Research Article

A Novel Multicriteria Group Decision-Making Approach with Hesitant Picture Fuzzy Linguistic Information

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Received 9 September 2019; Revised 7 December 2019; Accepted 17 December 2019; Published 14 February 2020

1.Introduction

In recent years, as one of the useful tools to solve MCGDM problems, the fuzzy linguistic set-based methods [1] have been a focus of research. Along with more and more complex decision-making environment, many scholars proposed different extensions of fuzzy linguistic term sets. In order to avoid the information loss for computing with words, Herrera and Martinez [2] put forward a 2-tuple fuzzy linguistic representation model. Then, some different 2-tuple fuzzy linguistic sets-based approaches are proposed to handle different decision-making problems [3–5]. Focusing on aggregating the criteria information, Wei [6] used 2-tuple linguistic information-based generalized weighted average and geometric operators, Wan [7] developed the 2-tuple linguistic hybrid aggregation operator, and Liu et al. [8] developed 2-tuple linguistic Heronian mean operators; these methods enormously enriched the decision-making theory. However, in some decision circumstances, interval number can better express the uncertainty of decision-maker information. Thus, uncertain linguistic set-based approaches [9–12] were proposed. Subsequently, to extend the 2-tuple fuzzy set with uncertain linguistic set, Zhang [13] defined several interval-valued 2-tuple linguistic aggregation operators, and Liu et al. [14] defined some new operation rules for 2-tuple linguistic term set and developed Bonferroni mean operators. All of the above approaches were illustrated by the practical application cases, and their effectiveness was demonstrated.

However, in some cases, the membership degree cannot be defined by one specific value; thus, Wang and Li [15] proposed intuitionistic fuzzy linguistic set. Then, intuitionistic fuzzy linguistic set-based approaches [16–19] were proposed. Due to the different demands of practical
decision-making problems, the intuitionistic fuzzy linguistic set is extended by many scholars. With regard to preference relation, Zhang et al. [20] defined 2-tuple intuitionistic fuzzy linguistic preference relation. For the information aggregation operator, Liu et al. [21] defined the hesitant intuitionistic fuzzy linguistic weighted average (HIFLWA) operator, and Meng et al. [22] developed interval-valued intuitionistic uncertain linguistic hybrid Shapley operators. In addition, Tan et al. [23] put forward an extended single-valued neutrosophic projection-based qualitative flexible aggregation operator, and Meng et al. [22] developed interval-valued linguistic preference relation. For the information aggregation operator, Liu et al. [21] defined the hesitant intuitionistic fuzzy linguistic weighted average (HIFLWA) operator, and Meng et al. [22] developed interval-valued intuitionistic uncertain linguistic hybrid Shapley operators. In addition, Tan et al. [23] put forward an extended single-valued neutrosophic projection-based qualitative flexible aggregation operator.

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Considering that decision-maker may be hesitant to give the evaluation value using the linguistic term, hesitant fuzzy linguistic set [24] was defined. Then, hesitant fuzzy linguistic set-based methods [25–27] were proposed. In order to effectively address practical decision-making issues, many extensive approaches are conducted. Zhou et al. [28] extended hesitant linguistic set with evidence reasoning. Wang et al. [29] extended the hesitant linguistic set with interval number and defined the interval-valued hesitant linguistic set. Pang et al. [30] extended the hesitant fuzzy set with probability distribution and defined probability linguistic term sets. Lin et al. [31] extended the hesitant fuzzy linguistic term with probability distribution and interval number and proposed a probability uncertainty hesitant fuzzy linguistic set. All the above methods based on the extension of hesitant fuzzy linguistic set are verified by an illustration example and the effectiveness is demonstrated.

Although these existing linguistic set-based approaches can demonstrate the effectiveness for solving the practical MCGDM issues, under some decision-making situations, they cannot be applied effectively. For example, one listed company carried on portfolio optimization, and there are four plans for the voting by the stakeholder. Some stakeholders may give “vote for,” some stakeholders may give “vote against,” some stakeholders may give “abstain,” and the rest of the stakeholders may refuse to vote. For such kind of decision-making problems, it is required to propose a new method.

Picture fuzzy set (PFS) was firstly proposed by Cătălin [32], which is the extension of the intuitionistic fuzzy set. After then, in order to solve the MCGDM voting problems, several picture fuzzy set-based MCGDM methods have been proposed [33–38]. With regard to measurement, Singh et al. developed the approach with picture fuzzy correlation coefficients [39], Wei et al. [40] introduced a novel method to solve MCGDM voting problems using cross-entropy of the picture fuzzy set. As for aggregation operators, Wei et al. [41] used the picture 2-tuple linguistic-based operators, including Bonferroni mean operator, weighted averaging operator, ordered weighted averaging operator, and hybrid averaging operator [42]. In addition, Nie et al. [43] put forward the MCGDM voting method based on 2-tuple linguistic picture preference relation and applied to the selection of voting the proxy advisor firm by a stakeholder. Nevertheless, the proposed picture fuzzy set-based methods are enriched and used to well solve the MCGDM issues. In the practical cases, such as the students rank their teacher’s teaching performance, it is common for students to provide the assessment value with positive, indeterminacy, negative, or refusal attitude information. For example, “I cannot give the teaching performance with the score “outstanding excellent,” but it is ok for the “excellent” or “I refuse to give any evaluation value.” For such kind of practical MCGDM problems, they cannot be appropriately solved by the existing fuzzy linguistic set-based MCGDM approaches. Thus, it is necessary to extend the existing fuzzy linguistic set to HPFLSs.

HPFLSs can elaborate the advantages of both hesitant linguistic set and picture fuzzy set, which are more suitable for the practical hesitant case of the decision-makers with positive indeterminacy, negative, and refusal information. For example, ten students are required to give the score for teaching quality (one of the criteria of teaching performance) for mathematics. Three students refuse to give any evaluation value, seven students oppose to give the teaching performance with the score “outstanding excellent,” but they agree for the “excellent.” Thus, if the evaluation term set is \( S = \{s_4 = \text{excellent}, s_3 = \text{excellent}, s_5 = \text{merit}, s_2 = \text{average}, s_1 = \text{pass}, s_0 = \text{failure}\} \), then the evaluation score can be collected as \( \langle\{s_4\}, \{0.7, 0.0\}\rangle, \{s_3\}, \{0.0, 0.7\}\rangle \) with HPFLSs.

In the following, the contribution of this paper is presented:

1. In view of the defects of operation rules appeared in the methods of linguistic term set (LTS) and picture fuzzy linguistic set methods, the new operation rules based on sigmoid and equivalent transformation functions are proposed to overcome the existing limitation.

2. The new definition of hesitant picture fuzzy linguistic sets (HPFLSs) is introduced, it is more flexible to describe the two more possible linguistic terms for voting decision-making than picture fuzzy linguistic sets.

3. In order to solve MCGDM voting problems under the completely unknown criteria weight environment, the max-min deviation model for HPFLSs is constructed, and then the criteria weighted is computed by using Lagrange functions.

4. In order to develop an effective approach with HPFLSs to solve practical decision-making problems, the average weighted operator of HPFLS (HPFLSWA) and hesitant picture fuzzy linguistic weighted geometric operator (HPFLSWSG) are developed in this paper. Meanwhile, the comparison methods for HPFLSs are introduced. Then, based on the proposed operators and comparison approaches, an effective multicriteria group decision-making (MCMD) voting method under HPFLSs is developed.

The rest of this paper is organized as follows: several definitions and operation laws related to PFS and TLSs are reviewed in Section 2. In Section 3, the definition and novel basic operation rules of HPFLSs are introduced, and the
HPFLSWA and HPFLSWG operators are developed and min-max deviation method, which is used to gain the criteria weight for completely unknown, is conducted; moreover, the ordering comparison methods for HPFLSs are presented. Section 4 provides a voting method of MCGDM based on the HPFLSWA and HPFLSWG operators, and its steps are described in detail. In Section 5, the real teaching performance evolution from Hubei University of Automotive Technology is used to illustrate the effectiveness of the proposed method, and according to the decision-maker preference, how to select the approximate value of parameter \( \theta \) is discussed. Conclusions are drawn in Section 6.

2. Preliminaries

In this section, the definitions of PFS, 2-tuple fuzzy linguistic sets (2TFLs), picture 2-tuple fuzzy linguistic sets (P2TFLSs), and 2-tuple linguistic picture fuzzy sets (2TLPSs) are presented, and the operation rules of P2TFLSs are reviewed to lay the groundwork for later analysis.

2.1. Definitions of PFS, 2TFLSs, P2TFLSs, and 2TLPSs

**Definition 1** (see [32]). Let \( X \) be a universe space, and a PFS is defined as

\[
A = \{ (x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X \},
\]

where \( \mu_A(x) \in [0, 1] \) is called the degree of positive membership of \( x \) in \( A \), \( \eta_A(x) \in [0, 1] \) is called the degree of neutral membership of \( x \) in \( A \), \( \nu_A(x) \in [0, 1] \) is called the degree of negative membership of \( x \) in \( A \), and \( \mu_A(x), \eta_A(x), \) and \( \nu_A(x) \) satisfy the condition \( 0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1 \). Moreover, \( \pi_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)) \) can be called the degree of refusal.

Let \( S = \{ s_j \mid j = 1, 2, \ldots, g \} \) be a linguistic term set, and symbolic method aggregation linguistic information obtains a value \( \beta \in [0, g) \), and if \( \beta \notin [0, g) \), then an approximation function \( \text{app}_\beta(\cdot) \) is used to express the index result of \( S \).

**Definition 2** (see [41]). Let \( \beta \) be the aggregation result of the indices of a set of labels assessed in a linguistic term set \( S \), i.e., the results of symbolic aggregation operation, and \( \beta \in [1, t] \) be the \( t \) cardinality of \( S \). Let \( i = \text{round}(\beta) \) and \( \alpha = \beta - i \) be two values such that \( i \in [1, t] \) and \( \alpha \in [0, 0.5] \); then, \( \alpha \) is called the symbolic translation.

Based on the PFS and Definition 2, Wei et al. [41] defined the picture 2-tuple fuzzy linguistic set.

**Definition 3**. A picture 2-tuple fuzzy linguistic set \( A \) in \( X \) is defined as

\[
A = \left\{ (s_{h(x)}, \rho), (\mu_A(x), \eta_A(x), \nu_A(x)) \right\} \mid x \in X \},
\]

where \( \mu_A(x) \in [0, 1], \eta_A(x) \in [0, 1], \) and \( \nu_A(x) \in [0, 1] \) with the condition \( 0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1 \), \( \forall x \in X \), \( s_{h(x)} \in S \), and \( \rho \in [0.5, 0.5] \), the numbers \( \mu_A(x), \eta_A(x), \) and \( \nu_A(x) \) represent the positive membership degree, neutral membership degree, and negative membership degree of the element \( x \) to linguistic variable \( \left( s_{h(x)}, \rho \right) \), respectively. Then, \( \pi_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)) \) could be called the refusal membership degree of the element \( x \) to linguistic variable \( \left( s_{h(x)}, \rho \right) \).

For convenience to collect the data in the selection of the proxy advisor firm problems, Nie et al. [43] defined the concept of 2-tuple linguistic picture fuzzy sets (2TLPSs).

**Definition 4**. Let \( S = \{ s_j \mid j = 0, 1, 2, \ldots, g \} \) be a linguistic set and \( \overline{S} = \{ s_{\mu}, s_{\eta}, s_\nu \mid s_\mu \in [0, 0.5], s_\eta \in [0.5, 0.5] \} \) be a linguistic 2-tuple fuzzy set. Suppose that \( (s_{\mu}, s_{\eta}), (s_{\eta}, s_{\eta}), (s\eta, s_\eta) \) and \( (s_\eta, s_\eta) \in S \), if \( 0 \leq \mu + \eta + \nu \leq g \), then \( \tau = \{(s_{\mu}, s_{\eta}), (s_{\mu}, s_{\eta}), (s\eta, s_{\eta}) \} \) is called 2TLPSs, and \( (s_{\mu}, s_{\eta}), (s_{\eta}, s_{\eta}), \) and \( (s\eta, s_{\eta}) \) could be called the positive membership degree, neutral membership degree, and negative membership degree of \( \tau \), respectively.

2.2. Operation Rules of Linguistic Terms Sets (LTSS) and 2TLPSs

Motivated by the aggregation function of t-norm, the operations of LTSS and 2TLPSs are constructed, as follows.

**Definition 5** (see [32]). Let \( a_1 \) and \( a_2 \) be two 2TLPSs, and the operation rules of 2TLPSs can be defined as follows:

1. \( a_1 + a_2 = \langle (\mu_1 + \mu_2, \eta_1 + \eta_2, \nu_1 + \nu_2) \rangle \)
2. \( a_1 \odot a_2 = \langle (\mu_1 \odot \mu_2, \eta_1 \odot \eta_2, \nu_1 \odot \nu_2) \rangle \)
3. \( \lambda a_1 = \langle (\lambda \mu_1, \lambda \eta_1, \lambda \nu_1) \rangle \)
4. \( a_1^\lambda = \langle (\mu_1^\lambda, \nu_1^\lambda) \rangle \), \( (\mu_1^\lambda)^1, 1 - (1 - \eta_1^\lambda)^1, 1 - (1 - \nu_1^\lambda)^1 > 0 \)

However, in some typical circumstances, the results obtained by the above operation rules can be unreasonable, and the examples are shown as follows.

**Example 1**. Suppose that two alternatives are evaluated under three criteria, the evaluation values are represented in the form of 2TLPSs as \( a_{11} = \langle (s_0, 0), (1, 0) \rangle \), \( a_{12} = \langle (s_0, 0), (0.01, 0) \rangle \), \( a_{13} = \langle (s_0, 0), (0.01, 0) \rangle \), and \( a_{21} = \langle (s_0, 0), (0.0, 0.0, 0) \rangle \), \( a_{22} = \langle (s_0, 0), (0.0, 0.0, 0) \rangle \), \( a_{23} = \langle (s_0, 0), (0.0, 0.0, 0) \rangle \). The criteria weight vector is \( w = (0.3, 0.3, 0.4) \). Obviously, the aggregated result of alternative \( a_1 \) is superior than that of \( a_1 \). However, in accordance with the above operation rules, the results are \( w_1 \odot a_{11} \odot w_2 \odot a_{12} \odot w_3 \odot a_{13} = \langle (s_0, 0), (1, 0) \rangle \) and \( w_1 \odot a_{21} \odot w_2 \odot a_{22} \odot w_3 \odot a_{23} = \langle (s_0, 0), (0.9, 0.0) \rangle \), which indicate that the aggregated result of alternative \( a_1 \) is superior than that of \( a_1 \). Thus, this result cannot be accepted.

**Example 2**. Assume that experts give three criteria score by 2TLPS for two alternatives that \( a_{11} = \langle (s_0, 0), (0.8, 0.0, 0) \rangle \), \( a_{12} = \langle (s_0, 0), (0.8, 0.0, 0) \rangle \), \( a_{13} = \langle (s_0, 0), (0.0, 0.5, 0) \rangle \), and \( a_{21} = \langle (s_0, 0), (0.3, 0.0, 0) \rangle \), \( a_{22} = \langle (s_0, 0), (0.3, 0.0, 0) \rangle \), \( a_{23} = \langle (s_0, 0), (0.0, 0.3) \rangle \). The criteria weight vector is

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*Note: The contents above are a brief summary of the provided text and may not capture all the details or nuances of the original document.*
\( w = (0.3, 0.3, 0.4) \), and it is easy to see that \( a_1 \) should be superior to \( a_2 \). However, according to the operation rules of Definition 5, the results are \((a_1)_{\text{HS}} = (a_2)_{\text{HS}} = (a_3)_{\text{HS}} = (s_1, 0, 0, 0, 0)\) and \((a_2)_{\text{HS}} = (a_3)_{\text{HS}} = (s_1, 0, 0, 0, 0)\), which indicate that the alternative \( a_1 \) is equal to alternative \( a_2 \). So, the result is not reasonable.

**Definition 6** (see [44]). Let \( s = \{s_1 | f = -r, \ldots, -1, 0, 1, \ldots, \tau\} \) be a LTS, \( s_a \) and \( s_b \) be two linguistic terms, \( g \) and \( g^{-1} \) be two equivalent transformation function of linguistic terms, and \( \lambda \) be a real number; then,

1. \( s_a + s_b = g^{-1}(g(s_a) + g(s_b)) - g(s_a) \)
2. \( s_a \otimes s_b = g^{-1}(g(s_a) \otimes g(s_b)) \)
3. \( \lambda s_a = g^{-1}(1 - (1 - g(s_a))^\lambda) \)
4. \( (s_a)^\lambda = g^{-1}((g(s_a))^\lambda) \)

Although the above operation rules can overcome some defects existed in the previous paper, it still can yield the unreasonable results under some typical situations. The example is similar to the above Example 2; thus, it is omitted here.

### 3. Hesitant Picture Fuzzy Linguistic Sets (HPFLSs)

In this section, a new concept of HPFLSs is introduced firstly. Subsequently, motivated by the operation rules of the linguistic term set \([44]\) and picture 2-tuple linguistic term sets \([41]\), to avoid the existing limitations pointed out in Examples 1 and 2, the novel operation laws of HPFLSs are proposed. Next, two aggregation operators of HPFLSs are introduced, and the related properties are discussed.

#### 3.1. Hesitant Picture Fuzzy Linguistic Sets (HPFLSs)

**Definition 7.** Let \( S = \{s_j | j = 0, 1, 2, \ldots, m\} \) be an LTS, a HPFLS is defined as

\[
H_s = \left\{ \left(\left( s_j \right)^k, \left( \mu_j^k, \eta_j^k, \upsilon_j^k \right) \right) | i \in (0, 1, 2, \ldots, m); k = 1, 2, \ldots, \alpha \right\}
\]

(3)

where \( \mu_j^k \in [0, 1] \), \( \eta_j^k \in [0, 1] \), \( \upsilon_j^k \in [0, 1] \), and \( 0 \leq \mu_j^k + \eta_j^k + \upsilon_j^k \leq 1 \). \( \mu_j^k \), \( \eta_j^k \), and \( \upsilon_j^k \) represent the positive membership, indeterminacy membership, and negative membership of the linguistic term \( s_j \), respectively; \( \pi_j^k = 1 - (\mu_j^k + \eta_j^k + \upsilon_j^k) \) is the refusal membership of the linguistic term \( s_j \). And its complementary set is \( H_s^c = \left\{ \left(\left( s_j \right)^k, (\pi_j^k, \eta_j^k, \upsilon_j^k) \right) | i \in (0, 1, 2, \ldots, m); k = 1, 2, \ldots, \alpha \right\} \).


In this subsection, two equivalent transformation functions \( f \) and \( g \) are defined. Then, the novel operation laws of HPFLSs are introduced based on the defined functions.

**Definition 8.** Let \( S = \{s_j | j = 0, 1, 2, \ldots, m\} \) be an LTS; then, the linguistic term \( s_j \) can be equivalently transformed to \( \alpha \) according to the function \( g \), and \( \alpha \) can be equivalently transformed to the linguistic term \( s_j \) according to the function \( g^{-1} \) as follows:

\[
g: [0, m] \to [a, b], \quad g(s_j) = \frac{1}{1 + e^{-\theta_1/(1 - a/m)}},
\]

\[
g^{-1}: [a, b] \to [0, m], \quad g^{-1}(a) = \frac{m}{m/a + \theta_1},
\]

where \( \theta_1 > 0, \theta_1 = \theta/m, \) and \( \theta \) are the adjustment parameters described in Definition 9.

**Definition 9.** For any \( x \in [0, 1] \), \( x \) can be equivalently transformed to \( \beta \) according to the function \( f \), and \( \beta \) can be equivalently transformed to \( x \) according to the function \( f^{-1} \) as follows:

\[
f: [0, 1] \to [c, d], \quad f(x) = \frac{1}{1 + e^{-\theta(x-0.5)}} = \beta,
\]

\[
f^{-1}: [c, d] \to [0, 1], \quad f^{-1}(\beta) = 0.5 - \ln(1 - \beta/\theta)/\varphi = x,
\]

where \( \theta > 0 \), by default, \( \theta = 16 \).

**Definition 10.** Let \( H_{s_1} = \{\langle (s_j^k), (\mu_j^k, \eta_j^k, \upsilon_j^k) \rangle | i \in (0, 1, 2, \ldots, m), k = 0, 1, 2, \ldots, \alpha \} \) and \( H_{s_2} = \{\langle (s_j^k), (\mu_j^k, \eta_j^k, \upsilon_j^k) \rangle | i \in (0, 1, 2, \ldots, m), k = 1, 2, \ldots, \alpha \} \) be two HPFLSs, \( f \), \( f^{-1} \) and \( g \), \( g^{-1} \) be the equivalent transformation functions, and \( \alpha > 0 \); then, the operation rules can be defined below:

1. \( H_{s_1} \odot H_{s_2} = \left\{ \left( \left( s_j^k \right)^*, (\mu_j^*, \eta_j^*, \upsilon_j^*) \right) \right\} \)

\[
\left( f^{-1}(1 - (1 - f(\mu_j^k))) \right) \left( 1 - f(\mu_j^k) \right), f^{-1}(f(\eta_j^k)) f^{-1}(f(\eta_j^k)), f^{-1}(f(\mu_j^k)) f^{-1}(f(\mu_j^k))
\]

(4)

(5)

(6)

(7)
Based on the above operation rules in Definition 10, we have concluded the following Theorem 1.

**Theorem 1.** Let $H_{s_1}, H_{s_2}, H_{s_3}$ be three HPFLSs, and $\lambda_i > 0, \lambda_j > 0, \lambda_k > 0,$ then the following properties are true.

1. $\lambda_1 H_{s_1} = H_{s_1}$ if and only if $\lambda_1 = 1$
2. $\lambda_1 H_{s_1} \oplus \lambda_2 H_{s_2} = \lambda_2 H_{s_2} \oplus \lambda_1 H_{s_1}$
3. $(\lambda_1 H_{s_1} \oplus \lambda_2 H_{s_2}) \oplus \lambda_3 H_{s_3} = \lambda_1 H_{s_1} \oplus (\lambda_2 H_{s_2} \oplus \lambda_3 H_{s_3})$
4. $\lambda_1 H_{s_1} \otimes (\lambda_2 H_{s_2}) = \lambda_2 H_{s_2} \otimes (\lambda_1 H_{s_1})$
5. $(\lambda_1 H_{s_1} \otimes (\lambda_2 H_{s_2})) \otimes (\lambda_3 H_{s_3}) = \lambda_1 H_{s_1} \otimes (\lambda_2 H_{s_2} \otimes (\lambda_3 H_{s_3}))$
6. $(H_{s_1}^i)^k \oplus (H_{s_2}^k)^i = (H_{s_2}^k)^i \oplus (H_{s_1}^i)^k$
7. $(H_{s_1}^i)^k \oplus (H_{s_2}^k)^i \oplus (H_{s_2}^i)^k = (H_{s_1}^i)^k \oplus ((H_{s_2}^k)^i \oplus (H_{s_2}^i)^k)$
8. $(H_{s_1}^i)^k \otimes (H_{s_2}^k)^i = (H_{s_2}^k)^i \otimes (H_{s_1}^i)^k$

**Example 3.** Let $S = \{s_j | J = 0, 1, 2, \ldots, 4\}$ be an LTS, $a_1$ and $a_2$ be two alternatives, $c = \{c_1, c_2, c_3\}$ be the criteria set, and $w = (0.2, 0.2, 0.6)^T$ be the weight vector. The corresponding evaluation values are presented in Table 1.

According to Definitions 8–10, we have $\theta_1 = 4, \theta = 16$, and $m/2 = 2$. By using the operation rules (1) and (3) in Definition 10, we obtain the aggregation results of $a_1$ is obtained as $\langle S_9, (0.8, 0, 0)\rangle$, and that of $a_2$ is $\langle S_4, (0.7976, 0.00)\rangle$. The aggregation results show that $a_1 > a_2$. Similarly, by using the operation rules (2) and (4) in Definition 10, we obtain the aggregation results for $a_1$ as $\langle S_0, (0.8, 0, 0)\rangle$ and the aggregation results for $a_2$ as $\langle S_0, (0.4399, 0.00)\rangle$. The aggregation results show that $a_1 > a_2$ as well. Intuitively, $a_1$ should be superior than $a_2$ according to the evaluation values shown in Table 1. Thus, the above results state that our proposed operation rule can effectively overcome the defects as pointed out in Examples 1 and 2. The detailed computing process of the aggregation results is omitted here.

### 3.3. Aggregation Operators of HPFLSs

In this subsection, two aggregation operators of HPFLSs are presented and can be utilized to solve the real-life decision-making problems.

**Table 1:** The evaluation values with HPFLSs.

<table>
<thead>
<tr>
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<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
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<tbody>
<tr>
<td>$a_1$</td>
<td>$\langle S_9, (0.8, 0, 0)\rangle$</td>
<td>$\langle S_9, (0.8, 0, 0)\rangle$</td>
<td>$\langle S_9, (0.8, 0, 0)\rangle$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$\langle S_4, (1.0, 0, 0)\rangle$</td>
<td>$\langle S_0, (0.1, 0, 0)\rangle$</td>
<td>$\langle S_0, (0.1, 0, 0)\rangle$</td>
</tr>
</tbody>
</table>

**Definition 11.** Let $H_{s_1} = \{\{(s_{i_1}^\varphi), (\mu_{s_1}^\varphi, \eta_{s_1}^\varphi, \nu_{s_1}^\varphi)\} | i_\varphi \in (0, 1, 2, \ldots, m); k_\varphi = 1, 2, \ldots, \alpha_\varphi, \varphi = 1, 2, \ldots, n\}$ be $n$ HPFLSs, and their weight vector is $w = (w_1, w_2, \ldots, w_n)^T$. Then, the hesitant picture fuzzy linguistic weighted average operator (HPFLWA) can be defined as follows:

$$\text{HPFLWA}(H_{s_1}, H_{s_2}, \ldots, H_{s_n}) = w_1 H_{s_1} \oplus w_2 H_{s_2} \oplus \cdots \oplus w_n H_{s_n}$$

**Definition 12.** Let $H_{s_1} = \{\{(s_{i_1}^\varphi), (\mu_{s_1}^\varphi, \eta_{s_1}^\varphi, \nu_{s_1}^\varphi)\} | i_\varphi \in (0, 1, 2, \ldots, m); k_\varphi = 1, 2, \ldots, \alpha_\varphi, \varphi = 1, 2, \ldots, n\}$ be $n$ HPFLSs, and their weight vector is $w = (w_1, w_2, \ldots, w_n)^T$. Then, the hesitant picture fuzzy linguistic weighted geometric operator (HPFLWG) is referred to as

$$\text{HPFLWG}(H_{s_1}, H_{s_2}, \ldots, H_{s_n}) = (H_{s_1})^{w_1} \otimes (H_{s_2})^{w_2} \cdots \otimes (H_{s_n})^{w_n}$$

Based on Definition 11 and the operation rules described in Definition 10, Theorem 2 can be gained.

**Theorem 2.** For $n$ HPFLSs, $H_{s_1} = \{(s_{i_1}^\varphi), (\mu_{s_1}^\varphi, \eta_{s_1}^\varphi, \nu_{s_1}^\varphi)\} | i_\varphi \in (0, 1, 2, \ldots, m); k_\varphi = 1, 2, \ldots, \alpha_\varphi, \varphi = 1, 2, \ldots, n$ and their weight vector is $w = (w_1, w_2, \ldots, w_n)^T$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, and the aggregation results can be obtained by the HPFLWA operator as

$$\text{HPFLWA}(H_{s_1}, H_{s_2}, \ldots, H_{s_n}) = \left\langle \bigcup_{k_1=1,2, \ldots, \alpha_1}^{k_2=1,2, \ldots, \alpha_2}^{k_n=1,2, \ldots, \alpha_n} \left\{ \left( g^{-1}\left( 1 - \prod_{\varphi=1}^{m} \left( 1 - g(s_{i_1}^\varphi) \eta_{s_1}^{k_1} \right) w_k \right) \right) , f^{-1}\left( 1 - \prod_{\varphi=1}^{m} \left( 1 - f(\mu_{s_1}^{k_2}) \nu_{s_1}^{k_2} \right) w_k \right) , f^{-1}\left( \prod_{\varphi=1}^{m} \left( f(\eta_{s_1}^{k_3}) \right) \right) \right\} \right\rangle.$$
Proof. (1) When $n = 2$, the following result can be gained:

\[
\begin{align*}
\omega_1 \otimes H_{s_1} &= \left\{ \bigcup_{k_i=1,2,\ldots,n_1} \left\{ \left( \omega_1 \otimes s_{1_i}^k, \left( f^{-1} \left( 1 - (1 - f(\mu_{s_1}^{k_i}))^{w_1} \right) \right) \right) \right\} \right\}, \\
\omega_2 \otimes H_{s_1} &= \left\{ \bigcup_{k_i=1,2,\ldots,n_1} \left\{ \left( \omega_2 \otimes s_{1_i}^k, \left( f^{-1} \left( 1 - (1 - f(\mu_{s_1}^{k_i}))^{w_1} \right) \right) \right) \right\} \right\},
\end{align*}
\]

then

\[
\begin{align*}
\omega_1 \otimes H_{s_1} + \omega_2 \otimes H_{s_1} &= \left\{ \bigcup_{k_i=1,2,\ldots,n_1, k_j=1,2,\ldots,n_2} \left\{ \left( \omega_1 \otimes s_{1_i}^k + \omega_2 \otimes s_{1_i}^k, \left( f^{-1} \left( 1 - f \left( f^{-1} \left( 1 - (1 - f(\mu_{s_1}^{k_i}))^{w_1} \right) \right) \right) \right) \right) \right\} \right\},
\end{align*}
\]

\[
\begin{align*}
&= \left\{ \bigcup_{k_i=1,2,\ldots,n_1, k_j=1,2,\ldots,n_2} \left\{ \left( \omega_1 \otimes s_{1_i}^k + \omega_2 \otimes s_{1_i}^k, \left( f^{-1} \left( 1 - f \left( f^{-1} \left( 1 - (1 - f(\mu_{s_1}^{k_i}))^{w_1} \right) \right) \right) \right) \right) \right\} \right\}.
\end{align*}
\]

(2) Assume that equation (8) holds for any $n$; thus, we have

\[
\begin{align*}
\omega_1 \otimes H_{s_1} + \omega_2 \otimes H_{s_2} + \cdots + \omega_n \otimes H_{s_n}
= &\left\{ \bigcup_{k_i=1,2,\ldots,n_1, k_j=1,2,\ldots,n_2, \ldots, k_n=1,2,\ldots,n_n} \left\{ \left( \omega_1 \otimes s_{1_i}^k + \omega_2 \otimes s_{1_i}^k + \cdots + \omega_n \otimes s_{1_i}^k, \left( f^{-1} \left( 1 - f \left( f^{-1} \left( 1 - (1 - f(\mu_{s_1}^{k_i}))^{w_1} \right) \right) \right) \right) \right) \right\} \right\}.
\end{align*}
\]
(3) When $\varphi = n + 1$, the following results are acquired:

$$w_1 \otimes H_{s_1} + w_2 \otimes H_{s_2} + \cdots + w_n \otimes H_{s_n} + w_{n+1} \otimes H_{s_{n+1}}$$

$$= \left\{ \left( g^{-1} \left( g^{-1} \left( 1 - \prod_{q=1}^{n} \left( 1 - g(s_{p_q})^{w_q} \right) \right) \right) + \left( g^{-1} \left( 1 - g(s_{n+1})^{w_{n+1}} \right) \right) \right) \right\}.$$

$$= \left\{ \left( f^{-1} \left( 1 - f \left( f^{-1} \left( 1 - \prod_{q=1}^{n} \left( 1 - f(\mu_{p_q})^{w_q} \right) \right) \right) \right) \right) \right\}.$$  

$$f^{-1} \left( f^{-1} \left( \prod_{q=1}^{n} \left( f(\eta_{p_q}^{k_q}) \right)^{w_q} \right) \right) \otimes f^{-1} \left( f^{-1} \left( \prod_{q=1}^{n} \left( f(\eta_{p_q}) \right)^{w_q} \right) \right) - f^{-1} \left( f^{-1} \left( \prod_{q=1}^{n} \left( f(\eta_{p_q}) \right)^{w_q} \right) \right).$$

So, the equation holds for any $n$. The proof is completed now.

Based on Definition 12 and the operation rules of Definition 10, Theorem 3 can be obtained. □

Theorem 3. For $n$ HPFLSSs, $H_{s_p} = \langle \langle s_{p_q} \rangle, (\mu_{p_q}, \eta_{p_q}) \rangle \mid i_p \in (0, 1, 2, \ldots, m); k_p = 0, 1, 2, \ldots, a_p \rangle$, $\varphi = 1, 2, \ldots, n$; their weight vector is $w = (w_1, w_2, \ldots, w_n)^T$; for any $j, w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, and the aggregation results can be calculated by the HPFLWG operator:

$$\text{HPFLGA}\left(H_{s_1}, H_{s_2}, \ldots, H_{s_n}\right)$$

$$= \left\{ \left( g^{-1} \left( g^{-1} \left( 1 - \prod_{q=1}^{n} \left( 1 - g(s_{p_q})^{w_q} \right) \right) \right) \right) \right\}.$$  

$$f^{-1} \left( f^{-1} \left( \prod_{q=1}^{n} \left( f(\eta_{p_q}) \right)^{w_q} \right) \right) \otimes f^{-1} \left( f^{-1} \left( \prod_{q=1}^{n} \left( f(\eta_{p_q}) \right)^{w_q} \right) \right) - f^{-1} \left( f^{-1} \left( \prod_{q=1}^{n} \left( f(\eta_{p_q}) \right)^{w_q} \right) \right).$$

$$= \left\{ \left( f^{-1} \left( f^{-1} \left( \prod_{q=1}^{n} \left( f(\eta_{p_q}) \right)^{w_q} \right) \right) \right) \right\}.$$
Proof.

(1) When $n = 2$, in accordance with the operation rules in Definition 10, the following processing is presented:
In the real-life decision-making circumstance, usually, the criteria weight is completely unknown due to the time pressure. Thus, the min-max deviation method is used to identify the criteria weight.

\[
(H_{s_1})^{w_1} \otimes (H_{s_2})^{w_2} \otimes \cdots \otimes (H_{s_n})^{w_n} \otimes (H_{s_{n+1}})^{w_{n+1}}
\]

\[
= \bigcup_{k_1=1,2,\ldots,n_1}^{k_1=1,2,\ldots,n_1} \bigg\{ \bigg( \sum_{q=1}^{n} (g^{1}((- \sum_{q=1}^{n} (g(\kappa_q^p))^{w_q}))) \otimes g^{1}((- \sum_{q=1}^{n} (g(S_{s_{n+1}}^{w_{n+1}})))) \bigg) \bigg\}
\]

\[
\begin{align*}
&\left(f^{-1}(f^{-1}\left(\prod_{q=1}^{n} (f(\mu_q^p + \eta_q^p))^{w_q}\right)) \otimes f^{-1}(f(\mu_{s_{n+1}} + \eta_{s_{n+1}}))^{w_{n+1}})) \right) \\
&\quad - f^{-1}(f^{-1}\left(\prod_{q=1}^{n} (f(\eta_q^p))^{w_q}\right)) \otimes f^{-1}(f(\eta_{s_{n+1}}))^{w_{n+1}})) \right) \bigg) \bigg) \\
&\quad \otimes f^{-1}(f(\eta_{s_{n+1}}))^{w_{n+1}}))) \\
&\quad - f^{-1}(1 - f^{-1}\left(1 - \prod_{q=1}^{n} (1 - f(\gamma_q^p))^{w_q}\right))) \bigg) \bigg) \\
&\quad \bigg(1 - f^{-1}(f^{-1}(1 - \prod_{q=1}^{n} (1 - f(\gamma_{s_{n+1}})))^{w_{n+1}})) \bigg) \bigg)
\end{align*}
\]

\[
(16)
\]

Thus, the equation holds for any \( n \), and the proof is completely down.

3.4. Criteria Weight Determination. In the real-life decision-making circumstance, usually, the criteria weight is completely unknown due to the time pressure. Thus, the min-max deviation method is used to identify the criteria weight.

\[
D_{c_q} = \frac{1}{m - 1} \sum_{i=1}^{m} \sum_{q \neq q'}^{m} \left| (H_{c_q} - H_{c_{q'}}) \right|
\]

\[
= \frac{1}{m - 1} \sum_{i=1}^{m} \sum_{q \neq q'}^{m} \left( \sum_{k_{c_q}=1}^{a_{c_q}} i_{c_q} - a_{c_q} \right) + \sum_{k_{c_q}=1}^{a_{c_q}} \sum_{k_{c_{q'}}=1}^{a_{c_{q'}}} \mu_{c_{q'}} - \sum_{k_{c_q}=1}^{a_{c_q}} \mu_{c_q} + \sum_{k_{c_q}=1}^{a_{c_q}} \eta_{c_{q'}} - \sum_{k_{c_q}=1}^{a_{c_q}} \eta_{c_q} + \sum_{k_{c_q}=1}^{a_{c_q}} \kappa_{c_{q'}} - \sum_{k_{c_q}=1}^{a_{c_q}} \kappa_{c_q}\right)
\]

\[
(17)
\]

The large \( D_{c_q} \) indicates the corresponding criteria are more important. Thus, the following model is used to determine the criteria weight:
\[
\text{(M)}: \begin{cases}
\max \quad D_c(u) = \sum_{q=1}^{n} w_q D_{c_q} \\
\text{s.t.} \quad \sum_{q=1}^{n} w_q^2 = 1, w_q \geq 0, q = 1, 2, \ldots, n.
\end{cases}
\]

(18)

Next, the above model (M) is solved by building the Lagrange function:

\[
L(u, \varepsilon) = \sum_{q=1}^{n} w_q D_{c_q} + \frac{\varepsilon}{2} \left( \sum_{q=1}^{n} w_q^2 - 1 \right).
\]

(19)

3.5. Comparison Methods for HPFLSs. The score function and accuracy function for HPFLSs are introduced. Then, based on the functions, the comparison approaches are provided.

**Definition 13.** For any HPFLSs, \( H_s = [\langle \hat{s}_i \rangle, \langle \mu^k_i, \nu^k_i, \nu^2_i \rangle] \) \( i \in \{0, 1, 2, \ldots, m\}; k = 1, 2, \ldots, a \), and its score function is defined as \( S(H_s) = s_q \), where \( \varepsilon = \sum_{k=1}^{a} i^k \otimes 1 + \mu^k_i - \nu^2_i/2/\alpha \) and \( k^i \) is the maximum subscript of linguistic term in linguistic term set \( S^i \).

For any two HPFLSs \( H_{s_1} \) and \( H_{s_2} \), if \( S(H_{s_1}) > S(H_{s_2}) \), then \( H_{s_1} > H_{s_2} \); if \( S(H_{s_1}) < S(H_{s_2}) \), then \( H_{s_1} < H_{s_2} \); and if \( S(H_{s_1}) = S(H_{s_2}) \), then \( H_{s_1} = H_{s_2} \). For such case, the accuracy function is conducted as below.

**Definition 14.** Let \( H_s \) be one HPFLS \( H_s = [\langle \hat{s}_i \rangle, \langle \mu^k_i, \nu^k_i, \nu^2_i \rangle] \) \( i \in \{0, 1, 2, \ldots, m\}; k = 1, 2, \ldots, a \), and then, the accuracy function of \( H_s \) is defined as \( A(H_s) = 1_q \), where \( y = \sum_{k=1}^{a} i^k \otimes 1 + \mu^k_i - \nu^2_i/2/\alpha \). For any two HPFLSs \( H_{s_1} \) and \( H_{s_2} \), if \( A(H_{s_1}) > A(H_{s_2}) \), then \( H_{s_1} < H_{s_2} \); if \( A(H_{s_1}) = A(H_{s_2}) \), then \( H_{s_1} = H_{s_2} \).

According to Definitions 13 and 14, we have the following conclusion.

**Conclusion 1.** Let \( H_{s_1} \) and \( H_{s_2} \) be two HPFLSs; then

1. if \( S(H_{s_1}) > S(H_{s_2}) \), then \( H_{s_1} > H_{s_2} \)
2. if \( S(H_{s_1}) < S(H_{s_2}) \), then \( H_{s_1} < H_{s_2} \)
3. if \( S(H_{s_1}) = S(H_{s_2}) \), then
   a. if \( A(H_{s_1}) > A(H_{s_2}) \), then \( H_{s_1} > H_{s_2} \)
   b. if \( A(H_{s_1}) = A(H_{s_2}) \), then \( H_{s_1} = H_{s_2} \)
   c. if \( A(H_{s_1}) < A(H_{s_2}) \), then \( H_{s_1} < H_{s_2} \)

In order to compute the partial derivatives of \( w_q \) and \( \varepsilon \), we have

\[
\frac{\partial (w_q, \varepsilon)}{w_q} = D_{c_q} + \varepsilon w_q = 0,
\]

(20)

\[
\frac{\partial (w_q, \varepsilon)}{\varepsilon} = \sum_{q=1}^{n} w_q^2 = 1 = 0.
\]

By solving (20), the following equation to determine the weight vector is acquired:

\[
w_q = \left( \frac{1}{m - 1} - \sum_{e=1}^{m} \sum_{e=1}^{m} \left( \frac{\sum_{k=q}^{m} k_q \eta_{1e} - \sum_{k=q}^{m} k_q \eta_{2e}}{\sum_{k=q}^{m} k_q \eta_{1e} - \sum_{k=q}^{m} k_q \eta_{2e}} \right) \right) \left( \frac{\sum_{k=q}^{m} k_q \eta_{1e} - \sum_{k=q}^{m} k_q \eta_{2e}}{\sum_{k=q}^{m} k_q \eta_{1e} - \sum_{k=q}^{m} k_q \eta_{2e}} \right).
\]

(21)

4. MCGDM Approach with Hesitant Picture Fuzzy Linguistic Sets (HPFLSs)

In this section, an effective approach for solving MCGDM problems with HPFLSs is presented. The detailed steps are introduced below. In this sake, the hierarchical structure of decision-making procedure of the proposed approach is presented in Figure 1.

4.1. Problem Description. There are the collection of \( m \) alternatives \( A = \{a_1, a_2, \ldots, a_m\} \) and \( k \) participants/experts, and \( E = \{e_1, e_2, \ldots, e_k\} \). \( C = \{c_1, c_2, \ldots, c_n\} \) is the criteria set, and \( w = \{w_1, w_2, \ldots, w_n\} \) is the weight vector of criteria. Due to the complex and uncertainty practical situation, the criteria weight cannot be reasonably given by DMs, and it is required to solve such case that the weight of criteria is completely unknown.

In the decision-making procedure, the uncertain assessment value of the alternative \( A_i \) with respect to criterion \( C_j \) given by all DMs is presented in the form of linguistic terms. Then, the final hesitant picture fuzzy linguistic evaluation can be obtained by collecting all linguistic terms and represented as the following decision matrix:

\[
\tilde{M} = \left( H_{s_j} \right)_{mn} = \begin{bmatrix}
H_{s_{j1}} & H_{s_{j2}} & \cdots & H_{s_{jn}} \\
H_{s_{j1}} & H_{s_{j2}} & \cdots & H_{s_{jn}} \\
\vdots & \vdots & \ddots & \vdots \\
H_{s_{j1}} & H_{s_{j2}} & \cdots & H_{s_{jn}}
\end{bmatrix}
\]

(22)

4.2. Decision-Making Procedure. Based on the above description, a useful approach based on HPFLS information is developed to handling practical MCGDM problems, in
Determine the aggregation value of each alternative
Compute the score or accuracy value of the aggregation value of each alternative
Rank all alternatives in accordance with score or accuracy value

Figure 1: The hierarchical structure of the proposed HPFLS-based MCGDM approach.

which the criteria weight is completely unknown, and the assessment value of each alternative are collected as HPFLSs. The decision-making steps are summarized as follows:

Step 1: Normalize the evaluation value in the decision matrix.

The criteria in the decision matrix should be identified and distinguished as benefit-type and cost-type criteria. The evaluation value of benefit-type criteria does not necessarily be changed, whereas for the cost-type criteria, it is required to normalize the evaluation using the complementary set.

The decision matrix can be normalized by the following formula:

\[
\tilde{M}_{ij} = \begin{cases} 
\bar{M}_{ij}, & C_j \in B_S, \\
\tilde{M}_{ij}^c, & C_j \in C_S,
\end{cases}
\]

where \(B_S\) is the collection of benefit-type criteria, \(C_S\) is the collection of cost-type criteria, and \(\tilde{M}_{ij}^c\) is the complementary set of \(\tilde{M}_{ij}\).

The normalized decision matrix can be denoted by \(\tilde{M} = (\tilde{M}_{ij})_{m \times n}\).

Step 2: Compute criteria weights.

Equation (21) is utilized to compute the weight vector \(w = [w_1, w_2, \ldots, w_n]^T\) of criteria set \(C = \{c_1, c_2, \ldots, c_n\}\). Hence, every criterion weight value is determined.

Step 3: Determine the aggregation value of each alternative \(a_i\).

In this step, the \(HPFLWA\) or \(HPFLWG\) operator is used to determine the aggregation value of alternative \(a_i\). Then, the \(HPFLWA\) or \(HPFLWG\) aggregation value is obtained by equation (8) or (13).

Step 4: Compute the score or accuracy value of the aggregation value of each alternative \(a_i\).

Based on Definitions 14 and 15 given in Section 3.5, the score and accuracy value of each alternative \(a_i\) are acquired. The value \(S(H_x)\) is bigger, and the alternative \(a_i\) is better. If the value \(S(H_x)\) of any two alternatives is equal, then the alternative with the bigger value \(A(H_x)\) is better.

Step 5: Rank all alternatives in accordance with score or accuracy value.

Based on the increasing value of \(S(H_x)\), the bigger \(S(H_x)\) is, the better the alternative \(a_i\) will be. If the value \(S(H_x)\) of any two alternatives is equal, then \(A(H_x)\) needs to be used. Based on the increasing value of \(A(H_x)\), the bigger \(A(H_x)\) is, the better the alternative \(a_i\) will be.

5. Practical Application

In this section, the teaching performance evaluation by the students from Hubei University of Automotive Technology with HPFLSs is introduced and also the further analysis is carried on.

5.1. Practical Application Illustration. Practical application at Hubei University of Automotive Technology is adopted as the illustration of the presented method. Hubei University of Automotive Technology includes many schools, such as school of economic and management, school of automotive engineering, and school of machinery design and automation. The university always pays high attention to ensure the teaching performance of teacher for every course. And some scoring rules have been drawn up to ensure the teaching quantity. For example, students grade the teaching performance of teachers’ course. According to the signed agreement, our proposed approach is used to help handle the student score and rank the teaching performance. In this paper, for convenience, the teaching performance assessment values of four teachers’ courses were chosen as the alternative \(a_j (j = 1, 2, 3, 4)\). Meanwhile, some more important criteria are discussed in detailed; at final, four key assessment criteria are selected: planning and preparation \((c_1)\), communication and interaction \((c_2)\), manage the learning environment \((c_3)\), and professionalism \((c_4)\). The linguistic term set \(S = \{s_5 = \text{outstanding excellent}; s_4 = \text{excellent}; s_3 = \text{merit}; s_2 = \text{average}; s_1 = \text{pass}; s_0 = \text{failure}\}\) is used during the evaluation process. In order to acquire the trustworthy evaluation value, the students anonymously give their comments on websites; in addition, students have the rights to refuse to provide any comment. For convenience, the process of transforming from students’ comments to picture fuzzy linguistic term information is bypassed. Finally, the HPFSLs, which are collected from the picture fuzzy linguistic term value, are shown in Table 2.
Table 2: Decision matrix of collective value $M$.

<table>
<thead>
<tr>
<th></th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$\langle {s_3}, (0.58, 0.12, 0.1)\rangle, \langle {s_2}, (0.22, 0.1, 0)\rangle$</td>
<td>$\langle {s_3}, (0.66, 0.1, 0.04)\rangle, \langle {s_4}, (0.1, 0.2, 0)\rangle$</td>
<td>$\langle {s_3}, (0.8, 0.0, 0)\rangle, \langle {s_4}, (0.2, 0.0)\rangle$</td>
<td>$\langle {s_3}, (0.72, 0.08, 0)\rangle, \langle {s_4}, (0.31, 0.09, 0)\rangle$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$\langle {s_3}, (0.53, 0.18, 0.09)\rangle, \langle {s_2}, (0.21, 0.1, 0)\rangle$</td>
<td>$\langle {s_3}, (0.68, 0.08, 0.04)\rangle, \langle {s_4}, (0.3, 0.0)\rangle$</td>
<td>$\langle {s_3}, (0.53, 0.2, 0.07)\rangle, \langle {s_4}, (0.31, 0.09, 0)\rangle$</td>
<td>$\langle {s_3}, (0.51, 0.19, 0.1)\rangle, \langle {s_4}, (0.3, 0.1, 0)\rangle$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$\langle {s_3}, (0.48, 0.1, 0.02)\rangle, \langle {s_2}, (0.6, 0.0)\rangle$</td>
<td>$\langle {s_3}, (0.63, 0.2, 0.07)\rangle, \langle {s_4}, (0.3, 0.06, 0)\rangle$</td>
<td>$\langle {s_3}, (0.71, 0.09, 0)\rangle, \langle {s_4}, (0.51, 0.09, 0)\rangle$</td>
<td>$\langle {s_3}, (0.7, 0.0, 0.1)\rangle, \langle {s_4}, (0.3, 0.0)\rangle$</td>
</tr>
<tr>
<td>$a_4$</td>
<td>$\langle {s_3}, (0.52, 0.3, 0.08)\rangle, \langle {s_2}, (0.3, 0.2, 0)\rangle$</td>
<td>$\langle {s_3}, (0.56, 0.2, 0.04)\rangle, \langle {s_4}, (0.61, 0.1, 0)\rangle$</td>
<td>$\langle {s_3}, (0.5, 0.2, 0.1)\rangle, \langle {s_4}, (0.63, 0.07, 0)\rangle$</td>
<td>$\langle {s_3}, (0.63, 0.08, 0.09)\rangle, \langle {s_4}, (0.5, 0.1, 0)\rangle$</td>
</tr>
</tbody>
</table>
Step 1. Normalize the decision matrix.
Due to none of the criterion is considered as cost criteria, the decision matrix of the collection value is not necessary to be normalized.

Step 2. Compute criteria weight vector.
Equation (21) is utilized to compute the weight vector, and the obtained results are

\[
a_1 = \{(s_1), (0.6643, 0.0897, 0.0486)\},\{(s_5), (0.6405, 0.0907, 0.0486)\},\{(s_9), (0.6163, 0.0897, 0.0486)\},\\]
\[
\{(s_{10}), (0.5902, 0.0907, 0.0486)\},\{(s_{14}), (0.5795, 0.1344, 0.0307)\},\{(s_{18}), (0.5498, 0.1354, 0.0307)\},\\]
\[
\{(s_{20}), (0.5157, 0.1344, 0.0307)\},\{(s_{24}), (0.4691, 0.1354, 0.0307)\},\{(s_{26}), (0.6324, 0.0836, 0.0179)\},\\]
\[
\{(s_{28}), (0.6073, 0.0846, 0.0179)\},\{(s_{30}), (0.5811, 0.0836, 0.0179)\},\{(s_{34}), (0.5516, 0.0846, 0.0179)\},\\]
\[
\{(s_{36}), (0.5389, 0.1282, 0.0000)\},\{(s_{38}), (0.5012, 0.1292, 0.0000)\},\{(s_{40}), (0.4777, 0.1282, 0.0000)\},\\]
\[
\{(s_{42}), (0.2129, 0.1292, 0.0000)\},\\]
\[
\begin{align*}
a_2 &= \{(s_{5}), (0.6032, 0.1390, 0.0654)\},\{(s_{14}), (0.5977, 0.1300, 0.0555)\},\{(s_{18}), (0.5932, 0.1232, 0.0553)\},\\]
& \quad \{(s_{20}), (0.5876, 0.1142, 0.0454)\},\{(s_{24}), (0.4777, 0.1031, 0.0472)\},\{(s_{26}), (0.4659, 0.0942, 0.0374)\},\\]
& \quad \{(s_{28}), (0.4553, 0.0873, 0.0372)\},\{(s_{30}), (0.4401, 0.0783, 0.0274)\},\{(s_{32}), (0.5807, 0.1144, 0.0378)\},\\]
& \quad \{(s_{34}), (0.5747, 0.1054, 0.0279)\},\{(s_{36}), (0.5699, 0.0986, 0.0278)\},\{(s_{38}), (0.5637, 0.0896, 0.0179)\},\\]
& \quad \{(s_{40}), (0.4176, 0.0785, 0.0198)\},\{(s_{42}), (0.3920, 0.0696, 0.0100)\},\{(s_{44}), (0.3625, 0.0627, 0.0098)\},\\]
& \quad \{(s_{46}), (0.2850, 0.0538, 0.0000)\},
\end{align*}
\]
\[
\begin{align*}
a_3 &= \{(s_{5}), (0.6134, 0.1333, 0.0472)\},\{(s_{9}), (0.5895, 0.1631, 0.0373)\},\{(s_{13}), (0.5850, 0.1333, 0.0472)\},\\]
& \quad \{(s_{17}), (0.5586, 0.1631, 0.0373)\},\{(s_{21}), (0.5338, 0.0706, 0.0161)\},\{(s_{25}), (0.4983, 0.1004, 0.0062)\},\\]
& \quad \{(s_{29}), (0.4907, 0.0706, 0.0161)\},\{(s_{33}), (0.4348, 0.1004, 0.0062)\},\{(s_{37}), (0.6404, 0.1024, 0.0410)\},\\]
& \quad \{(s_{39}), (0.6179, 0.1322, 0.0310)\},\{(s_{41}), (0.6137, 0.1024, 0.0410)\},\{(s_{43}), (0.5898, 0.1322, 0.0310)\},\\]
& \quad \{(s_{45}), (0.5682, 0.0398, 0.0100)\},\{(s_{47}), (0.5398, 0.0696, 0.0000)\},\{(s_{51}), (0.5341, 0.0398, 0.0100)\},\\]
& \quad \{(s_{53}), (0.4988, 0.0696, 0.0000)\},
\end{align*}
\]
\[
\begin{align*}
a_4 &= \{(s_{5}), (0.5491, 0.2184, 0.0648)\},\{(s_{9}), (0.5349, 0.2204, 0.0556)\},\{(s_{13}), (0.5680, 0.1995, 0.0501)\},\\]
& \quad \{(s_{17}), (0.5551, 0.2016, 0.0410)\},\{(s_{21}), (0.5736, 0.1735, 0.0462)\},\{(s_{25}), (0.5610, 0.1755, 0.0372)\},\\]
& \quad \{(s_{29}), (0.5908, 0.1547, 0.0319)\},\{(s_{33}), (0.5790, 0.1567, 0.0229)\},\{(s_{37}), (0.5244, 0.1880, 0.0412)\},\\]
& \quad \{(s_{39}), (0.5077, 0.1900, 0.0321)\},\{(s_{41}), (0.5457, 0.1692, 0.0268)\},\{(s_{43}), (0.5312, 0.1712, 0.0178)\},\\]
& \quad \{(s_{45}), (0.5518, 0.1432, 0.0232)\},\{(s_{47}), (0.5378, 0.1452, 0.0142)\},\\]
& \quad \{(s_{53}), (0.5705, 0.1244, 0.0090)\},\{(s_{55}), (0.5577, 0.1264, 0.0000)\}.
\end{align*}
\]

\[w = \{0.3038, 0.4478, 0.1443, 0.0999\}^T.\]  \hspace{1cm} (24)

Step 3. Obtain the aggregation value of alternative \(a_i\).
By utilizing equation (8), the aggregation value of each alternative \(a_i\) can be calculated, and the results are shown below:
By utilizing equation (13), the aggregation value of each alternative \( a_i \) can be calculated, and the results are shown below:

\[
a_1 = \{ (s_{5.0000}, (0.6503, 0.0897, 0.0602)), (s_{4.6229}, (0.5040, 0.0907, 0.0602)), (s_{4.5381}, (0.4077, 0.0897, 0.0602)), \\
(\ldots)
\]

\[
a_2 = \{ (s_{4.2363}, (0.3866, 0.0907, 0.0602)), (s_{4.2363}, (0.2904, 0.1344, 0.0498)), (s_{4.1783}, (0.2754, 0.1354, 0.0498)), \\
(\ldots)
\]

\[
a_3 = \{ (s_{4.2363}, (0.50197, 0.1333, 0.0556)), (s_{4.2363}, (0.4794, 0.1631, 0.0446)), (s_{4.5381}, (0.5022, 0.1333, 0.0556)), \\
(\ldots)
\]

\[
a_4 = \{ (s_{4.0000}, (0.5323, 0.2184, 0.0706)), (s_{4.6229}, (0.4919, 0.2204, 0.0637)), (s_{4.5381}, (0.5512, 0.1955, 0.0578)), \\
(\ldots)
\]
Step 4. Compute the score or accuracy value of the aggregation value of each alternative $a_i$.

The alternative score value can be obtained based on Definitions 13 and 14 given in Section 3.5, and the results are as follows.

For the aggregation result with HPFLWA operator, we have the score values

\[
\begin{align*}
    s_{a_1} &= 3.4293, \\
    s_{a_2} &= 2.8791, \\
    s_{a_3} &= 3.4524, \\
    s_{a_4} &= 3.4180.
\end{align*}
\]

(27)

For the aggregation result with HPFLWG operator, we obtain

\[
\begin{align*}
    s_{a_1} &= 2.7751, \\
    s_{a_2} &= 2.2932, \\
    s_{a_3} &= 3.0336, \\
    s_{a_4} &= 3.0730.
\end{align*}
\]

(28)

Step 5. Rank all alternatives in accordance with score or accuracy value.

Based on the increasing score or accuracy value of the aggregation result with HPFLWA operator, the final ranking order of alternatives is $a_3 > a_1 > a_4 > a_2$, the best alternative is $a_3$, and the worst alternative is $a_2$. The above results clearly show that the alternative $a_3$ is the best teaching performance teacher.

Based on the increasing score or accuracy value of the aggregation result with HPFLWG operator, the final ranking order of alternatives is $a_3 > a_1 > a_4 > a_2$, the best alternative is $a_3$, and the worst alternative is $a_1$. The above results clearly show that the alternative $a_3$ is the best teaching performance teacher.

5.2. Further Discussion. In this subsection, a sensitivity analysis is conducted based on the different representative values of parameter $\theta$, and it indicates whether the ranking results are varied or not when different representative values of parameter $\theta$ are provided. The detailed results are shown in Table 3.

From the ranking results presented in Table 3, it is obvious that the ranking result is sensitive to the value of parameter $\theta$ for both HPFLWA and HPFLWG operators. If the HPFLWA operator is used, the best teaching performance alternative is $a_3$ and the worst teaching performance alternative is $a_2$; however, if the HPFLWG operator is used, the best teaching performance alternative is $a_3$ by decreasing the value of parameter $\theta$; with increasing the value of parameter $\theta$, the best teaching performance alternative is changed as $a_4$, and the worst teaching performance alternative is $a_2$. For both of operators, the worst alternative is remaining unchangeable with the various value of parameter $\theta$.

In addition, the trend of different values of parameter $\theta$, which can lead to different final score results $s_{a_i}$ for

\[
\begin{array}{c|c|c}
\text{Value of parameter } \theta & \text{HPFLWA} & \text{HPFLWG} \\
\hline
\theta = 1 & a_1 > a_2 > a_3 > a_4 & a_1 > a_2 > a_3 > a_4 \\
\theta = 5 & a_3 > a_4 > a_1 > a_2 & a_1 > a_2 > a_3 > a_4 \\
\theta = 16 & a_4 > a_3 > a_2 > a_1 & a_1 > a_2 > a_3 > a_4 \\
\theta = 32 & a_3 > a_4 > a_2 > a_1 & a_1 > a_2 > a_3 > a_4 \\
\theta = 60 & a_4 > a_3 > a_2 > a_1 & a_1 > a_2 > a_3 > a_4 \\
\end{array}
\]

Table 3: Sensitivity analysis based on different values of parameter $\theta$.

HPFLWA and HPFLWG operators, is presented schematically in Figures 2 and 3.

From the results presented in Figure 2, it is apparent that the best teaching performance alternative is $a_3$ and the worst teaching performance alternative is $a_2$ in most cases. However, if the value of parameter $\theta$ is less than 13, $a_4$ is superior than $a_1$; if the value of parameter $\theta$ is greater than 13, $a_1$ is superior than $a_4$. The main reason is, with the increasing value of parameter $\theta$, the aggregation results obtained by the HPFLWA operator are approaching to the max limited value, which is slightly less than the max value in the evaluation value for the corresponding alternative. With
decreasing value of parameter $\theta$, the aggregation results obtained by the HPFLWA operator are approaching to the min limited value, which is slightly greater than the min value in the evaluation value for the alternative. For example, the minimum and maximum values of $\mu$ appeared in the evaluation value for alternative $a_1$ are 0.1 and 0.8; however, the min and max values of the membership degree of "vote for" $\mu$ appeared in the evaluation value for alternative $a_2$ are 0.3 and 0.63 (Table 1). Thus, with the increasing value of parameter $\theta$, the trend of $a_1$ should be superior than $a_2$. In addition, from the overview of final score results, different values of parameter $\theta$ can critically affect the ranking order of alternatives.

From the results presented in Figure 3, it can be seen that the best alternative is $a_4$, and the worst alternative is $a_2$, in all circumstances. With the increasing value of parameter $\theta$, the score value of all alternatives is decreasing and approaching to the limited value. However, the ranking order of alternatives is slightly changed; that is, if the value of parameter $\theta$ is less than 48, $a_4$ is superior than $a_3$, and if the value of parameter $\theta$ is greater than 48, $a_3$ is superior than $a_4$. The main reason is similar with the reason appeared in the HPFLWG operator. From the overview of final order results, the different value of parameter $\theta$ can affect the score value of alternatives and affect the ranking order of alternatives.

From the above analysis results appeared in Figures 2 and 3, we know that different values of parameter $\theta$ can affect the score value and final ranking order. In order to better elaborate the effectiveness of our proposed method in the real teaching performance evaluation, the value of parameter $\theta$ can be selected according to the following circumstances:

1. If DMs prefer to pay high attention to the max values of the membership degree of "vote for" and linguistic term, the HPFLWA operator is used or maximum values of membership degree of and "vote against" and linguistic term, the HPFLWG operator is used. The value of parameter $\theta$ can be greater than 20 and less than 50.

2. If the DMs prefer to pay high attention to min values of the membership degree of "voting for" and linguistic term, the HPFLWA operator is used or minimum values of membership degree and "vote against" and linguistic term, the HPFLWG operator is used. The value of parameter $\theta$ can be assigned with less than 10 and greater than 5.

3. Otherwise, $\theta$ can be assigned with any value except for the values appeared in above (1)-(2). By default, $\theta$ could be 16.

5.3. Comparative Analysis. In order to further verify the effectiveness of the proposed method, the comparative analysis based on same teaching performance ranking is carried out with some representative methods, which are proposed by Wei et al. [42]. For convenience to comparison, the HPFLSs have been transformed to P2TLSs. According to the analysis in Section 5.2, four representative values are selected and assigned for the parameter $\theta$, which are $\theta = 5$, $\theta = 16$, $\theta = 32$, and $\theta = 64$. Then, the comparative ranking orders are obtained by the different methods and presented in Table 4.

Table 4 shows that the final ranking order generated by Wei et al. [42] is slightly different from the proposed method. The difference might mainly be caused by the operation rule limitation as pointed out in Examples 1–3. However, for the proposed method, if the different values of parameter $\theta$ are used, the final order of alternatives is slightly distinct; in some cases, the alternative $a_1$ is superior than alternative $a_4$, and in some cases, the alternative $a_4$ is superior than alternative $a_1$. This is caused by the characteristics of the sigmoid function used in the operation rules. Normally, in order to elaborate the effectiveness of the sigmoid function in the proposed operation rules, $\theta$ can be assigned with different values as discussed in Section 5.2.

In summary, for all obtained results by different methods, the worst alternative is $a_2$, the best alternative is $a_1$, $a_2$, or $a_4$, and the ranking orders are different.

In Table 5, the comparative analysis of the proposed method versus the method of Wei et al. [42] is provided, and the advantages of the proposed method include four parts of the parameter, and the disadvantage is the part of time of complexity. The detailed description is presented in Table 5. According to the comparative analysis and further discussion in Section 5.2, it can be concluded that the proposed method possesses the following advantages:

1. The proposed method can overcome the defects of operation rules existed in the previous methods, such as the methods of P2TLSs and TLS.

2. The proposed method can bring into fully play the advantages of HPFLSs, and it is more flexible to express DMs’ preferences under group decision-making environments.

3. HPFLSs can be more feasible to express the hesitation of DMs than P2TLSs, and the HPFWA and HPFWG operators are developed based on the sigmoid functions. The proposed approach can effectively elaborate the advantages of HPFLSs and the
HPFWA and HPFWG operators; thus, it can effectively solve the MCGDM problems.

(4) Under different real decision-making environments, the DMs can pay higher attention to specific aspect such as membership degree of “vote for” or “vote against,” and the proposed approach can be more flexible and reasonable to adjust the ranking results by the parameter $\theta$ and give more acceptable results in accordance with the real demands of DMs.

**6. Conclusion and Further Direction**

Considering the real-life decision-making environment, different DMs could have the different attitudes (e.g., attitude for support, neutral, oppose, and refusal) for the evaluation value of criteria or alternative under the hesitant fuzzy linguistic environment. In this regard, the paper proposed the hesitant picture fuzzy linguistic group decision-making approaches that are based on aggregation operators, which are weighted average (WA) and geometric (WG) operators.

HPFLSs can provide the DMs with the opportunity to express the evaluation value of hesitant linguistic with a set of crisp number membership degree for criteria of alternative for avoiding attitude information losing. As the extension of PFSs is based on hesitant linguistic term sets, it can be more suitable and flexible to solve MCGDM voting problems. It can bring into fully play the advantages of hesitant linguistic term sets and PFSs.

In the proposed approach, improved operation laws are developed based on the sigmoid function, and two equivalent transformations functions $f$ and $g$ are defined. It can overcome the defects of operation rules appeared in the previous PFS method. To illustrate the effectiveness and performance of the proposed method, a real case of teaching performance evaluation form Hubei University of Automotive Industry is adopted. In addition, the comparison analysis of the ranking result obtained from the proposed method versus the method of Wei et al. [42] is presented in detail; meanwhile, the sensitive analysis of the proposed method is conducted, and the results showed that the proposed method can be more flexible and effective than that of Wei et al. [42]. Afterward, a comparison of the proposed method versus the method of Wei et al. [42] is performed based on six attributes, including the approach of uncertainty modeling, operation laws, weight of criteria, rank flexibility, experts’ weights, and time of complexity. The comparison results show that the proposed method has the advantages at four attributes.

To highlight the advantages of HPFLSs and WA, WG operator, HPFWA, and HPFWG operators are proposed and related theorem are discussed; meanwhile, the max-min deviation methods to identify the completely unknown criteria weight under hesitant picture fuzzy linguistic environment are presented, and the comparison method based on score and accuracy functions for HPFLSs are introduced. In addition, an approach is constructed to solve the teaching performance evaluation problem. The best and worst alternatives can be easily acquired.

The main contribution of this paper is developing an improved operation rule to overcome the defects of operation rules appeared in the previous PFS-based MCGDM method. In this way, a novel sigmoid function-based operation rule is developed. In order to verify the effectiveness of the proposed method, the optimal value selection of parameter $\theta$ is analyzed. Thus, the practical MCGDM voting problems can be addressed using our proposed method under the HPFLS environment. However, the limitation in the proposed method is that the DMs are assigned with the same weight; for most cases, it is required to allocate the different weights for different DMs. Additionally, under the most real-life MCGDM environment, DMs more prefer to provide the evaluation value with simple crisp number or linguistic term; thus, how to transfer the simple crisp number and linguistic term to HPFLSs is necessary to studying in the future. Also, developing the decision system based on the proposed HPFLS MCGDM approaches is more encouraged to solve the real-life MCGDM problem.
Data Availability

The hesitant picture fuzzy linguistic data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (nos. 71701065 and 71271218) and the Science Foundation of Hubei Province (grant no. 2016CFB519).

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