Research Article

A High-Order Numerical Method for a Nonlinear System of Second-Order Boundary Value Problems

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This paper is concerned with a high-order numerical scheme for nonlinear systems of second-order boundary value problems (BVPs). First, by utilizing quasi-Newton’s method (QNM), the nonlinear system can be transformed into linear ones. Based on the standard Lobatto orthogonal polynomials, we introduce a high-order Lobatto reproducing kernel method (LRKM) to solve these linear equations. Numerical experiments are performed to investigate the reliability and efficiency of the presented method.

1. Introduction

Nonlinear systems of second-order BVPs are widely used in applied physics, mechanical engineering, biology, etc. In this article, we mainly focus on numerical solutions for such problems. In general, these systems can be described by

\[
\begin{align*}
\mathcal{F}(u,v) &= f, & \quad x \in (a,b), \\
\mathcal{G}(u,v) &= g, & \quad x \in (a,b), \\
\end{align*}
\]

in which \(\mathcal{F}\) and \(\mathcal{G}\) represent the nonlinear terms, \(a_i(x), b_i(x) \in C[a,b]\) for \(i = 1, 2, 3\), and \(f, g\) are given functions defined in \([a,b]\).

Concerning the existence and uniqueness of solution of problem (1), we refer to [1–3]. Nevertheless, to the best of our knowledge, research studies on numerical methods for nonlinear systems of second-order BVPs were seldom reported. Recently, several techniques for approximate solutions of such systems have been presented. Arqub and Abo-Hammour [4] suggested a continuous genetic algorithm to deal with second-order linear systems. In [5, 6], the authors combined homotopy perturbation methods and iterative RKM to solve problem (1). In [7–9], Peng et al. proposed an alternative iterative method called symplectic method to solve nonlinear BVPs. Furthermore, methods such as sinc-collocation methods and Jacobi matrix methods were also considered in [10, 11].

In this work, QNM [12] and simplified reproducing kernel method (SRKM) [13] are combined to design a numerical method for solving nonlinear systems (1). It is worth noting that QNM acts directly on operator equations to linearize nonlinear problems by Fréchet derivative. Here we extend QNM to solve nonlinear systems (1). Many improved versions of RKM have been developed to solve...
linear second-order BVPs [14–21]. However, the high-order RKM has been seldom discussed. Motivated by designing RKM of high-order accuracy, we establish a reproducing kernel space with polynomial form by applying the well-known Lobatto orthogonal polynomials. NDM converges fast and LRMK avoids the Schmidt orthogonalization process and uses very few points. Therefore, QDM-LRKM is efficient and simple to implement.

The article is built up as follows. Several fundamental definitions about Fréchet derivative and Hilbert spaces are recalled in Section 2. The QNM-LRKM is introduced in Section 3. Some numerical examples validate the robustness and effectiveness in Section 4. Finally, a summary is provided in the last section.

2. Preliminaries

To simplify our discussion and demonstrate the idea clearly, we rewrite system (1) into a system with homogenous boundary conditions and the domain defined on [−1, 1]. Actually, by a scaling from [a, b] to [−1, 1], that is, \[ x = ((b - a)/2)t + ((b + a)/2), \] we can derive a system defined on [−1, 1] as follows:

\[
\begin{align*}
\ddot{u}'' + \ddot{a}_1(t)\ddot{u} + \ddot{a}_2(t)v'' + \ddot{a}_3(t)v + \dddot{F}(\ddot{u}; \ddot{v}) &= \dddot{f}(t); \quad t \in (-1, 1), \\
\dddot{v}'' + \dddot{b}_1(t)v' + \dddot{b}_2(t)u'' + \dddot{b}_3(t)u' + \dddot{G}(\dddot{u}; \dddot{v}) &= \dddot{g}(t); \quad t \in (-1, 1), \\
\ddot{u}(-1) &= \ddot{u}(1) = 0, \quad \ddot{v}(-1) = \ddot{v}(1) = 0,
\end{align*}
\]

where \( \ddot{a}_1(t) = ((b - a)/2)a_1(x(t)), \ddot{a}_2(t) = a_2(x(t)), \ddot{a}_3(t) = ((b - a)/2)a_3(x(t)), \ddot{b}_1(t) = ((b - a)/2)b_1(x(t)), \ddot{b}_2(t) = b_2(x(t)), \ddot{b}_3(t) = ((b - a)/2)b_3(x(t)), \dddot{F}(\ddot{u}; \ddot{v}) = ((b - a)^2/4)F(\dddot{u}; \dddot{v}), \dddot{G}(\dddot{u}; \dddot{v}) = ((b - a)^2/4)G(\dddot{u}; \dddot{v}) \]

and \( f(x(t)) \) and \( g(t) = ((b - a)^2/4)g(x(t)) \).

2.1. Several Reproducing Kernel Spaces and Hilbert Spaces.

Denote by \( \mathcal{S}_n \) the space of polynomial functions with degrees no more than \( n \). Let \( P_n \) and \( L_n \) be the well-known Legendre and Lobatto polynomials, respectively. To be more precise,

\[
P_0(t) = 1, \quad P_1(t) = t, \quad P_n(t) = \frac{2n - 1}{n} t P_{n-1}(t) - \frac{n - 1}{n} P_{n-2}(t), \quad n = 2, 3, \ldots,
\]

satisfying the orthogonality condition on \([−1, 1] \), which means

\[
(P_m, P_n) = \int_{-1}^{1} P_m(t)P_n(t)dt = \begin{cases} \frac{2}{2n + 1}, & m \neq n, \\ 0, & m = n. \end{cases}
\]

(4)

Furthermore, Lobatto polynomials are defined by

\[
L_0(t) = \frac{1 - t}{2}, \quad L_1(t) = \frac{1 + t}{2},
\]

\[
L_n(t) = \int_{-1}^{t} P_{n-1}(s)ds = \frac{2}{2n + 1} (P_n - P_{n-2}),
\]

\[ n = 2, 3, \ldots. \]

Clearly, Lobatto polynomials satisfy

\[
L_n(-1) = L_n(1) = 0, \quad n = 2, 3, \ldots
\]

Let \( L^2[−1, 1] = \left\{ u \mid \int_{-1}^{1} u^2(x)dx < \infty \right\} \) and \( W_n[−1, 1] \) be the standard linear inner product space of polynomials with degrees not exceeding \( n \), namely,

\[
\langle u, v \rangle_{W_n} = \int_{-1}^{1} uvdx, \quad \forall u, v \in W_n[−1, 1],
\]

and the norm

\[
\|u\|_{W_n[−1, 1]} = \sqrt{\langle u, u \rangle_{W_n}}, \quad \forall u \in W_n[−1, 1].
\]

(8)

From Theorem 3.7 of [22], we obtain that the inner space \( W_n[−1, 1] \) is a Hilbert space as \( W_n[−1, 1] \) is a finite dimension closed subspace of \( L^2[−1, 1] \).

Definition 1

\[
W_n[−1, 1] = \{ u \mid u \in W_n[−1, 1], \quad u(-1) = u(1) = 0 \}.
\]

(9)

Let \( \{e_i\}_{i=2}^{n} \) be the standard orthonormal basis functions obtained by the Gram–Schmidt process of \( \{L_i\}_{i=2}^{n} \).

Lemma 1. \( W_n[−1, 1] \) is a reproducing kernel space with kernel function

\[
K(s, t) = \sum_{i=2}^{n} e_i(t)e_i(s).
\]

(10)

Definition 2

\[
W_n[−1, 1] = W_n[−1, 1] \oplus W_n[−1, 1] = \left\{ (u_1, u_2)^T \mid u_1, u_2 \in W_n[−1, 1] \right\}.
\]

(11)

We equip the following inner product and norm for \( W_n[−1, 1] \):

\[
\langle (u_1, u_2), (v_1, v_2) \rangle = \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle, \quad \| (u_1, u_2) \| = \sqrt{\langle (u_1, u_2), (u_1, u_2) \rangle}.
\]
\[ \langle U, V \rangle_{W_n} = \sum_{i=1}^{2} \langle u_i, v_i \rangle_{W_i}, \quad U, V \in W_n \subseteq \mathbb{L}^2, \]

\[ \| U \|_{W_n}^2 = \| u_1 \|_{W_1}^2 + \| u_2 \|_{W_2}^2, \quad U \in W_n. \]  \tag{12}

Under such definitions, we obtain that \( W_n \subseteq \mathbb{L}^2 \) is a Hilbert space. Similarly, we can define a subspace \( W_0^\prime \subseteq \mathbb{L}^2 \) of \( W_n \subseteq \mathbb{L}^2 \),

\[ W_n \subseteq \mathbb{L}^2 = W_0^\prime \otimes W_0^\prime, \]  \tag{13}

2.2. Fréchet Derivative. Let \( X \) and \( Y \) be Banach spaces with norms \( \| \cdot \|_X \) and \( \| \cdot \|_Y \), respectively. \( \mathcal{F} \) denotes an operator from \( X \) to \( Y \).

**Definition 3** (see [23]). We say that the linear operator \( \mathcal{A} \) is the Fréchet derivative of \( \mathcal{F} \) at \( u_0 \in X \) if

\[ \lim_{h \to 0} \frac{\| \mathcal{F}(u_0 + h) - \mathcal{F}(u_0) - \mathcal{A}(h) \|_Y}{\| h \|_X} = 0, \]  \tag{14}

and \( \mathcal{A} \) is denoted by \( \mathcal{F}'(u_0) \).

Let \( \mathcal{F}: X_1 \times X_2 \to Y \), \( \mathcal{A}_1: X_1 \times X_2 \to Y \), and \( \mathcal{A}_2: X_1 \times X_2 \to Y \), where \( X_1 \) and \( X_2 \) are also Banach spaces with norms \( \| \cdot \|_{X_1} \) and \( \| \cdot \|_{X_2} \).

**Definition 4**. We say that the linear operator \( \mathcal{A}_1 \) is the partial Fréchet derivative of \( \mathcal{F} \) on \( u \) at \( (u_0, v_0) \in X_1 \times X_2 \) if

\[ \lim_{h \to 0} \frac{\| \mathcal{F}(u_0 + h, v_0) - \mathcal{F}(u_0, v_0) - \mathcal{A}_1(h, v_0) \|_Y}{\| h \|_{X_1}} = 0, \]  \tag{15}

and \( \mathcal{A}_1 \) is denoted by \( \mathcal{F}_1'(u_0, v_0) \).

Similarly, we can define the partial Fréchet derivative of \( \mathcal{F} \) at \( (u_0, v_0) \in X_1 \times X_2 \) with respect to \( v \) by \( \mathcal{F}_v'(u_0, v_0) \).

3. QNM-SRKM for Solving Equation (2)

In this section, we present the QNM-SRKM to solve the nonlinear system (2).

3.1. Analysis of QNM. By equation (1), we define \( \mathcal{L}_{ij}: L_{x}^2 \longrightarrow L_{x}^2 \) as

\[ \mathcal{L}_{11}: \tilde{u} \longrightarrow \tilde{u}'' + \tilde{a}_1 \tilde{u}', \quad \mathcal{L}_{12}: \tilde{v} \longrightarrow \tilde{a}_2 \tilde{v}' + \tilde{a}_3 \tilde{v}, \]

\[ \mathcal{L}_{21}: \tilde{u} \longrightarrow \tilde{b}_1 \tilde{u}', \quad \mathcal{L}_{22}: \tilde{v} \longrightarrow \tilde{b}_2 \tilde{v}' + \tilde{b}_3 \tilde{v}. \]  \tag{16}

Let

\[ \mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}, \]

\[ \mathcal{N}(U) = \begin{pmatrix} \mathcal{F}(\tilde{u}, \tilde{v}) \\ \mathcal{F}(\tilde{u}, \tilde{v}) \end{pmatrix}. \]  \tag{17}

One can correspondingly rewrite equation (2) into the following operator form:

\[ \begin{cases} \mathcal{L}U + \mathcal{M}U = F, \\ U(-1) = 0, U(1) = 0, \end{cases} \]  \tag{18}

where \( U = (\tilde{u}, \tilde{v})^T \) and \( F = (\tilde{f}, \tilde{g})^T \in L_{x}^2 [-1, 1] \).

**Theorem 1.** The Fréchet derivative of \( \mathcal{N} \) at \( U_0 \) is

\[ \mathcal{N}'(U_0) = \begin{pmatrix} \mathcal{F}_{u}(u_0, v_0) & \mathcal{F}_{v}(u_0, v_0) \\ \mathcal{F}_{u}(u_0, v_0) & \mathcal{F}_{v}(u_0, v_0) \end{pmatrix} U, \]  \tag{19}

To show Fréchet derivative of the nonlinear operator more clearly, we introduce two examples.

**Example 1.** Let \( \mathcal{N}'U = \left( \begin{array}{c} x^2 + u^2 \\ u + x^2 v^2 \end{array} \right) \), then

\[ \mathcal{N}'(U_0)U = \left( \begin{array}{c} 2u_0(x) x \mathcal{J} x \mathcal{J} \mathcal{F} + 2x^2 v_0(x) \end{array} \right) U, \] where \( \mathcal{J} \) is identity operator.

**Example 2.** Let \( \mathcal{N}'U = \left( \begin{array}{c} u'' + x e^{u} + v^3 \\ v'' + u^3 + x \sin v \end{array} \right) \), then

\[ \mathcal{N}'(U_0)U = \left( \begin{array}{c} u'' \\ v'' \end{array} \right) + \left( \begin{array}{c} x e^{u_0} 3v_0^2 \\ 2u_0^3 + x \sin v_0 \end{array} \right) U. \]

Now, we will introduce the QNM to linearize the operator equation (18).

The Quasi-Newton’s Scheme: Finding \( \{ U_k \}_{k=1}^{\infty} \) such that

\[ \begin{cases} \mathcal{L}U_{k+1} + \mathcal{M}U_{k+1} = F, \\ U(1) = 0, \end{cases} \]  \tag{20}

3.2. Analysis of LRKM. By improving the RKM, an effective and simple algorithm is derived to solve the linear system (20). For convenience, we rewrite (20) into the following form:

\[ \begin{cases} \mathcal{L}U = F, \\ U(-1) = 0, U(1) = 0. \end{cases} \]  \tag{21}

**Lemma 2.** \( \mathcal{L} \) is linear and bounded.

**Proof.** Obviously, \( \mathcal{L}_{11}, \mathcal{L}_{12}, \mathcal{L}_{21}, \) and \( \mathcal{L}_{22} \) are linear operators. For \( \forall U \in W_{n} \), we have

\[ \| \mathcal{L}U \|_{L_{x}^2}^2 \leq \sum_{i=1}^{2} \left( \| \mathcal{L}_{i1}U \|_{L_{x}^2}^2 + \| \mathcal{L}_{i2}V \|_{L_{x}^2}^2 \right), \]

\[ \leq 2 \sum_{i=1}^{2} \left( \left( \| \mathcal{L}_{i1}U \|_{W_{n}} + \| \mathcal{L}_{i2}V \|_{W_{n}} \right)^2 \right) \]

\[ \leq 2 \sum_{i=1}^{2} \left( \| \mathcal{L}_{i1}U \|_{W_{n}}^2 + \| \mathcal{L}_{i2}V \|_{W_{n}}^2 \right) \]

\[ \leq 2 \sum_{i=1}^{2} \left( \| \mathcal{L}_{i1}U \|_{L_{x}^2}^2 + \| \mathcal{L}_{i2}V \|_{L_{x}^2}^2 \right) \cdot \| U \|_{W_{n}}. \]  \tag{22}

The boundedness of \( \| \mathcal{L}_{i1} \|_{L_{x}^2}^2 \) results in the boundedness of the operator \( \mathcal{L} \).
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wherein \( x_i \) \( i = 1, 2, \ldots, n \) are different points in \([-1, 1]\), \( L^* \) is the conjugate operator of \( L \), and \( R \) is the reproducing kernel of \( L_2 [-1, 1] \).

**Lemma 3.** \( \Psi_{i1}(x) = \left( \frac{L_{11} K(x, x_i)}{L_{12} K(x, x_i)}, \frac{L_{12} K(x, x_i)}{L_{22} K(x, x_i)} \right), \quad \Psi_{i2}(x) = \left( \frac{L_{21} K(x, x_i)}{L_{22} K(x, x_i)} \right), \quad i = 1, 2, \ldots, n. \)

**Proof.** From the reproducibility of \( R \), we immediately deduce that

\[
\Psi_{i1}(x) = R^* \left( \frac{R(x, x_i)}{R(x, x_i)} \right) = \left( \frac{L_{11}^* R(x, x_i)}{L_{12}^* R(x, x_i)} \right), \quad \Psi_{i2}(x) = \left( \frac{L_{12}^* R(x, x_i)}{L_{22}^* R(x, x_i)} \right),
\]

Similarly, we have \( \Psi_{i2}(x) = \left( \frac{L_{21} K(x, x_i)}{L_{22} K(x, x_i)} \right), \quad i = 1, 2, \ldots, n. \)

**Lemma 4.** \( \{\Psi_{11}, \Psi_{21}, \ldots, \Psi_{n1}, \Psi_{12}, \Psi_{22}, \ldots, \Psi_{n2}\} \) are linearly independent in \( W_0^n [-1, 1] \).

Considering the numerical implement, we apply the SRKM proposed in [12, 13] to solve equation (21). We first denote a finite-dimensional subspace \( S_{2n} \) by

\[
S_{2n} = \text{span}\{\Psi_{21}\} \cup \{\Psi_{12}\}. \tag{25}
\]

Let \( \tilde{U}_n \) be the projection of \( U_n \) onto \( S_{2n} \) obtained by \( \text{proj}_{S_{2n}} : W_\infty \rightarrow S_{2n} \) as the projection operator. Then, \( \tilde{U}_n \) converges to \( U \) in the sense of \( \|U - \tilde{U}_n\|_{W_\infty} \rightarrow 0 \) as \( n \rightarrow \infty. \)

As \( \tilde{U}_n \in W_n [-1, 1] \), there exits undetermined coefficients \( a_{i1}, a_{i2} \) \( i = 1, 2, \ldots, n \) such that

\[
\tilde{U}_n(x) = \sum_{i=1}^{n} a_{i1} \Psi_{i1}(x) + \sum_{i=1}^{n} a_{i2} \Psi_{i2}(x). \tag{27}
\]

The LRKM reads as follows: Finding \( \tilde{U} \in W_0^n [-1, 1] \) such that

\[
\begin{align*}
L_{11} \tilde{u}(x_i) + L_{12} \tilde{v}(x_i) &= f_1(x_i), \quad i = 1, 2, \ldots, n, \\
L_{21} \tilde{u}(x_i) + L_{22} \tilde{v}(x_i) &= f_2(x_i), \quad i = 1, 2, \ldots, n.
\end{align*} \tag{28}
\]

Once \( \tilde{U}_n \) is obtained, we can get an approximate solution \( U_n \) of \( U \) by applying scaling from \([-1, 1]\) to \([a, b]\):

\[
U_n(x) = \tilde{U} \left( \frac{2}{b-a} x - \frac{b+a}{b-a} \right). \tag{29}
\]

Next, we will discuss the convergence of LRKM.

**Theorem 2.** \( \tilde{u}_n(x) \) and \( \tilde{v}_n(x) \) uniformly converge to \( \tilde{u}(x) \) and \( \tilde{v}(x) \) on \([-1, 1] \) as \( n \rightarrow \infty \), respectively.

**Proof.**

\[
\begin{align*}
\|\tilde{u}_n - \tilde{u}\| &\leq \|\tilde{u}_n - \tilde{u}\|_{W_2} + \|\tilde{u} - \tilde{u}\|_{W_2} + \|\tilde{u} - \tilde{u}\|_{W_2} + \|\tilde{u} - \tilde{u}\|_{W_2} \\
&\leq \sqrt{\|\tilde{u}_n - \tilde{u}\|_{W_2}^2 + \|\tilde{u} - \tilde{u}\|_{W_2}^2 + \|\tilde{u} - \tilde{u}\|_{W_2}^2 + \|\tilde{u} - \tilde{u}\|_{W_2}^2}.
\end{align*} \tag{30}
\]

From the continuity of \( K \), we have \( \|K\|_{W_2} \leq M. \) Notice that \( \|\tilde{U}_n - \tilde{U}\|_{W_2} \rightarrow 0 \) as \( n \rightarrow \infty \), and we conclude that

\[
\|\tilde{u}_n(x) - \tilde{u}(x)\| \leq M \|\tilde{U}_n - \tilde{U}\| \rightarrow 0. \tag{31}
\]

Analogously, \( \|\tilde{v}_n(x) - \tilde{v}(x)\| \leq M \|\tilde{U}_n - \tilde{U}\| \rightarrow 0. \) The proof is completed.

**4. Numerical Experiments**

**Example 3.** Let us consider a linear problem suggested in [5, 10]:

\[
\begin{align*}
\frac{d^2 u}{dx^2}(x) + xu(x) + xu(x) + v(x) + 2xv(x) &= f(x), & x \in [0, 1], \\
2u(x) + x^2u(x) + v(x) + v(x) &= g(x), & x \in [0, 1], \\
u(0) = u(1) = 0, & v(0) = v(1) = 0.
\end{align*} \tag{32}
\]

Here, \( f(x) = \pi \cos(\pi x) - 2(1 + \cos(x) + 4x - 2x^2 - 4\sin(x) + 2\sin(\pi x), \quad g(x) = 1 - \pi^2 \sin(x) + 2(-x^2 + x^2 + x - 3)\sin(x) - 4\cos(x). \) The exact solution is \( u(x) = 2(1 - x)\sin(x), \quad v(x) = \sin(\pi x). \) Applying the LRKM, absolute errors are presented in Table 1 and Figure 1. Besides, we compare our method with RKM [5] and the sinc-collocation method [10] in Table 2. Numerical results show that LRKM has high accuracy in solving the linear equation, but uses the least number of points.

**Example 4.** Let us consider a nonlinear system proposed in [5, 10]:

\[
\begin{align*}
\frac{d^2 u}{dx^2}(x) + xu(x) + x^2u(x) + x^2u(x) &= f, & x \in [0, 1], \\
\frac{d^2 v}{dx^2}(x) + v(x) + x^2u(x) + \sin(x)v(x) &= g, & x \in [0, 1], \\
u(0) = u(1) = 0, & v(0) = v(1) = 0.
\end{align*} \tag{33}
\]

where \( f = 2x\sin(\pi x) + x^2 - 2x^4 + x^2 - 2 \) and \( g = x^3(1 - x) + (1 - \pi^2)\sin(\pi x) + \sin(\pi x)\sin^2(\pi x). \) The exact solution is \( u = u = x^2; \quad v = \sin(\pi x). \) Selecting initial functions \( u_0, v_0 \) as the polynomial that satisfies the boundary conditions and applying the technique of QNM-LRKM three times, we depict numerical results of \( u(x_i) \) and \( v(x_i) \) in Table 3 and Figure 2. Numerical comparison with homotopy analysis and collocation method for problem 2 is shown in Table 4. Numerical results indicate that our method uses fewer nodes but obtains the more accurate numerical solution. We take \( n = 11 \) in LRKM (28) and different iteration steps...
Table 1: Absolute errors for Example 3.

<table>
<thead>
<tr>
<th>n</th>
<th>|u - u|_{C}</th>
<th>|v - v|_{C}</th>
<th>|u' - u'|_{C}</th>
<th>|v' - v'|_{C}</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3.32 × 10^{-4}</td>
<td>2.09 × 10^{-3}</td>
<td>1.51 × 10^{-3}</td>
<td>1.44 × 10^{-2}</td>
</tr>
<tr>
<td>7</td>
<td>1.80 × 10^{-6}</td>
<td>9.60 × 10^{-5}</td>
<td>1.85 × 10^{-5}</td>
<td>2.18 × 10^{-4}</td>
</tr>
<tr>
<td>9</td>
<td>9.44 × 10^{-9}</td>
<td>1.20 × 10^{-7}</td>
<td>1.13 × 10^{-7}</td>
<td>1.89 × 10^{-6}</td>
</tr>
<tr>
<td>11</td>
<td>4.21 × 10^{-10}</td>
<td>7.28 × 10^{-9}</td>
<td>1.28 × 10^{-9}</td>
<td>2.86 × 10^{-8}</td>
</tr>
</tbody>
</table>

Table 2: Comparison with RKM [5] and the sinc-collocation method [10] for Example 3.

<table>
<thead>
<tr>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>|u - u|_{C}</td>
<td>|v - v|_{C}</td>
<td>|u - u|_{C}</td>
</tr>
<tr>
<td>0.08</td>
<td>8.0 × 10^{-4}</td>
<td>1.9 × 10^{-3}</td>
<td>3.0 × 10^{-4}</td>
</tr>
<tr>
<td>0.24</td>
<td>1.9 × 10^{-3}</td>
<td>5.1 × 10^{-3}</td>
<td>8.5 × 10^{-4}</td>
</tr>
<tr>
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<td>6.9 × 10^{-3}</td>
<td>3.5 × 10^{-4}</td>
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<td>2.4 × 10^{-3}</td>
<td>6.9 × 10^{-3}</td>
<td>2.6 × 10^{-4}</td>
</tr>
<tr>
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<td>5.2 × 10^{-3}</td>
<td>2.0 × 10^{-4}</td>
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</tr>
<tr>
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<td>2.0 × 10^{-4}</td>
<td>8.0 × 10^{-4}</td>
<td>2.6 × 10^{-3}</td>
</tr>
</tbody>
</table>

Table 3: Absolute errors for Example 4.

<table>
<thead>
<tr>
<th>n</th>
<th>|u - u|_{C}</th>
<th>|v - v|_{C}</th>
<th>|u' - u'|_{C}</th>
<th>|v' - v'|_{C}</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2.92 × 10^{-5}</td>
<td>2.55 × 10^{-4}</td>
<td>1.50 × 10^{-3}</td>
<td>1.46 × 10^{-2}</td>
</tr>
<tr>
<td>7</td>
<td>3.62 × 10^{-7}</td>
<td>1.95 × 10^{-5}</td>
<td>1.88 × 10^{-5}</td>
<td>2.12 × 10^{-4}</td>
</tr>
<tr>
<td>9</td>
<td>1.84 × 10^{-9}</td>
<td>9.84 × 10^{-8}</td>
<td>1.14 × 10^{-7}</td>
<td>1.88 × 10^{-6}</td>
</tr>
<tr>
<td>11</td>
<td>2.67 × 10^{-10}</td>
<td>2.43 × 10^{-9}</td>
<td>2.73 × 10^{-9}</td>
<td>4.42 × 10^{-8}</td>
</tr>
</tbody>
</table>

Figure 1: Absolute error \|u - u_0\| (a) and \|v - v_0\| (b) for Example 3.

Figure 2: Absolute error \|u - u_0\| (a) and \|v - v_0\| (b) for Example 4.
Table 4: Comparison with homotopy method and collocation method for Example 4.

<table>
<thead>
<tr>
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<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$|u - u|_{C}$</td>
<td>$|v - v|_{C}$</td>
<td>$|u - u|_{C}$</td>
</tr>
<tr>
<td>0.08</td>
<td>$1.4 \times 10^{-4}$</td>
<td>$2.4 \times 10^{-4}$</td>
<td>$5.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.24</td>
<td>$4.4 \times 10^{-2}$</td>
<td>$2.3 \times 10^{-2}$</td>
<td>$1.4 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.40</td>
<td>$6.7 \times 10^{-3}$</td>
<td>$8.9 \times 10^{-4}$</td>
<td>$2.1 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.56</td>
<td>$9.3 \times 10^{-3}$</td>
<td>$1.4 \times 10^{-3}$</td>
<td>$2.2 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.72</td>
<td>$4.9 \times 10^{-3}$</td>
<td>$3.1 \times 10^{-3}$</td>
<td>$1.8 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.88</td>
<td>$8.6 \times 10^{-3}$</td>
<td>$1.6 \times 10^{-3}$</td>
<td>$9.0 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.96</td>
<td>$7.1 \times 10^{-3}$</td>
<td>$9.8 \times 10^{-3}$</td>
<td>$3.0 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

$k$, and absolute errors are shown in Table 5. Numerical results indicate that the convergence order of NDM is nearly 2.

5. Conclusions

This paper presents an efficient numerical method for solving second-order nonlinear system of BVPs. The algorithm is easy to implement. Numerical results verify that the QNM-LRKM is a reliable numerical technique of high-order accuracy. Precisely, NDM ensures the speed of iteration, while LRKM guarantees the accuracy of the algorithm.

Data Availability

No datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References


