i) + C'_3 \psi(x_{i+1}) \neq O(\Delta x). \tag{12} \]

Thus, step (3) cannot become first order, and it cannot calculate the affect of diffusion effectively. Thus, the numerical schemes (2) and (3) cannot become the second order.

To obtain accurate numerical schemes on an irregular grid, a new approximation method of $\frac{\partial \psi}{\partial x}|_{x_{i+1/2}}$ is given

\[ \begin{pmatrix} \ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} x_{i-1} - x_{i+1/2} \\ x_i - x_{i+1/2} \\ x_{i+1} - x_{i+1/2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{14} \]

It is easy to see that the rank of the coefficient matrix and the extended matrix for linear equations (11) are the same, and the linear equations (11) have infinite solutions. Here, the least-norm solutions are used (see Appendix). Then, we have the approximation for $\frac{\partial \psi}{\partial x}$:

\[ \frac{\partial \psi}{\partial x}|_{x_{i+1/2}} = \psi_{i-1} \cdot \kappa_{i+1/2}^{k'_1} + \psi_{i+1} \cdot \kappa_{i+1/2}^{k'_2} + \psi_{i+2} \cdot \kappa_{i+1/2}^{k'_3} + O(\Delta x). \tag{15} \]

Because $\bar{u}$ is a continuous function, we have

\[ \bar{C}_1 \psi(x_{i-1}) \Delta t + \bar{C}_2 \psi(x_i) \Delta t + \bar{C}_3 \psi(x_{i+1}) \Delta t \sim O(\Delta x)(\Delta x \rightarrow 0). \tag{18} \]

Thus, using (12) instead of (5) will enable step (3) to reach the first order, whereupon it would be able to calculate the affect of diffusion effectively. Furthermore, (2) and (3) can achieve second-order accuracy.

Before we begin the discussion for a two-dimensional situation, we have another proposition to state.

**Proposition 2.** The compatibility of the first-order numerical scheme is important for the MPDATA method.

Generally speaking, the MPDATA schemes (2) and (3) belong to the class of staggered central-upwind schemes, which are excellent tools for solving various complex problems on regular domains. However, in practice complicated geometries exist, and the use of irregular triangular meshes could be advantageous or even unavoidable [7]. Staggered central-upwind schemes are sensitive to mesh effects. For example, Küther proved that Lax–Friedrichs-type finite-volume schemes estimate the error to be of the order of 0.25 [9, 10].

**For the first-order scheme,**

\[ \psi_i^{n+1} = \psi_i^n - \{ F(\psi_i^n, \psi_{i+1}^n, u_{i+1/2}^n) - F(\psi_i^n, \psi_{i-1}^n, u_{i-1/2}^n) \}. \tag{19} \]

With the Taylor expansion $\psi_{i-1}$, $\psi_i$, $\psi_{i+1}$, and $\psi_{i+2}$ at the point $\psi_{i+1/2}$, we have

\[ F(\psi_i, \psi_{i+1}, u_{i+1/2}) - F(\psi_{i-1}, \psi_{i-1/2}, u_{i-1/2}) = (u_{i+1/2} - u_{i-1/2}) \frac{\Delta t}{4\Delta x} \frac{\partial \psi}{\partial x} \left[ (\Delta x_i + \Delta x_{i+1})(u_{i+1/2} - u_{i+1/2}) + (\Delta x_i + \Delta x_i)(u_{i-1/2} - u_{i-1/2}) \right] + O(\Delta x^2). \tag{20} \]
Using this information, the control volume can be constructed as illustrated in Figure 2.

Based on the abovementioned equation, it is clear that if the mesh is distorted, a few points are incompatible. When \( u \) is not a discontinuous function, equation (21) implies that scheme (21) is largely incompatible.

We computed the irregular grid by carrying out a comparative simulation in Section 6.1. We results in Tables 1–3 indicate that the standard MPDATA method reduces the order when calculated on an irregular grid.

2.2. New MPDATA for Irregular Grids in One-Dimensional Problems. Based on the abovementioned analysis, we propose a new MPDATA method for irregular grids in the one-dimensional situation. One of the grid elements being studied is denoted as \( T_j \). After time step \( \Delta t \), the discontinuity around \( x_{j+1/2} \) can be described as \( a_{j+1/2}^m \Delta t \) and \( a_{j+1/2}^n \Delta t \) for inward \( T_j \), where \( a_{j+1/2}^m \Delta t \) is around \( x_{j+1/2} \) and \( a_{j+1/2}^n \Delta t \) is around \( x_{j+1/2} \). For outward, \( a_{j-1/2}^n \Delta t \) is around \( x_{j-1/2} \) and \( a_{j-1/2}^m \Delta t \) is around \( x_{j-1/2} \). Using this information, the control volume can be constructed as follows. As shown in Figure 2, the inside domain \( D_j \) is the boundary domain \( D_{j+1/2} \), which can be divided into the inward domain \( D_{j+1/2}^m \) and \( D_{j+1/2}^n \) (the control volume can be described as that illustrated in Figure 2).

Let \( u_j^i \) be the velocity in domain \( D_{j-1/2}^i \) and \( u_j^R \) be the velocity in domain \( D_{j+1/2}^R \). For the advection equations (1),

\[
\frac{\partial \psi}{\partial t} + \frac{\partial (\psi u)}{\partial x} = 0
\]

On the other hand, we can have the relationship

\[
\frac{1}{|T_j|} \int_{T_j} \psi(x, t') \, dx = \int_{T_j} \psi^n(x) \, dx
\]

The time derivative of \( \psi_j(t) \) is expressed with the help of (24) as

\[
\frac{d}{dt} \psi_j(t) = \lim_{\Delta t \to 0} \frac{\psi_{j+1} - \psi_j}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{|T_j|} \int_{T_j} \psi_{j+1} - \psi_j \, dx
\]

Because the width of the Riemann fans is bounded by \( (a_{j+1/2}^n + a_{j+1/2}^m) \Delta t \) or \( (a_{j-1/2}^n + a_{j-1/2}^m) \Delta t \), we obtain

**Table 1:** Former MPDATA for regular grids.

<table>
<thead>
<tr>
<th>N</th>
<th>( L^1 ) order</th>
<th>( L^2 ) order</th>
<th>( L^3 ) order</th>
<th>( L^4 ) order</th>
<th>( L^5 ) order</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>4.33E − 02</td>
<td>—</td>
<td>3.330E − 02</td>
<td>—</td>
<td>3.31E − 02</td>
</tr>
<tr>
<td>32</td>
<td>1.115E − 02</td>
<td>1.96</td>
<td>8.513E − 03</td>
<td>1.97</td>
<td>9.022E − 03</td>
</tr>
<tr>
<td>64</td>
<td>2.717E − 03</td>
<td>2.04</td>
<td>2.132E − 03</td>
<td>2.00</td>
<td>2.545E − 03</td>
</tr>
<tr>
<td>128</td>
<td>6.648E − 04</td>
<td>2.03</td>
<td>5.280E − 04</td>
<td>2.01</td>
<td>6.412E − 04</td>
</tr>
<tr>
<td>256</td>
<td>1.640E − 04</td>
<td>2.02</td>
<td>1.308E − 04</td>
<td>2.01</td>
<td>1.596E − 04</td>
</tr>
</tbody>
</table>

**Table 2:** Former MPDATA for irregular grids.

<table>
<thead>
<tr>
<th>N</th>
<th>( L^1 ) order</th>
<th>( L^2 ) order</th>
<th>( L^3 ) order</th>
<th>( L^4 ) order</th>
<th>( L^5 ) order</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>2.181E − 01</td>
<td>—</td>
<td>1.834E − 01</td>
<td>—</td>
<td>2.471E − 01</td>
</tr>
<tr>
<td>32</td>
<td>6.887E − 02</td>
<td>1.66</td>
<td>6.200E − 02</td>
<td>1.57</td>
<td>1.070E − 01</td>
</tr>
<tr>
<td>64</td>
<td>2.188E − 02</td>
<td>1.65</td>
<td>2.176E − 02</td>
<td>1.51</td>
<td>4.905E − 02</td>
</tr>
<tr>
<td>128</td>
<td>7.550E − 03</td>
<td>1.53</td>
<td>8.901E − 03</td>
<td>1.29</td>
<td>2.385E − 02</td>
</tr>
<tr>
<td>256</td>
<td>3.099E − 03</td>
<td>1.28</td>
<td>4.125E − 03</td>
<td>1.11</td>
<td>1.814E − 02</td>
</tr>
</tbody>
</table>

The discontinuity propagate with inward and outward \( T_j \) local speeds, which can be estimated by

\[
\begin{align*}
\alpha_{in}^m &= -\min\left\{-u_j^{i-1}, -u_j^i, 0\right\}, \\
\alpha_{in}^n &= -\min\left\{u_j^{i+1}, u_j^i, 0\right\}, \\
\alpha_{out}^m &= \max\left\{-u_j^{i-1}, -u_j^i, 0\right\}, \\
\alpha_{out}^n &= \max\left\{u_j^{i+1}, u_j^i, 0\right\}.
\end{align*}
\]
\[
\psi^{n+1}_{j+1/2}(x) = \psi^{n+1}_{j+1/2} + O(\Delta t),
\]

(28)

\[
\frac{d}{dt}\psi_j(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{D_{j+1/2}} \psi^{n+1}_j(x)dx + \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{D_j} \psi^{n+1}_j(x)dx
\]

+ \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{D_{j+1/2}} \psi^n_j(x)dx - \lim_{\Delta t \to 0} \frac{1}{\Delta t} \psi^n_j
\]

(29)

With the help of (24) and (25), we obtain

\[
\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left( \frac{D_j}{T_j} \psi^n_j - \psi^n_j \right)
\]

(30)

\[
= -\frac{a^{in}_j \psi^n_j + a^{in}_j \psi^n_j}{T_j} - \frac{(u^n_j - u^n_j)\psi^n_j}{T_j} + O(\Delta x),
\]

and

\[
\lim_{\Delta t \to 0} \psi^{n+1}_{j+1/2} = \frac{a^{in}_j \psi^n_j + a^{in}_j \psi^n_j}{a^{in}_j + a^{in}_j} - \frac{-u^n_j \psi^n_j + u^n_{j+1} \psi^n_j}{a^{in}_j + a^{in}_j} + O(\Delta x).
\]

(31)

With the help of the forward Euler method, the substitution of (30) and (31) in (29) results in the new first-order scheme:

\[
\psi^{n+1}_j = \psi^n_j - \frac{\Delta t}{\Delta x_j} \left[ H_{j+1/2}(\psi_j, \psi_{j+1}, u) - H_{j-1/2}(\psi_{j-1}, \psi_j, u) \right],
\]

(32)

where

\[
H_{j+1/2} = a^{out}_{j+1/2} \frac{\psi^n_{j+1/2}}{a^{out}_{j+1/2} + a^{out}_{j+1/2}} - a^{out}_{j+1/2} \frac{\psi^n_{j+1/2}}{a^{out}_{j+1/2} + a^{out}_{j+1/2}} (\psi^n_{j+1/2} - \psi^n_{j+1/2}),
\]

(33)

Using the Taylor expansion \(x_{j+1}, x_j\) at \(x_{j+1/2}\)

\[
\psi^n_{j+1} = \psi^n_{j+1/2} + \frac{1}{2} \Delta x_{j+1/2} \frac{\partial}{\partial x} \psi^n_{j+1/2} + O(\Delta x^2),
\]

(34)

and \(x_{j-1}, x_{j-1/2}\)

\[
\psi^n_{j-1} = \psi^n_{j-1/2} - \frac{1}{2} \Delta x_{j-1/2} \frac{\partial}{\partial x} \psi^n_{j-1/2} + O(\Delta x^2),
\]

(35)

Taking those Taylor expressions (34)–(37) into \(H_{j+1/2}\) and \(H_{j-1/2}\), we can obtain
\[ H_{j+1/2} = a_{JR}^{out} u_j^R + a_{JR}^{in} u_{j+1}^L \psi_{j+1/2}^{n+1/2} \]
\[ + \left[ \frac{-a_{JR}^{out} u_j^R \Delta x_j + a_{JR}^{in} u_{j+1}^L \Delta x_{j+1}}{a_{JR}^{out} + a_{JR}^{in}} \frac{1}{2} - \frac{\Delta x_{j+1} + \Delta x_j}{2} \frac{a_{JR}^{out} a_{JR}^{in}}{a_{JR}^{out} + a_{JR}^{in}} \right] \frac{\partial}{\partial x} \psi_{j+1/2}^{n} + O(\Delta x^2), \]
\[ H_{j-1/2} = a_{ JL}^{out} u_j^L + a_{ JL}^{in} u_{j-1}^R \psi_{j-1/2}^{n+1/2} \]
\[ + \left[ \frac{a_{ JL}^{out} u_j^L \Delta x_j - a_{ JL}^{in} u_{j-1}^R \Delta x_{j-1}}{a_{ JL}^{out} + a_{ JL}^{in}} \frac{1}{2} - \frac{\Delta x_{j-1} + \Delta x_j}{2} \frac{a_{ JL}^{out} a_{ JL}^{in}}{a_{ JL}^{out} + a_{ JL}^{in}} \right] \frac{\partial}{\partial x} \psi_{j-1/2}^{n} + O(\Delta x^2). \]

With the Taylor expansions on time,
\[ \psi_{j+1/2}^{n+1} = \psi_j^{n+1/2} + \frac{1}{2} \frac{\partial \psi}{\partial t} \bigg|_j \Delta t^{n+1/2} + O(\Delta t^2), \]
\[ \psi_{j-1/2}^{n+1/2} = \psi_j^{n+1/2} - \frac{1}{2} \frac{\partial \psi}{\partial t} \bigg|_j \Delta t^{n+1/2} + O(\Delta t^2) \]
\[ \psi_{j+1/2}^{n} = \psi_j^{n+1/2} + \frac{1}{2} \Delta t \frac{\partial \psi}{\partial x} \bigg|_j + O(\Delta t^2). \]

Scheme (32) becomes
\[ \frac{\partial \psi_{j+1/2}^{n+1}}{\partial t} = \frac{1}{\Delta x_j} \left( \frac{a_{JR}^{out} u_j^R + a_{JR}^{in} u_{j+1}^L}{a_{JR}^{out} + a_{JR}^{in}} \psi_{j+1/2}^{n+1/2} \right) - \frac{a_{ JL}^{out} u_j^L + a_{ JL}^{in} u_{j-1}^R}{a_{ JL}^{out} + a_{ JL}^{in}} \psi_{j-1/2}^{n+1/2} + \frac{1}{\Delta x_j} \bar{E}, \]

where
\[ \bar{E} := \bar{u}_{j+1/2} \psi_{j+1/2}^{n+1/2} - \bar{u}_{j-1/2} \psi_{j-1/2}^{n+1/2}, \]
\[ \bar{u}_{j+1/2} := \frac{1}{\psi_{j+1/2}^{n+1/2}} \left[ \frac{a_{JR}^{out} u_j^R + a_{JR}^{in} u_{j+1}^L}{a_{JR}^{out} + a_{JR}^{in}} \frac{1}{2} \Delta t \frac{\partial}{\partial x} (u \psi_{j+1/2}^{n+1/2}) \right] + \frac{1}{\psi_{j+1/2}^{n+1/2}} \left[ \frac{-a_{JR}^{out} u_j^R \Delta x_j + a_{JR}^{in} u_{j+1}^L \Delta x_{j+1}}{a_{JR}^{out} + a_{JR}^{in}} \frac{1}{2} - \frac{\Delta x_{j+1} + \Delta x_j}{2} \frac{a_{JR}^{out} a_{JR}^{in}}{a_{JR}^{out} + a_{JR}^{in}} \right] \frac{\partial}{\partial x} \psi_{j+1/2}^{n}, \]
\[ \bar{u}_{j-1/2} := \frac{1}{\psi_{j-1/2}^{n+1/2}} \left[ \frac{a_{ JL}^{out} u_j^L + a_{ JL}^{in} u_{j-1}^R}{a_{ JL}^{out} + a_{ JL}^{in}} \frac{1}{2} \Delta t \frac{\partial}{\partial x} (u \psi_{j-1/2}^{n+1/2}) \right] + \frac{1}{\psi_{j-1/2}^{n+1/2}} \left[ \frac{a_{ JL}^{out} u_j^L \Delta x_j - a_{ JL}^{in} u_{j-1}^R \Delta x_{j-1}}{a_{ JL}^{out} + a_{ JL}^{in}} \frac{1}{2} - \frac{\Delta x_{j-1} + \Delta x_j}{2} \frac{a_{ JL}^{out} a_{ JL}^{in}}{a_{ JL}^{out} + a_{ JL}^{in}} \right] \frac{\partial}{\partial x} \psi_{j-1/2}^{n}. \]

When \( u \) is continuous around \( x_{j-1/2} \) and \( x_{j+1/2} \), noticing one of \( a_{out} \) or \( a_{in} \) is zero. Then,
\[ \frac{a_{JR}^{out} u_j^R + a_{JR}^{in} u_{j+1}^L}{a_{JR}^{out} + a_{JR}^{in}} \Rightarrow u_{j+1/2}, \]
\[ \frac{a_{ JL}^{out} u_j^L + a_{ JL}^{in} u_{j-1}^R}{a_{ JL}^{out} + a_{ JL}^{in}} \Rightarrow u_{j-1/2}. \]
Thus, equation (40) can be viewed as
\[ \frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial x} + \frac{\partial \bar{u} \psi}{\partial x}. \]

Here, the second term of (43) is the "pseudoadvection term." (32) is the second-order scheme of equation (43). Therefore, we have the new MPDATA for irregular grids:
\[ \psi_{j+1/2} = \psi_j - \frac{1}{\Delta x_j} \left( H_{j+1/2}(\psi_j, \psi_{j+1}, u) - H_{j-1/2}(\psi_j, \psi_{j-1}, u) \right) \frac{\Delta t}{\Delta x_j}, \]
\[ \psi_{j-1/2} = \psi_j - \frac{1}{\Delta x_j} \left( H_{j+1/2}(\psi_j, \psi_{j+1}, u) - H_{j-1/2}(\psi_j, \psi_{j-1}, u) \right) \frac{\Delta t}{\Delta x_j}, \]

Because the new MPDATA scheme depends on the two-step scheme, we can analyze the positivity of (32) instead of
the schemes (44) and (45). Thus, we have the following claim for the new MPDATA schemes (44) and (45).

**Theorem 1.** Consider system (1) and the central-upwind discrete scheme (29), where \(a_{i+1/2}^{in}, a_{i+1/2}^{out}, a_{j+1/2}^{out} \) and \(a_{j}^{out} \) are defined in (22) and (23). Then, for all \( j \), \( \bar{\psi}_{j}^{n+1} \geq 0 \), provided that \( \Delta t \leq \Delta x/\alpha \), where \( \Delta x := \min_{j} \Delta x_{j} \) and \( \alpha := \max \{ \max \{ u_{j+1}, u_{j} \} \} \).

\[
\bar{\psi}_{j+1} = \bar{\psi}_{j} - \frac{\Delta t}{\Delta x_{j}} \left[ H_{j+1/2} \left( \bar{\psi}_{j}, \bar{\psi}_{j+1}, u_{j} \right) - H_{j-1/2} \left( \bar{\psi}_{j-1}, \bar{\psi}_{j}, u_{j} \right) \right]
\]

\[
= \left[ \frac{1}{2} - \lambda_{j} a_{j}^{out} \left( \frac{a_{j}^{in} + u_{j}^{R}}{a_{j}^{in} + a_{j}^{out}} \right) \right] \bar{\psi}_{j}^{n} + \left[ \frac{1}{2} + \lambda_{j} a_{j}^{out} \left( \frac{a_{j}^{in} - u_{j}^{L}}{a_{j}^{in} + a_{j}^{out}} \right) \right] \bar{\psi}_{j+1}^{n} + \lambda_{j} a_{j}^{in} \left( \frac{u_{j}^{R} - a_{j}^{out}}{a_{j}^{in} + a_{j}^{out}} \right) \bar{\psi}_{j}^{n-1}.
\]

(47)

In the above equation (47), under \( \bar{\psi}_{j}^{n} \geq 0 \), it is easy to see that

\[
\left[ \frac{1}{2} - \lambda_{j} a_{j}^{out} \left( \frac{a_{j}^{in} + u_{j}^{R}}{a_{j}^{in} + a_{j}^{out}} \right) \right] \bar{\psi}_{j}^{n} \geq 0,
\]

\[
\lambda_{j} a_{j}^{in} \left( \frac{u_{j}^{R} - a_{j}^{out}}{a_{j}^{in} + a_{j}^{out}} \right) \bar{\psi}_{j}^{n-1} \geq 0,
\]

\[
\left[ \frac{1}{2} + \lambda_{j} a_{j}^{out} \left( \frac{a_{j}^{in} - u_{j}^{L}}{a_{j}^{in} + a_{j}^{out}} \right) \right] \bar{\psi}_{j+1}^{n} \geq 0,
\]

\[
\left[ \frac{1}{2} - \lambda_{j} a_{j}^{out} \left( \frac{a_{j}^{in} + u_{j}^{R}}{a_{j}^{in} + a_{j}^{out}} \right) \right] \bar{\psi}_{j}^{n} \geq 0, \quad \text{when} \quad \lambda_{j} \leq \frac{1}{2 a_{j}^{out}}
\]

Thus, we can claim that, for all \( j \), \( \bar{\psi}_{j}^{n+1} \geq 0 \), provided that \( \Delta t \leq \Delta x/\alpha \), where \( \Delta x := \min \{ \Delta x_{j} \} \) and \( \alpha := \max \{ \max \{ u_{j+1}, u_{j} \} \} \).

2.3. New MPDATA for Irregular Grids in a Two-Dimensional Problem. For the two-dimensional advection problem,

\[
\frac{\partial \psi}{\partial t} = -v \cdot (\nabla \psi),
\]

(49)

where \( \psi(x, t) \) is a nondiffusive scalar field assumed to be nonnegative at \( t = 0 \), and the \( v = v(x) \) is the prescribed flow. The finite-volume MPDATA is considered on equation (48). A triangular partition \( T_{j} \) is used on the computational domain. The area of \( T_{j} \) is \( |T_{j}| \). All domains around the triangular cell \( T_{j} \) are illustrated in Figure 3. The neighborhood cell near \( T_{j} \) is denoted as \( T_{j+k}, k = 1, 2, 3 \). The three edges of cell \( T_{j} \) are denoted as \( S_{j+i} \). The discontinuous long edges \( S_{j+k} \) can move inward, in which case \( T_{j} \) is denoted as \( a_{j}^{in} \Delta t \), and outward such that \( T_{j} \) is represented as \( a_{j}^{out} \Delta t \). The inside domain is \( D_{j} \). The three nonconvex domains surrounding the corners are \( E_{jk}, k = 1, 2, 3 \).

\[
\frac{\bar{\psi}_{j}^{n+1} - \bar{\psi}_{j}^{n}}{\Delta t} = \frac{1}{|T_{j}|} \sum_{k=1}^{3} h_{jk} \left( a_{jk}^{in} \bar{v}_{jk} \bar{\psi}_{jk}^{n} + a_{jk}^{out} \bar{v}_{jk} \bar{\psi}_{jk}^{n} \right) \cdot \mathbf{n}_{jk}
\]

\[
+ \frac{1}{|T_{j}|} \sum_{k=1}^{3} h_{jk} a_{jk}^{in} a_{jk}^{out} \left( \bar{\psi}_{jk}^{n} - \bar{\psi}_{jk}^{n+1} \right),
\]

(52)

where \( \bar{v}_{jk}^{n} \) and \( \bar{v}_{jk}^{n+1} \) are advection velocities in domain \( D_{jk}^{i} \) and domain \( D_{jk}^{o} \). Here, (52) is a first-order scheme. Then, expanding \( \psi_{jk} \) and \( \psi_{jk}^{n} \) at the middle point of the edge intersecting face \( S_{jk} \),

\[
\psi_{jk} = \psi_{S_{j}} \left( \theta_{jk} \psi_{S} \right) + O(\Delta x^{2} + \Delta y^{2}),
\]

(53)

where \( \theta_{jk} = (x_{jk} - x_{S})/(\partial x) + (y_{jk} - y_{S})/(\partial y) \).
\[ \nabla \psi = \psi_{S_j} + \left( \psi_T \right)_{S_j} + O(\Delta x^2 + \Delta y^2), \quad (54) \]

where \( \psi = (x_j - x_k)(\partial / \partial x) + (y_j - y_k)(\partial / \partial y) \). We substitute (53) and (54) into (52), and we have

\[
\begin{align*}
\frac{\psi^{n+1} - \psi^n}{\Delta t} &= -1 \sum_{j=1}^3 h_{jk} \left( \frac{a_{jk}^n \nabla \psi(n) \left| n \right| + a_{jk}^{out} \nabla \psi(n) \left| n \right|}{a_{jk} + a_{jk}^{out}} \right) \cdot n_{jk} \\
&+ \frac{1}{|T_j|} \sum_{j=1}^3 h_{jk} \left( \frac{a_{jk}^n \nabla \psi(n) \left| n \right| + a_{jk}^{out} \nabla \psi(n) \left| n \right|}{a_{jk} + a_{jk}^{out}} \right) \cdot n_{jk} \\
&+ \frac{1}{|T_j|} \sum_{j=1}^3 h_{jk} \left( \frac{a_{jk}^n \nabla \psi(n) \left| n \right| + a_{jk}^{out} \nabla \psi(n) \left| n \right|}{a_{jk} + a_{jk}^{out}} \right) \cdot n_{jk} \\
&+ \frac{1}{|T_j|} \sum_{j=1}^3 h_{jk} \left( \frac{a_{jk}^n \nabla \psi(n) \left| n \right| + a_{jk}^{out} \nabla \psi(n) \left| n \right|}{a_{jk} + a_{jk}^{out}} \right) \cdot n_{jk}
\end{align*}
\]

Expanding \( \psi^{n+1} \) and \( \psi^n \) at \( \psi^{n+1/2} \),

\[
\begin{align*}
\psi^{n+1} &= \psi^{n+1/2} + \frac{1}{2} \frac{\partial \psi}{\partial t} \Delta t + O(\Delta t^2), \\
\psi^n &= \psi^{n+1/2} - \frac{1}{2} \frac{\partial \psi}{\partial t} \Delta t + O(\Delta t^2) \\
&= \psi^{n+1/2} + \frac{1}{2} \Delta t (\psi \nabla \psi)^{n+1/2} + \frac{1}{2} \Delta t (\nabla \psi) \psi^{n+1/2} + O(\Delta t^2).
\end{align*}
\]

Then, we have the relationship

\[
\begin{align*}
\frac{\partial \psi}{\partial t} &= -1 \sum_{j=1}^3 h_{jk} \left( \frac{a_{jk}^n \nabla \psi(n) \left| n \right| + a_{jk}^{out} \nabla \psi(n) \left| n \right|}{a_{jk} + a_{jk}^{out}} \right) \cdot n_{jk} + \text{Error},
\end{align*}
\]

where

\[
\text{Error} = -1 \sum_{j=1}^3 h_{jk} \left[ \frac{a_{jk}^n \nabla \psi(n) \left| n \right| + a_{jk}^{out} \nabla \psi(n) \left| n \right|}{a_{jk} + a_{jk}^{out}} \frac{1}{2} \Delta t (\psi \nabla \psi) \left| S_j \right| \right] \cdot n_{jk} \\
&- \frac{1}{|T_j|} \sum_{j=1}^3 h_{jk} \left[ \frac{a_{jk}^n \nabla \psi(n) \left| n \right| + a_{jk}^{out} \nabla \psi(n) \left| n \right|}{a_{jk} + a_{jk}^{out}} \frac{1}{2} \Delta t (\nabla \psi) \left| S_j \right| \right] \cdot n_{jk} \\
&- \frac{1}{|T_j|} \sum_{j=1}^3 h_{jk} \left[ \frac{a_{jk}^n \nabla \psi(n) \left| n \right| + a_{jk}^{out} \nabla \psi(n) \left| n \right|}{a_{jk} + a_{jk}^{out}} \right] \cdot n_{jk} \\
&+ \frac{1}{|T_j|} \sum_{j=1}^3 h_{jk} \left[ \frac{a_{jk}^n a_{jk}^{out}}{a_{jk} + a_{jk}^{out}} \left( \psi \nabla \psi \right) \left| S_j \right| - \left( \psi \nabla \psi \right) \left| S_j \right| \right] \cdot n_{jk}. 
\]
Noticing that
\[
\frac{a_{\text{in}} v_{jk} + a_{\text{out}} v_{jk}}{a_{\text{in}} + a_{\text{out}}} \rightarrow \mathbf{v}|_{S_j},
\]  
(59)

(59) can be understood as the average value of \( \mathbf{v} \) in the small rectangle \( D_{jk} \) (see Figure 3). Furthermore, we can assume \( t = n\Delta t \), and (57) can be written as

\[
\frac{\partial}{\partial t} v_{jk} := -\frac{1}{\psi} \left( a_{\text{in}} v_{jk} + a_{\text{out}} v_{jk} \right) \nabla \cdot \mathbf{v} |_{S_j} + \frac{1}{\psi} \nabla \cdot (v_{jk} \nabla \psi) |_{S_j} + \frac{1}{\psi} \left( a_{\text{in}} a_{\text{out}} \right) \nabla \cdot (\nabla \psi) |_{S_j}.
\]  
(61)

Here, the gradient \( \nabla \psi \) at the edges can be obtained by the combination of the physical values around the edge. The mesh structure, which is used to obtain the gradient \( \nabla \psi |_{S_j} \), around the midpoint of an edge \( S_{ji} \) is described in Figure 4.

Here, we use \( \psi_{j1} \) to represent the physical value at point \( M(j1) \). Using the Taylor expression, expand \( \nabla \psi, \nabla j1, \nabla j2, \nabla j3, \nabla j11, \) and \( \nabla j12 \) at point \( M(j1) \) to second-order, and we have

\[
\begin{align*}
\nabla \psi_{j1} &= \psi_{S_j} + (x_{j1} - x_{S_j}) \frac{\partial}{\partial x} \psi_{S_j} + (y_{j1} - y_{S_j}) \frac{\partial}{\partial y} \psi_{S_j} + O(\Delta x^2 + \Delta y^2), \\
\nabla \psi_{j2} &= \psi_{S_j} + (x_{j2} - x_{S_j}) \frac{\partial}{\partial x} \psi_{S_j} + (y_{j2} - y_{S_j}) \frac{\partial}{\partial y} \psi_{S_j} + O(\Delta x^2 + \Delta y^2), \\
\nabla \psi_{j3} &= \psi_{S_j} + (x_{j3} - x_{S_j}) \frac{\partial}{\partial x} \psi_{S_j} + (y_{j3} - y_{S_j}) \frac{\partial}{\partial y} \psi_{S_j} + O(\Delta x^2 + \Delta y^2), \\
\n\nabla \psi_{j11} &= \psi_{S_j} + (x_{j11} - x_{S_j}) \frac{\partial}{\partial x} \psi_{S_j} + (y_{j11} - y_{S_j}) \frac{\partial}{\partial y} \psi_{S_j} + O(\Delta x^2 + \Delta y^2), \\
\n\nabla \psi_{j12} &= \psi_{S_j} + (x_{j12} - x_{S_j}) \frac{\partial}{\partial x} \psi_{S_j} + (y_{j12} - y_{S_j}) \frac{\partial}{\partial y} \psi_{S_j} + O(\Delta x^2 + \Delta y^2).
\end{align*}
\]  
(62)

Next, use the relationship

\[
\begin{align*}
\sum_{i=0}^{5} a_i \psi_{S_i} &= 0, \\
\sum_{i=0}^{5} a_i (y_i - y_{S_j}) &= 0, \\
\sum_{i=0}^{5} a_i (x_{j1} - x_{S_j}) &= 1,
\end{align*}
\]  
(63)

for \( \partial / \partial y \psi_{S_j} \). Both relationship (63) and (64) are linear equations. We can use the method showed in A1 to obtain the least-norm solutions such as in the one-dimensional
have the following results for the one-step scheme (52).

**Proof.**

\[
\psi_j^t = \psi_j^n - \frac{\Delta t}{T_j} \sum_{k=1}^{3} H(\psi_j^t, \psi_j^n, v_{jk}, n_{jk})h_{jk},
\]

\[
\psi_j^{t+1} = \psi_j^t - \frac{\Delta t}{T_j} \sum_{k=1}^{3} H(\psi_j^t, \psi_j^n, v_{jk}, n_{jk})h_{jk},
\]

where

\[
H(\psi_j^t, \psi_j^n, v_{jk}, n_{jk}) = \left( \frac{a_{jk}^m v_{jk} + a_{jk}^m v_{jk}}{a_{jk}^m + a_{jk}^m} \right) \cdot n_{jk}
\]

\[
- \frac{a_{jk}^m a_{jk}^m}{a_{jk}^m + a_{jk}^m} (\psi_j^t - \psi_j^n).
\]

Then, we can say that \(\psi_j^{t+1}\) is a convex combination of \(\psi_j^n, M_1, M_2,\) and \(M_3\), which implies if \(\psi_j^n, M_1, M_2,\) and \(M_3\) are nonnegative, \(\psi_j^{t+1}\) is nonnegative. Noticing the identical relation,

\[
\sum_{k=1}^{3} h_{jk} H\left(\frac{\psi_j^n \psi_j^t v_{jk}}{v_{j2}}, n_{jk}\right) = \sum_{k=1}^{3} h_{jk} \left[ \frac{a_{jk}^m v_{jk} + a_{jk}^m v_{jk}}{a_{jk}^m + a_{jk}^m} \right] \cdot n_{jk}
\]

\[
= 0.
\]

Incorporating (72) into (70), we obtain

\[
M_2 = \frac{h_{j1}}{\sum_{k=1}^{3} h_{jk}} M_{21} + \frac{h_{j2}}{\sum_{k=1}^{3} h_{jk}} M_{22} + \frac{h_{j3}}{\sum_{k=1}^{3} h_{jk}} M_{23},
\]

where

\[
M_1 = \psi_j^n - \frac{4\Delta t}{T_j} \left[ H(\psi_j^n, \psi_j^n, v_{j1}, n_{j1})h_{j1} + H\left(\frac{\psi_j^n \psi_j^t v_{j2}}{v_{j2}, n_{j2}}\right) h_{j2} \right],
\]

\[
M_3 = \psi_j^n - \frac{4\Delta t}{T_j} \left[ H\left(\frac{\psi_j^n \psi_j^t v_{j3}}{v_{j3}, n_{j3}}\right) h_{j3} + H(\psi_j^n, \psi_j^n, v_{j3}, n_{j3}) h_{j3} \right].
\]
3. Numerical Results

3.1. One-Dimensional Accuracy Test. We consider the equation that expresses the linear scalar conservation law:

$$\frac{\partial}{\partial t} \psi + \frac{\partial}{\partial x} \psi = 0. \quad (76)$$

Here, the periodic boundary condition is used, and the exact solution is

$$\psi(x, t) = 1 + 0.5 \sin(\pi(x - t)). \quad (77)$$

The computational domain is $[0, 2]$, and the final time $T = 2.0$. The initial condition is $\psi(x, 0)$. To solve this problem, we use three approaches for simulation for the purpose of comparison. First, the simulation is conducted on regular grids with the aforementioned former MPDATA proposed by Smolarkiewicz and Margolin [3]. Second, the same algorithm, MPDATA by Smolarkiewicz, is used on irregular grids. Lastly, the simulation is carried out on irregular grids with the new MPDATA method. The construction of these irregular grids is illustrated in Figure 5. Here, the left-most cell size is assumed to be $\Delta x$, in which case the second left-most cell size is $2\Delta x$, the third left-most cell size is $3\Delta x$, and so on. In Figure 6, the convergence order of the former MPDATA (Figure 6(a)) and the new MPDATA (Figure 6(b)) is compared with different number of grids. For the sake of distinction, a set of parallel dashed lines with a slope of $-2$ is drawn in Figure 6. It can be seen from the figure that the new MPDATA can achieve second-order convergence, while the former scheme cannot reach the second-order accuracy. The simulation results of all three methods with multiple group cells are presented in Tables 1–3. $N$ is the number of cells in the simulation. Table 1 contains the results of the simulation of the former MPDATA for regular grids, indicating that the former MPDATA scheme for regular grid simulation is a useful tool and that $L^1$, $L^2$, and $L^\infty$ tend to the second-order norm as the mesh increases. The simulation results obtained with the former MPDATA on irregular grids are listed in Table 2. A comparison of Tables 1 and 2 reveals that the former MPDATA is useful for regular grids but that it fails for irregular grids. Table 3 is devoted to the simulation results of the new MPDATA on irregular grids, showing that $L^1$, $L^2$, and $L^\infty$ tend to second-order as the mesh increases.
3.2. Two-Dimensional Accuracy Test. The two-dimensional accuracy test is taken on the linear convection equation:

$$\frac{\partial \psi}{\partial t} + v_0 \cdot \nabla \psi = 0, \quad v_0 = (0.5, 0.5).$$

(78)

The periodic boundary condition is used as the one-dimensional situation. The exact solution is taken as

$$\psi(x, y, t) = 1 + 0.2 \sin(x + y - t).$$

(79)

The computational domain is $[0,2\pi] \times [0,2\pi]$. The computational time is $T = 2.0$. The periodic boundary condition is used. The computational domain with $32 \times 32$ is shown in Figure 7 and is surrounded by ghost cells. The simulation results are shown in Figure 8. Figure 8(a) shows the simulation result with $128 \times 128$ cells, which implies that our scheme can accommodate the situation of an unstructured grid well. The results in Figure 8(b) were simulated at a different resolution. The results confirm that our schemes can attain second-order accuracy as the mesh increases. In Table 4, we give the test results of the accuracy of the new MPDATA for two-dimensional situation. It can be seen that the new MPDATA can reached second order.
3.3. Cosine Bell Test for Two-Dimensional Situations. The cosine bell test was suggested by Huang et al. [11]. The cosine bell with an unsmooth boundary and all nonnegative values was used to examine the spurious oscillations as well as the damping effects of both schemes in the 2D case. The initial distribution of the cosine bell is given by

\[
\psi(x, y, 0) = \begin{cases} 
\frac{1}{2} \left(1 + \cos \frac{\pi r}{r_0}\right), & r \leq r_0, \\
0, & r > r_0,
\end{cases}
\]

(a)

(b)

Figure 9: Initial computational area.

Figure 10: Computational results at time (a) \(T = 1\); (b) \(T = 2\).

Table 4: New MPDATA for irregular grids.

<table>
<thead>
<tr>
<th>N</th>
<th>(L^1) error</th>
<th>(L^1) order</th>
<th>(L^2) error</th>
<th>(L^2) order</th>
<th>(L^\infty) error</th>
<th>(L^\infty) order</th>
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<tr>
<td>16 \times 16</td>
<td>6.623E − 01</td>
<td>—</td>
<td>1.242E − 01</td>
<td>—</td>
<td>3.892E − 02</td>
<td>—</td>
</tr>
<tr>
<td>32 \times 32</td>
<td>1.968E − 01</td>
<td>1.75</td>
<td>3.799E − 02</td>
<td>1.70</td>
<td>1.657E − 02</td>
<td>1.23</td>
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<td>64 \times 64</td>
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<td>2.01</td>
<td>9.651E − 03</td>
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<td>5.945E − 03</td>
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</tr>
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<td>128 \times 128</td>
<td>1.152E − 02</td>
<td>2.07</td>
<td>2.255E − 03</td>
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<td>1.504E − 03</td>
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<tr>
<td>256 \times 256</td>
<td>2.721E − 03</td>
<td>2.08</td>
<td>5.382E − 04</td>
<td>2.06</td>
<td>3.735E − 04</td>
<td>2.01</td>
</tr>
</tbody>
</table>

3.3. Cosine Bell Test for Two-Dimensional Situations. The cosine bell test was suggested by Huang et al. [11]. The cosine bell with an unsmooth boundary and all nonnegative values was used to examine the spurious oscillations as well as the damping effects of both schemes in the 2D case. The initial distribution of the cosine bell is given by
where \( r = \sqrt{(x-x_0)^2 + (y-y_0)^2} \), \( r = 0.5 \), and \( x_0 = y_0 \). Irregular grids were used in the simulations. The initial values are synoptically shown in Figure 9 for a 32 \( \times \) 32 area. The final time of the simulation is \( T = 2 \). In our test, the cosine bell passes through areas of dense grid and areas of sparse grid. A mesh of 300 \( \times \) 300 is used in the simulation in Figure 10. Figure 10(a) shows the results at time \( T = 1 \), whereas those in Figure 10(b) were obtained at time \( T = 2 \). These results enabled us to conclude that our schemes are able to solve this problem without obtaining negative values.

4. Conclusion and Discussion

We combined the MPDATA method with the semidiscrete central difference strategy. We presented a positive definite shape-preserving scheme for complex grid structures. The MPDATA method proposed by Smolarkiewicz et al. is a useful tool for atmospheric dynamics simulation. However, their method is problematic as it experiences loss of precision when computing on an irregular grid. The reasons for this problem are two fold: the first is the numerical discretization of the boundary derivative in the finite-volume method; the other is that the numerical format is developed based on a staggered grid. Our approach overcomes these two problems in combination with the multipoint Taylor expansion method. We thus obtain the boundary derivative approximation method that does not depend on the grid structure. Combined with the central-upwind scheme, we propose a positive definite advective scheme for complex grid structures. On this basis, we carried out the positive definite analysis and numerical test of the new numerical scheme.

In this paper, starting from the one-dimensional MPDATA [3], we found that the existing MPDATA method [3–5] could not reach the second-order calculation accuracy required by the design for nonuniform grids, and then through improvement, we obtained the MPDATA method that could reach the second-order accuracy for the irregular grid. Here, we need to explain that the irregular grid studied in this paper is different from the unstructured grid. The irregular grid we studied specifically refers to the grid with irregular grid we studied specifically refers to the grid with unstructured mesh algorithms can achieve two precision for uniform unstructured grids but cannot achieve second precision for nonuniform irregular grids.

In addition, in order to explain that the existing MPDATA methods cannot achieve second-order accuracy for nonuniform irregular grids, the one-dimensional case of traditional MPDATA methods is mainly discussed here. However, in the case of two-dimensional MPDATA method, there are similar accuracy problems. The most direct point is in the calculation of boundary derivatives. The unified MPDTA scheme still adopts the calculation of the difference between physical quantities of adjacent grids which is similar to one-dimensional MPDATA [4, 5]:

\[
\left( \frac{\partial \Phi}{\partial x_i} \right)_j = \frac{1}{V_j + V_i} \left( \sum_{m=1}^{l(i)} \Theta_i^{m} s_{m} + \sum_{m'=1}^{l(i')} \Theta_{m'}^{m} s_{m'} \right),
\]

where \( \Theta_i^{m} = 0.5(\Phi_i + \Phi_m) \).

Appendix

Least-Norm Solution

The linear equation

\[
A^n x^n = b^m,
\]

satisfy the following conditions:

(i) The rank of matrix \( A \) is the same as the rank of extend matrix \([A; b]\)

(ii) The rank of matrix \( A \) is less than \( n \)

Thus, there are infinite solutions to linear equations (A.1). The least-norm solution is the minimum value of

\[
\min\|x\|,
\]

which also satisfies equation (A.1). Without loss of generality, we assume the rank of \( A \) is \( m \). Then, the matrix \( A \) can be decomposed as

\[
A = [R^{m}; D^{m,n-m}].
\]

Then, \( x \) can be written as

\[
x = [x_r, A^{n-m}].
\]

Thus, (A.1) becomes

\[
Rx_r + Dx_D = b,
\]

which leads

\[
x_r = R^{-1}b - R^{-1}Dx_D.
\]

In addition, we have

\[
\|x\|^2 = \|x_r\|^2 + \|x_D\|^2\]

\[
= (R^{-1}b - R^{-1}Dx_D)^2 + x_D^T x_D.
\]

Then, the derivative of \( \|x\| \) is obtained to determine the extreme values:

\[
V\|x\| = 2[D^T (R^{-1}Dx_D - R^{-1}b) + x_D].
\]

In the extreme value point, the least-norm solution is found:

\[
x_D = (I - D^T R^{-1} D)^{-1} D^T R^{-1} b,
\]

\[
x_r = R^{-1}b - R^{-1}D(I - D^T R^{-1} D)^{-1} D^T R^{-1} b.
\]

Data Availability

The data in this paper are all open, and this paper is mainly engaged in theoretical analysis; numerical simulation is an auxiliary means of verification.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
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