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Research Article

A Novel Multicriteria Group Decision-Making Approach with Hesitant Picture Fuzzy Linguistic Information

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In real-life group decision-making environment, different decision-makers (DMs) might be hesitant to provide their evaluation by more than two linguistic terms with positive, neutral, negative, and refusal information (e.g., attitude for support, neutral, oppose, and refusal) for criteria of alternative. In order to solve such kinds of decision-making problems, a novel definition of hesitant picture fuzzy linguistic sets (*HPFLSs*) is introduced, and the HPFLS-based methods are developed. Considering the operation laws appeared in previous papers are not well suitable for HPFLS operation, the novel operation laws of *HPFLSs* are developed. Then, two aggregation operators of *HPFLSs* are developed, including hesitant picture fuzzy linguistic weighted average operator (*HPFLSWA*) and hesitant picture fuzzy linguistic weighted geometric operator (*HPFLSGA*). Meanwhile, the related proofs are given in detail. Additionally, the comparison method of score and accuracy functions is provided to rank the alternative. Finally, a real-life case of teaching performance evaluation is used to verify the proposed methods. The same case-based comparisons are further conducted between the proposed method and previous methods. The results showed that the proposed method can well overcome the lack of operation rules appeared in previous multicriteria group decision-making (*MCGDM*) methods and demonstrate effectiveness.

1. Introduction

In recent years, as one of the useful tools to solve *MCGDM* problems, the fuzzy linguistic set-based methods [1] have been a focus of research. Along with more and more complex decision-making environment, many scholars proposed different extensions of fuzzy linguistic term sets. In order to avoid the information loss for computing with words, Herrera and Martinez [2] put forward a 2-tuple fuzzy linguistic representation model. Then, some different 2-tuple fuzzy linguistic sets-based approaches are proposed to handle different decision-making problems [3–5]. Focusing on aggregating the criteria information, Wei [6] used 2-tuple linguistic information-based generalized weighted average and geometric operators, Wan [7] developed the 2-tuple linguistic hybrid aggregation operator, and Liu et al. [8] developed 2-tuple linguistic Heronian mean operators; these methods enormously enriched the decision-making theory. However, in some decision circumstances, interval number can better express the uncertainty of decision-maker information. Thus, uncertain linguistic set-based approaches [9–12] were proposed. Subsequently, to extend the 2-tuple fuzzy set with uncertain linguistic set, Zhang [13] defined several interval-valued 2-tuple linguistic aggregation operators, and Liu et al. [14] defined some new operation rules for 2-tuple linguistic term set and developed Bonferroni mean operators. All of the above approaches were illustrated by the practical application cases, and their effectiveness was demonstrated.

However, in some cases, the membership degree cannot be defined by one specific value; thus, Wang and Li [15] proposed intuitionistic fuzzy linguistic set. Then, intuitionistic fuzzy linguistic set-based approaches [16–19] were proposed. Due to the different demands of practical

decision-making problems, the intuitionistic fuzzy linguistic set is extended by many scholars. With regard to preference relation, Zhang et al. [20] defined 2-tuple intuitionistic fuzzy linguistic preference relation. For the information aggregation operator, Liu et al. [21] defined the hesitant intuitionistic fuzzy linguistic weighted average (HIFLWA) operator, and Meng et al. [22] developed interval-valued intuitionistic uncertain linguistic hybrid Shapley operators. In addition, Tan et al. [23] put forward an extended single-valued neutrosophic projection-based qualitative flexible multicriteria decision-making method. All the mentioned approaches above can yield the effectiveness.

Considering that decision-maker may be hesitant to give the evaluation value using the linguistic term, hesitant fuzzy linguistic set [24] was defined. Then, hesitant fuzzy linguistic set-based methods [25–27] were proposed. In order to effectively address practical decision-making issues, many extensive approaches are conducted. Zhou et al. [28] extended hesitant linguistic set with evidence reasoning. Wang et al. [29] extended the hesitant linguistic set with interval number and defined the interval-valued hesitant linguistic set. Pang et al. [30] extended the hesitant fuzzy set with probability distribution and defined probability linguistic term sets. Lin et al. [31] extended the hesitant fuzzy linguistic term with probability distribution and interval number and proposed a probability uncertainty hesitant fuzzy linguistic set. All the above methods based on the extension of hesitant fuzzy linguistic set are verified by an illustration example and the effectiveness is demonstrated.

Although these existing linguistic set-based approaches can demonstrate the effectiveness for solving the practical MCGDM issues, under some decision-making situations, they cannot be applied effectively. For example, one listed company carried on portfolio optimization, and there are four plans for the voting by the stakeholder. Some stakeholders may give "vote for," some stakeholders may give "vote against," some stakeholders may give "abstain," and the rest of the stakeholders may refuse to vote. For such kind of decision-making problems, it is required to propose a new method.

Picture fuzzy set (PFS) was firstly proposed by Cường [32], which is the extension of the intuitionistic fuzzy set. After then, in order to solve the MCGDM voting problems, several picture fuzzy set-based MCGDM methods have been proposed [33–38]. With regard to measurement, Singh et al. developed the approach with picture fuzzy correlation coefficients [39], Wei et al. [40] introduced a novel method to solve MCGDM voting problems using cross-entropy of the picture fuzzy set. As for aggregation operators, Wei et al. [41] used the picture 2-tuple linguistic-based operators, including Bonferroni mean operator, weighted averaging operator, ordered weighted averaging operator, and hybrid averaging operator [42]. In addition, Nie et al. [43] put forward the MCGDM voting method based on 2-tuple linguistic picture preference relation and applied to the selection of voting the proxy advisor firm by a stakeholder. Nevertheless, the proposed picture fuzzy set-based methods are enriched and used to well solve the MCGDM issues. In the practical cases, such as the students rank their teacher's teaching performance, it is common for students to provide the assessment value with positive, indeterminacy, negative, or refusal attitude information. For example, "I cannot give the teaching performance with the score "outstanding excellent," but it is ok for the "excellent"" or "I refuse to give any evaluation value." For such kind of practical MCGDM problems, they cannot be appropriately solved by the existing fuzzy linguistic set-based MCGDM approaches. Thus, it is necessary to extend the existing fuzzy linguistic set to HPFLSs.

HPFLSs can elaborate the advantages of both hesitant linguistic set and picture fuzzy set, which are more suitable for the practical hesitant case of the decision-makers with positive indeterminacy, negative, and refusal information. For example, ten students are required to give the score for teaching quality (one of the criteria of teaching performance) for mathematics. Three students refuse to give any evaluation value, seven students oppose to give the teaching performance with the score "outstanding excellent," but they agree for the "excellent." Thus, if the evaluation term set is $S = \{s_5 = \text{outstading excellent}, s_4 = \text{excellent}, s_3 = \text{merit}, s_2 = \text{average}, s_1 = \text{pass}, s_0 = \text{failure}\}$, then the evaluation score can be collected as $\langle\{(s_4), (0.7, 0, 0)\}, \{(s_5), (0, 0, 0.7)\}\rangle$ with HPFLSs.

In the following, the contribution of this paper is presented:

(1) In view of the defects of operation rules appeared in the methods of linguistic term set (LTS) and picture fuzzy linguistic set methods, the new operation rules based on sigmoid and equivalent transformation functions are proposed to overcome the existing limitation.

(2) The new definition of hesitant picture fuzzy linguistic sets (HPFLSs) is introduced, it is more flexible to describe the two more possible linguistic terms for voting decision-making than picture fuzzy linguistic sets.

(3) In order to solve MCGDM voting problems under the completely unknown criteria weight environment, the max-min deviation model for HPFLSs is constructed, and then the criteria weighted is computed by using Lagrange functions.

(4) In order to develop an effective approach with HPFLSs to solve practical decision-making problems, the average weighted operator of HPFLS (*HPFLSWA*) and hesitant picture fuzzy linguistic weighted geometric operator (*HPFLSWG*) are developed in this paper. Meanwhile, the comparison methods for HPFLSs are introduced. Then, based on the proposed operators and comparison approaches, an effective multicriteria group decision-making (MCGDM) voting method under HPFLSs is developed.

The rest of this paper is organized as follows: several definitions and operation laws related to PFS and TLSs are reviewed in Section 2. In Section 3, the definition and novel basic operation rules of HPFLSs are introduced, and the

HPFLSWA and HPFLSWG operators are developed and min-max deviation method, which is used to gain the criteria weight for completely unknown, is conducted; moreover, the ordering comparison methods for HPFLSs are presented. Section 4 provides a voting method of MCGDM based on the HPFLSWA and HPFLSWG operators, and its steps are described in detail. In Section 5, the real teaching performance evolution from Hubei University of Automotive Technology is used to illustrate the effectiveness of the proposed method, and according to the decision-maker preference, how to select the approximate value of parameter θ is discussed. Conclusions are drawn in Section 6.

2. Preliminaries

In this section, the definitions of PFS, 2-tuple fuzzy linguistic sets (2TFLSs), picture 2-tuple fuzzy linguistic sets (P2TFLSs), and 2-tuple linguistic picture fuzzy sets (2TLPFSs) are presented, and the operation rules of P2TFLSs are reviewed to lay the groundwork for later analysis.

2.1. Definitions of PFS, 2TFLSs, P2TFLSs, and 2TLPFSs

Definition 1 (see [32]). Let X be a universe space, and a PFS is defined as

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}, \tag{1}$$

where $\mu_A(x) \in [0,1]$ is called the degree of positive membership of x in A, $\eta_A(x) \in [0,1]$ is called the degree of neutral membership of x in A, $\nu_A(x) \in [0,1]$ is called the degree of negative membership of x in A, and $\mu_A(x)$, $\eta_A(x)$, and $\nu_A(x)$ satisfy the condition $0 \le \mu_A(x) + \eta_A(x) + \nu_A(x) \le 1$. Moreover, $\pi_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ can be called the degree of refusal.

Let $S = \{s_j \mid j = 0, 1, 2, \ldots, g\}$ be a linguistic term set, and symbolic method aggregation linguistic information obtains a value $\beta \in [0, g]$, and if $\beta \notin [0, g]$, then an approximation function $(\mathrm{app}_2(\cdot))$ is used to express the index result of S.

Definition 2 (see [41]). Let β be the aggregation result of the indices of a set of labels assessed in a linguistic term set S, i.e., the results of symbolic aggregation operation, and $\beta \in [1, t]$ be the t cardinality of S. Let $i = \mathrm{round}(\beta)$ and $\alpha = \beta - i$ be two values such that $i \in [1, t]$ and $\alpha \in [-0.5, 0.5)$; then, α is called the symbolic translation.

Based on the PFS and Definition 2, Wei et al. [41] defined the picture 2-tuple fuzzy linguistic set.

Definition 3. A picture 2-tuple fuzzy linguistic set A in X is defined as

$$A = \left\{\left(\left(s_{\theta(x)}, \rho\right), \left(\mu_A(x), \eta_A(x), \nu_A(x)\right)\right) \mid x \in X\right\}, \tag{2}$$

where $\mu_A(x) \in [0,1]$, $\eta_A(x) \in [0,1]$, and $\nu_A(x) \in [0,1]$ with the condition $0 \le \mu_A(x) + \eta_A(x) + \nu_A(x) \le 1$, $\forall x \in X$, $s_{\theta(x)} \in s$, and $\rho \in [-0.5, 0.5]$, the numbers $\mu_A(x)$, $\eta_A(x)$, and $\nu_A(x)$ represent the positive membership degree, neutral membership degree, negative membership degree of the element x to linguistic variable $(s_{\theta(x)}, \rho)$, respectively. Then, $\pi_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ could be called the refusal membership degree of the element x to linguistic variable $(s_{\theta(x)}, \rho)$.

For convenience to collect the data in the selection of the proxy advisor firm problems, Nie et al. [43] defined the concept of 2-tuple linguistic picture fuzzy sets (2TLPFSs).

Definition 4. Let $S = \{s_j \mid j = 0, 1, 2, \ldots, g\}$ be a linguistic set and $\overline{S} = \{(s_q, \alpha_q) \mid s_q \in S, \alpha_q \in [0.5, 0.5]\}$ be a linguistic 2-tuple fuzzy set. Suppose that (s_u, α_u), (s_η, α_η), and $(s_\nu, \alpha_\nu) \in \overline{S}$, if $0 \le \mu + \eta + \nu \le g$, then $\overline{r} = [(s_u, \alpha_u), (s_\eta, \alpha_\eta), (s_\nu, \alpha_\nu)]$ is called 2TLPFSs, and (s_u, α_u), (s_η, α_η), and (s_ν, α_ν) could be called the positive membership degree, neutral membership degree, and negative membership degree of $\overline{r}$, respectively.

2.2. Operation Rules of Linguistic Terms Sets (LTSs) and 2TLPFSs.

Motivated by the aggregation function of t-norm, the operations of LTSs and 2TLPFSs are constructed, as follows.

Definition 5 (see [32]). Let a_1 and a_2 be two 2TLPFSs, and the operation rules of 2TLPFSs can be defined as follows:

(1) $a_1 + a_2 = \langle \Delta(\Delta^{-1}(s_{\theta(a_1)}, \rho_{a_1}) + \Delta^{-1}(s_{\theta(a_2)}, \rho_{a_2})), (\mu_{a_1} + \mu_{a_2} - \mu_{a_1}\mu_{a_2}, \eta_{a_1}\eta_{a_2}, \nu_{a_1}\nu_{a_2})\rangle$

(2) $a_1 \otimes a_2 = \langle \Delta(\Delta^{-1}(s_{\theta(a_1)}, \rho_{a_1}) \otimes \Delta^{-1}(s_{\theta(a_2)}, \rho_{a_2})), (\mu_{a_1}\mu_{a_2}, \eta_{a_1} + \eta_{a_2} - \eta_{a_1}\eta_{a_2}, \nu_{a_1} + \nu_{a_2} - \nu_{a_1}\nu_{a_2})\rangle$

(3) $\lambda a_1 = \langle \Delta(\lambda\Delta^{-1}(s_{\theta(a_1)}, \rho_{a_1})), (1 - (1 - \mu_{a_1})^\lambda, (\eta_{a_1})^\lambda, (\nu_{a_1})^\lambda)\rangle, \lambda > 0$

(4) $a_1^\lambda = \langle \Delta(\Delta^{-1}(s_{\theta(a_1)}, \rho_{a_1})^\lambda), ((\mu_{a_1})^\lambda, 1 - (1 - \eta_{a_1})^\lambda, 1 - (1 - \nu_{a_1})^\lambda)\rangle, \lambda > 0$

However, in some typical circumstances, the results obtained by the above operation rules can be unreasonable, and the examples are shown as follows.

Example 1. Suppose that two alternatives are evaluated under three criteria, the evaluation values are represented in the form of 2TLPFS as $a_{11} = \langle (s_0, 0), (1, 0, 0)\rangle$, $a_{12} = \langle (s_0, 0), (0.01, 0, 0)\rangle$, $a_{13} = \langle (s_0, 0), (0.01, 0, 0)\rangle$, and $a_{21} = \langle (s_0, 0), (0.9, 0, 0)\rangle$, $a_{22} = \langle (s_0, 0), (0.9, 0, 0)\rangle$, $a_{23} = \langle (s_0, 0), (0.9, 0, 0)\rangle$. The criteria weight vector is $w = (0.3, 0.3, 0.4)$. Obviously, the aggregated result of alternative a_2 is superior than that of a_1. However, in accordance with the above operation rules, the results are $w_1 \otimes a_{11} \oplus w_2 \otimes a_{12} \oplus w_2 \otimes a_{13} = \langle (s_0, 0), (1, 0, 0)\rangle$ and $w_1 \otimes a_{21} \oplus w_2 \otimes a_{22} \oplus w_3 \otimes a_{23} = \langle (s_0, 0), (0.9, 0, 0)\rangle$, which indicate that the aggregated result of alternative a_1 is superior than that of a_2. Thus, this result cannot be accepted.

Example 2. Assume that experts give three criteria score by 2TLPFS for two alternatives that $a_{11} = \langle (s_1, 0), (0.8, 0, 0)\rangle$, $a_{12} = \langle (s_1, 0), (0.8, 0, 0)\rangle$, $a_{13} = \langle (s_1, 0), (0, 0.5, 0)\rangle$, and $a_{21} = \langle (s_1, 0), (0.3, 0, 0)\rangle$, $a_{22} = \langle (s_1, 0), (0.3, 0, 0)\rangle$, $a_{23} = \langle (s_1, 0), (0, 0, 0.3)\rangle$. The criteria weight vector is

$w = (0.3, 0.3, 0.4)$, and it is easy to see that a_1 should be superior than a_2. However, according to the operation rules of Definition 5, the results are $(a_{11})^{w_1} \otimes (a_{12})^{w_2} \otimes (a_{13})^{w_3} = \langle (s_1, 0), (0, 0, 0) \rangle$ and $(a_{21})^{w_1} \otimes (a_{22})^{w_2} \otimes (a_{23})^{w_3} = \langle (s_1, 0), (0, 0, 0) \rangle$, which indicate that the alternative a_1 is equal to alternative a_2. So, the result is not reasonable.

Definition 6 (see [44]). Let $s = \{s_t \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a LTS, s_α and s_β be two linguistic terms, g and g^{-1} be two equivalent transformation function of linguistic terms, and λ be a real number; then,

(1) $s_\alpha + s_\beta = g^{-1}(g(s_\alpha) + g(s_\beta) - g(s_\alpha) \otimes g(s_\beta))$
(2) $s_\alpha \otimes s_\beta = g^{-1}(g(s_\alpha) \otimes g(s_\beta))$
(3) $\lambda s_\alpha = g^{-1}(1 - (1 - g(s_\alpha))^\lambda)$
(4) $(s_\alpha)^\lambda = g^{-1}((g(s_\alpha))^\lambda)$

Although the above operation rules can overcome some defects existed in the previous paper, it still can yield the unreasonable results under some typical situations. The example is similar to the above Example 2; thus, it is omitted here.

3. Hesitant Picture Fuzzy Linguistic Sets (HPFLSs)

In this section, a new concept of HPFLSs is introduced firstly. Subsequently, motivated by the operation rules of the linguistic term set [44] and picture 2-tuple linguistic term sets [41], to avoid the existing limitations pointed out in Examples 1 and 2, the novel operation laws of HPFLSs are proposed. Next, two aggregation operators of HPFLSs are introduced, and the related properties are discussed.

3.1. Hesitant Picture Fuzzy Linguistic Sets (HPFLSs)

Definition 7. Let $S = \{s_J \mid J = 0, 1, 2, \ldots, m\}$ be an LTS, a *HPFLS* is defined as

$$H_s = \left\{ \left\langle \left(s_i^k\right), \left(\mu_i^k, \eta_i^k, \nu_i^k\right) \right\rangle \mid i \in (0, 1, 2, \ldots, m); k = 1, 2, \ldots, \alpha \right\}, \tag{3}$$

where $\mu_i^k \in [0, 1]$, $\eta_i^k \in [0, 1]$, $\nu_i^k \in [0, 1]$, and $0 \le \mu_i^k + \eta_i^k + \nu_i^k \le 1$. μ_i^k, η_i^k, and ν_i^k represent the positive membership, indeterminacy membership, and negative membership of the linguistic term s_i^k, respectively; $\pi_i^k = 1 - (\mu_i^k + \eta_i^k + \nu_i^k)$ is the refusal membership of the linguistic term s_i^k. And its complementary set is $H_s^c = \{\langle (s_i^k), (\nu_i^k, \eta_i^k, \mu_i^k) \rangle \mid i \in (0, 1, 2, \ldots, m); k = 1, 2, \ldots, \alpha\}$.

3.2. Novel Operation Rules of HPFLSs. In this subsection, two equivalent transformation functions f and g are defined. Then, the novel operation laws of *HPFLSs* are introduced based on the defined functions.

Definition 8. Let $S = \{s_J \mid J = 0, 1, 2, \ldots, m\}$ be an LTS; then, the linguistic term s_J can be equivalently transformed to α according to the function g, and α can be equivalently transformed to the linguistic term s_J according to the function g^{-1} as follows:

$$g\colon [0, m] \longrightarrow [a, b] \subset [0, 1], \quad g\left(s_J\right) = \frac{1}{1 + e^{-\theta_1 (J - m/2)}} = \alpha,$$
$$g^{-1}\colon [a, b] \longrightarrow [0, m], \quad g^{-1}(\alpha) = s_{m/2 - \ln^{(1-\alpha)/\alpha}/\theta_1} = s_J, \tag{4}$$

where $\theta_1 > 0$, $\theta_1 = \theta/m$, and θ are the adjustment parameters described in Definition 9.

Definition 9. For any $x \in [0, 1]$, x can be equivalently transformed to β according to the function f, and β can be equivalently transformed to x according to the function f^{-1} as follows:

$$f\colon [0, 1] \longrightarrow [c, d] \subset [0, 1], \quad f(x) = \frac{1}{1 + e^{-\theta(x - 0.5)}} = \beta,$$
$$f^{-1}\colon [c, d] \longrightarrow [0, 1], \quad f^{-1}(\beta) = 0.5 - \ln^{(1-\beta)/\beta}/\varphi = x, \tag{5}$$

where $\theta > 0$, by default, $\theta = 16$.

Definition 10. Let $H_{s_1} = \{\langle (s_{i_1}^{k_1}), (\mu_{i_1}^{k_1}, \eta_{i_1}^{k_1}, \nu_{i_1}^{k_1}) \rangle \mid i_1 \in (0, 1, 2, \ldots, m), k_1 = 0, 1, 2, \ldots, \alpha_1\}$ and $H_{s_2} = \{\langle (s_{i_2}^{k_2}), (\mu_{i_2}^{k_2}, \eta_{i_2}^{k_2}, \nu_{i_2}^{k_2}) \rangle \mid i_2 \in (0, 1, 2, \ldots, m), k_2 = 1, 2, \ldots, \alpha_2\}$ be two *HPFLSs*, f, f^{-1} and g, g^{-1} be the equivalent transformation functions, and $\lambda > 0$; then, the operation rules can be defined below:

(1) $H_{s_1} \oplus H_{s_2} = \langle \cup_{\substack{k_1=0,1,\ldots,\alpha_1; \\ k_2=0,1,\ldots,\alpha_2}} \{(s_{i_1}^{k_1} \oplus s_{i_2}^{k_2}),$
$(f^{-1}(1 - (1 - f(\mu_{i_1}^{k_1})) \quad (1 - f(\mu_{i_2}^{k_2}))), f^{-1}(f(\eta_{i_1}^{k_1})$
$f(\eta_{i_2}^{k_2})), f^{-1}(f(\eta_{i_1}^{k_1} + \nu_{i_1}^{k_1}) f(\eta_{i_2}^{k_2} + \nu_{i_2}^{k_2})) - f^{-1}(f(\eta_{i_1}^{k_1})$
$f(\eta_{i_2}^{k_2})))\}\rangle$

(2) $H_{s_1} \otimes H_{s_2} = \langle \cup_{\substack{k_1=0,1,\ldots,\alpha_1; \\ k_2=0,1,\ldots,\alpha_2}} \{(s_{i_1}^{k_1} \otimes s_{i_2}^{k_2}),$
$(f^{-1}(f(\mu_{i_1}^{k_1} + \eta_{i_1}^{k_1}) f(\mu_{i_2}^{k_2} + \eta_{i_2}^{k_2})) -$
$f^{-1}(f(\eta_{i_1}^{k_1}) f(\eta_{i_2}^{k_2})), f^{-1}(f(\eta_{i_1}^{k_1}) f(\eta_{i_2}^{k_2})),$
$f^{-1}(1 - (1 - f(\nu_{i_1}^{k_1}))(1 - f(\nu_{i_2}^{k_2}))))\}\rangle$

(3) $\lambda H_{s_1} = \langle \cup_{k_1=1,2,\ldots,\alpha_1} \{(\lambda \otimes s_{i_1}^{k_1}),$
$(f^{-1}(1 - (1 - f(\mu_{i_1}^{k_1}))^\lambda), f^{-1}((f(\eta_{i_1}^{k_1}))^\lambda),$
$f^{-1}((f(\eta_{i_1}^{k_1} + \nu_{i_1}^{k_1}))^\lambda) - f^{-1}((f(\eta_{i_1}^{k_1}))^\lambda))\}\rangle$

(4) $(H_{s_1})^\lambda = \langle \cup_{k_1=1,2,\ldots,\alpha_1} \{((s_{i_1}^{k_1})^\lambda),$
$(f^{-1}((f(\mu_{i_1}^{k_1} + \eta_{i_1}^{k_1}))^\lambda) - f^{-1}((f(\eta_{i_1}^{k_1}))^\lambda),$
$f^{-1}((f(\eta_{i_1}^{k_1}))^\lambda), f^{-1}(1 - (1 - f(\nu_{i_1}^{k_1}))^\lambda))\}\rangle$

where $s_{i_1}^{k_1} \oplus s_{i_2}^{k_2} = g^{-1}(g(s_{i_1}^{k_1}) + g(s_{i_1}^{k_1}) - g(s_{i_1}^{k_1}) g(s_{i_1}^{k_1}))$, $s_{i_1}^{k_1} \otimes s_{i_2}^{k_2} = g^{-1}(g(s_{i_1}^{k_1}) \otimes g(s_{i_2}^{k_2}))$, $\lambda \otimes s_{i_1}^{k_1} = g^{-1}(1 - (1 - g(s_{i_1}^{k_1}))^\lambda)$, and $(s_{i_1}^{k_1})^\lambda = g^{-1}((g(s_{i_1}^{k_1}))^\lambda)$.

Based on the above operation rules in Definition 10, we have concluded the following Theorem 1.

Theorem 1. *Let $H_{s_1}, H_{s_2}, H_{s_3}$ be three HPFLS, and $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$, then, the following properties are true.*

(1) $\lambda_1 H_{s_1} = H_{s_1}$, *if and only if* $\lambda_1 = 1$

(2) $\lambda_1 H_{s_1} \oplus \lambda_2 H_{s_2} = \lambda_2 H_{s_2} \oplus \lambda_1 H_{s_1}$

(3) $(\lambda_1 H_{s_1} \oplus \lambda_2 H_{s_2}) \oplus \lambda_3 H_{s_3} = \lambda_1 H_{s_1} \oplus (\lambda_2 H_{s_2} \oplus \lambda_3 H_{s_3})$

(4) $\lambda_1 H_{s_1} \otimes (\lambda_2 H_{s_2}) = \lambda_2 H_{s_2} \otimes (\lambda_1 H_{s_1})$

(5) $(\lambda_1 H_{s_1} \otimes (\lambda_2 H_{s_2})) \otimes (\lambda_3 H_{s_3}) = \lambda_1 H_{s_1} \otimes (\lambda_2 H_{s_2} \otimes (\lambda_3 H_{s_3}))$

(6) $(H_{s_1})^{\lambda_1} \oplus (H_{s_2})^{\lambda_2} = (H_{s_2})^{\lambda_2} \oplus (H_{s_1})^{\lambda_1}$

(7) $(H_{s_1})^{\lambda_1} \oplus (H_{s_2})^{\lambda_2} \oplus (H_{s_3})^{\lambda_3} = (H_{s_1})^{\lambda_1} \oplus ((H_{s_2})^{\lambda_2} \oplus (H_{s_3})^{\lambda_3})$

(8) $(H_{s_1})^{\lambda_1} \otimes (H_{s_2})^{\lambda_2} = (H_{s_2})^{\lambda_2} \otimes (H_{s_1})^{\lambda_1}$

Example 3. Let $S = \{s_J \mid J = 0, 1, 2, \ldots, 4\}$ be an LTS, a_1 and a_2 be two alternatives, $c = \{c_1, c_2, c_3\}$ be the criteria set, and $w = (0.2, 0.2, 0.6)^T$ be the weight vector. The corresponding evaluation values are presented in Table 1.

According to Definitions 8–10, we have $\theta_1 = 4$, $\theta = 16$, and $m/2 = 2$. By using the operation rules (1) and (3) in Definition 10, the aggregation results of a_1 is obtained as $\langle (S_4), (0.8, 0, 0) \rangle$, and that of a_2 is $\langle (S_{2.3437}), (0.7796, 0, 0) \rangle$. The aggregation results show that $a_1 \succ a_2$. Similarly, by using the operation rules (2) and (4) in Definition 10, we obtain the aggregation results for a_1 as $\langle (S_4), (0.8, 0, 0) \rangle$ and the aggregation results for a_2 as $\langle (S_{0.4003}), (0.4399, 0, 0) \rangle$. The aggregation results show that $a_1 \succ a_2$ as well. Intuitively, a_1 should be superior than a_2 according to the evaluation values shown in Table 1. Thus, the above results state that our proposed operation rule can effectively overcome the defects as pointed out in Examples 1 and 2. The detailed computing process of the aggregation result is omitted here.

3.3. Aggregation Operators of HPFLSs. In this subsection, two aggregation operators of HPFLSs are presented and can be utilized to solve the real-life decision-making problems.

Table 1: The evaluation values with HPFLSs.

	c_1	c_2	c_3
a_1	$\langle (S_4), (0.8, 0, 0) \rangle$	$\langle (S_4), (0.8, 0, 0) \rangle$	$\langle (S_4), (0.8, 0, 0) \rangle$
a_2	$\langle (S_4), (1, 0, 0) \rangle$	$\langle (S_0), (0.1, 0, 0) \rangle$	$\langle (S_0), (0.1, 0, 0) \rangle$

Definition 11. Let $H_{s_\varphi} = \{\langle (s_{i_\varphi}^{k_\varphi}), (\mu_{i_\varphi}^{k_\varphi}, \eta_{i_\varphi}^{k_\varphi}, \nu_{i_\varphi}^{k_\varphi}) \rangle \mid i_\varphi \in (0, 1, 2, \ldots, m); k_\varphi = 1, 2, \ldots, \alpha_\varphi\} (\varphi = 1, 2, \ldots, n)$ be n HPFLSs, and their weight vector is $w = (w_1, w_2, \ldots, w_n)^T$. Then, the hesitant picture fuzzy linguistic weighted average operator (*HPFLWA*) can be defined as follows:

$$HPFLWA(H_{s_1}, H_{s_2}, \ldots, H_{s_n}) = w_1 H_{s_1} \oplus w_1 H_{s_2} \oplus, \ldots, \oplus w_n H_{s_n} = \sum_{j=1}^{n} w_j H_{s_j}. \tag{6}$$

Definition 12. Let $H_{s_\varphi} = \{\langle (s_{i_\varphi}^{k_\varphi}), (\mu_{i_\varphi}^{k_\varphi}, \eta_{i_\varphi}^{k_\varphi}, \nu_{i_\varphi}^{k_\varphi}) \rangle \mid i_\varphi \in (0, 1, 2, \ldots, m); k_\varphi = 1, 2, \ldots, \alpha_\varphi\} (\varphi = 1, 2, \ldots, n)$ be n HPFLSs, and their weight vector is $w = (w_1, w_2, \ldots, w_n)^T$. Then, the hesitant picture fuzzy linguistic weighted geometric operator (*HPFLWG*) is referred to as

$$HPFLGA(H_{s_1}, H_{s_2}, \ldots, H_{s_n}) = (H_{s_1})^{w_1} \otimes (H_{s_2})^{w_2} \otimes, \ldots, \otimes (H_{s_n})^{w_n} = \prod_{j=1}^{n} (H_{s_j})^{w_j}. \tag{7}$$

Based on Definition 11 and the operation rules described in Definition 10, Theorem 2 can be gained.

Theorem 2. *For n HPFLSs, $H_{s_\varphi} = \{\langle (s_{i_\varphi}^{k_\varphi}), (\mu_{i_\varphi}^{k_\varphi}, \eta_{i_\varphi}^{k_\varphi}, \nu_{i_\varphi}^{k_\varphi}) \rangle \mid i_\varphi \in (0, 1, 2, \ldots, m); k_\varphi = 1, 2, \ldots, \alpha_\varphi\}$, $\varphi = 1, 2, \ldots, n$; and their weight vector is $w = (w_1, w_2, \ldots, w_n)^T$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, and the aggregation results can be obtained by the HPFLWA operator as*

$$\begin{aligned} &HPFLWA(H_{s_1}, H_{s_2}, \ldots, H_{s_n}) \\ &= \Bigg\langle \bigcup_{\substack{k_1=1,2,\cdots,\alpha_1 \\ k_2=1,2,\cdots,\alpha_2 \\ \vdots \\ k_n=1,2,\cdots,\alpha_n}} \Bigg\{ \left(g^{-1}\left(1 - \prod_{\varphi=1}^{n} \left(1 - g\left(s_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right)\right), \left(f^{-1}\left(1 - \prod_{\varphi=1}^{n} \left(1 - f\left(\mu_{i_\varphi}^{k\varphi}\right)\right)^{w_\varphi}\right), f^{-1}\left(\prod_{\varphi=1}^{n} \left(f\left(\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right), \\ &f^{-1}\left(\prod_{\varphi=1}^{n} \left(f\left(\eta_{i_\varphi}^{k_\varphi} + \nu_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right) - f^{-1}\left(\prod_{\varphi=1}^{n} \left(f\left(\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right)\right) \Bigg\} \Bigg\rangle. \end{aligned} \tag{8}$$

Proof. (1) When $n = 2$, the following result can be gained:

$$\begin{aligned} w_1 \otimes H_{s_1} &= \left\langle \bigcup_{k_1=1,2,\cdots,\alpha_1} \left\{ \left(w_1 \otimes s_{i_1}^{k_1}\right), \left(f^{-1}\left(1-\left(1-f\left(\mu_{i_1}^{k_1}\right)\right)^{w_1}\right), f^{-1}\left(\left(f\left(\eta_{i_1}^{k_1}\right)\right)^{w_1}\right), f^{-1}\left(\left(f\left(\eta_{i_1}^{k_1}+\nu_{i_1}^{k_1}\right)\right)^{w_1}\right) - f^{-1}\left(\left(f\left(\eta_{i_1}^{k_1}\right)\right)^{w_1}\right)\right)\right\}\right\rangle, \\ w_2 \otimes H_{s_2} &= \left\langle \bigcup_{k_2=1,2,\cdots,\alpha_2} \left\{ \left(w_2 \otimes s_{i_2}^{k_2}\right), \left(f^{-1}\left(1-\left(1-f\left(\mu_{i_2}^{k_2}\right)\right)^{w_2}\right), f^{-1}\left(\left(f\left(\eta_{i_2}^{k_2}\right)\right)^{w_2}\right), f^{-1}\left(\left(f\left(\eta_{i_2}^{k_2}+\nu_{i_2}^{k_2}\right)\right)^{w_2}\right) - f^{-1}\left(\left(f\left(\eta_{i_2}^{k_2}\right)\right)^{w_2}\right)\right)\right\}\right\rangle, \end{aligned} \tag{9}$$

then

$$\begin{aligned} & w_1 \otimes H_{s_1} + w_2 \otimes H_{s_2} \\ &= \left\langle \bigcup_{\substack{k_1=1,2,\ldots,\alpha_1;\\ k_2=1,2,\ldots,\alpha_2}} \left\{ \left(w_1 \otimes s_{i_1}^{k_1} + w_2 \otimes s_{i_2}^{k_2}\right), \left(f^{-1}\left(1-\left(1-f\left(f^{-1}\left(1-\left(1-f\left(\mu_{i_1}^{k_1}\right)\right)^{w_1}\right)\right)\right)\right) \otimes \left(1-f\left(f^{-1}\left(1-\left(1-f\left(\mu_{i_2}^{k_2}\right)\right)^{w_2}\right)\right)\right), \right.\right. \\ & \quad f^{-1}\left(f\left(f^{-1}\left(\left(f\left(\eta_{i_1}^{k_1}\right)\right)^{w_1}\right)\right) \otimes f\left(f^{-1}\left(\left(f\left(\eta_{i_2}^{k_2}\right)\right)^{w_2}\right)\right)\right), \\ & \quad \left.\left. f^{-1}\left(f\left(f^{-1}\left(\left(f\left(\eta_{i_1}^{k_1}+\nu_{i_1}^{k_1}\right)\right)^{w_1}\right)\right) \otimes f\left(f^{-1}\left(\left(f\left(\eta_{i_2}^{k_2}+\nu_{i_2}^{k_2}\right)\right)^{w_2}\right)\right)\right) - f^{-1}\left(f\left(f^{-1}\left(\left(f\left(\eta_{i_1}^{k_1}\right)\right)^{w_1}\right)\right) \otimes f\left(f^{-1}\left(\left(f\left(\eta_{i_2}^{k_2}\right)\right)^{w_2}\right)\right)\right)\right)\right\}\right\rangle \\ &= \left\langle \bigcup_{\substack{k_1=1,2,\ldots,\alpha_1;\\ k_2=1,2,\ldots,\alpha_2}} \left\{ \left(g^{-1}\left(g\left(g^{-1}\left(1-\left(1-g\left(s_{i_1}^{k_1}\right)\right)^{w_1}\right)\right) + g\left(g^{-1}\left(1-\left(1-g\left(s_{i_2}^{k_2}\right)\right)^{w_2}\right)\right) - g\left(g^{-1}\left(1-\left(1-g\left(s_{i_1}^{k_1}\right)\right)^{w_1}\right)\right)\right.\right.\right. \right. \\ & \quad \left.\left.\left. \otimes g\left(g^{-1}\left(1-\left(1-g\left(s_{i_2}^{k_2}\right)\right)^{w_2}\right)\right)\right)\right), \left(f^{-1}\left(\left(1-\left(1-f\left(\mu_{i_1}^{k_1}\right)\right)^{w_1} \otimes \left(1-f\left(\mu_{i_2}^{k_2}\right)\right)^{w_2}\right), f^{-1}\left(\left(f\left(\eta_{i_1}^{k_1}\right)\right)^{w_1}\right.\right.\right. \\ & \quad \left.\left.\left.\left. \otimes \left(f\left(\eta_{i_2}^{k_2}\right)\right)^{w_2}\right), f^{-1}\left(\left(f\left(\eta_{i_1}^{k_1}+\nu_{i_1}^{k_1}\right)\right)^{w_1} \otimes \left(f\left(\eta_{i_2}^{k_2}+\nu_{i_2}^{k_2}\right)\right)^{w_2}\right) - f^{-1}\left(\left(f\left(\eta_{i_1}^{k_1}\right)\right)^{w_1} \otimes \left(f\left(\eta_{i_2}^{k_2}\right)\right)^{w_2}\right)\right)\right\}\right\rangle \\ &= \left\langle \bigcup_{\substack{k_1=1,2,\ldots,\alpha_1\\ k_2=1,2,\ldots,\alpha_2}} \left\{ \left(g^{-1}\left(1-\prod_{\varphi=1}^{2}\left(1-g\left(s_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right)\right), \left(f^{-1}\left(1-\prod_{\varphi=1}^{2}\left(1-f\left(\mu_{i_\varphi}^{k\varphi}\right)\right)^{w_\varphi}\right), f^{-1}\left(\prod_{\varphi=1}^{2}\left(f\left(\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right), \right.\right. \\ & \quad \left.\left. f^{-1}\left(\prod_{\varphi=1}^{2}\left(f\left(\eta_{i_\varphi}^{k_\varphi}+\nu_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right) - f^{-1}\left(\prod_{\varphi=1}^{2}\left(f\left(\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right)\right)\right\}\right\rangle. \end{aligned} \tag{10}$$

(2) Assume that equation (8) holds for any n; thus, we have

$$\begin{aligned} & w_1 \otimes H_{s_1} + w_2 \otimes H_{s_2} + \cdots + w_n \otimes H_{s_n} \\ &= \left\langle \bigcup_{\substack{k_1=1,2,\cdots,\alpha_1\\ k_2=1,2,\cdots,\alpha_2\\ \vdots\ \vdots\ \vdots\ \ddots\ \vdots\\ k_n=1,2,\cdots,\alpha_n}} \left\{ \left(g^{-1}\left(1-\prod_{\varphi=1}^{n}\left(1-g\left(s_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right)\right), \left(f^{-1}\left(1-\prod_{\varphi=1}^{n}\left(1-f\left(\mu_{i_\varphi}^{k\varphi}\right)\right)^{w_\varphi}\right), f^{-1}\left(\prod_{\varphi=1}^{n}\left(f\left(\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right), \right.\right. \\ & \quad \left.\left. f^{-1}\left(\prod_{\varphi=1}^{n}\left(f\left(\eta_{i_\varphi}^{k_\varphi}+\nu_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right) - f^{-1}\left(\prod_{\varphi=1}^{n}\left(f\left(\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right)\right)\right\}\right\rangle. \end{aligned} \tag{11}$$

(3) When $\varphi = n+1$, the following results are acquired:

$$
\begin{aligned}
&w_1 \otimes H_{s_1} + w_2 \otimes H_{s_2} + \cdots + w_n \otimes H_{s_n} + w_{n+1} \otimes H_{s_{n+1}} \\
&= \Bigg\langle \bigcup_{\substack{k_1=1,2,\cdots,\alpha_1 \\ k_2=1,2,\cdots,\alpha_2 \\ \vdots\ \vdots\ \vdots\ \ddots\ \vdots \\ k_n=1,2,\cdots,\alpha_n \\ k_{n+1}=1,2,\cdots\alpha_{n+1}}} \Bigg\{ \Bigg(g^{-1}\Bigg(g^{-1}\Bigg(1 - \prod_{\varphi=1}^{n} \Big(1 - g\Big(s_{i_\varphi}^{k_\varphi} \Big)\Big)^{w_\varphi} \Bigg) + g\Bigg(g^{-1}\Bigg(1 - \Big(1 - g\Big(s_{i_{n+1}}^{k_{n+1}} \Big)\Big)^{w_{n+1}} \Bigg)\Bigg) \\
&\qquad - g^{-1}\Bigg(1 - \prod_{\varphi=1}^{n} \Big(1 - g\Big(s_{i_\varphi}^{k_\varphi} \Big)\Big)^{w_\varphi} \Bigg) \otimes g\Bigg(g^{-1}\Bigg(1 - \Big(1 - g\Big(s_{i_{n+1}}^{k_{n+1}} \Big)\Big)^{w_{n+1}} \Bigg)\Bigg)\Bigg)\Bigg), \\
&\Bigg(f^{-1}\Bigg(1 - \Bigg(1 - f\Bigg(f^{-1}\Bigg(1 - \prod_{\varphi=1}^{n} \Big(1 - f\Big(\mu_{i_\varphi}^{k\varphi} \Big)\Big)^{w_\varphi} \Bigg)\Bigg)\Bigg) \otimes \Bigg(1 - f\Bigg(f^{-1}\Bigg(1 - \Big(1 - f\Big(\mu_{i_{n+1}}^{k_{n+1}} \Big)\Big)^{w_{n+1}} \Bigg)\Bigg)\Bigg)\Bigg), \\
&f^{-1}\Bigg(f\Bigg(f^{-1}\Bigg(\prod_{\varphi=1}^{n} \Big(f\Big(\eta_{i_\varphi}^{k_\varphi} \Big)\Big)^{w_\varphi} \Bigg)\Bigg) \otimes f\Bigg(f^{-1}\Bigg(\Big(f\Big(\eta_{i_{n+1}}^{k_{n+1}} \Big)\Big)^{w_{n+1}} \Bigg)\Bigg)\Bigg), \\
&f^{-1}\Bigg(f\Bigg(f^{-1}\Bigg(\prod_{\varphi=1}^{n} \Big(f\Big(\eta_{i_\varphi}^{k_\varphi} \Big) + f\Big(\nu_{i_\varphi}^{k_\varphi} \Big)\Big)^{w_\varphi} \Bigg)\Bigg) \otimes f\Bigg(f^{-1}\Bigg(\Big(f\Big(\eta_{i_{n+1}}^{k_{n+1}} + \nu_{i_{n+1}}^{k_{n+1}} \Big)\Big)^{w_{n+1}} \Bigg)\Bigg)\Bigg) - f^{-1}\Bigg(f\Bigg(f^{-1}\Bigg(\prod_{\varphi=1}^{n} \Big(f\Big(\eta_{i_\varphi}^{k_\varphi} \Big)\Big)^{w_\varphi} \Bigg)\Bigg) \\
&\otimes f\Bigg(f^{-1}\Bigg(\Big(f\Big(\eta_{i_{n+1}}^{k_{n+1}} \Big)\Big)^{w_{n+1}} \Bigg)\Bigg)\Bigg)\Bigg)\Bigg\}\Bigg\rangle \\
&= \Bigg\langle \bigcup_{\substack{k_1=1,2,\cdots,\alpha_1 \\ k_2=1,2,\cdots,\alpha_2 \\ \vdots\ \vdots\ \vdots\ \ddots\ \vdots \\ k_n=1,2,\cdots,\alpha_n \\ k_{n+1}=1,2,\cdots,\alpha_{n+1}}} \Bigg\{ \Bigg(g^{-1}\Bigg(1 - \prod_{\varphi=1}^{n+1} \Big(1 - g\Big(s_{i_\varphi}^{k_\varphi} \Big)\Big)^{w_\varphi} \Bigg)\Bigg), \Bigg(f^{-1}\Bigg(1 - \prod_{\varphi=1}^{n+1} \Big(1 - f\Big(\mu_{i_\varphi}^{k\varphi} \Big)\Big)^{w_\varphi} \Bigg), f^{-1}\Bigg(\prod_{\varphi=1}^{n+1} \Big(f\Big(\eta_{i_\varphi}^{k_\varphi} \Big)\Big)^{w_\varphi} \Bigg), \\
&f^{-1}\Bigg(\prod_{\varphi=1}^{n+1} \Big(f\Big(\eta_{i_\varphi}^{k_\varphi} + \nu_{i_\varphi}^{k_\varphi} \Big)\Big)^{w_\varphi} \Bigg) - f^{-1}\Bigg(\prod_{\varphi=1}^{n+1} \Big(f\Big(\eta_{i_\varphi}^{k_\varphi} \Big)\Big)^{w_\varphi} \Bigg)\Bigg)\Bigg\}\Bigg\rangle.
\end{aligned}
\tag{12}
$$

So, the equation holds for any n. The proof is completed now.

Based on Definition 12 and the operation rules of Definition 10, Theorem 3 can be obtained. □

Theorem 3. *For n HPFLSs, $H_{s_\varphi} = \Big\{ \langle (s_{i_\varphi}^{k_\varphi}), (\mu_{i_\varphi}^{k_\varphi}, \eta_{i_\varphi}^{k_\varphi}, \nu_{i_\varphi}^{k_\varphi}) \rangle \mid i_\varphi \in (0,1,2,\cdots,m); k_\varphi = 0,1,\cdots,\alpha_\varphi \Big\}$, $\varphi = 1,2,\cdots,n$; their weight vector is $w = (w_1, w_2, \cdots, w_n)^T$; for any j, $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, and the aggregation results can be calculated by the HPFLWG operator:*

$$
\begin{aligned}
&HPFLGA\big(H_{s_1}, H_{s_2}, \cdots, H_{s_n}\big) \\
&= \Bigg\langle \bigcup_{\substack{k_1=1,2,\cdots,\alpha_1 \\ k_2=1,2,\cdots,\alpha_2 \\ \vdots\ \vdots\ \vdots\ \ddots\ \vdots \\ k_n=1,2,\cdots,\alpha_n}} \Bigg\{ \Bigg(g^{-1}\Bigg(\prod_{\varphi=1}^{n} \Big(g\Big(s_{i_\varphi}^{k_\varphi} \Big)\Big)^{w_\varphi} \Bigg)\Bigg), \Bigg(f^{-1}\Bigg(\prod_{\varphi=1}^{n} \Big(f\Big(\mu_{i_\varphi}^{k\varphi} + \eta_{i_\varphi}^{k_\varphi} \Big)\Big)^{w_\varphi} \Bigg) - f^{-1}\Bigg(\prod_{\varphi=1}^{n} \Big(f\Big(\eta_{i_\varphi}^{k_\varphi} \Big)\Big)^{w_\varphi} \Bigg), \\
&f^{-1}\Bigg(\prod_{\varphi=1}^{n} \Big(f\Big(\eta_{i_\varphi}^{k_\varphi} \Big)\Big)^{w_\varphi} \Bigg), f^{-1}\Bigg(1 - \prod_{\varphi=1}^{n} \Big(1 - f\Big(\nu_{i_\varphi}^{k_\varphi} \Big)\Big)^{w_\varphi} \Bigg)\Bigg)\Bigg\}\Bigg\rangle.
\end{aligned}
\tag{13}
$$

Proof.

(1) When $n = 2$, in accordance with the operation rules in Definition 10, the following processing is presented:

$$
\begin{aligned}
\left(H_{s_1}\right)^{w_1} &= \Bigg\langle \bigcup_{k_1=1,2,\ldots,\alpha_1} \Bigg\{\left(\left(s_{i_1}^{k_1}\right)^{w_1}\right), \Bigg(f^{-1}\left(\left(f\left(\mu_{i_1}^{k_1}+\eta_{i_1}^{k_1}\right)\right)^{w_1}\right) - f^{-1}\left(\left(f\left(\eta_{i_1}^{k_1}\right)\right)^{w_1}\right), f^{-1}\left(\left(f\left(\eta_{i_1}^{k_1}\right)\right)^{w_1}\right), \\
&\quad f^{-1}\left(1-\left(1-f\left(\nu_{i_1}^{k_1}\right)\right)^{w_1}\right)\Bigg)\Bigg\}\Bigg\rangle, \\
\left(H_{s_2}\right)^{w_2} &= \Bigg\langle \bigcup_{k_2=1,2,\ldots,\alpha_2} \Bigg\{\left(\left(s_{i_2}^{k_2}\right)^{w_2}\right), \Bigg(f^{-1}\left(\left(f\left(\mu_{i_2}^{k_2}+\eta_{i_2}^{k_2}\right)\right)^{w_2}\right) - f^{-1}\left(\left(f\left(\eta_{i_2}^{k_2}\right)\right)^{w_2}\right), f^{-1}\left(\left(f\left(\eta_{i_2}^{k_2}\right)\right)^{w_2}\right), \\
&\quad f^{-1}\left(1-\left(1-f\left(\nu_{i_2}^{k_2}\right)\right)^{w_2}\right)\Bigg)\Bigg\}\Bigg\rangle, \\
\left(H_{s_1}\right)^{w_1} \otimes \left(H_{s_2}\right)^{w_2} &= \Bigg\langle \bigcup_{\substack{k_1=0,1,\ldots,\alpha_1; \\ k_2=0,1,\ldots,\alpha_2}} \Bigg\{\left(\left(s_{i_1}^{k_1}\right)^{w_1} \otimes \left(s_{i_2}^{k_2}\right)^{w_2}\right), \Bigg(f^{-1}\left(f\left(f^{-1}\left(\left(f\left(\mu_{i_1}^{k_1}+\eta_{i_1}^{k_1}\right)\right)^{w_1}\right)\right) \otimes f\left(f^{-1}\left(\left(f\left(\mu_{i_2}^{k_2}+\eta_{i_2}^{k_2}\right)\right)^{w_2}\right)\right)\right) \\
&\quad - f^{-1}\left(f\left(f^{-1}\left(\left(f\left(\eta_{i_1}^{k_1}\right)\right)^{w_1}\right)\right) \otimes f\left(f^{-1}\left(\left(f\left(\eta_{i_2}^{k_2}\right)\right)^{w_2}\right)\right)\right), \\
&\quad f^{-1}\left(f\left(f^{-1}\left(\left(f\left(\eta_{i_1}^{k_1}\right)\right)^{w_1}\right)\right) \otimes f\left(f^{-1}\left(\left(f\left(\eta_{i_2}^{k_2}\right)\right)^{w_2}\right)\right)\right), f^{-1}\left(1-\left(1-f\left(f^{-1}\left(1-\left(1-f\left(\nu_{i_1}^{k_1}\right)\right)^{w_1}\right)\right)\right)\right. \\
&\quad \left.\otimes\left(1-f\left(f^{-1}\left(1-\left(1-f\left(\nu_{i_2}^{k_2}\right)\right)^{w_2}\right)\right)\right)\right)\Bigg)\Bigg\}\Bigg\rangle \\
&= \Bigg\langle \bigcup_{\substack{k_1=0,1,\ldots,\alpha_1; \\ k_2=0,1,\ldots,\alpha_2}} \Bigg\{\left(g^{-1}\left(g\left(g^{-1}\left(\left(g\left(s_{i_1}^{k_1}\right)\right)^{w_1}\right)\right) \otimes g\left(g^{-1}\left(\left(g\left(s_{i_2}^{k_2}\right)\right)^{w_2}\right)\right)\right)\right), \\
&\quad \left(f^{-1}\left(\prod_{\varphi=1}^{2}\left(f\left(\mu_{i_\varphi}^{k\varphi}+\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right) - f^{-1}\left(\prod_{\varphi=1}^{2}\left(f\left(\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right), f^{-1}\left(\prod_{\varphi=1}^{2}\left(f\left(\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right), \right. \\
&\quad \left. f^{-1}\left(1-\prod_{\varphi=1}^{2}\left(1-f\left(\nu_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right)\right)\Bigg\}\Bigg\rangle \\
&= \Bigg\langle \bigcup_{\substack{k_1=0,1,\ldots,\alpha_1; \\ k_2=0,1,\ldots,\alpha_2}} \Bigg\{\left(g^{-1}\left(\prod_{\varphi=1}^{2}\left(g\left(s_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right)\right), \left(f^{-1}\left(\prod_{\varphi=1}^{2}\left(f\left(\mu_{i_\varphi}^{k\varphi}+\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right) - f^{-1}\left(\prod_{\varphi=1}^{2}\left(f\left(\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right), \right. \\
&\quad \left. f^{-1}\left(\prod_{\varphi=1}^{2}\left(f\left(\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right), f^{-1}\left(1-\prod_{\varphi=1}^{2}\left(1-f\left(\nu_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right)\right)\Bigg\}\Bigg\rangle.
\end{aligned}
\tag{14}
$$

Assume that the equation holds for any n:

$$
\begin{aligned}
\left(H_{s_1}\right)^{w_1} \otimes \left(H_{s_2}\right)^{w_2} \otimes \cdots \otimes \left(H_{s_n}\right)^{w_n} &= \Bigg\langle \bigcup_{\substack{k_1=1,2,\cdots,\alpha_1 \\ k_2=1,2,\cdots,\alpha_2 \\ \vdots\ \vdots\ \vdots\ \ddots\ \vdots \\ k_n=1,2,\cdots,\alpha_n}} \Bigg\{\left(g^{-1}\left(\prod_{\varphi=1}^{n}\left(g\left(s_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right)\right), \left(f^{-1}\left(\prod_{\varphi=1}^{n}\left(f\left(\mu_{i_\varphi}^{k\varphi}+\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right)\right. \\
&\quad \left. - f^{-1}\left(\prod_{\varphi=1}^{n}\left(f\left(\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right), f^{-1}\left(\prod_{\varphi=1}^{n}\left(f\left(\eta_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right), f^{-1}\left(1-\prod_{\varphi=1}^{n}\left(1-f\left(\nu_{i_\varphi}^{k_\varphi}\right)\right)^{w_\varphi}\right)\right)\Bigg\}\Bigg\rangle.
\end{aligned}
\tag{15}
$$

(3) When $\varphi = n+1$, the following results are acquired:

$$
\begin{aligned}
&\left(H_{s_1}\right)^{w_1} \otimes \left(H_{s_2}\right)^{w_2} \otimes \cdots \otimes \left(H_{s_n}\right)^{w_n} \otimes \left(H_{s_{n+1}}\right)^{w_{n+1}} \\
&= \Bigg\langle \bigcup_{\substack{k_1=1,2,\cdots,\alpha_1 \\ k_2=1,2,\cdots,\alpha_2 \\ \vdots\ \vdots\ \vdots\ \ddots\ \vdots \\ k_n=1,2,\cdots,\alpha_n \\ k_{n+1}=1,2,\cdots,\alpha_{n+1}}} \Bigg\{ \left(g^{-1}\left(g\left(g^{-1}\left(\prod_{\varphi=1}^{n} \left(g\left(s_{i_\varphi}^{k_\varphi} \right) \right)^{w_\varphi} \right) \right) \otimes g\left(g^{-1}\left(\left(g\left(s_{i_{n+1}}^{k_{n+1}} \right) \right)^{w_{n+1}} \right) \right) \right), \\
&\left(f^{-1}\left(f\left(f^{-1}\left(\prod_{\varphi=1}^{n} \left(f\left(\mu_{i_\varphi}^{k\varphi} + \eta_{i_\varphi}^{k_\varphi} \right) \right)^{w_\varphi} \right) \right) \otimes f\left(f^{-1}\left(\left(f\left(\mu_{i_{n+1}}^{k_{n+1}} + \eta_{i_{n+1}}^{k_{n+1}} \right) \right)^{w_{n+1}} \right) \right) \right) \right. \\
&- f^{-1}\left(f\left(f^{-1}\left(\prod_{\varphi=1}^{n} \left(f\left(\eta_{i_\varphi}^{k_\varphi} \right) \right)^{w_\varphi} \right) \right) \otimes f\left(f^{-1}\left(\left(f\left(\eta_{i_{n+1}}^{k_{n+1}} \right) \right)^{w_{n+1}} \right) \right) \right), f^{-1}\left(f\left(f^{-1}\left(\prod_{\varphi=1}^{n} \left(f\left(\eta_{i_\varphi}^{k_\varphi} \right) \right)^{w_\varphi} \right) \right) \right. \\
&\left. \otimes f\left(f^{-1}\left(\left(f\left(\eta_{i_{n+1}}^{k_{n+1}} \right) \right)^{w_{n+1}} \right) \right) \right), \\
&\left. f^{-1}\left(1 - \left(1 - f\left(f^{-1}\left(1 - \prod_{\varphi=1}^{n} \left(1 - f\left(\nu_{i_\varphi}^{k_\varphi} \right) \right)^{w_\varphi} \right) \right) \right) \otimes \left(1 - f\left(f^{-1}\left(1 - \left(1 - f\left(\nu_{i_{n+1}}^{k_{n+1}} \right) \right)^{w_{n+1}} \right) \right) \right) \right) \right) \Bigg\} \Bigg\rangle \\
&= \Bigg\langle \bigcup_{\substack{k_1=1,2,\cdots,\alpha_1 \\ k_2=1,2,\cdots,\alpha_2 \\ \vdots\ \vdots\ \vdots\ \ddots\ \vdots \\ k_n=1,2,\cdots,\alpha_n \\ k_{n+1}=1,2,\cdots,\alpha_{n+1}}} \Bigg\{ \left(g^{-1}\left(\prod_{\varphi=1}^{n+1} \left(g\left(s_{i_\varphi}^{k_\varphi} \right) \right)^{w_\varphi} \right) \right), \left(f^{-1}\left(\prod_{\varphi=1}^{n+1} \left(f\left(\mu_{i_\varphi}^{k\varphi} + \eta_{i_\varphi}^{k_\varphi} \right) \right)^{w_\varphi} \right) - f^{-1}\left(\prod_{\varphi=1}^{n+1} \left(f\left(\eta_{i_\varphi}^{k_\varphi} \right) \right)^{w_\varphi} \right), \right. \\
&\left. f^{-1}\left(\prod_{\varphi=1}^{n+1} \left(f\left(\eta_{i_\varphi}^{k_\varphi} \right) \right)^{w_\varphi} \right), f^{-1}\left(1 - \prod_{\varphi=1}^{n+1} \left(1 - f\left(\nu_{i_\varphi}^{k_\varphi} \right) \right)^{w_\varphi} \right) \right) \Bigg\} \Bigg\rangle.
\end{aligned}
\tag{16}
$$

Thus, the equation holds for any n, and the proof is completely down. □

3.4. Criteria Weight Determination. In the real-life decision-making circumstance, usually, the criteria weight is completely unknown due to the time pressure. Thus, the min-max deviation method is used to identify the criteria weight. Obviously, the criteria with the max-deviation should be assigned with the largest weight; the criteria with the min-deviation should be given the smallest weight. In this subsection, according to the deviation method constructed by the hamming distance, the deviation of the alternative $(a_t = 1, 2, \ldots, m)$ to other alternative with respect to a certain criteria $(c_q = 1, 2, \ldots, n)$ can be calculated by

$$
\begin{aligned}
D_{c_q} &= \frac{1}{m-1} \sum_{t=1}^{m} \sum_{e=1, e\neq t}^{m} \left(\left| H_{s_{tq}} - H_{s_{eq}} \right| \right) \\
&= \frac{1}{m-1} \sum_{t=1}^{m} \sum_{e=1, e\neq t}^{m} \left(\left| \sum_{k_{tq}=1}^{\alpha_{tq}} i_{tq} - \sum_{k_{eq}=1}^{\alpha_{eq}} i_{eq} \right| + \left| \sum_{k_{tq}=1}^{\alpha_{tq}} \mu_{i_{tq}}^{k_{tq}} - \sum_{k_{eq}=1}^{\alpha_{eq}} \mu_{i_{eq}}^{k_{eq}} \right| + \left| \sum_{k_{tq}=1}^{\alpha_{tq}} \eta_{i_{tq}}^{k_{tq}} - \sum_{k_{eq}=1}^{\alpha_{eq}} \eta_{i_{eq}}^{k_{eq}} \right| + \left| \sum_{k_{tq}=1}^{\alpha_{tq}} \nu_{i_{tq}}^{k_{tq}} - \sum_{k_{eq}=1}^{\alpha_{eq}} \nu_{i_{eq}}^{k_{eq}} \right| \right).
\end{aligned}
\tag{17}
$$

The large D_{c_q} indicates the corresponding criteria are more important. Thus, the following model is used to determine the criteria weight:

$$(\mathrm{M})\begin{cases} \max \quad D_c(w) = \sum_{q=1}^{n} w_q D_{c_q} \\ \text{s.t.} \quad \sum_{q=1}^{n} w_q^2 = 1, w_q \geq 0, q = 1, 2, \ldots, n. \end{cases} \tag{18}$$

Next, the above model (M) is solved by building the Lagrange function:

$$L(w, \varepsilon) = \sum_{q=1}^{n} w_q D_{c_q} + \frac{\varepsilon}{2}\left(\sum_{q=1}^{n} w_q^2 - 1\right). \tag{19}$$

In order to compute the partial derivatives of w_q and ε, we have

$$\begin{cases} \dfrac{\partial(w_q, \varepsilon)}{w_q} = D_{c_q} + \varepsilon w_q = 0, \\ \dfrac{\partial(w_q, \varepsilon)}{\varepsilon} = \sum_{q=1}^{n} w_q^2 - 1 = 0. \end{cases} \tag{20}$$

By solving (20), the following equation to determine the weight vector is acquired:

$$w_q = \frac{1/m - 1\sum_{t=1}^{m}\sum_{e=1,e\neq t}^{m}\left(\begin{array}{l}\left|\sum_{k_{tq=1}}^{\alpha_{tq}} i_{tq} - \sum_{k_{eq}=1}^{\alpha_{eq}} i_{eq}\right| + \left|\sum_{k_{tq=1}}^{\alpha_{tq}} \mu_{i_{tq}}^{k_{tq}} - \sum_{k_{eq}=1}^{\alpha_{eq}} \mu_{i_{eq}}^{k_{eq}}\right| + \\ \left|\sum_{k_{tq=1}}^{\alpha_{tq}} \eta_{i_{tq}}^{k_{tq}} - \sum_{k_{eq}=1}^{\alpha_{eq}} \eta_{i_{eq}}^{k_{eq}}\right| + \left|\sum_{k_{tq=1}}^{\alpha_{tq}} \nu_{i_{tq}}^{k_{tq}} - \sum_{k_{eq}=1}^{\alpha_{eq}} \nu_{i_{eq}}^{k_{eq}}\right|\end{array}\right)}{\sum_{q=1}^{n}\left(1/m - 1\sum_{t=1}^{m}\sum_{e=1,e\neq t}^{m}\left(\begin{array}{l}\left|\sum_{k_{tq=1}}^{\alpha_{tq}} i_{tq} - \sum_{k_{eq}=1}^{\alpha_{eq}} i_{eq}\right| + \left|\sum_{k_{tq=1}}^{\alpha_{tq}} \mu_{i_{tq}}^{k_{tq}} - \sum_{k_{eq}=1}^{\alpha_{eq}} \mu_{i_{eq}}^{k_{eq}}\right| + \\ \left|\sum_{k_{tq=1}}^{\alpha_{tq}} \eta_{i_{tq}}^{k_{tq}} - \sum_{k_{eq}=1}^{\alpha_{eq}} \eta_{i_{eq}}^{k_{eq}}\right| + \left|\sum_{k_{tq=1}}^{\alpha_{tq}} \nu_{i_{tq}}^{k_{tq}} - \sum_{k_{eq}=1}^{\alpha_{eq}} \nu_{i_{eq}}^{k_{eq}}\right|\end{array}\right)\right)}. \tag{21}$$

3.5. Comparison Methods for HPFLSs. The score function and accuracy function for HPFLSs are introduced. Then, based on the functions, the comparison approaches are provided.

Definition 13. For any HPFLSs, $H_s = \{\langle(s_i^k), (\mu_i^k, \eta_i^k, \nu_i^k)\rangle \mid i \in (0, 1, 2, \ldots, m); k = 1, 2, \ldots, \alpha\}$, and its score function is defined as $S(H_s) = s_{\overline{x}}$, where $\overline{x} = \sum_{k=1}^{\alpha} i^k \otimes 1 + \mu_i^k - \nu_i^k/2/\alpha$ and i^k is the maximum subscript of linguistic term in linguistic term set S_i^k.

For any two HPFLSs H_{s_1} and H_{s_2}, if $S(H_{s_1}) > S(H_{s_2})$, then $H_{s_1} \succ H_{s_2}$; that is, H_{s_1} is superior than H_{s_2}, and vice versa. If $S(H_{s_1}) = S(H_{s_2})$, then H_{s_1} and H_{s_2} cannot be distinguished. For such case, the accuracy function is conducted as below.

Definition 14. Let H_s be one HPFLS $H_s = \{\langle(s_i^k), (\mu_i^k, \eta_i^k, \nu_i^k)\rangle \mid i \in (0, 1, 2, \ldots, m); k = 1, 2, \ldots, \alpha\}$, and then, the accuracy function of H_s is defined as $A(H_s) = s_y$, where $y = \sum_{k=1}^{\alpha} i^k \otimes 1 + \mu_i^k + \eta_i^k - \nu_i^k/2/\alpha$. For any two HPFLSs H_{s_1} and H_{s_2}, if $A(H_{s_1}) > A(H_{s_2})$, then H_{s_1} is superior than H_{s_2}; if $A(H_{s_1}) < A(H_{s_2})$, then H_{s_2} is superior than H_{s_1}; if $A(H_{s_1}) = A(H_{s_2})$, then $H_{s_1} = H_{s_2}$.

According to Definitions 13 and 14, we have the following conclusion.

Conclusion 1. Let H_{s_1} and H_{s_2} be two HPFLSs; then

(1) if $S(H_{s_1}) > S(H_{s_2})$, then $H_{s_1} \succ H_{s_2}$
(2) if $S(H_{s_1}) < S(H_{s_2})$, then $H_{s_1} \prec H_{s_2}$
(3) if $S(H_{s_1}) = S(H_{s_2})$, then
  (a) if $A(H_{s_1}) > A(H_{s_2})$, then $H_{s_1} \succ H_{s_2}$
  (b) if $A(H_{s_1}) \prec A(H_{s_2})$, then $H_{s_1} \prec H_{s_2}$
  (c) if $A(H_{s_1}) = A(H_{s_2})$, then $H_{s_1} = H_{s_2}$

4. MCGDM Approach with Hesitant Picture Fuzzy Linguistic Sets (HPFLSs)

In this section, an effective approach for solving MCGDM problems with HPFLSs is presented. The detailed steps are introduced below. In this sake, the hierarchical structure of decision-making procedure of the proposed approach is presented in Figure 1.

4.1. Problem Description. There are the collection of m alternatives $A = \{a_1, a_2, \ldots, a_m\}$ and k participants/experts, and $E = \{e_1, e_2, \ldots, e_k\}$. $C = \{c_1, c_2, \ldots, c_n\}$ is the criteria set, and $w = \{w_1, w_2, \ldots, w_n\}^T$ is the weight vector of criteria. Due to the complex and uncertainty practical situation, the criteria weight cannot be reasonably given by DMs, and it is required to solve such case that the weight of criteria is completely unknown.

In the decision-making procedure, the uncertain assessment value of the alternative A_i with respect to criterion C_j given by all DMs is presented in the form of linguistic terms. Then, the final hesitant picture fuzzy linguistic evaluation can be obtained by collecting all linguistic terms and represented as the following decision matrix:

$$\widetilde{M} = \left(H_{s_{ij}}\right)_{m\times n} = \begin{bmatrix} H_{s_{11}} & H_{s_{12}} & \cdots & H_{s_{1n}} \\ H_{s_{21}} & H_{s_{22}} & \cdots & H_{s_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ H_{s_{m1}} & H_{s_{22}} & \cdots & H_{s_{mn}} \end{bmatrix}. \tag{22}$$

4.2. Decision-Making Procedure. Based on the above description, a useful approach based on HPFLS information is developed to handling practical MCGDM problems, in

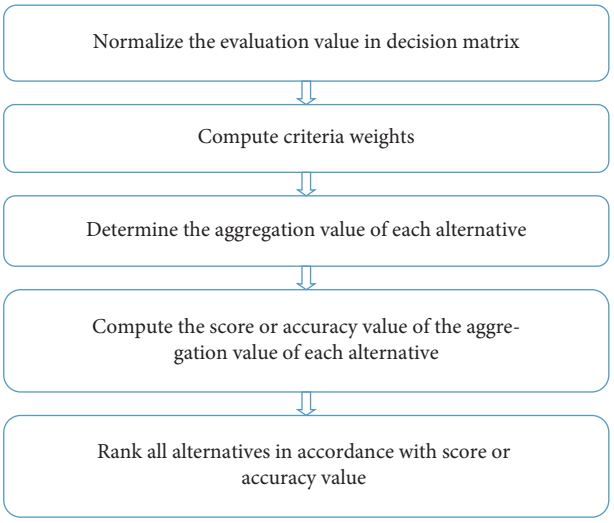

Figure 1: The hierarchical structure of the proposed HPFLS-based MCGDM approach.

which the criteria weight is completely unknown, and the assessment value of each alternative are collected as HPFLSs. The decision-making steps are summarized as follows:

Step 1: Normalize the evaluation value in the decision matrix.

The criteria in the decision matrix should be identified and distinguished as benefit-type and cost-type criteria. The evaluation value of benefit-type criteria does not necessarily be changed, whereas for the cost-type criteria, it is required to normalize the evaluation using the complementary set.

The decision matrix can be normalized by the following formula:

$$\widehat{\widetilde{M}}_{ij} = \begin{cases} \widetilde{M}_{ij}, & C_j \in B_S, \\ \widetilde{M}_{ij}^c, & C_j \in C_S, \end{cases} \tag{23}$$

where B_S is the collection of benefit-type criteria, C_S is the collection of cost-type criteria, and $\widetilde{M}_{ij}^c$ is the complementary set of $\widetilde{M}_{ij}^c$.

The normalized decision matrix can be denoted by $\widehat{M} = (\widehat{\widetilde{M}}_{ij})m \times n$.

Step 2: Compute criteria weights.

Equation (21) is utilized to compute the weight vector $w = \{w_1, w_2, \ldots, w_n\}^T$ of criteria set $C = \{c_1, c_2, \ldots, c_n\}$. Hence, every criterion weight value is determined.

Step 3: Determine the aggregation value of each alternative a_i.

In this step, the *HPFLWA* or *HPFLWG* operator is used to determine the aggregation value of alternative a_i. Then, the *HPFLWA* or *HPFLWG* aggregation value is obtained by equation (8) or (13).

Step 4: Compute the score or accuracy value of the aggregation value of each alternative a_i.

Based on Definitions 14 and 15 given in Section 3.5, the score and accuracy value of each alternative a_i are acquired. The value $S(H_{s_i})$ is bigger, and the alternative a_i is better. If the value $S(H_{s_i})$ of any two alternatives is equal, then the alternative with the bigger value $A(H_{s_i})$ is better.

Step 5: Rank all alternatives in accordance with score or accuracy value.

Based on the increasing value of $S(H_{s_i})$, the bigger $S(H_{s_i})$ is, the better the alternative a_i will be. If the value $S(H_{s_i})$ of any two alternatives is equal, then $A(H_{s_i})$ needs to be used. Based on the increasing value of $A(H_{s_i})$, the bigger $A(H_{s_i})$ is, the better the alternative a_i will be.

5. Practical Application

In this section, the teaching performance evaluation by the students from Hubei University of Automotive Technology with HPFLSs is introduced and also the further analysis is carried on.

5.1. Practical Application Illustration. Practical application at Hubei University of Automotive Technology is adopted as the illustration of the presented method. Hubei University of Automotive Technology includes many schools, such as school of economic and management, school of automotive engineering, and school of machinery design and automation. The university always pays high attention to ensure the teaching performance of teacher for every course. And some scoring rules have been drawn up to ensure the teaching quantity. For example, students grade the teaching performance of teachers' course. According to the signed agreement, our proposed approach is used to help handle the student score and rank the teaching performance. In this paper, for convenience, the teaching performance assessment values of four teachers' courses were chosen as the alternative $a_j\,(j = 1, 2, 3, 4)$. Meanwhile, some more important criteria are discussed in detailed; at final, four key assessment criteria are selected: planning and preparation (c_1), communication and interaction (c_2), manage the learning environment (c_3), and professionalism (c_4). The linguistic term set $S = \{s_5 = \text{outstading excellent}; s_4 = \text{excellent}; s_3 = \text{merit}; s_2 = \text{average}; s_1 = \text{pass}; s_0 = \text{failure}\}$ is used during the evaluation process. In order to acquire the trustworthy evaluation value, the students anonymously give their comments on websites; in addition, students have the rights to refuse to provide any comment. For convenience, the process of transforming from students' comments to picture fuzzy linguistic term information is bypassed. Finally, the HPFSLSs, which are collected from the picture fuzzy linguistic term value, are shown in Table 2.

Table 2: Decision matrix of collective value M.

	c_1	c_2	c_3	c_4
a_1	$\langle\{(s_5), (0.58, 0.12, 0.1)\}, \{(s_4), (0.22, 0.1, 0)\}\rangle$	$\langle\{(s_5), (0.66, 0.1, 0.04)\}, \{(s_4), (0.1, 0.2, 0)\}\rangle$	$\langle\{(s_5), (0.8, 0, 0)\}, \{(s_4), (0.2, 0, 0)\}\rangle$	$\langle\{(s_5), (0.72, 0.08, 0)\}, \{(s_4), (0.31, 0.09, 0)\}\rangle$
a_2	$\langle\{(s_5), (0.53, 0.18, 0.09)\}, \{(s_3), (0.21, 0.1, 0)\}\rangle$	$\langle\{(s_4), (0.68, 0.08, 0.04)\}, \{(s_3), (0.3, 0, 0)\}\rangle$	$\langle\{(s_5), (0.53, 0.2, 0.07)\}, \{(s_4), (0.31, 0.09, 0)\}\rangle$	$\langle\{(s_5), (0.51, 0.19, 0.1)\}, \{(s_4), (0.3, 0.1, 0)\}\rangle$
a_3	$\langle\{(s_5), (0.48, 0.1, 0.02)\}, \{(s_4), (0.6, 0, 0)\}\rangle$	$\langle\{(s_5), (0.63, 0.2, 0.07)\}, \{(s_4), (0.3, 0.06, 0)\}\rangle$	$\langle\{(s_5), (0.71, 0.09, 0)\}, \{(s_4), (0.51, 0.09, 0)\}\rangle$	$\langle\{(s_5), (0.7, 0, 0.1)\}, \{(s_4), (0.3, 0, 0)\}\rangle$
a_4	$\langle\{(s_5), (0.52, 0.3, 0.08)\}, \{(s_4), (0.3, 0.2, 0)\}\rangle$	$\langle\{(s_5), (0.56, 0.2, 0.04)\}, \{(s_4), (0.61, 0.1, 0)\}\rangle$	$\langle\{(s_5), (0.5, 0.2, 0.1)\}, \{(s_4), (0.63, 0.07, 0)\}\rangle$	$\langle\{(s_5), (0.63, 0.08, 0.09)\}, \{(s_4), (0.5, 0.1, 0)\}\rangle$

Step 1. Normalize the decision matrix.

Due to none of the criterion is considered as cost criteria, the decision matrix of the collection value is not necessary to be normalized.

Step 2. Compute criteria weight vector.

Equation (21) is utilized to compute the weight vector, and the obtained results are

$$w = \{0.3038, 0.4478, 0.1443, 0.0999\}^T. \tag{24}$$

Step 3. Obtain the aggregation value of alternative a_i.

By utilizing equation (8), the aggregation value of each alternative a_i can be calculated, and the results are shown below:

$$\begin{aligned}
a_1 = \langle &\{(s_5), (0.6643, 0.0897, 0.0486)\}, \{(s_{4.9003}), (0.6405, 0.0907, 0.0486)\}, \{(s_{4.8560}), (0.6163, 0.0897, 0.0486)\}, \\
&\{(s_{4.7563}), (0.5902, 0.0907, 0.0486)\}, \{(s_{4.5529}), (0.5795, 0.1344, 0.0307)\}, \{(s_{4.4531}), (0.5498, 0.1354, 0.0307)\}, \\
&\{(s_{4.4087}), (0.5157, 0.1344, 0.0307)\}, \{(s_{4.3088}), (0.4691, 0.1354, 0.0307)\}\{(s_{4.6926}), (0.6324, 0.0836, 0.0179)\}, \\
&\{(s_{4.5928}), (0.6073, 0.0846, 0.0179)\}, \{(s_{4.5485}), (0.5811, 0.0836, 0.0179)\}, \{(s_{4.4487}), (0.5516, 0.0846, 0.0179)\}, \\
&\{(s_{4.2450}), (0.5389, 0.1282, 0.0000)\}, \{(s_{4.1449}), (0.5012, 0.1292, 0.0000)\}, \{(s_{4.1003}), (0.4477, 0.1282, 0.0000)\}, \\
&\{(s_{4.0000}), (0.2129, 0.1292, 0.0000)\}\rangle, \\
a_2 = \langle &\{(s_{4.5529}), (0.6032, 0.1390, 0.0654)\}, \{(s_{4.4531}), (0.5977, 0.1300, 0.0555)\}, \{(s_{4.4087}), (0.5932, 0.1232, 0.0553)\}, \\
&\{(s_{4.3088}), (0.5876, 0.1142, 0.0454)\}, \{(s_{4.1284}), (0.4777, 0.1031, 0.0472)\}, \{(s_{4.0281}), (0.4659, 0.0942, 0.0374)\}, \\
&\{(s_{3.9835}), (0.4553, 0.0873, 0.0372)\}, \{(s_{3.8828}), (0.4401, 0.0783, 0.0274)\}, \{(s_{3.9521}), (0.5807, 0.1144, 0.0378)\}, \\
&\{(s_{3.8513}), (0.5747, 0.1054, 0.0279)\}, \{(s_{3.8064}), (0.5699, 0.0986, 0.0278)\}, \{(s_{3.7049}), (0.5637, 0.0896, 0.0179)\}, \\
&\{(s_{3.5201}), (0.4176, 0.0785, 0.0198)\}, \{(s_{3.4159}), (0.3920, 0.0696, 0.0100)\}, \{(s_{3.3691}), (0.3625, 0.0627, 0.0098)\}, \\
&\{(s_{3.2621}), (0.2850, 0.0538, 0.0000)\}\rangle \\
a_3 = \langle &\{(s_{5.0000}), (0.6134, 0.1333, 0.0472)\}, \{(s_{4.9003}), (0.5895, 0.1631, 0.0373)\}, \{(s_{4.8560}), (0.5850, 0.1333, 0.0472)\}, \\
&\{(s_{4.7563}), (0.5586, 0.1631, 0.0373)\}, \{(s_{4.5529}), (0.5338, 0.0706, 0.0161)\}, \{(s_{4.4531}), (0.4983, 0.1004, 0.0062)\}, \\
&\{(s_{4.4087}), (0.4907, 0.0706, 0.0161)\}, \{(s_{4.3088}), (0.4348, 0.1004, 0.0062)\}, \{(s_{4.6926}), (0.6404, 0.1024, 0.0410)\}, \\
&\{(s_{4.5928}), (0.6179, 0.1322, 0.0310)\}, \{(s_{4.5485}), (0.6137, 0.1024, 0.0410)\}, \{(s_{4.4487}), (0.5898, 0.1322, 0.0310)\}, \\
&\{(s_{4.2450}), (0.5682, 0.0398, 0.0100)\}, \{(s_{4.1449}), (0.5398, 0.0696, 0.0000)\}, \{(s_{4.1003}), (0.5341, 0.0398, 0.0100)\}, \\
&\{(s_{4.0000}), (0.4988, 0.0696, 0.0000)\}\rangle, \\
a_4 = \langle &\{(s_{5.0000}), (0.5491, 0.2184, 0.0648)\}, \{(s_{4.9003}), (0.5349, 0.2204, 0.0556)\}, \{(s_{4.8560}), (0.5680, 0.1995, 0.0501)\}, \\
&\{(s_{4.7563}), (0.5551, 0.2016, 0.0410)\}, \{(s_{4.5529}), (0.5736, 0.1735, 0.0462)\}, \{(s_{4.4531}), (0.5610, 0.1755, 0.0372)\}, \\
&\{(s_{4.4087}), (0.5908, 0.1547, 0.0319)\}, \{(s_{4.3088}), (0.5790, 0.1567, 0.0229)\}, \{(s_{4.6926}), (0.5244, 0.1880, 0.0412)\}, \\
&\{(s_{4.5928}), (0.5077, 0.1900, 0.0321)\}, \{(s_{4.5485}), (0.5457, 0.1692, 0.0268)\}, \{(s_{4.4487}), (0.5312, 0.1712, 0.0178)\}, \\
&\{(s_{4.2450}), (0.5518, 0.1432, 0.0232)\}, \{(s_{4.1449}), (0.5378, 0.1452, 0.0142)\}, \{(s_{4.1003}), (0.5705, 0.1244, 0.0090)\}, \\
&\{(s_{4.0000}), (0.5577, 0.1264, 0.0000)\}\rangle.
\end{aligned} \tag{25}$$

By utilizing equation (13), the aggregation value of each alternative a_i can be calculated, and the results are shown below:

$$
\begin{aligned}
a_1 = \langle & \{(s_{5.0000}), (0.6503, 0.0897, 0.0602)\}, \{(s_{4.6229}), (0.5040, 0.0907, 0.0602)\}, \{(s_{4.5381}), (0.4077, 0.0897, 0.0602)\}, \\
& \{(s_{4.4042}), (0.3866, 0.0907, 0.0602)\}, \{(s_{4.2363}), (0.2904, 0.1344, 0.0498)\}, \{(s_{4.1783}), (0.2754, 0.1354, 0.0498)\}, \\
& \{(s_{4.1556}), (0.2384, 0.1344, 0.0498)\}, \{(s_{4.1098}), (0.2250, 0.1354, 0.0498)\}, \{(s_{4.3414}), (0.3915, 0.0836, 0.0211)\}, \\
& \{(s_{4.2629}), (0.3729, 0.0846, 0.0211)\}, \{(s_{4.2335}), (0.3301, 0.0836, 0.0211)\}, \{(s_{4.1759}), (0.3155, 0.0846, 0.0211)\}, \\
& \{(s_{4.0837}), (0.2307, 0.1282, 0.0000)\}, \{(s_{4.0467}), (0.2175, 0.1292, 0.0000)\}, \{(s_{4.0316}), (0.1842, 0.1282, 0.0000)\}, \\
& \{(s_{4.0000}), (0.1716, 0.1292, 0.0000)\}\rangle, \\
a_2 = \langle & \{(s_{4.2363}), (0.5906, 0.1390, 0.0704)\}, \{(s_{4.1783}), (0.4645, 0.1300, 0.0619)\}, \{(s_{4.1556}), (0.4477, 0.1232, 0.0641)\}, \\
& \{(s_{4.1098}), (0.4206, 0.1142, 0.0545)\}, \{(s_{3.2665}), (0.3214, 0.1031, 0.0617)\}, \{(s_{3.2634}), (0.3166, 0.0942, 0.0518)\}, \\
& \{(s_{3.2621}), (0.3173, 0.0873, 0.0544)\}, \{(s_{3.2590}), (0.3133, 0.0783, 0.0431)\}, \{(s_{3.3677}), (0.3552, 0.1144, 0.0462)\}, \\
& \{(s_{3.3635}), (0.3475, 0.1054, 0.0331)\}, \{(s_{3.3617}), (0.3471, 0.0986, 0.0366)\}, \{(s_{3.3576}), (0.3412, 0.0896, 0.0211)\}, \\
& \{(s_{3.0945}), (0.2768, 0.0785, 0.0329)\}, \{(s_{3.0926}), (0.2739, 0.0696, 0.0163)\}, \{(s_{3.0918}), (0.2754, 0.0627, 0.0208)\}, \\
& \{(s_{3.0899}), (0.2727, 0.0538, 0.0000)\}\rangle, \\
a_3 = \langle & \{(s_{5.0000}), (0.5197, 0.1333, 0.0556)\}, \{(s_{4.6229}), (0.4794, 0.1631, 0.0446)\}, \{(s_{4.5381}), (0.5022, 0.1333, 0.0556)\}, \\
& \{(s_{4.4042}), (0.4642, 0.1631, 0.0446)\}, \{(s_{4.2363}), (0.3832, 0.0706, 0.0258)\}, \{(s_{4.1783}), (0.3521, 0.1004, 0.0069)\}, \\
& \{(s_{4.1556}), (0.3809, 0.0706, 0.0258)\}, \{(s_{4.1098}), (0.3498, 0.1004, 0.0069)\}, \{(s_{4.3414}), (0.5673, 0.1024, 0.0526)\}, \\
& \{(s_{4.2629}), (0.5244, 0.1322, 0.0409)\}, \{(s_{4.2335}), (0.5457, 0.1024, 0.0526)\}, \{(s_{4.1759}), (0.5061, 0.1322, 0.0409)\}, \\
& \{(s_{4.0837}), (0.4157, 0.0398, 0.0208)\}, \{(s_{4.0467}), (0.3847, 0.0696, 0.0000)\}, \{(s_{4.0316}), (0.4134, 0.0398, 0.0208)\}, \\
& \{(s_{4.0000}), (0.3823, 0.0696, 0.0000)\}\rangle, \\
a_4 = \langle & \{(s_{5.0000}), (0.5323, 0.2184, 0.0706)\}, \{(s_{4.6229}), (0.4919, 0.2204, 0.0637)\}, \{(s_{4.5381}), (0.5512, 0.1955, 0.0578)\}, \\
& \{(s_{4.4042}), (0.5108, 0.2016, 0.0493)\}, \{(s_{4.2363}), (0.5531, 0.1735, 0.0619)\}, \{(s_{4.1783}), (0.5225, 0.1755, 0.0539)\}, \\
& \{(s_{4.1556}), (0.5719, 0.1547, 0.0470)\}, \{(s_{4.1098}), (0.5413, 0.1567, 0.0367)\}, \{(s_{4.3414}), (0.3967, 0.1880, 0.0519)\}, \\
& \{(s_{4.2629}), (0.3902, 0.1900, 0.0424)\}, \{(s_{4.2335}), (0.4155, 0.1692, 0.0340)\}, \{(s_{4.1759}), (0.4091, 0.1712, 0.0211)\}, \\
& \{(s_{4.0837}), (0.4390, 0.1432, 0.0398)\}, \{(s_{4.0467}), (0.4327, 0.1452, 0.0282)\}, \{(s_{4.0316}), (0.4577, 0.1244, 0.0174)\}, \\
& \{(s_{4.0000}), (0.4514, 0.1264, 0.0000)\}\rangle.
\end{aligned}
\tag{26}
$$

Step 4. Compute the score or accuracy value of the aggregation value of each alternative a_i.

The alternative score value can be obtained based on Definitions 13 and 14 given in Section 3.5, and the results are as follows,

For the aggregation result with *HPFLWA* operator, we have the score values

$$\begin{aligned} s_{a_1} &= 3.4293, \\ s_{a_2} &= 2.8791, \\ s_{a_3} &= 3.4524, \\ s_{a_4} &= 3.4180. \end{aligned} \tag{27}$$

For the aggregation result with *HPFLWG* operator, we obtain

$$\begin{aligned} s_{a_1} &= 2.7751, \\ s_{a_2} &= 2.2932, \\ s_{a_3} &= 3.0336, \\ s_{a_4} &= 3.0730. \end{aligned} \tag{28}$$

Step 5. Rank all alternatives in accordance with score or accuracy value.

Based on the increasing score or accuracy value of the aggregation result with *HPFLWA* operator, the final ranking order of alternatives is $a_3 \succ a_1 \succ a_4 \succ a_2$, the best alternative is a_3, and the worst alternative isa_2. The above results clearly show that the alternative a_2 is the best teaching performance teacher.

Based on the increasing score or accuracy value of the aggregation result with *HPFLWG* operator, the final ranking order of alternatives is $a_4 \succ a_3 \succ a_1 \succ a_2$, the best alternative is a_4, and the worst alternative is a_2. The above results clearly show that the alternative a_3 is the best teaching performance teacher.

5.2. Further Discussion. In this subsection, a sensitivity analysis is conducted based on the different representative values of parameter θ, and it indicates whether the ranking results are varied or not when different representative values of parameter θ are provided. The detailed results are shown in Table 3.

From the ranking results presented in Table 3, it is obvious that the ranking result is sensitive to the value of parameter θ for both *HPFLWA* and *HPFLWG* operators. If the *HPFLWA* operator is used, the best teaching performance alternative is a_3 and the worst teaching performance alternative is a_2; however, if the *HPFLWG* operator is used, the best teaching performance alternative is a_3 by decreasing the value of parameter θ; with increasing the value of parameter θ, the best teaching performance alternative is changed as a_4, and the worst teaching performance alternative is a_2. For both of operators, the worst alternative is remaining unchangeable with the various value of parameter θ.

In addition, the trend of different values of parameter θ, which can lead to different final score results s_{a_i} for *HPFLWA* and *HPFLWG* operators, is presented schematically in Figures 2 and 3.

Table 3: Sensitivity analysis based on different values of parameter θ.

Value of parameter θ	Rank results	
	HPFLWA	*HPFLWG*
$\theta = 1$	$a_3 \succ a_4 \succ a_1 \succ a_2$	$a_4 \succ a_3 \succ a_1 \succ a_2$
$\theta = 5$	$a_3 \succ a_4 \succ a_1 \succ a_2$	$a_4 \succ a_3 \succ a_1 \succ a_2$
$\theta = 16$	$a_3 \succ a_1 \succ a_4 \succ a_2$	$a_4 \succ a_3 \succ a_1 \succ a_2$
$\theta = 32$	$a_3 \succ a_1 \succ a_4 \succ a_2$	$a_4 \succ a_3 \succ a_1 \succ a_2$
$\theta = 60$	$a_3 \succ a_1 \succ a_4 \succ a_2$	$a_3 \succ a_4 \succ a_1 \succ a_2$

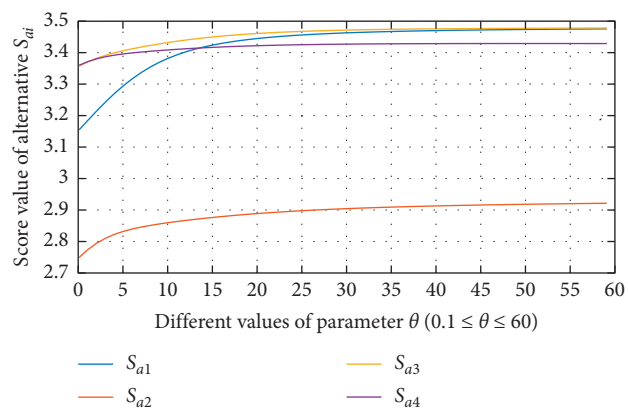

Figure 2: Final score results s_{a_i} for the *HPFLWA* aggregation value with different values of parameter θ.

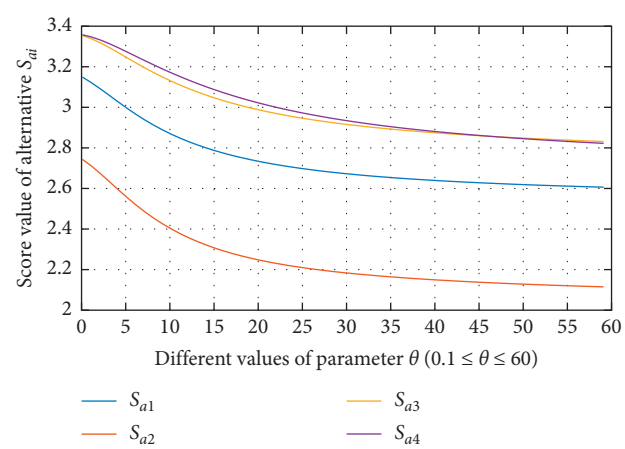

Figure 3: Final score results s_{a_i} for the *HPFLWG* aggregation value with different values of parameter θ.

From the results presented in Figure 2, it is apparent that the best teaching performance alternative is a_3 and the worst teaching performance alternative is a_2 in most cases. However, if the value of parameter θ is less than 13, a_4 is superior than a_1; if the value of parameter θ is greater than 13, a_1 is superior than a_4. The main reason is, with the increasing value of parameter θ, the aggregation results obtained by the *HPFLWA* operator are approaching to the max limited value, which is slightly less than the max value in the evaluation value for the corresponding alternative. With

Table 4: Comparative order ranking obtained by different approaches.

Methods	Operator	Parameter	Rank results
Wei et al. [42]	*P2TLWA*	None	$a_1 \succ a_3 \succ a_4 \succ a_2$
	P2TLGA	None	$a_1 \succ a_3 \succ a_4 \succ a_2$
The proposed methods	*HPFLWA*	$\theta = 5$	$a_3 \succ a_4 \succ a_1 \succ a_2$
		$\theta = 16$, $\theta = 32$, $\theta = 60$	$a_3 \succ a_1 \succ a_4 \succ a_2$
	HPFLWG	$\theta = 5$, $\theta = 16$, $\theta = 32$	$a_4 \succ a_3 \succ a_1 \succ a_2$
		$\theta = 60$	$a_3 \succ a_4 \succ a_1 \succ a_2$

decreasing value of parameter θ, the aggregation results obtained by the *HPFLWA* operator are approaching to the min limited value, which is slightly greater than the min value in the evaluation value for the alternative. For example, the minimum and maximum values of μ appeared in the evaluation value for alternative a_1 are 0.1 and 0.8; however, the min and max values of the membership degree of "vote for" μ appeared in the evaluation value for alternative a_4 are 0.3, and 0.63 (Table 1). Thus, with the increasing value of parameter θ, the trend of a_1 should be superior than a_4. In addition, from the overview of final score results, different values of parameter θ can critically affect the ranking order of alternatives.

From the results presented in Figure 3, it can be seen that the best alternative is a_4, and the worst alternative is a_2 in all circumstances. With the increasing value of parameter θ, the score value of all alternatives is decreasing and approaching to the limited value. However, the ranking order of alternatives is slightly changed; that is, if the value of parameter θ is less than 48, a_4 is superior than a_3, and if the value of parameter θ is greater than 48, a_3 is superior than a_4. The main reason is similar with the reason appeared in the *HPFLWG* operator. From the overview of final order results, the different value of parameter θ can affect the score value of alternatives and affect the ranking order of alternatives.

From the above analysis results appeared in Figures 2 and 3, we know that different values of parameter θ can affect the score value and final ranking order. In order to better elaborate the effectiveness of our proposed method in the real teaching performance evaluation, the value of parameter θ can be selected according to the following circumstances:

(1) If DMs prefer to pay high attention to the max values of the membership degree of "vote for" and linguistic term, the *HPFLWA* operator is used or maximum values of membership degree of and "vote against" and linguistic term, the *HPFLWG* operator is used. The value of parameter θ can be greater than 20 and less than 50.

(2) If the DMs prefer to pay high attention to min values of the membership degree of "voting for" and linguistic term, the *HPFLWA* operator is used or minimum values of membership degree and "vote against" and linguistic term, the *HPFLWG* operator is used. The value of parameter θ can be assigned with less than 10 and greater than 5.

(3) Otherwise, θ can be assigned with any value except for the values appeared in above (1)-(2). By default, θ could be 16.

5.3. Comparative Analysis. In order to further verify the effectiveness of the proposed method, the comparative analysis based on same teaching performance ranking is carried out with some representative methods, which are proposed by Wei et al. [42]. For convenience to comparison, the *HPFLSs* have been transformed to *P2TLSs*. According to the analysis in Section 5.2, four representative values are selected and assigned for the parameter θ, which are $\theta = 5$, $\theta = 16$, $\theta = 32$, and $\theta = 64$. Then, the comparative ranking orders are obtained by the different methods and presented in Table 4.

Table 4 shows that the final ranking order generated by Wei et al. [42] is slightly different from the proposed method. The difference might mainly be caused by the operation rule limitation as pointed out in Examples 1–3. However, for the proposed method, if the different values of parameter θ are used, the final order of alternatives is slightly distinct; in some cases, the alternative a_1 is superior than alternative a_4, and in some cases, the alternative a_4 is superior than alternative a_1. This is caused by the characteristics of the sigmoid function used in the operation rules. Normally, in order to elaborate the effectiveness of the sigmoid function in the proposed operation rules, θ can be assigned with different values as discussed in Section 5.2. In summary, for all obtained results by different methods, the worst alternative is a_2, the best alternative is a_1, a_2, or a_4, and the ranking orders are different.

In Table 5, the comparative analysis of the proposed method versus the method of Wei et al. [42] is provided, and the advantages of the proposed method include four parts of the parameter, and the disadvantage is the part of time of complexity. The detailed description is presented in Table 5.

According to the comparative analysis and further discussion in Section 5.2, it can be concluded that the proposed method possesses the following advantages:

(1) The proposed method can overcome the defects of operation rules existed in the previous methods, such as the methods of *P2TLSs* and *TLS*.

(2) The proposed method can bring into fully play the advantages of *HPFLSs*, and it is more flexible to express DMs' preferences under group decision-making environments.

(3) *HPFLSs* can be more feasible to express the hesitation of DMs than *P2TLSs*, and the *HPFWA* and *HPFWG* operators are developed based on the sigmoid functions. The proposed approach can effectively elaborate the advantages of *HPFLSs* and the

Table 5: Comparative analysis of the proposed method versus the method of Wei et al. [42].

Comparison parameter	Comparisons results
Uncertainty modeling	Because both of methods are based on picture fuzzy linguistics sets, they can be adequate to cope with uncertainty circumstance under the MCGDM environment. However, the proposed method elaborates the advantages of hesitant fuzzy set and picture fuzzy linguistic set and is better to describe the hesitant degree and risk attitude information of the decision-maker.
Operation laws	The operation laws in Wei et al. [42] still have some lacks; the proposed method can overcome these lacks.
Weights of criteria	The criteria weight vector in the proposed approach is determined by the max-min hamming distance deviation method, it is more objective and precise than Wei et al.'s [42] approach which did not consider the weight of criteria determination process.
Rank flexibility	The proposed method can be more flexible to support the different membership degrees of "vote for" or "vote against" by providing different values of parameter θ and gives more acceptable ranking results versus Wei et al.'s [42] method which can only provide one fixed ranking result in accordance with the specific value of each criteria of alternative.
Experts' weights	Both of methods have not provided the expert weight determination approach because in the voting environment, each expert has been assigned with equal weight by default.
Time of complexity	The proposed approaches include the parameter θ and weight of criteria determination process, which has more time complexity in computation compared with Wei et al.'s [42] approach.

HPFWA and *HPFWG* operators; thus, it can effectively solve the MCGDM problems.

(4) Under different real decision-making environments, the DMs can pay higher attention to specific aspect such as membership degree of "vote for" or "vote against," and the proposed approach can be more flexible and reasonable to adjust the ranking results by the parameter θ and give more acceptable results in accordance with the real demands of DMs.

6. Conclusion and Further Direction

Considering the real-life decision-making environment, different DMs could have the different attitudes (e.g., attitude for support, neutral, oppose, and refusal) for the evaluation value of criteria or alternative under the hesitant fuzzy linguistic environment. In this regard, the paper proposed the hesitant picture fuzzy linguistic group decision-making approaches that are based on aggregation operators, which are weighted average (WA) and geometric (WG) operators.

HPFLSs can provide the DMs with the opportunity to express the evaluation value of hesitant linguistic with a set of crisp number membership degree for criteria of alternative for avoiding attitude information losing. As the extension of PFSs is based on hesitant linguistic term sets, it can be more suitable and flexible to solve MCGDM voting problems. It can bring into fully play the advantages of hesitant linguistic term sets and PFSs.

In the proposed approach, improved operation laws are developed based on the sigmoid function, and two equivalent transformations functions f and g are defined. It can overcome the defects of operation rules appeared in the previous PFS method. To illustrate the effectiveness and performance of the proposed method, a real case of teaching performance evaluation form Hubei University of Automotive Industry is adopted. In addition, the comparison analysis of the ranking result obtained from the proposed method versus the method of Wei et al. [42] is presented in detail; meanwhile, the sensitive analysis of the proposed method is conducted, and the results showed that the proposed method can be more flexible and effective than that of Wei et al. [42]. Afterward, a comparison of the proposed method versus the method of Wei et al. [42] is performed based on six attributes, including the approach of uncertainty modeling, operation laws, weight of criteria, rank flexibility, experts' weights, and time of complexity. The comparison results show that the proposed method has the advantages at four attributes.

To highlight the advantages of *HPFLSs* and WA, WG operator, *HPFWA*, and *HPFWG* operators are proposed and related theorem are discussed; meanwhile, the max-min deviation methods to identify the completely unknown criteria weight under hesitant picture fuzzy linguistic environment are presented, and the comparison method based on score and accuracy functions for *HPFLSs* are introduced. In addition, an approach is constructed to solve the teaching performance evaluation problem. The best and worst alternatives can be easily acquired.

The main contribution of this paper is developing an improved operation rule to overcome the defects of operation rules appeared in the previous PFS-based MCGDM method. In this way, a novel sigmoid function-based operation rule is developed. In order to verify the effectiveness of the proposed method, the optimal value selection of parameter θ is analyzed. Thus, the practical MCGDM voting problems can be addressed using our proposed method under the *HPFLS* environment. However, the limitation in the proposed method is that the DMs are assigned with the same weight; for most cases, it is required to allocate the different weights for different DMs. Additionally, under the most real-life MCGDM environment, DMs more prefer to provide the evaluation value with simple crisp number or linguistic term; thus, how to transfer the simple crisp number and linguistic term to HPFLSs is necessary to studying in the future. Also, developing the decision system based on the proposed *HPFLS* MCGDM approaches is more encouraged to solve the real-life MCGDM problem.

Data Availability

The hesitant picture fuzzy linguistic data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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