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*Research Article*

# Stability of Two Weakly Coupled Elastic Beams with Partially Local Damping

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In this paper, the stability of two weakly coupled elastic beams connected vertically by a spring is investigated via the frequency domain method and the multiplier technique. When the two beams have partially local damping, the operator $\mathscr{A}$ is obtained via variable conversion, and it generating a semigroup is proved, then we obtain that the semigroup is exponentially stable by reduction to absurdity.

## 1. Introduction

Artificial intelligence (AI) has been around and penetrated into all fields, such as research, production, and life [1–5]. Scientists pay more attention to energy, materials, and environment [6–10]. The stability study of the coupling control system in space vehicles is one of the important research subjects in the control field in recent years. Much attention has been paid to research the stability of control systems using semigroup theory [11]. Reference [12] is for coupled heat-wave system. References [13, 14] are for wave equations. Reference [15] is for second-order hyperbolic operators. References [16–19] are about polynomial stability of systems, and references [20–33] are about exponential stability. The authors of [33–35] considered weakly coupled evolution equations of wave-Petrowsky, wave-wave, and Kirchhoff–Petrowsky for its asymptotic stability and boundary controllability. The case of strongly coupled system was studied by Lasiecka [36], and she obtained the strong stability for the open-loop systems with polynomial energy decay rate.

A viscoelastic microcomposite beam reinforced by various distributions of boron nitride nanotubes with initial geometrical imperfection has been described in [37], and the nonlinear static, buckling, and vibration are analyzed by using the finite element method. The bending and dynamic behavior of functionally graded plates resting on visco-Pasternak foundations is studied in [38]. Using a simple quasi-3D hyperbolic theory, the dynamic behavior of functionally graded plates is concerned in [39]. Using a hyperbolic shear deformation theory, the static and dynamic behaviors of functionally graded beams is studied in [40]. A dynamic study of functionally graded plates resting on elastic foundation is considered in [41]. Thermomechanical analysis of functionally graded sandwich plates resting on a two-parameter elastic foundation is studied in [42]. The free vibration of FG plates resting on elastic foundations is modeled by two-dimensional (2D) and quasi-three-dimensional (quasi-3D) shear deformation theories in [43]. Input-to-state stability for a class of discrete-time nonlinear input-saturated switched descriptor systems with unstable subsystems is discussed in [44]. In [45], nonlinear dynamic behavior of the winding hoisting rope under head sheave axial wobbles is concerned. A dynamic model of a mine hoisting system with constant length and variable length is analyzed in [46]. Liu and Rao [47] studied the stability of a weakly coupled and partially damped system. They obtained a sharp polynomial decay rate, when compared with Alabau-Boussouira's results. Recently, they also obtained the exact boundary controllability of this system with the control acted only on one equation. The Timoshenko beam equation with locally distributed Kelvin–Voigt damping is considered in [48]. A wave equation with local Kelvin–Voigt damping is proposed in [49], and the semigroup corresponding to the system is eventually differentiable. The behavior of slow or exponential decay is analyzed about elastic material with voids in [50]. The stability of an elastic string system with local Kelvin–Voigt damping is studied in [51].

The above research leads us to study a new problem. Consider a system of two beams connected vertically by a spring. In engineering construction, there are elastic beams everywhere, and it is of great significance to study the elastic beam system. If two beams subject to local damping or only one beam subjects to local damping, what is the energy decay rate of every system? The coupling terms are local which model the location of the spring in this case. Such a coupling is even weaker than the one studied by Liu and Rao. Hence, finding the energy decay rate is a challenge. For reader's convenience, we include these two frequency domain conditions here. The first one is the frequency domain characterization of exponential decay, which was obtained by Prss [52]. The second one is of polynomial energy decay rate for a $\mathbf{C}_0$-semigroup of contraction, which was obtained by Liu and Rao [53].

In this paper, the stability of weakly coupled elastic beam system with damping control by using the semigroup theoretical frequency domain multiplier method is studied. By variable conversion, the elastic beam control system is transformed into first-order evolution equations and a linear operator is obtained, and the linear operator-producing semigroup is proved. When the two beams have local damping control, using reduction to absurdity, from the local dissipation to the global dissipation, the exponential stability of the semigroup generated from the linear operator is proved. The method in this paper can be employed to handle other elastic beam systems in the future.

In this paper, we need some definitions and lemmas which are as follows.

*Definition 1* (see [19]). Let $\mathscr{H}$ be a real or complex Hilbert space and we define $(,)$ is the inner product of $\mathscr{H}$ and $\|\cdot\|$ is the norm of $\mathscr{H}$. Let $\mathscr{A}$ be a dense linear operator on $\mathscr{H}$, that is, $\mathscr{D}(\mathscr{A}) \subseteq \mathscr{H} \longrightarrow \mathscr{H}$, then $\mathscr{A}$ is dissipative, and for any $x \in \mathscr{D}(\mathscr{A})$, we get $\operatorname{Re}(\mathscr{A}x, x) \leq 0$.

*Definition 2* (see [19]). $e^{\mathscr{A}t}$ is exponentially stable if there are normal numbers $M$ and $\alpha$ which make

$$\left\| e^{\mathscr{A}t} \right\| \leq M e^{-\alpha t}, \quad \forall t \geq 0. \tag{1}$$

**Lemma 1** (see [19]). *Linear operator $\mathscr{A}$ can generate $\mathbf{C}_0$ semigroup $\mathbf{S}(t)$ on Hilbert space $\mathscr{H}$ if it satisfies the following:*

*(1) $\mathscr{D}(\mathscr{A})$ is dense on Hilbert space $\mathscr{H}$*

*(2) $\mathscr{A}$ is dissipative*

*(3) $0 \in \rho(\mathscr{A})$*

**Lemma 2** (see [19]). *A $\mathbf{C}_0$ semigroup $e^{\mathscr{A}t}$ of contractions on a Hilbert space $\mathscr{H}$ is exponentially stable if and only if*

$$\begin{gathered} \rho(\mathscr{A}) \supseteq \{ i\beta, \beta \in R \} \equiv \mathrm{iR}, \\ \overline{\lim_{|\beta| \longrightarrow +\infty}} \left\| (i\beta I - \mathscr{A})^{-1} \right\| < +\infty. \end{gathered} \tag{2}$$

## 2. Model Description

Consider the system of two beams connected vertically by a spring. When both upper and lower beams have local damping control. The physical model of weakly coupled elastic beam control system is given in Figure 1.

The system is governed by the following equations:

$$\rho_1 u_{tt} = -a_1 u_{xxxx} - k(x)(u - y) - \delta k(x) u_t, \tag{3}$$

$$\rho_2 y_{tt} = -a_2 y_{xxxx} + k(x)(u - y) - \delta k(x) y_t, \tag{4}$$

where $u$ and $y$ are the displacement of upper and lower beams. $x \in (0, l)$ and $t \in [0, +\infty)$; $\rho_1$, $\rho_2$, $a_1$, and $a_2$ are positive physical constants; $k(x) \geq 0$ in $(a, b) \subset (0, l)$ is a locally supported smooth function, which represents the position and elasticity of the spring. We assume $k(x) \in C^2$, and there exists a constant $c$ such that $|k''(x)| \leq c k(x)$ and $|k'(x)| \leq c k(x)$; $\delta > 0$ is the damping coefficient of the system.

The two beams (3) and (4) have local damping and their boundary conditions are

$$\begin{gathered} u_{xx}(0, t) = u_{xx}(l, t) = u_{xxx}(0, t) = u_{xxx}(l, t) = 0, \\ y(0, t) = y(l, t) = y_x(0, t) = y_x(l, t) = 0. \end{gathered} \tag{5}$$

To convert the system into a first-order evolution equation, here we denote

$$\begin{gathered} v = u_t, \\ w = y_t, \end{gathered} \tag{6}$$

and the state variable vector is $\mathbf{z} \equiv \mathbf{z}(t) = (u, v, y, w)^{\mathrm{T}}$; then, systems (3) and (4) can be rewritten as the following form:

$$\begin{aligned} u_t &= v, \\ v_t &= \frac{1}{\rho_1} \left[ -a_1 D^4 u - k(x)(u - y) - \delta k(x) v \right], \\ y_t &= w, \\ w_t &= \frac{1}{\rho_2} \left[ -a_2 D^4 y + k(x)(u - y) - \delta k(x) w \right]. \end{aligned} \tag{7}$$

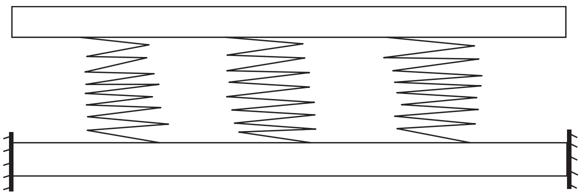

Figure 1: The figure of two weakly coupled elastic beams.

Here, we have used the notation $D^i = \partial^i/\partial x^i$, and the state space is

$$\mathscr{H} = H^2(0,l) \times L^2(0,l) \times H_0^2(0,l) \times L^2(0,l). \tag{8}$$

The Hilbert space $\mathscr{H}$ is equipped with the inner product which induces the energy norm:

$$\|\mathbf{z}\|_{\mathrm{H}}^2 = a_1\|u''\|^2 + a_2\|y''\|^2 + \rho_1\|v\|^2 + \rho_2\|w\|^2 + \|k^{1/2}(x)(u-y)\|^2. \tag{9}$$

Here and after, $\|\cdot\|$, $\prime$, and $\langle\cdot,\cdot\rangle$ denote the $L^2(0,l)$ norm, derivative, and inner product, respectively.

Define a linear operator $\mathscr{A}\colon \mathscr{H} \longrightarrow \mathscr{H}$ by

$$\mathrm{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ -\dfrac{a_1D^4 + k(x)}{\rho_1} & -\dfrac{\delta k(x)}{\rho_1} & \dfrac{k(x)}{\rho_1} & 0 \\ 0 & 0 & 0 & I \\ \dfrac{k(x)}{\rho_2} & 0 & -\dfrac{a_2D^4 + k(x)}{\rho_2} & -\dfrac{\delta k(x)}{\rho_2} \end{pmatrix}, \tag{10}$$

with

$$\mathscr{D}(\mathscr{A}) = \left\{z \in \mathscr{H} \mid u, y \in H^4, v \in H^2, w \in H_0^2, u'' \in H_0^2\right\}. \tag{11}$$

Thus, equations (3) and (4) are transformed into a first-order evolution on the Hilbert space:

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathscr{A}\mathbf{z},\ \mathbf{z}(0) = \mathbf{z}_0. \tag{12}$$

## 3. Main Result

**Theorem 1.** *The operator $\mathscr{A}$ generates a $\mathbf{C}_0$ semigroup $\mathbf{S}(t)$ of contractions on $\mathscr{H}$.*

*Proof.* It is clear that $\mathscr{D}(\mathscr{A})$ is dense in $\mathscr{H}$. By a straight forward calculation,

$$\begin{aligned} \operatorname{Re}\langle \mathrm{A}\mathbf{z},\mathbf{z}\rangle_{\mathrm{H}} &= a_1\int_0^l v''\overline{u}''\,\mathrm{d}x + \int_0^l \left[-a_1D^4u - k(x)(u-y) - \delta k(x)v\right]\overline{v}\,\mathrm{d}x \\ &\quad + a_2\int_0^l y''\overline{w}''\,\mathrm{d}x + \int_0^l \left[-a_2D^4y + k(x)(u-y) - \delta k(x)w\right]\overline{w}\,\mathrm{d}x \\ &= -\delta\left(\|k^{1/2}(x)v\|^2 + \|k^{1/2}(x)w\|^2\right) \le 0. \end{aligned} \tag{13}$$

Hence, $\mathscr{A}$ is dissipative. It is easy to show that, for any $F = (f_1,\cdots,f_4)^T \in H$,

$$\mathscr{A}\mathbf{z} = F, \tag{14}$$

has unique solution $z \in \mathscr{D}(\mathscr{A})$, and

$$\|F\|_{\mathrm{H}}^2 = a_1\|f_1''\|^2 + a_2\|f_3''\|^2 + \rho_1\|f_2\|^2 + \rho_2\|f_4\|^2 + \|k^{1/2}(x)(f_1 - f_3)\|^2. \tag{15}$$

In fact, from the first and third equations of (14), we get $v = f_1 \in \mathscr{H}^2$ and $w = f_3 \in \mathscr{H}_0^2$.

Substituting them into the second and fourth equations in (14), we have

$$-a_1D^4u - k(x)(u-y) - \delta k(x)f_1 = \rho_1 f_2, \tag{16}$$

$$-a_2D^4y + k(x)(u-y) - \delta k(x)f_3 = \rho_2 f_4. \tag{17}$$

Taking the inner product with (16) and $-u$ and with (17) and $-y$, respectively, we obtain

$$a_1\langle D^4u, u\rangle + \langle k(x)(u-y), u\rangle + \delta\langle k(x)f_1, u\rangle = \rho_1\langle f_2, -u\rangle, \tag{18}$$

$$a_2\langle D^4y, y\rangle - \langle k(x)(u-y), y\rangle + \delta\langle k(x)f_3, y\rangle = \rho_2\langle f_4, -y\rangle. \tag{19}$$

Suppose there exist infinitesimal constants $\varepsilon, \varepsilon_1$, and $\varepsilon_2$ and

$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}. \tag{20}$$

Taking the boundary conditions to (18) and (19), we have

$$\begin{aligned} &a_1\|u''\|^2 + \langle k(x)(u-y), u\rangle + \delta\langle k(x) f_1, u\rangle \\ &= \rho_1\langle f_2, -u\rangle \le \rho_1\|f_2\|\|u\| \le \frac{\rho_1}{2\varepsilon_1}\|f_2\|^2 + \frac{\rho_1\varepsilon_1}{2}\|u\|^2, \end{aligned} \tag{21}$$

$$\begin{aligned} &a_2\|y''\|^2 - \langle k(x)(u-y), y\rangle + \delta\langle k(x) f_3, y\rangle \\ &= \rho_2\langle f_4, -y\rangle \le \rho_2\|f_4\|\|y\| \le \frac{\rho_2}{2\varepsilon_2}\|f_4\|^2 + \frac{\rho_2\varepsilon_2}{2}\|y\|^2. \end{aligned} \tag{22}$$

Adding (21) and (22) and because $\varepsilon$ is an infinitesimal constant, we have

$$\begin{aligned} &a_1\|u''\|^2 + a_2\|y''\|^2 + \|k^{1/2}(x)(u-y)\|^2 + \delta\langle k(x) f_1, u\rangle \\ &\quad + \delta\langle k(x) f_3, y\rangle \\ &\le \frac{\rho_1}{2\varepsilon}\|f_2\|^2 + \frac{\rho_2}{2\varepsilon}\|f_4\|^2. \end{aligned} \tag{23}$$

Also, we have

$$\delta\langle k(x) f_1, u\rangle \le \frac{\delta}{2\varepsilon}\|f_1\|^2 + \frac{\delta\varepsilon}{2}\|k(x)u\|^2, \tag{24}$$

$$\delta\langle k(x) f_3, y\rangle \le \frac{\delta}{2\varepsilon}\|f_3\|^2 + \frac{\delta\varepsilon}{2}\|k(x)y\|^2. \tag{25}$$

Suppose there exist positive constants $c_1, c_2, c_3, c_4$, and $C$ independent of $\mathbf{z}$ and $F$ and

$$C = \max\{c_1, c_2, c_3, c_4\}, \tag{26}$$

by Poincare's inequalities, we have

$$\rho_1\|v\|^2 \le c_1 a_1\|f_1''\|^2, \tag{27}$$

$$\rho_2\|w\|^2 \le c_2 a_2\|f_3''\|^2. \tag{28}$$

Combining (23)–(28) yields

$$\begin{aligned} &a_1\|u''\|^2 + a_2\|y''\|^2 + \|k^{1/2}(x)(u-y)\|^2 + \rho_1\|v\|^2 + \rho_2\|w\|^2 \\ &\quad + \delta\langle k(x) f_1, u\rangle + \delta\langle k(x) f_3, y\rangle \\ &\le c_3\left(a_1\|f_1''\|^2 + a_2\|f_3''\|^2 + \rho_1\|f_2\|^2 + \rho_2\|f_4\|^2\right). \end{aligned} \tag{29}$$

By Poincare's inequalities, (29) yields

$$\|\mathbf{z}\|_{\mathscr{H}}^2 \le c_4\|F\|_{\mathscr{H}}^2 - \delta\langle k(x) f_1, u\rangle - \delta\langle k(x) f_3, y\rangle. \tag{30}$$

Then, by (24), (25), and Poincare's inequalities, we have

$$\|\mathbf{z}\|_{\mathscr{H}}^2 \le C\|F\|_{\mathscr{H}}^2, \tag{31}$$

thus $0 \in \rho(\mathrm{A})$. By Lemma 1, the proof is completed. □

**Theorem 2.** *The semigroup $\mathbf{S}(t)$, generated by the operator $\mathscr{A}$, defined in (10), is exponentially stable, i.e., there exist two positive constants $\alpha$ and $M$ such that*

$$\|\mathbf{S}(t)\| \le M e^{-\alpha t}, \forall t > 0. \tag{32}$$

*Proof.* By Lemma 2, it suffices to verify

$$\rho(\mathrm{A}) \supseteq \{i\beta, \beta \in \mathbf{R}\} \equiv i\mathbf{R}, \tag{33}$$

$$\overline{\lim_{|\beta| \longrightarrow +\infty}}\left\|(\mathbf{i}\beta I - \mathscr{A})^{-1}\right\| < +\infty. \tag{34}$$

We use reduction to absurdity to prove (33). If (33) is false, then there exists $\beta \in R$ and $\beta \neq 0$, $i\beta$ is the spectral point of $\mathscr{A}$. Because $\mathscr{A}^{-1}$ is dense, $i\beta$ is the eigenvalue of operator $\mathscr{A}$; then, there exists vector

$$\mathbf{z} = (u, v, y, w)^{\mathrm{T}} \in \mathscr{D}(\mathscr{A}), \|\mathbf{z}\|_{\mathscr{H}} = 1, \tag{35}$$

such that

$$\|(\mathbf{i}\beta\mathbf{I} - \mathscr{A})\mathbf{z}\|_{\mathscr{H}} = 0, \tag{36}$$

i.e.,

$$\mathbf{i}\beta u - v = 0, \text{ in } \mathscr{H}^2, \tag{37}$$

$$\mathbf{i}\beta\rho_1 v + a_1 D^4 u + k(x)(u-y) + \delta k(x) v = 0, \text{ in } \mathscr{L}^2, \tag{38}$$

$$\mathbf{i}\beta y - w = 0, \text{ in } \mathscr{H}_0^2, \tag{39}$$

$$\mathbf{i}\beta\rho_2 w + a_2 D^4 y - k(x)(u-y) + \delta k(x) w = 0, \text{ in } \mathscr{L}^2. \tag{40}$$

Taking the inner product of (36) with $z$ in $H$ and taking its real part

$$\mathrm{Re}\langle(\mathbf{i}\beta\mathbf{I} - \mathrm{A})\mathbf{z}, \mathbf{z}\rangle_{\mathrm{H}} = -\delta\|k^{1/2}(x)v\|^2 - \delta\|k^{1/2}(x)w\|^2 = 0, \tag{41}$$

yields that

$$\|k^{1/2}(x)v\| = 0, \tag{42}$$

$$\|k^{1/2}(x)w\| = 0. \tag{43}$$

Taking (42) and (43) into (37) and (39), we obtain

$$\begin{aligned} &\|k^{1/2}(x)u\| = 0, \\ &\|k^{1/2}(x)y\| = 0. \end{aligned} \tag{44}$$

Taking (37) into (38) and (39) into (40), we can easily deduce from (38) and (40) that

$$a_1 D^4 u - \beta^2\rho_1 u = 0, \tag{45}$$

$$a_2 D^4 y - \beta^2\rho_2 y = 0. \tag{46}$$

If there exists $x_0 \in [a, b] \subset (0, l)$,

$$u(x_0, t) = 0, y(x_0, t) = 0. \tag{47}$$

According to the existence-uniqueness theorem of solutions to ordinary differential equations, (45) and (46) have unique solution, respectively, as follows:

$$u(x,t) = 0, x \in (0,l),\quad y(x,t) = 0, x \in (0,l). \tag{48}$$

Then, we obtain

$$v(x,t) = 0, x \in (0,l),\quad w(x,t) = 0, x \in (0,l), \tag{49}$$

i.e.,

$$\|\mathbf{z}\|_{\mathcal{H}} = 0. \tag{50}$$

which contradicts with $\|z\|_{\mathcal{H}} = 1$; thus, the proof of $iR \subset \rho(A)$ is completed.

Now, we use reduction to absurdity to prove (34). If (34) is false, then there exists a sequence $z_n \in \mathcal{D}(\mathcal{A})$, $z_n = (u_n, v_n, y_n, w_n)^T$ with $\|z_n\|_{\mathcal{H}} = 1$ and a sequence $\beta_n \in R$ with $\beta_n \longrightarrow \infty$ as $n \longrightarrow \infty$ such that

$$\|(\mathbf{i}\beta_n \mathbf{I} - A)\mathbf{z}_n\|_{\mathcal{H}} \longrightarrow 0, \tag{51}$$

i.e.,

$$\mathbf{i}\beta_n u_n - v_n = f_n \longrightarrow 0, \text{ in } \mathcal{H}^2, \tag{52}$$

$$\mathbf{i}\beta_n\rho_1 v_n + a_1 D^4 u_n + k(x)(u_n - y_n) + \delta k(x) v_n = g_n \longrightarrow 0, \text{ in } \mathcal{L}^2, \tag{53}$$

$$\mathbf{i}\beta_n y_n - w_n = T_n \longrightarrow 0, \text{ in } \mathcal{H}_0^2, \tag{54}$$

$$\mathbf{i}\beta_n\rho_2 w_n + a_2 D^4 y_n - k(x)(u_n - y_n) + \delta k(x) w_n = S_n \longrightarrow 0, \text{ in } \mathcal{L}^2. \tag{55}$$

Our goal is to prove $\|\mathbf{z}_n\|_{\mathcal{H}}^2 = 0$, which contradicts with $\|\mathbf{z}_n\|_{\mathcal{H}}^2 = 1$.

*Step 1.* Local attenuation

Taking the inner product of (51) with $\mathbf{z}_n$ in $\mathcal{H}$ and then taking its real part

$$\mathrm{Re}\langle(\mathbf{i}\beta_n\mathbf{I} - \mathrm{A})\mathbf{z}_n, \mathbf{z}_n\rangle_{\mathcal{H}} = -\delta\|k^{1/2}(x)v_n\|^2 - \delta\|k^{1/2}(x)w_n\|^2 \longrightarrow 0. \tag{56}$$

yields that

$$\|k^{1/2}(x)v_n\| \longrightarrow 0,\quad \|k^{1/2}(x)w_n\| \longrightarrow 0. \tag{57}$$

From (52) and (54), we obtain

$$\|k^{1/2}(x)\beta_n u_n\| \longrightarrow 0,\quad \|k^{1/2}(x)\beta_n y_n\| \longrightarrow 0. \tag{58}$$

i.e.,

$$\|k^{1/2}(x)u_n\| \longrightarrow 0,\quad \|k^{1/2}(x)y_n\| \longrightarrow 0. \tag{59}$$

Taking the inner product of (53) with $k^6(x)u_n$ and (55) with $k^6(x)y_n$, respectively, because of $k(x) \in C^2$, $k^6(x)u_n$ and $k^6(x)y_n$ are bounded, we obtain

$$\langle \mathbf{i}\beta_n\rho_1 v_n + a_1 D^4 u_n + k(x)(u_n - y_n) + \delta k(x) v_n, k^6(x)u_n\rangle \longrightarrow 0, \tag{60}$$

$$\langle \mathbf{i}\beta_n\rho_2 w_n + a_2 D^4 y_n - k(x)(u_n - y_n) + \delta k(x) w_n, k^6(x)y_n\rangle \longrightarrow 0, \tag{61}$$

by $k(x)$, and we can easily deduce from (57) and (59) that

$$\begin{aligned}
&\langle k(x)u_n, k^6(x)u_n\rangle \le C\|k^{1/2}(x)u_n\| \longrightarrow 0,\\
&\langle k(x)y_n, k^6(x)y_n\rangle \le C\|k^{1/2}(x)y_n\| \longrightarrow 0.\\
&\langle \delta k(x)v_n, k^6(x)u_n\rangle \longrightarrow 0,\\
&\langle \delta k(x)w_n, k^6(x)y_n\rangle \longrightarrow 0,\\
&|\langle k(x)y_n, k^6(x)u_n\rangle| = |\langle k(x)u_n, k^6(x)y_n\rangle| \longrightarrow 0.
\end{aligned} \tag{62}$$

By (60) and (61), we can obtain that

$$\langle \mathbf{i}\beta_n\rho_1 v_n + a_1 D^4 u_n, k^6(x)u_n\rangle \longrightarrow 0,\quad \langle \mathbf{i}\beta_n\rho_2 w_n + a_2 D^4 y_n, k^6(x)y_n\rangle \longrightarrow 0. \tag{63}$$

Because

$$\langle \mathbf{i}\beta_n\rho_1 v_n, k^6(x)u_n\rangle = -\langle \mathbf{i}\rho_1 v_n, -ik^6(x)v_n\rangle = -\rho_1\|k^3(x)v_n\|^2 \longrightarrow 0, \tag{64}$$

$$\langle \mathbf{i}\beta_n\rho_2 w_n, k^6(x)y_n\rangle = \langle \mathbf{i}\rho_2 w_n, -ik^6(x)w_n\rangle = -\rho_2\|k^3(x)w_n\|^2 \longrightarrow 0. \tag{65}$$

Thus,

$$\langle a_1 D^4 u_n, k^6(x)u_n\rangle = a_1\left[\langle u_n'', (k^6)''(x)u_n\rangle + 2\mathrm{Re}\langle u_n'', (k^6)'(x)u_{nn}'\rangle + \langle u_n'', k^6(x)u_n''\rangle\right] \longrightarrow 0, \tag{66}$$

$$\langle a_2 D^4 y_n, k^6(x)y_n\rangle = a_2[\left[\langle y_n'', (k^6)''(x)y_n\rangle + 2\mathrm{Re}\langle y_n'', (k^6)'(x)y_n'\rangle + \langle y_n'', k^6(x)y_n''\rangle\right] \longrightarrow 0. \tag{67}$$

Because

$$\langle u_n'', (k^6)''(x)u_n\rangle \le C\|k^3(x)u_n''\|\|k^{1/2}(x)u_n\| \longrightarrow 0, \tag{68}$$

$$\langle y_n'', (k^6)''(x)y_n\rangle \le C\|k^3(x)y_n''\|\|k^{1/2}(x)y_n\| \longrightarrow 0, \tag{69}$$

integrating by part, we obtain that

$$\left|\mathrm{Re}\langle u_n'', (k^6)'(x)u_n'\rangle\right| \le c\left(\|k^3(x)u_n''\|\|k^{1/2}(x)u_n\| + \|k^{1/2}(x)u_n\|^2\right) \longrightarrow 0, \tag{70}$$

$$\left|\mathrm{Re}\langle y_n'', (k^6)'(x)y_n'\rangle\right| \le c\left(\|k^3(x)y_n''\|\|k^{1/2}(x)y_n\| + \|k^{1/2}(x)y_n\|^2\right) \longrightarrow 0. \tag{71}$$

From (68) to (71), we now take them into (66) and (67) to obtain that

$$\|k^3(x)u_n''\| \longrightarrow 0, \tag{72}$$

$$\|k^3(x)y_n''\| \longrightarrow 0. \tag{73}$$

Because $k(x)$ is continuous and $k(x) \ge 0$ in $(a,b) \subset (0,l)$ and there exists a constant $c$ such that $|k''(x)| \le ck(x)$ and $|k\prime(x)| \le ck(x)$, we can easily deduce from (57) that

$$\begin{gathered}\|k^3(x)v_n\| \longrightarrow 0,\\ \|k^3(x)w_n\| \longrightarrow 0.\end{gathered} \tag{74}$$

*Step 2.* From local dissipation to global dissipation

Here, were going to use the multiplier method to prove

$$\begin{gathered}u_n'' \longrightarrow 0,\\ v_n \longrightarrow 0,\\ y_n'' \longrightarrow 0,\\ w_n \longrightarrow 0 \text{ in } (0,l).\end{gathered} \tag{75}$$

Taking (52) into (53) and (54) into (55), respectively, we can easily deduce from (53) and (55) that

$$a_1 D^4 u_n - \beta_n^2 \rho_1 u_n = g_n + \mathbf{i}\beta_n \rho_1 f_n, \tag{76}$$

$$a_2 D^4 y_n - \beta_n^2 \rho_2 y_n = S_n + \mathbf{i}\beta_n \rho_2 T_n. \tag{77}$$

Let $q(x) \in C^2$ be a real function, which will be chosen later. Taking the inner product of (76) with $q(x)u_n'$ and (77) with $q(x)y_n'$ in $L^2$, respectively, integrating by part, we obtain that

$$\begin{aligned}&\mathrm{Re}\langle a_1 D^4 u_n - \beta_n^2 \rho_1 u_n, q(x)u_n'\rangle\\ &= 3a_1 \int_0^l q'(x)|u_n''|^2 dx + 2\mathrm{Re}\left(a_1 \int_0^l q''(x)u_n'\overline{u}_n'' dx\right)\\ &\quad - \beta_n^2\rho_1 q(x)|u_n|^2\big|_0^l + \beta_n^2\rho_1 \int_0^l q'(x)|u_n|^2 dx\\ &= 2\langle g_n, q(x)u_n\rangle - 2\langle i\beta_n\rho_1 (f_n q(x))', u_n\rangle,\end{aligned} \tag{78}$$

$$\begin{aligned}&\mathrm{Re}\langle a_2 D^4 y_n - \beta_n^2 \rho_2 y_n, q(x)y_n'\rangle\\ &= 3a_2 \int_0^l q'(x)|y_n''|^2 dx + 2\mathrm{Re}\left(a_2 \int_0^l q''(x)y_n'\overline{y}_n'' dx\right)\\ &\quad - a_2 q(x)|y_n''|^2\big|_0^l + \beta_n^2\rho_2 \int_0^l q'(x)|y_n|^2 dx\\ &= 2\langle S_n, q(x)y_n'\rangle - 2\langle i\beta_n\rho_2 (T_n q(x))', y_n\rangle.\end{aligned} \tag{79}$$

Because $u_n'$ and $\beta_n u_n$ are uniformly bounded in $L^2$ and $y_n'$ and $\beta_n y_n$ are also uniformly bounded in $L^2$, the terms on the right-hand side of (78) and (79) converge to zero. Taking $q(x) = x$, we deduce from (78) and (79) that

$$3a_1\|u_n''\|^2 + \rho_1\|v_n\|^2 - l\beta_n^2\rho_1|u_n(l)|^2 \longrightarrow 0, \tag{80}$$

$$3a_2\|y_n''\|^2 + \rho_2\|w_n\|^2 - la_2|y_n''(l)|^2 \longrightarrow 0. \tag{81}$$

We now take $q(x) = \int_0^x k^6(s)ds$ into (78) and (79) to obtain that

$$3a_1\|k^3(x)u_n''\|^2 + \rho_1\|k^3(x)v_n\|^2 + 2\mathrm{Re}\left(a_1 \int_0^l (k^6)'(x)u_n'\overline{u}_n'' dx\right) - q(l)\beta_n^2\rho_1|u_n(l)|^2 \longrightarrow 0, \tag{82}$$

$$3a_2\|k^3(x)y_n''\|^2 + \rho_2\|k^3(x)w_n\|^2 + 2\mathrm{Re}\left(a_2 \int_0^l (k^6)'(x)y_n'\overline{y}_n'' dx\right) - q(l)a_2|y_n''(l)|^2 \longrightarrow 0. \tag{83}$$

Taking (70), (72), and (74) into (82) and taking (71), (73), and (76) into (83), we obtain

$$q(l)\beta_n^2\rho_1\left|u_n(l)\right|^2 \longrightarrow 0, \tag{84}$$

$$q(l)a_2\left|y_n''(l)\right|^2 \longrightarrow 0, \tag{85}$$

i.e.,

$$\beta_n^2\rho_1\left|u_n(l)\right|^2 \longrightarrow 0, \tag{86}$$

$$a_2\left|y_n''(l)\right|^2 \longrightarrow 0. \tag{87}$$

Taking (86) and (87) into (80) and (81), we obtain

$$a_1\left\|u_n''\right\|^2 + \rho_1\left\|v_n\right\|^2 \longrightarrow 0, \tag{88}$$

$$a_2\left\|y_n''\right\|^2 + \rho_2\left\|w_n\right\|^2 \longrightarrow 0. \tag{89}$$

From (59), (88), and (89), we obtain $\|\mathbf{z}_n\|_{\mathscr{H}}^2 = 0$, which contradicts with $\|\mathbf{z}_n\|_H^2 = 1$. Thus, the proof is completed.

## 4. Conclusion

In this paper, sufficient findings are provided for the exponential stability of weakly coupled elastic beam system with damping control by using the semigroup theoretical frequency domain multiplier method. By variable conversion, the elastic beam control system is transformed into first-order evolution equations and a linear operator is obtained, and the linear operator-producing semigroup is proved. When the two beams have local damping control, from the local dissipation to the global dissipation, the exponential stability of the semigroup generated from the linear operator is proved by reduction to absurdity. The method in this paper can be employed to handle other elastic beam systems in the future.

## Data Availability

The datasets used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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