Research Article

Pinning Synchronization of Complex Dynamical Networks with Variable-Delayed Coupling by Periodically Intermittent Control

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This paper studied the adaptive pinning synchronization in complex networks with variable-delay coupling via periodically intermittent control. Theoretical analysis is included by means of Lyapunov functions and linear matrix inequalities (LMI) to make all nodes reach complete synchronization. Moreover, the synchronization criteria do not impose any restriction on the size of time delay. Numerical examples including the regular, Watts–Strogatz and scale-free BA random topological architecture are provided to illustrate the importance of our theoretical analysis.

1. Introduction

Complex dynamical networks have been intensively studied [1, 2] over the last few years due to their potential applications in various fields of the real world, such as in the biological systems, scientific citation web, social networks, and electrical power grids and thus became an important part of our daily life.

Since Pecora and Carroll [3] found chaos synchronization in 1990, it has become a focal research topic recently. Due to the synchronization not only can well explain many natural phenomena observed, such as synchronized firefly flashing and swarming of fishes, but also has many practical potential applications such as image processing, the operation of unmanned aerial vehicle, and secure communication. However, the complex network cannot synchronize by itself, so many control methods have been developed, such as linear and nonlinear feedback control, time-delay feedback control, sliding mode control [4], adaptive control [5], pinning control [6], impulsive control [7, 8], and intermittent control [9–12]. Especially, discontinuous feedback

controls, including impulsive control [13] and intermittent control, have been attracting much attention since they are practically and easily implemented in some engineering domains. However, the intermittent control is different from the impulsive control because impulsive control is activated only at some isolated moments, while the intermittent control has a nonzero control width.

Intermittent control was first introduced to control chaos systems by Zochowski [9] in 2000. It is more efficient when the system output is measured intermittently rather than continuously. There are some novel synchronization criteria in complex networks with delay or nondelay coupling by intermittent control. Li et al. [14] studied the inner synchronization of delayed nonlinear chaotic systems by intermittent control. Further, Mei et al. [10] used driver and response systems to produce the finite-time synchronization for the complex systems without delay. In [12], the authors investigated the synchronization problem of stochastic perturbed complex systems with time-varying delays. On the contrary, complex networks have a large number of nodes in the real world, and it is usually impractical to control a

complex network by adding the controllers to all nodes. To reduce the number of controlled nodes, pinning control, in which controllers are only applied to partial nodes, is introduced [15–20]. In [21], the authors discussed the synchronization for coupled dynamical networks with mixed delays and uncertain parameters using pinning control and intermittent control. By means of intermittent control, the authors [22] studied finite-time synchronization for a class of reaction-diffusion neural networks by small domains on their spatial boundaries, associated with an interaction graph. Li et al. [23] and Xu et al. [24] discussed the synchronization problem of general complex networks with fractional-order dynamical networks by periodically intermittent pinning control.

Moreover, in [11, 15], the authors investigated the synchronization of complex networks with delays by pinning periodically intermittent control; especially, they assumed that the control width needs to be larger than the time delay or the time-varying delays should be differentiable, and their derivatives are less than 1. Motivated by the above discussions, we removed these constraints in our results. In this paper, we give the complex systems with both time-varying delays and nondelay couplings, and using pinning control and periodically intermittent control methods, some novel criteria for pinning synchronization are derived. Numerical examples including the regular, Watts–Strogatz and scale-free BA random topological architecture are provided to illustrate the importance of our theoretical analysis.

The rest of the paper is organized as follows: in Section 2, we propose a general complex dynamical network model; some preliminaries and lemmas are given. In Section 3, some pinning adaptive synchronization criteria for the general complex dynamical networks with delay coupling are given. Numerical examples are given in Section 4. Finally, we draw our conclusion in Section 5.

2. Preliminaries and Mathematical Models

Consider a generally controlled complex delayed dynamical system consisting of N nodes, with each node being of n dimensions, which is described by

$$\begin{aligned} \dot{x}_i(t) = & Ax_i(t) + f(x_i(t)) + c_0 \sum_{j=1}^N b_{ij} \Gamma x_j(t) \\ & + c_1 \sum_{j=1}^N \bar{b}_{ij} \Gamma x_j(t - \tau(t)), \end{aligned} \quad (1)$$

where $1 \leq i \leq N$, $x_i(t) = (x_{i1}(t), \dots, x_{in}(t))^T \in \mathbb{R}^n$ is the state variable of node i , $A \in \mathbb{R}^{n \times n}$ is a given constant matrix, and $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuously differentiable function describing the nonlinear dynamics of the single node. Here, c_0 and c_1 are two parameters of the nondelay and delay coupling strength, respectively, $\Gamma = (\gamma_{ij}) \in \mathbb{R}^{n \times n}$ is an inner-

coupling matrix, and $\tau(t)$ is the coupled delay and bounded by a known constant, i.e., $0 < \tau(t) < \tau$. $B = (b_{ij})_{N \times N}$ and $\bar{B} = (\bar{b}_{ij})_{N \times N}$ represent the adjacency configuration of the network with the nondelay and delay couplings, respectively; if there is a link from node i to node j , then $b_{ij} > 0$ (or $\bar{b}_{ij} > 0$) ($j \neq i$); otherwise, $b_{ij} = 0$ (or $\bar{b}_{ij} = 0$). Moreover, $b_{ii} = -\sum_{j \neq i} b_{ij}$ and $\bar{b}_{ii} = -\sum_{j \neq i} \bar{b}_{ij}$.

As we know, the complex network cannot synchronize by itself; then, we add the adaptive controller $v_i(t)$ as follows:

$$\begin{aligned} \dot{x}_i(t) = & Ax_i(t) + f(x_i(t)) + c_0 \sum_{j=1}^N b_{ij} \Gamma x_j(t) \\ & + c_1 \sum_{j=1}^N \bar{b}_{ij} \Gamma x_j(t - \tau(t)) + v_i(t). \end{aligned} \quad (2)$$

Hereafter, let $s(t) = s(t; t_0, x_0) \in \mathbb{R}^m$ be a solution of the node system $\dot{x}(t) = f(x(t))$. Then, $s(t)$ is a synchronous solution of controlled complex delayed dynamical system (2). Note that $s(t)$ may be an equilibrium point, a periodic orbit, or a chaotic attractor.

Define error vectors as

$$e_i(t) = x_i(t) - s(t), \quad 1 \leq i \leq N. \quad (3)$$

According to system (2), the error system is described by

$$\begin{aligned} \dot{e}_i(t) = & Ae_i + f(x_i(t)) - f(y_i(t)) + c_0 \sum_{j=1}^N b_{ij} \Gamma e_j(t) \\ & + c_1 \sum_{j=1}^N \bar{b}_{ij} \Gamma e_j(t - \tau(t)) + v_i(t), \end{aligned} \quad (4)$$

where $1 \leq i \leq N$.

Lemma 1 (Schur complement, see [25]). *The linear matrix inequality (LMI) is as follows:*

$$\begin{pmatrix} A(x) & B(x) \\ B^T(x) & C(x) \end{pmatrix} > 0, \quad (5)$$

where $A(x) = A^T(x)$, $C(x) = C^T(x)$ is equivalent to one of the following conditions:

- (1) $A(x) > 0$, $C(x) - B^T(x)A^{-1}(x)B(x) > 0$
- (2) $C(x) > 0$, $A(x) - B(x)C^{-1}(x)B^T(x) > 0$

Lemma 2 (see [26]). *Let X and Y be arbitrary n -dimensional real vectors, K be a positive definite matrix, and $P \in \mathbb{R}^{m \times m}$. Then, the following matrix inequality holds:*

$$2X^T P Y \leq X^T P K^{-1} X + Y^T K Y. \quad (6)$$

3. Main Results

In the following, assume that $\Gamma \neq 0$ and $\|\Gamma\|_2 = \gamma > 0$. Denote ρ_{ii} as the minimum eigenvalue of the matrix $(\Gamma + \Gamma^T)/2$. Let

$\hat{B}^s = (\hat{B} + \hat{B}^T)/2$, and its eigenvalues are expressed as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$, where $\hat{B}$ is a modifying matrix of B via replacing the diagonal elements b_{ii} by $(\rho_{\min}/\gamma)b_{ii}$. Note that, generally, $\hat{B}$ does not possess the property of zero row sums.

To realize the network synchronization, the controllers v_i should guide the error vectors $e_i(t)$ to approach zero as t goes to infinity as

$$\lim_{t \rightarrow \infty} \|e_i(t)\|_2 = 0, \quad 1 \leq i \leq N. \quad (7)$$

Choose the adaptive controllers as follows:

$$v_i(t) = \begin{cases} -p_i(t)e_i(t), & 1 \leq i \leq l, \quad t \in [mT, (m+h)T), \\ 0, & l+1 \leq i \leq N, \quad t \in [mT, (m+h)T), \\ 0, & 1 \leq i \leq N, \quad t \in [(m+h)T, (m+1)T), \end{cases} \quad (8)$$

where $\dot{p}_i(t) = \exp\{a_1 t\} q_i e_i^T(t) e_i(t)$, q_i , and a_1 are positive constants. $T > 0$ denotes the control period, $0 < h < 1$ is the rate of control duration called control rate, and $m = 0, 1, 2, \dots$

Theorem 1. Suppose that $\|f(x_i) - f(y_i)\|_2 \leq \beta \|e_i\|_2$. If there exists a positive constant a_1, a_2 , and k such that

$$\left(\alpha + \beta + \frac{a_1}{2} + \frac{c_1 \gamma [(N-1)\bar{b}_1 + k\bar{b}_2]}{2k} \right) I_N + c_0 \gamma \hat{B}^s - D < 0,$$

$$\alpha + \beta - \frac{a_2 - a_1}{2} + \frac{c_1 \gamma [(N-1)\bar{b}_1 + k\bar{b}_2]}{2k} + c_0 \gamma \lambda_1 < 0,$$

$$\xi - a_2(1-h) > 0, \quad (9)$$

where $\|A\|_2 = \alpha$, $D = \text{diag}(\underbrace{p, \dots, p}_l, \underbrace{0, \dots, 0}_{N-l})$, $p > 0$,

$q = c_1[(N-1)k\bar{b}_1 + \bar{b}_2\gamma]$, $\bar{b}_1 = \max\{\bar{b}_{ij}, j \neq i\}$, $a_2 > a_1 > q$, $\bar{b}_2 = \max\{\bar{b}_{ii}\}$, and $\xi > 0$ is the smallest real root of the equation $-a_1 + \xi + q \exp(\xi\tau) = 0$, then the synchronized manifold $\bigcup_{i=1}^N (s(t), s(t), \dots, s(t))$ of controlled complex delayed dynamical system (2) is globally asymptotically stable under periodical intermittent controllers (8).

Proof. Construct the candidate Lyapunov function as follows:

$$V(t) = \frac{1}{2} \sum_{i=1}^N e_i^T(t) e_i(t) + \frac{1}{2} \sum_{i=1}^l \exp(-a_1 t) \frac{(p_i(t) - p)^2}{q_i}, \quad (10)$$

and when $mT \leq t \leq mT + h$, calculating the time derivative of $V(t)$ along the trajectories of (4), one has

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^N e_i^T(t) \dot{e}_i(t) + \sum_{i=1}^l \exp(-a_1 t) \frac{(p_i(t) - p) \dot{p}_i(t)}{q_i} - \frac{a_1}{2} \sum_{i=1}^l \exp(-a_1 t) \frac{(p_i(t) - p)^2}{q_i} \\ &= \sum_{i=1}^N e_i^T(t) (Ae_i + f(x_i) - f(s)) + c_0 \sum_{i=1}^N \sum_{j=1}^N b_{ij} e_i^T(t) \Gamma e_j(t) + c_1 \sum_{i=1}^N \sum_{j=1}^N \bar{b}_{ij} e_i^T(t) \Gamma e_j(t - \tau(t)) \\ &\quad - \sum_{i=1}^l p_i(t) e_i^T(t) e_i(t) + \sum_{i=1}^l \exp(-a_1 t) \frac{(p_i(t) - p) \dot{p}_i(t)}{q_i} - \frac{a_1}{2} \sum_{i=1}^l \exp(-a_1 t) \frac{(p_i(t) - p)^2}{q_i} \\ &\leq (\alpha + \beta) \sum_{i=1}^N e_i^T(t) e_i(t) + c_0 \sum_{i=1}^N \sum_{j=1}^N b_{ij} e_i^T(t) \Gamma e_j(t) + c_1 \sum_{i=1}^N \sum_{j=1}^N \bar{b}_{ij} e_i^T(t) \Gamma e_j(t - \tau(t)) \\ &\quad - \sum_{i=1}^l p_i(t) e_i^T(t) e_i(t) - \frac{a_1}{2} \sum_{i=1}^l \exp(-a_1 t) \frac{(p_i(t) - p)^2}{q_i}. \end{aligned} \quad (11)$$

From Lemma 2, one can find

$$\begin{aligned}
 2 \sum_{i=1}^N \sum_{j=1}^N \bar{b}_{ij} e_i^T \Gamma e_j(t - \tau(t)) &\leq \sum_{i=1}^N \sum_{j=1, j \neq i}^N \bar{b}_{ij} [e_i^T(t) \Gamma K^{-1} e_i(t) + e_j(t - \tau(t))^T K e_j(t - \tau(t))] \\
 &\quad + 2 \sum_{i=1}^N |\bar{b}_{ii}| \gamma \|e_i(t)\|_2 \|e_i(t - \tau)\|_2 \\
 &\leq \sum_{i=1}^N \sum_{j=1, j \neq i}^N \left[\frac{\bar{b}_1 \gamma}{k} e_i^T(t) e_i(t) + k \bar{b}_1 e_j^T(t - \tau(t)) e_j(t - \tau(t)) \right] \quad (12)
 \end{aligned}$$

$$\begin{aligned}
 &\quad + \sum_{i=1}^N |\bar{b}_{ii}| \gamma [e_i^T(t) e_i^T(t) + e_i^T(t - \tau(t)) e_i^T(t - \tau(t))] \\
 &= \left[\frac{(N-1) \bar{b}_1 \gamma}{k} + \bar{b}_2 \gamma \right] \sum_{i=1}^N e_i^T(t) e_i(t) + [(N-1) k \bar{b}_1 + \bar{b}_2 \gamma] \sum_{i=1}^N e_i^T(t - \tau(t)) e_i(t - \tau(t)), \\
 c_0 \sum_{i=1}^N \sum_{j=1}^N b_{ij} e_i^T \Gamma e_j(t) &\leq c_0 \sum_{i=1}^N \sum_{j=1, j \neq i}^N \gamma b_{ij} \|e_i(t)\|_2 \|e_j(t)\|_2 + c_0 \sum_{i=1}^N b_{ii} \rho_{\min} e_i^T(t) e_i(t), \quad (13)
 \end{aligned}$$

where $\|K\|_2 = k$, $\|\Gamma\|_2 = \gamma$, $\bar{b}_1 = \max\{\bar{b}_{ij}, j \neq i\}$, $\bar{b}_2 = \max\{\bar{b}_{ii}\}$, and ρ_{ii} is the minimum eigenvalue of the matrix $(\Gamma + \Gamma^T)/2$. It follows from (12) and (13) that

$$\begin{aligned}
 \dot{V}(t) &\leq e^T(t) \left[\left(\alpha + \beta + \frac{c_1 \gamma [(N-1) \bar{b}_1 + k \bar{b}_2]}{2k} \right) I_N + c_0 \gamma \hat{B}^s - D \right] e(t) \\
 &\quad + \frac{c_1 [(N-1) k \bar{b}_1 + \bar{b}_2 \gamma]}{2} e^T(t - \tau(t)) e(t - \tau(t)) - \frac{a_1}{2} \sum_{i=1}^l \exp(-a_1 t) \frac{(p_i(t) - p)^2}{q_i} \\
 &\leq e^T(t) \left[\left(\alpha + \beta + \frac{a_1}{2} + \frac{c_1 \gamma [(N-1) \bar{b}_1 + k \bar{b}_2]}{2k} \right) I_N + c_0 \gamma \hat{B}^s - D \right] e(t) \quad (14) \\
 &\quad + \frac{q}{2} e^T(t - \tau(t)) e(t - \tau(t)) - \frac{a_1}{2} \left[\sum_{i=1}^N e^T(t) e(t) + \sum_{i=1}^l \exp(-a_1 t) \frac{(p_i(t) - p)^2}{q_i} \right] \\
 &\leq -a_1 V(t) + q V(t - \tau(t)),
 \end{aligned}$$

where $e(t) = (\|e_1(t)\|_2, \|e_2(t)\|_2, \dots, \|e_N(t)\|_2)^T$, $D = \text{diag}(\underbrace{p, \dots, p}_l, \underbrace{0, \dots, 0}_{N-l})$, and $q = c_1[(N-1)kb_1 + b_2\gamma]$.

When $mT + h \leq t \leq (m+1)T$,

$$\begin{aligned}
 \dot{V}(t) &= \sum_{i=1}^N e_i^T(t) \dot{e}_i(t) - \frac{a_1}{2} \sum_{i=1}^l \exp(-a_1 t) \frac{(p_i(t) - p)^2}{q_i} \\
 &= \sum_{i=1}^N e_i^T(t) (Ae_i + f(x_i) - f(s)) + c_0 \sum_{i=1}^N \sum_{j=1}^N b_{ij} e_i^T(t) \Gamma e_j(t) + c_1 \sum_{i=1}^N \sum_{j=1}^N \bar{b}_{ij} e_i^T \Gamma e_j(t - \tau(t)) \\
 &\quad - \frac{a_1}{2} \sum_{i=1}^l \exp(-a_1 t) \frac{(p_i(t) - p)^2}{q_i} \\
 &\leq (\alpha + \beta) \sum_{i=1}^N e_i^T(t) e_i(t) + c_0 \sum_{i=1}^N \sum_{j=1}^N b_{ij} e_i^T \Gamma e_j(t) + c_1 \sum_{i=1}^N \sum_{j=1}^N \bar{b}_{ij} e_i^T \Gamma e_j(t - \tau(t)) \\
 &\quad - \frac{a_1}{2} \sum_{i=1}^l \exp(-a_1 t) \frac{(p_i(t) - p)^2}{q_i} \\
 &\leq e^T(t) \left[\left(\alpha + \beta + \frac{c_1 \gamma [(N-1)\bar{b}_1 + k\bar{b}_2]}{2k} \right) I_N + c_0 \gamma \hat{B}^s \right] e(t) \\
 &\quad + \frac{q}{2} e^T(t - \tau) e(t - \tau(t)) - \frac{a_1}{2} \sum_{i=1}^l \exp(-a_1 t) \frac{(p_i(t) - p)^2}{q_i} \\
 &\leq e^T(t) \left[\left(\alpha + \beta - \frac{a_2 - a_1}{2} + \frac{c_1 \gamma [(N-1)\bar{b}_1 + k\bar{b}_2]}{2k} \right) I_N + c_0 \gamma \hat{B}^s \right] e(t) + \frac{q}{2} e^T(t - \tau) e(t - \tau(t)) \\
 &\quad + \frac{a_2 - a_1}{2} \left[\sum_{i=1}^N e_i^T(t) e_i(t) + \sum_{i=1}^l \exp(-a_1 t) \frac{(p_i(t) - p)^2}{q_i} \right] \\
 &\leq (a_2 - a_1)V(t) + qV(t - \tau(t)).
 \end{aligned} \tag{15}$$

Namely, we have

$$\begin{cases} \dot{V}(t) \leq -a_1 V(t) + qV(t - \tau(t)), & t \in [mT, (m+h)T), \\ \dot{V}(t) \leq (a_2 - a_1)V(t) + qV(t - \tau(t)), & t \in [(m+h)T, (m+1)T). \end{cases} \tag{16}$$

In the following, we will prove that

$$V(t) \leq \sup_{-\tau \leq \theta \leq 0} V(\theta) \exp\{-[\xi - a_2(1-h)]t\}, \quad t \geq 0. \tag{17}$$

Denote $g(\xi) = -a_1 + \xi + q \exp(\xi\tau)$; as $a_1 > q > 0$, we have $g(0) < 0$, $g(\infty) > 0$, and $g'(\xi) > 0$. Using the continuity and the monotonicity of the function, $g(\xi) = 0$ has a unique

positive solution $\xi > 0$. Let $M = \sup_{-\tau \leq \theta \leq 0} V(\theta)$ and $W(t) = \exp\{\xi t\}V(t)$.

$Q_1(t) = W(t) - \epsilon M$, $\epsilon > 1$ is a constant. Obviously,

$$Q_1(t) < 0, \quad \text{for all } t \in [-\tau, 0]. \tag{18}$$

Next, we will prove that

$$Q_1(t) < 0, \quad \text{for all } t \in [0, hT]. \tag{19}$$

Otherwise, there exists $t_1 \in [0, hT]$ such that

$$Q_1(t_1) = 0, \dot{Q}_1(t_1) \geq 0, \text{ and } Q_1(t) < 0, -\tau \leq t < t_1. \tag{20}$$

Using (16) and (20), one obtains

$$\begin{aligned}
\dot{Q}_1(t_1) &= \xi \exp\{\xi t_1\} V(t_1) + \exp\{\xi t_1\} \dot{V}(t_1) \\
&\leq (\xi - a_1)W(t) + q \exp\{\xi \tau(t_1)\} W(t_1 - \tau(t_1)) \\
&< (-a_1 + \xi + q \exp(\xi \tau)) \varepsilon M = 0.
\end{aligned} \tag{21}$$

This leads to a contradiction with (20); hence, (19) holds. Now, we prove that, for $t \in [hT, T]$,

$$Q_2(t) = W(t) - \varepsilon M \exp\{a_2(t - hT)\} < 0. \tag{22}$$

Otherwise, there exists $t_2 \in [hT, T]$ such that

$$Q_2(t_2) = 0, \dot{Q}_2(t_2) \geq 0, \text{ and } Q_2(t) < 0, \quad hT \leq t < t_2. \tag{23}$$

Then,

$$\begin{aligned}
\dot{Q}_2(t_2) &= \xi W(t_2) + \exp\{\xi t_2\} \dot{V}(t_2) - a_2 \varepsilon M \exp\{a_1(t - hT)\} \\
&\leq (\xi + a_2 - a_1)W(t) + q \exp\{\xi \tau(t_2)\} W(t_2 - \tau(t_2)) \\
&\quad - a_2 \varepsilon M \exp\{a_2(t - hT)\}.
\end{aligned} \tag{24}$$

For $0 < \tau(t) < \tau$, if $hT \leq t_2 - \tau(t_2) < t_2$, then from (23), one has

$$W(t_2 - \tau(t_2)) < \varepsilon M \exp\{a_1(t_2 - hT)\}, \tag{25}$$

and if $-\tau \leq t_2 - \tau(t_2) < hT$, from (19), one obtains

$$W(t_2 - \tau(t_2)) < \varepsilon M < \varepsilon M \exp\{a_2(t_2 - hT)\}. \tag{26}$$

So,

$$\dot{Q}_2(t_2) < (-a_1 + \xi + q \exp(\xi \tau)) \varepsilon M \exp\{a_2(t_2 - hT)\} = 0. \tag{27}$$

This leads to a contradiction with (23); hence, (22) holds. According (19) and (22), one obtains

$$W(t) < \varepsilon M \exp\{a_2(1 - h)T\}, \text{ for all } t \in [-\tau, T]. \tag{28}$$

Similarly, we can prove that

$$\begin{aligned}
W(t) &< \varepsilon M \exp\{a_2(1 - h)T\}, \quad t \in [T, (1 + h)T], \\
W(t) &< \varepsilon M \exp\{a_2(t - 2h)T\}, \quad t \in [(1 + h)T, 2T].
\end{aligned} \tag{29}$$

By induction, we can derive the following estimation of $W(t)$ for any integer m :

$$\begin{aligned}
W(t) &< \varepsilon M \exp\{a_2 m(1 - h)T\}, \quad t \in [mT, (m + h)T], \\
W(t) &< \varepsilon M \exp\{a_2(t - (m + 1)h)T\}, \quad t \in [(m + h)T, (m + 1)T].
\end{aligned} \tag{30}$$

Since for any $t \geq 0$, there exists a nonnegative integer r such that $rT \leq t \leq (r + 1)T$; then, we have

$$\begin{aligned}
W(t) &< \varepsilon M \exp\{a_2 r(1 - h)T\} \leq \varepsilon M \exp\{a_2(1 - h)t\}, \\
&\quad t \in [rT, (r + h)T], \\
W(t) &< \varepsilon M \exp\{a_2(t - (r + 1)h)T\} \leq \varepsilon M \exp\{a_2(1 - h)t\}, \\
&\quad t \in [(r + h)T, (r + 1)T].
\end{aligned} \tag{31}$$

Let $\varepsilon \rightarrow 1$, and from the definition of $W(t)$, one has

$$V(t) \leq M \exp\{-[\xi - a_2(1 - h)]t\}. \tag{32}$$

As $V(t) = 1/2 \sum_{i=1}^N e_i^T(t) e_i(t) + 1/2 \sum_{i=1}^l \exp(-a_1 t) ((p_i(t) - p)^2 / q_i)$, one has

$$\|e(t)\|_2 \leq (2M)^{(1/2)} \exp\left\{-\frac{\xi - a_2(1 - h)}{2}t\right\}. \tag{33}$$

In view of $\xi - a_2(1 - h) > 0$, we can draw the conclusion. The proof is thus completed.

Let

$$\begin{aligned}
\eta &= \alpha + \beta + \frac{a_1}{2} + \frac{c_1 \gamma [(N - 1)\bar{b}_1 + k\bar{b}_2]}{2k}, \\
M &= \eta I_N + c_0 \gamma \tilde{B}^s - D = \begin{pmatrix} Q - \tilde{D} & G \\ G^T & \tilde{M} \end{pmatrix},
\end{aligned} \tag{34}$$

where $\tilde{D} = \text{diag}(\underbrace{p, \dots, p}_l)$, $\tilde{M}$ is obtained by removing the $1, 2, \dots, l$ row-column pairs of matrix M , and Q and G are matrices with appropriate dimensions.

According to Lemma 1, one can easily see that $M < 0$ is equivalent to $\tilde{M} < 0$ because of $Q - \tilde{D} - G\tilde{M}^{-1}G^T < 0$ by choosing $p > \lambda_{\max}(Q - G\tilde{M}^{-1}G^T)$, i.e., $M < 0$ is equivalent to $\tilde{M} < 0$ for sufficiently large p . Note that $\tilde{M} = \eta I_{N-l} + c\gamma \tilde{B}^s$, where $\tilde{B}_{i,j}^s = \tilde{B}_{(l+i),(l+j)}^s$, $i, j = 1, 2, \dots, N - l$. Then, one can immediately get the following corollary: $\square$

Corollary 1. Suppose that $\|f(x_i) - f(y_i)\|_2 \leq \beta \|e_i\|_2$, $\tau \leq h$ and $\tau \leq T - h$. If there exists a positive constant a_1, a_2 , and k such that

$$\begin{aligned}
\bar{\lambda}_l &= \lambda_{\max}(\tilde{B}^s) < -\frac{\eta}{c_0 \gamma}, \\
\alpha + \beta - \frac{a_2 - a_1}{2} + \frac{c_1 \gamma [(N - 1)\bar{b}_1 + k\bar{b}_2]}{2k} + c_0 \gamma \lambda_1 &< 0, \\
\xi - a_2(1 - h) &> 0,
\end{aligned} \tag{35}$$

where $\|A\|_2 = \alpha$, $q = c_1 [(N - 1)k\bar{b}_1 + \bar{b}_2 \gamma]$, $\bar{b}_1 = \max\{\bar{b}_{ij}, j \neq i\}$, $a_2 > a_1 > q$, $\bar{b}_2 = \max\{\bar{b}_{ii}\}$, and $\|I\|_2 = \gamma$, $\xi > 0$ is the smallest real root of the equation $-a_1 + \xi + q \exp(\xi \tau) = 0$, then

the synchronized manifold $\overbrace{(s(t), s(t), \dots, s(t))^T}^N$ of controlled complex delayed dynamical system (2) is globally asymptotically stable under periodical intermittent controllers (8).

Remark 1. The number of control nodes l can be chosen properly to adjust the synchronization efficiency of networks. However, it is noted that l by the theoretical prediction is only a sufficient condition but not a necessary one. In simulations, we will show that a small value of l can also lead to synchronization.

4. Numerical Simulation

In this section, a numerical example is used to verify the effectiveness of the proposed network synchronization criteria.

Consider the Lorenz oscillator model described by the following equation:

FIGURE 1: The chaotic attractor of the Lorenz system.

TABLE 1: Parameters in system (38) with different topological architecture.

Parameter	Regular $p = 0$	Small-world $p = 0.1$	Scale-free
$\bar{b}_2$	8	14	47
λ_1	0.8561	0.9388	0.6623
q	2.27601	2.3113	2.7439

$$\dot{x}_i(t) = Ax_i + f(x_i), \quad (36)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T \in \mathbb{R}^3$, $f(x_i) = (0, -x_{i1}(t)x_{i3}(t), x_{i1}(t)x_{i2}(t))^T$, and

$$A = \begin{pmatrix} -10 & 10 & 0 \\ 30 & -1 & 0 \\ 0 & 0 & -3/8 \end{pmatrix}. \quad (37)$$

It has been known that the Lorenz oscillators exhibit chaotic behavior, and Figure 1 shows it clearly.

Then, we consider controlled complex delayed dynamical system (2) consisting of 200 identical Lorenz systems, which are described by

$$\begin{aligned} \dot{x}_i(t) &= Ax_i + f(x_i) + c_0 \sum_{j=1}^{200} b_{ij} \Gamma x_j(t) + c_1 \sum_{j=1}^{200} \bar{b}_{ij} \Gamma x_j \\ &\quad \cdot (t - \tau) + v_i(t), \quad 1 \leq i \leq 200, \end{aligned} \quad (38)$$

where $\Gamma = \text{diag}\{1, 1.2, 1\}$ and $B = (b_{ij})_{200 \times 200}$ and $\bar{B} = (\bar{b}_{ij})_{200 \times 200}$ are symmetrically diffusive coupling matrices with $b_{ij} = 0$ (or $\bar{b}_{ij} = 0$) or 1 ($j \neq i$). Here, the coupling coefficient $c_0 = 30$.

As we know, Lorenz system is bounded. Here, we suppose $\|x_{i1}\| \leq 29$, $\|x_{i2}\| \leq 29$, $-1 \leq x_{i3} \leq 57$, $\|s_1\| \leq 29$, $\|s_2\| \leq 29$, $-1 \leq s_3 \leq 57$, and $1 \leq i \leq 200$.

$$\begin{aligned} \|f(x_i) - f(s)\|_2 &= \sqrt{(-x_{i1}x_{i3} + s_1s_3)^2 + (x_{i1}x_{i2} - s_1s_2)^2} \\ &= \sqrt{(-x_{i3}e_{i1} - s_1e_{i3})^2 + (x_{i2}e_{i1} + s_1e_{i2})^2} \\ &\leq 75.97\|e_i\|_2. \end{aligned} \quad (39)$$

Suppose that the network structure of equation (38) obeys the regular ($p=0$, $2m=8$)/small-world ($p=0.3$, $2m=8$) [1]/scale-free (SF) [2] distribution, respectively. The number of the nodes $N = 200$. Obviously, $\gamma = \|\Gamma\|_2 = 1.2$, $\alpha = \|A\|_2 = 31.91$, $\beta = 75.97$, $\bar{b}_1 = 1$, and $\rho_{\min} = 1$. Choosing the coupling coefficient $c_0 = 30$, $c_1 = 0.01$, $k = \sqrt{\gamma}$, $T = 0.2$, $h = 0.9$, and $\tau(t) = 1 + 0.1|\sin t| < \tau = 1.1$. By using the MATLAB LMI Toolbox and the corollary, one can obtain Table 1.

Please note that the average number of neighbours is the same between the small-world network with the connection probability $p = 0.1$ and the regular network ($p=0$), the number of neighbours becomes rand in the scale-free/small-world topological structure, and $\max\{\bar{b}_{ii}\}$ can be estimated by the statistic method.

As $a_1 - 8 < \xi < a_1 - 9$, then if we choose $a_1 = 45$ and $a_2 = 325$, it is easy to verify that the criteria in Theorem 1 are satisfied.

For the regular network ($p=0$), one has $\eta = 131.580$; from corollary, one can find $\lambda_{l^*} < -3.6531$. As $\lambda_{25} = -3.5984$ and $\lambda_{26} = -4.0041$, we can choose $l^* = 26$. By numerical simulation, the synchronization error quickly tends to zero for $l = 15 < l^*$. This further indicates that $l > l^*$ is only a sufficient condition, and a small value of l can also lead to synchronization.

The initial conditions of the numerical simulations are as follows: $x_i(0) = (4 + 0.5i, 5 + 0.5i, 6 + 0.5i)^T$ and $s(0) = (4, 5, 6)^T$. Synchronous errors are shown in Figure 2. And we choose different control periods, and the synchronous errors are shown in Figure 3. It is seen from the figures that small-world or scale-free networks can reach complete synchronization by pin-controlling fewer nodes than regular systems.

FIGURE 2: Synchronizability in system (38) with different topological architecture ($N = 200$) by pinning control: (a) regular networks with 8 neighbours, $l = 15$, (b) WS small-world networks with the connection probability $p = 0.1$, $l = 5$, and (c) scale-free networks, $l = 2$.

FIGURE 3: Synchronizability in system (38) with different periods: (a) $T = 0.2$; (b) $T = 0.5$.

5. Conclusions

In this paper, we investigate adaptive pinning synchronization in complex delayed dynamical networks with time-varying delays by intermittent control. Based on the Lyapunov stability theory and chaos control method, several adaptive synchronization criteria are obtained. Our results show that the control width does not need to be larger than the time delays, and there is no restriction on the size of time delays. Moreover, we also find that small-world or scale-free networks can reach complete synchronization by pin-controlling fewer nodes than regular systems.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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