Research Article

Coupled System of Nonlinear Fractional Langevin Equations with Multipoint and Nonlocal Integral Boundary Conditions

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This research paper is about the existence and uniqueness of the coupled system of nonlinear fractional Langevin equations with multipoint and nonlocal integral boundary conditions. The Caputo fractional derivative is used to formulate the fractional differential equations, and the fractional integrals mentioned in the boundary conditions are due to Atangana–Baleanu and Katugampola. The existence of solution has been proven by two main fixed-point theorems: O’Regan’s fixed-point theorem and Krasnoselskii’s fixed-point theorem. By applying Banach’s fixed-point theorem, we proved the uniqueness result for the concerned problem. This research paper highlights the examples related with theorems that have already been proven.

1. Introduction

Recently, many mathematical fields have been developed rapidly via fractional calculus. Different applications can be described by fractional equations involving fractional derivatives. Fractional calculus was an essential element in many recently published articles, such as a fractional biological population model, a fractional SISR-SI malaria disease model, a fractional Biswas–Milovic model, fractional wave equations, fractional reaction-diffusion equations, and nonlinear fractional shock wave equations. More recent published articles related with fractional calculus can be clearly found in [1–9]. Fractional differential equations have obtained a remarkable reputation among the mathematicians due to rapid development which is applicable in many fields such as mathematics, chemistry, and electronics. For more details, we refer to [10–16]. The coupled systems of fractional differential equations are mainly significant because such systems occur frequently in various scientific applications (see [17–19]).

The Langevin equations (first formulated by Langevin in 1908) have been done with accuracy in order to have a full description of evolution of physical phenomena in fluctuating environment [20]. There is a clear progress on fractional Langevin equations in physics (see [21, 22]). New results on Langevin equations under the variety of boundary value conditions have been published [23–26].

Different forms of fractional integral have been identified and employed in many different applications. Three of the fractional integrals will be used: Riemann–Liouville [10], Atangana and Baleanu [27, 28], and Ntouyas et al. [29, 30].

Recent paper [31] has discussed the existence and uniqueness of solutions obtained from boundary value conditions for nonlinear fractional differential equations for Riemann–Liouville type under the generalized nonlocal integral boundary condition. In addition, the authors in [32] have studied existence and uniqueness of the solution for a certain class of ordinary differential equations of Atangana–Baleanu fractional derivative.

In this paper, we modify the boundary value conditions of coupled systems of Langevin fractional differential equations of Caputo type into new boundary value conditions. So, we deal with the following coupled systems of nonlinear fractional Langevin equations of $\alpha$ and $\beta$ fractional orders:

$$
\begin{align*}
\frac{d^\alpha}{dt^\alpha}x(t) &= f(t, x(t), y(t)), \\
\frac{d^\beta}{dt^\beta}y(t) &= g(t, x(t), y(t)), \\
\end{align*}
$$
supplemented by the following:

\[
\begin{align*}
&\frac{cD^\beta}{\Gamma(\alpha+1)} f(t) = f_1(t, x_1(t), x_2(t)), \quad t \in [0, 1], \\
&\frac{cD^\beta}{\Gamma(\alpha+1)} g(t) = g_1(t, y_1(t), y_2(t)), \quad t \in [0, 1], \\
&\frac{cD^\beta}{\Gamma(\alpha+1)} h(t) = h_1(t), \quad t \in [0, 1],
\end{align*}
\]  

(1)

where \(cD\) is the Caputo fractional derivative of order \(0 < \alpha_i \leq 1 \quad \text{and} \quad 1 < \beta_i \leq 2\) for \(i = 1, 2\).

\[\begin{align*}
&x_1(0) = 0, \quad cD^\alpha x_1(t) = \Gamma(\alpha+1) \frac{d}{dt} I^\alpha x_1(t), \\
&x_2(0) = 0, \quad cD^\alpha x_2(t) = \Gamma(\alpha+1) \frac{d}{dt} I^\alpha x_2(t),
\end{align*}\]

(2)

provided the right-hand side exists.

\[\begin{align*}
&\text{Definition 2 (see [33]). The Riemann–Liouville fractional integral of order } \omega \text{ for a continuous function } f: (0, \infty) \to \mathbb{R} \text{ is given by} \\
&I^\omega f(r) = \int_0^r \frac{f(x)}{\Gamma(\omega)} dx, \quad \omega > 0, r > 0,
\end{align*}\]

(4)

provided the integral exists.

\[\begin{align*}
&\text{Lemma 1 (see [33]). If } cD^\beta f(x) \text{ is a continuous function on } (0, \infty), \text{ then} \\
&I^\beta cD^\beta f(x) = f(x) + \sum_{i=1}^n c_{i-1} x^{i-1}, \quad n - 1 < \beta \leq n,
\end{align*}\]

(5)

where \(c_{i-1} \in \mathbb{R}, \quad i = 1, 2, \ldots, n\).

\[\begin{align*}
&\text{Lemma 2 (see [33]). Let } \gamma > 0. \text{ Then,} \\
&I^\gamma x^\alpha = \frac{\Gamma(n+1)}{\Gamma(n+y+1)} x^{n+y}, \quad n > -1.
\end{align*}\]

(6)

\[\begin{align*}
&\text{Definition 3 (see [34]). The Katugampola fractional integral of order } \beta \text{ for a function } g \text{ defined on } (0, \infty) \text{ is given by the following formula:} \\
&\int^\beta g(x) = \int_0^\beta \frac{(x^\rho - s^\rho)^{\beta-1}}{\Gamma(\beta)} g(s) ds, \quad \beta > 0, \rho > 0,
\end{align*}\]

(7)

provided the right-hand side exists.

\[\begin{align*}
&\text{Definition 4 (see [35]). The AB } \text{"Atangana–Baleanu" fractional integral } \frac{AB}{\Gamma(\beta)} \text{ is defined by} \\
&\frac{AB}{\Gamma(\beta)} f(\theta) = \frac{1 - \theta}{B(\gamma)} f(\theta) + \frac{\theta}{B(\gamma)} \int_0^\theta (\theta - s)^{\beta-1} f(s) ds, \quad \theta > 0,
\end{align*}\]

(8)

provided the right-hand side exists whenever \(\gamma \in (0, 1)\) and \(f \in L^1(0, \infty)\). \(B(\gamma)\) is called a normalization function satisfying \(B(0) = B(1) = 1\).

\[\begin{align*}
&\text{Lemma 3. For } h_1, h_2 \in C([0, 1], \mathbb{R}), \text{ the coupled system} \\
&\left\{ \begin{array}{ll}
\frac{cD^\beta}{\Gamma(\alpha+1)} f_1(t) = h_1(t), & t \in [0, 1], \\
\frac{cD^\beta}{\Gamma(\alpha+1)} g_1(t) = g_1(t), & t \in [0, 1], \\
\frac{cD^\beta}{\Gamma(\alpha+1)} h_2(t) = h_2(t), & t \in [0, 1],
\end{array} \right.
\]

supplemented by

\[\begin{align*}
&\frac{cD^\beta}{\Gamma(\alpha+1)} f_2(t) = \frac{f(t)}{\Gamma(\omega)} f(t), \quad n - 1 < \beta \leq n,
\end{align*}\]

(3)
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The solution of (1) can be given as

$$
x_1(0) = 0, cD^{\alpha_i}x_1(0) = \Gamma(\alpha_i + 1) \int_0^{\eta_i} \frac{1 - \gamma_i (\eta_i^{\alpha_i} - \sigma_i^{\alpha_i})^{\gamma_i - 1} \sigma_i^{\gamma_i - 1} x_1(s) ds}{\Gamma(\gamma_i)},$$

$$
\sum_{j=1}^{m_i} a_{j\alpha} x_1(\xi_{j\alpha}) = \mu_{j\alpha} \left( \frac{1 - \gamma_i x_1(\eta_i)}{B(\gamma_i)} + \frac{\gamma_i}{B(\gamma_i)} \int_0^{\eta_i} \frac{\eta_i - \gamma_i (\eta_i^{\alpha_i} - \sigma_i^{\alpha_i})^{\gamma_i - 1} \sigma_i^{\gamma_i - 1} x_1(s) ds}{\Gamma(\gamma_i)} \right),$$

$$
x_2(0) = 0, cD^{\alpha_i}x_2(0) = \Gamma(\alpha_i + 1) \int_0^{\eta_i} \frac{1 - \gamma_i (\eta_i^{\alpha_i} - \sigma_i^{\alpha_i})^{\gamma_i - 1} \sigma_i^{\gamma_i - 1} x_2(s) ds}{\Gamma(\gamma_i)},$$

$$
\sum_{j=1}^{m_i} a_{j\beta} x_2(\xi_{j\beta}) = \mu_{j\beta} \left( \frac{1 - \gamma_i x_2(\eta_i)}{B(\gamma_i)} + \frac{\gamma_i}{B(\gamma_i)} \int_0^{\eta_i} \frac{\eta_i - \gamma_i (\eta_i^{\alpha_i} - \sigma_i^{\alpha_i})^{\gamma_i - 1} \sigma_i^{\gamma_i - 1} x_2(s) ds}{\Gamma(\gamma_i)} \right).$$

(10)

has a solution given by

$$
x_1(t) = \int_0^{t} \frac{(t-s)^{\alpha_i+\beta_i-1}h_1(s) ds}{\Gamma(\alpha_i + \beta_i)} - \lambda_i \int_0^{t} \frac{(t-s)^{\alpha_i-1}x_1(s) ds}{\Gamma(\alpha_i)} + S_1(t) \int_0^{t} \frac{1 - \gamma_i (\eta_i^{\alpha_i} - \sigma_i^{\alpha_i})^{\gamma_i - 1} \sigma_i^{\gamma_i - 1} x_1(s) ds}{\Gamma(\gamma_i)} + \frac{t^{\gamma_i + 1}}{S(\alpha_i + 1)} \Lambda_1(x_i),$$

$$
x_2(t) = \int_0^{t} \frac{(t-s)^{\alpha_i+\beta_i-1}h_2(s) ds}{\Gamma(\alpha_i + \beta_i)} - \lambda_2 \int_0^{t} \frac{(t-s)^{\alpha_i-1}x_2(s) ds}{\Gamma(\alpha_i)} + S_2(t) \int_0^{t} \frac{1 - \gamma_i (\eta_i^{\alpha_i} - \sigma_i^{\alpha_i})^{\gamma_i - 1} \sigma_i^{\gamma_i - 1} x_2(s) ds}{\Gamma(\gamma_i)} + \frac{t^{\gamma_i + 1}}{S(\alpha_i + 1)} \Lambda_2(x_i),$$

(11)

where

$$S_i(t) = \frac{t^{\gamma_i}}{S(\alpha_i + 1)} t^{\gamma_i + 1},$$

$$S(\alpha_i) = \sum_{j=1}^{m_i} a_{j\alpha} \xi_{j\alpha}^{\alpha_i},$$

(13)

$$i = 1, 2,$$

$$\Lambda_i(x_i) = \mu_i \left( \frac{1 - \gamma_i x_1(\eta_i)}{B(\eta_i)} + \frac{\gamma_i}{B(\eta_i)} \int_0^{\eta_i} \frac{\eta_i - \gamma_i (\eta_i^{\alpha_i} - \sigma_i^{\alpha_i})^{\gamma_i - 1} \sigma_i^{\gamma_i - 1} x_1(s) ds}{\Gamma(\gamma_i)} \right) - \sum_{j=1}^{m_i} a_{j\alpha} \int_0^{\xi_{j\alpha}} \frac{(\xi_{j\alpha} - s)^{\alpha_i-1} x_1(s) ds}{\Gamma(\alpha_i)}.$$

(14)

Proof. Clearly, by direct computation, both (11) and (12) are solutions to (1). Conversely, by using Lemma 1, the general solution of (1) can be given as

$$x_i(t) = \int_0^{t} \frac{(t-s)^{\alpha_i+\beta_i-1}h_i(s) ds}{\Gamma(\alpha_i + \beta_i)} + \lambda_i \int_0^{t} \frac{(t-s)^{\alpha_i-1}x_i(s) ds}{\Gamma(\alpha_i)} + \frac{t^{\gamma_i}}{S(\alpha_i + 1)} \Lambda_i(x_i) + c_{0i} \frac{t^{\gamma_i}}{\Gamma(\alpha_i + 1)} + c_{1i} \frac{t^{\gamma_i + 1}}{\Gamma(\alpha_i + 2)} + c_{2i},$$

(15)

and for $i = 1, 2$. The proof is completed.
For $i = 1$, both first and second boundary conditions, we obtain $c_{i1} = 0$ and $c_{i1} = \Gamma((a_i + 1) t) \int_0^t (\rho_1^{1+\gamma_1} \eta_1^{\rho_1 - \eta_1^{1+\gamma_1}} s^{\rho_1 - 1} x_1(s))/\Gamma(\gamma_1) ds$. Consequently, $c_{i1} = \Gamma((a_i + 1) t) / \Gamma(\gamma_1)$. By substituting $c_{i1}$ and $c_{i1}$ in (15), we get (11).

Similarly, in case of $i = 2$, the proof is done. □

Lemma 4. For each $i = 1, 2$, we have

$$\bar{S}_i = \max_{t \in [0, 1]} |S(t)| = \max \left\{ 1 - \frac{S(a_i)}{S(a_i + 1)}, \left( a_i + 1 \right)^{\alpha_i} \right\}. \tag{16}$$

Proof. For each $i = 1, 2$, we have

$$\bar{S}_i(t) = t^{\alpha_i} \left( a_i - (a_i + 1) \frac{S(a_i)}{S(a_i + 1)} t \right). \tag{17}$$

If $S(a_i)/S(a_i + 1) \leq 0$, then $\bar{S}_i(t) > 0$ for all $t \in [0, 1]$ which means that $\bar{S}_i(t)$ is increasing on $[0, 1]$ and so

$$\bar{S}_i(t) \leq \bar{S}_i(1) = 1 - \frac{S(a_i)}{S(a_i + 1)}. \tag{18}$$

If $S(a_i)/S(a_i + 1) > 0$, then $\bar{S}_i(t) > 0$ for all $t \in (0, t_0)$ and

$$\bar{S}_i(t) < 0 \text{ for all } t \in (t_0, 1) \text{ where}$$

$$t_0 = \frac{a_i - S(a_i)}{a_i + 1}. \tag{19}$$

These mean that $\bar{S}_i(t)$ is increasing on $(0, t_0)$ and decreasing on $(t_0, 1)$ and so

$$S_i(t) \leq S_i(t_0) = \frac{a_i^{\alpha_i}}{(a_i + 1)^{\alpha_i}} \left( \frac{S(a_i + 1)}{S(a_i)} \right)^{\alpha_i}. \tag{20}$$

The proof is done. □

3. Main Results

For each $i = 1, 2$, consider $X_i = \{ x_i(t) \mid x_i(t) \in C([0, 1], \mathbb{R}) \}$ is the Banach space of all continuous functions from $[0, 1]$ to all real numbers introduced by the norm $|x_i| = \sup_{t \in [0, 1]} |x_i(t)|$. Moreover, product space $(X_1 \times X_2, \| (x_1, x_2) \| )$ is a Banach space equipped with $\| x_i \| = \| x_i \|_1 + \| x_i \|_2$.

For convenience, we simplify the following expressions:

$$\Lambda_1 = \frac{|\lambda_1|}{\Gamma(a_1 + 1)} + \frac{s_1^{\gamma_1}}{\Gamma(y_1 + 1)} + \frac{1}{\Gamma(\alpha_1 + 1)} \int \frac{1}{\Gamma(\gamma_1)} + \frac{|\lambda_1|}{\Gamma(a_1 + 1)} \left[ 1 - \frac{S(a_1 + \beta_1)}{S(a_1 + 1)} \right], \tag{21}$$

$$\Lambda_2 = \frac{|\lambda_2|}{\Gamma(a_2 + 1)} + \frac{s_2^{\gamma_2}}{\Gamma(y_2 + 1)} + \frac{1}{\Gamma(\alpha_2 + 1)} \int \frac{1}{\Gamma(\gamma_2)} + \frac{|\lambda_2|}{\Gamma(a_2 + 1)} \left[ 1 + \frac{S(a_2 + \beta_2)}{S(a_2 + 1)} \right], \tag{22}$$

$$\Lambda_3 = \frac{1}{\Gamma(a_1 + \beta_1 + 1)} \left[ 1 - \frac{S(a_1 + \beta_1)}{S(a_1 + 1)} \right], \tag{23}$$

$$\Lambda_4 = \frac{1}{\Gamma(a_2 + \beta_2 + 1)} \left[ 1 + \frac{S(a_2 + \beta_2)}{S(a_2 + 1)} \right]. \tag{24}$$

Define the operator $T$: $X_1 \times X_2 \rightarrow X_1 \times X_2$ by

$$T(x_1, x_2)(t) = (U(x_1, x_2)(t), V(x_1, x_2)(t)), \tag{25}$$

where

$$U(x_1, x_2)(t) = \int_0^t (t - s)^{(\alpha_1 + \beta_1 - 1)} f_1(s, x_1, x_2) ds - \lambda_1 \int_0^t (t - s)^{\alpha_1 - 1} x_1(s) ds + s_1(t) \int_0^t (t - s)^{\alpha_1 - 1} x_1(s) ds \int_0^t \frac{\rho_1^{1+\gamma_1} \eta_1^{\rho_1 - \eta_1^{1+\gamma_1}} s^{\rho_1 - 1} x_1(s)}{\Gamma(\gamma_1)} ds$$

$$+ \frac{t^{\alpha_1 + 1}}{S(a_1 + 1)} \Lambda_1(x_1),$$

$$V(x_1, x_2)(t) = \int_0^t (t - s)^{(\alpha_2 + \beta_2 - 1)} f_2(s, x_1, x_2) ds - \lambda_2 \int_0^t (t - s)^{\alpha_2 - 1} x_2(s) ds + s_2(t) \int_0^t (t - s)^{\alpha_2 - 1} x_2(s) ds \int_0^t \frac{\rho_2^{1+\gamma_2} \eta_2^{\rho_2 - \eta_2^{1+\gamma_2}} s^{\rho_2 - 1} x_2(s)}{\Gamma(\gamma_2)} ds$$

$$+ \frac{t^{\alpha_2 + 1}}{S(a_2 + 1)} \Lambda_2(x_2). \tag{26}$$
\[ V(x_1, x_2)(t) = \int_0^t \frac{(t-s)^{\alpha_1} f_2(s, x_1, x_2)}{\Gamma(\alpha_2 + \beta_2)} \, ds - \lambda_2 \int_0^t \frac{(t-s)^{\alpha_1-1} x_3(s) \, ds}{\Gamma(\alpha_2)} + S_2(t) \int_0^t \frac{\rho_2^{\alpha_3} (\eta_2 - s)^{\nu - 1} x_2(s) \, ds}{\Gamma(\gamma_3)} \]

+ \frac{t^{\alpha_1+1}}{S(\alpha_2 + 1)} \lambda_2 (x_2). \tag{27}

By splitting both (26) and (27), we have

\[ U_1(x_1, x_2)(t) = \int_0^t \frac{(t-s)^{\alpha_1} f_1(s, x_1, x_2)}{\Gamma(\alpha_1 + \beta_1)} \, ds - \frac{t^{\alpha_1+1}}{S(\alpha_1 + 1)} \sum_{j=1}^{m_2} \int_0^t \frac{\xi_j (\xi_j - s)^{\alpha_1+1} f_1(s, x_1, x_2)}{\Gamma(\alpha_1 + \beta_1)} \, ds, \]

\[ U_2(x_1)(t) = -\lambda_1 \int_0^t \frac{(t-s)^{\alpha_1-1} x_1(s) \, ds}{\Gamma(\alpha_1)} + S_1(t) \int_0^t \frac{\rho_1^{\alpha_3} (\eta_1 - s)^{\nu - 1} x_1(s) \, ds}{\Gamma(\gamma_1)} \]

+ \frac{t^{\alpha_1+1}}{S(\alpha_1 + 1)} \left[ \mu_1 \frac{1 - y^2 x_1(\eta_2)}{B(\gamma_2)} + \frac{y_2}{B(\gamma_2)} \right] \int_0^t \frac{\rho_1^{\alpha_3} (\eta_1 - s)^{\nu - 1} x_1(s) \, ds}{\Gamma(\gamma_1)} + \lambda_1 \sum_{j=1}^{m_1} \int_0^t \frac{\xi_j (\xi_j - s)^{\alpha_1-1} x_1(s) \, ds}{\Gamma(\alpha_1)}. \tag{28}

Forthcoming theorems proof need more convenient dialog for the operator \( T \). So, it is required to rewrite it as follows:

\[ T(x_1, x_2)(t) = T_1(x_1, x_2)(t) + T_2(x_1, x_2)(t), \tag{29} \]

where

\[ T_1(x_1, x_2)(t) = \langle U_1(x_1, x_2), V_1(x_1, x_2) \rangle(t), \tag{30} \]

\[ T_2(x_1, x_2)(t) = \langle U_2(x_1), V_2(x_2) \rangle(t). \tag{31} \]

3.1. Existence via O’Regan’s Theorem. The first main result counts on O’Regan’s theorem which proves the existence of the solutions for (1) and (2).
(E₁): there exist nonnegative functions \( \Theta_1, \Theta_2 \in \mathbb{C}([0, 1], \mathbb{R}) \) and nondecreasing functions \( \varphi_i, \Phi_i : [0, \infty) \rightarrow [0, \infty) \) for each \( i = 1, 2 \) such that

\[
|f_1(t, x_1, x_2)| \leq \Theta_1(t) \left[ \varphi_1(\|x_1\|) + \Phi_2(\|x_2\|) \right], \quad \text{for all} (t, x_1, x_2) \in [0, 1] \times \mathbb{R}^2,
\]

\[
|f_2(t, x_1, x_2)| \leq \Theta_2(t) \left[ \varphi_1(\|x_1\|) + \Phi_2(\|x_2\|) \right], \quad \text{for all} (t, x_1, x_2) \in [0, 1] \times \mathbb{R}^2.
\]

(E₂): there exists
\[
\Delta = \max \{ \Lambda_1, \Lambda_2 \} < 1, \tag{33}
\]
where \( \Lambda_i \) for each \( i = 1, 2 \) are defined in (21) and (22), respectively.

Then, (1) and (2) have at least one solution on \([0, 1] \).

Proof. Consider \( V_r = \{(x_1, x_2) \in X_1 \times X_2 : \|(x_1, x_2)\| \leq r\} \) with fix radius \( r \) as

\[
|U_1(x_1, x_2)(t)| \leq \frac{\int_0^t (t-s)^{\alpha_i-1} |f_1(s, x_1, x_2)| ds}{\Gamma(\alpha_i)} + \frac{\int_0^\infty + 1 \sum_{j=1}^{m_1} |a_{j1}|}{S(\alpha_i + 1)} \int_0^{\xi_i} (\xi_i - s)^{\alpha_i-1} |f_1(s, x_1, x_2)| ds \]

\[
\leq \Theta_1(t) \left[ \varphi_1(\|x_1\|) + \Phi_2(\|x_2\|) \right] \Lambda_3. \tag{35}
\]

By taking the norm over both sides, we obtain
\[
\|U_1(x_1, x_2)\| \leq \|\Theta_1\| \|\varphi_1(\|x_1\|) + \Phi_2(\|x_2\|)\| \Lambda_3. \tag{36}
\]

Similarly,
\[
\|V_1(x_1, x_2)\| \leq \|\Theta_2\| \|\varphi_1(\|x_1\|) + \Phi_2(\|x_2\|)\| \Lambda_4. \tag{37}
\]

\[
|U_2(x_1)| \leq |\lambda_1| \int_0^t (t-s)^{\alpha_i-1} |x_1(s)| ds + S_1(t) \int_0^{\eta_1} \frac{(t-\xi_1 s)^{\gamma_i-1} |x_1(s)| ds}{\Gamma(\gamma_1)} + \frac{\int_0^\infty + 1 \sum_{j=1}^{m_1} |a_{j1}|}{S(\alpha_i + 1)} \int_0^{\xi_1} (\xi_1 - s)^{\alpha_i-1} |x_1(s)| ds \]

\[
\leq \Lambda_i \|x_1\|. \tag{39}
\]

By taking the norm over both sides, it yields
\[
\|U_2(x_1)\| \leq \Lambda_i \|x_1\|. \tag{40}
\]

Similarly, we can deduce that
\[
\|V_2(x_2)\| \leq \Lambda_2 \|x_2\|. \tag{41}
\]

which implies that
\[
\|T_2(x_1, x_2)\| \leq \Lambda_i \|x_1\| + \Lambda_2 \|x_2\| \leq \Delta r. \tag{42}
\]

Therefore, \( T_1 \) is uniformly bounded:
\[
\|T_1(x_1, x_2)\| \leq \Theta_1\|\varphi_1(\|x_1\|) + \Phi_2(\|x_2\|)\| \Lambda_3 + \Theta_2\|\varphi_1(\|x_1\|) + \Phi_2(\|x_2\|)\| \Lambda_4. \tag{38}
\]
\[
T_2 \text{ is bounded, i.e., } \forall (x_1, x_2) \in \mathbb{V}_r, \text{ and we have }
\]

\[
\|T_1(x_1, x_2)\| \leq \Theta_1\|\varphi_1(\|x_1\|) + \Phi_2(\|x_2\|)\| \Lambda_3 + \Theta_2\|\varphi_1(\|x_1\|) + \Phi_2(\|x_2\|)\| \Lambda_4 + \Delta r \leq r. \tag{43}
\]

Now, we shall show that \( T_1 \) is completely continuous, \( \forall t_1, t_2 \in [0, 1] \), then we have
\[
\left[ U_1(x_1, x_2)(t_2) - U_1(x_1, x_2)(t_1) \right] \leq \left\| \Theta_1 \left[ \varphi_1(\|r\|) + \varphi_2(\|r\|) \right] \right\| \left[ \int_{t_0}^{t_1} \frac{t_2 - s}{\Gamma(\alpha_1 + \beta_1)} ds - \int_{t_0}^{t_1} \frac{t_1 - s}{\Gamma(\alpha_1 + \beta_1)} ds \right] \\
+ \frac{(t_2^{\alpha_1+1} - t_1^{\alpha_1+1})}{S(a_1 + 1)} \sum_{j=1}^{m_0} a_j \int_{t_0}^{t_1} \frac{(t_j - s)}{\Gamma(\alpha_1 + \beta_1)} ds \\
\leq \left\| \Theta_1 \left[ \varphi_1(\|r\|) + \varphi_2(\|r\|) \right] \right\| \left[ (t_2^{\alpha_1+1} - t_1^{\alpha_1+1}) + \frac{S(a_1 + \beta_2)}{S(a_1 + 1)} (t_2^{\alpha_2+1} - t_1^{\alpha_2+1}) \right].
\]

Similarly,

\[
\left[ V_1(x_1, x_2)(t_2) - V_1(x_1, x_2)(t_1) \right] \leq \left\| \Theta_2 \left[ \varphi_1(\|r\|) + \varphi_2(\|r\|) \right] \right\| \left[ (t_2^{\alpha_2+1} - t_1^{\alpha_2+1}) + \frac{S(a_1 + \beta_2)}{S(a_1 + 1)} (t_2^{\alpha_1+1} - t_1^{\alpha_1+1}) \right],
\]

which shows the independence of the pair \((x_1, x_2)\) and \((t_2 - t_1) \to 0\). We conclude that \(T_1\) is equicontinuous. By Arzelà-Ascoli theorem, \(T_1(\overline{V})\) is relatively compact. Thus, \(T_1\) is completely continuous.

We shall show that \(T_2\) is a contraction mapping. Indeed, for any \((x_1, x_1')\), \((x_2, x_2') \in X \times Y\), we have

\[
\left\| U_2(x_2) - U_2(x_1) \right\| \leq \Lambda_2 \left\| x_2 - x_1 \right\|, \\
\left\| V_2(x_2) - V_2(x_1) \right\| \leq \Lambda_2 \left\| x_2' - x_1' \right\|.
\]

So, we can write

\[
\left\| T_2(x_2, x_1) - T_2(x_1, x_1) \right\| \leq \Lambda_2 \left\| x_2 - x_1 \right\| + \Lambda_2 \left\| x_2' - x_1' \right\|,
\]

which clarifies that \(T_2\) is a contraction mapping via (3.11). The last step is to show the first case of Lemma 5. By the way of contradiction, so we suppose \(\exists \theta (0, 1)\) and \((x_1, x_2) \in \partial \overline{V}_r\) such that \((x_1, x_2) \in \partial \overline{V}_r\). Then, we have \(\|x_1, x_2\| = r\), where

\[
\left\| x_1 \right\| \leq \Theta_1 \left[ \varphi_1(\|r\|) + \varphi_2(\|r\|) \right] A_3 + \Lambda_1 \left\| x_1 \right\|, \\
\left\| x_2 \right\| \leq \Theta_2 \left[ \varphi_1(\|r\|) + \varphi_2(\|r\|) \right] A_4 + \Lambda_2 \left\| x_2 \right\|,
\]

Example 1. Consider the following coupled system of the nonlinear fractional Langevin equation subject to both the nonlocal integro-multipoint of Atangana–Baaleau type and the nonlocal integral of Katugampola type as boundary value conditions.

\[
\begin{cases}
C^{3/2}D^{1/2} \left( CD^{1/2} + \frac{1}{101} \right) x(t) = f(t, x, y), & 0 < t < 1, \\
C^{4/3}D^{1/3} \left( CD^{1/3} + \frac{1}{110} \right) y(t) = g(t, x, y), & 0 < t < 1, \\
x(0) = 0, \quad C^{1/2}D^{1/2} x(0) = \Gamma \left[ \frac{3}{2} \right] \frac{1}{109} x \left( \frac{1}{109} \right), & 0 \leq \xi \leq 1, \\
y(0) = 0, \quad C^{1/3}D^{1/3} y(0) = \Gamma \left[ \frac{4}{3} \right] \frac{1}{103} y \left( \frac{1}{103} \right), & 0 \leq \xi \leq 1.
\end{cases}
\]
Obviously,

\[ a_1 = \frac{1}{2}, \quad a_2 = \frac{1}{3}, \quad \beta_1 = \frac{3}{2}, \quad \beta_2 = \frac{4}{3}, \quad \lambda_1 = \frac{1}{101}, \quad \lambda_2 = \frac{1}{110}, \quad \rho_1 = \frac{3}{2}, \quad \rho_2 = \frac{2}{3}, \quad \mu_1 = \frac{1}{105}, \quad \mu_2 = \frac{1}{115}, \quad \gamma_1 = \frac{1}{4}, \quad \gamma_2 = \frac{1}{10} \]

\[ \gamma_3 = \frac{5}{9}, \quad \eta_1 = \frac{3}{11}, \quad \eta_2 = \frac{1}{109}, \quad \eta_3 = \frac{1}{135}, \quad a_1 = \frac{1}{4}, \quad a_2 = \frac{3}{6}, \quad a_3 = \frac{3}{4}, \quad b_1 = \frac{1}{7}, \quad b_2 = \frac{5}{8}, \quad b_3 = \frac{3}{4}, \quad \xi_1 = \frac{1}{5} \]

\[ \xi_2 = \frac{2}{7}, \quad \xi_3 = \frac{5}{9}, \quad \xi_4 = \frac{1}{5}, \quad \xi_5 = \frac{2}{9}, \quad \xi_6 = \frac{4}{11}, \quad B\left(\frac{1}{10}\right) = \frac{99}{100}, \quad B\left(\frac{3}{11}\right) \]

\[ f(t, x, y) = \frac{t + 2}{5(t + 1)} \left[ \frac{|x(t)|^2}{3|x(t)| + 2} + \frac{|y(t)|}{2|y(t)| + 2} \right] \Rightarrow |f(t, x, y)| \leq \frac{t + 2}{5(t + 1)} \left[ \frac{|x(t)|^2}{2} + \frac{|y(t)|}{2} \right], \quad (52) \]

\[ g(t, x, y) = \frac{t + 2}{4(t + 3)} \left[ \frac{|x(t)|}{2|x(t)| + 3} + \frac{|y(t)|^2}{7|y(t)| + 3} \right] \Rightarrow |g(t, x, y)| \leq \frac{t + 2}{4(t + 3)} \left[ \frac{|x(t)|}{3} + \frac{|y(t)|^2}{3} \right], \quad (53) \]

From the shown data above, we have

\[ \Lambda_1 = 0.0883, \quad \Lambda_2 = 0.1502, \quad \Lambda_3 = 0.8559, \]

\[ \Lambda_4 = 1.1138, \quad \Delta = 0.1502. \]

Clearly, by applying Theorem 1 we have

\[ |A_2| f(t, x_1, x_2) \leq \Psi(t), \quad \forall (t, x_1, x_2) \in [0, 1] \times \mathbb{R}^2 \]

and \( \Psi \in C([0, 1], \mathbb{R}_+) \).

Then, the boundary value problem (1) and (2) has at least one solution on \([0, 1] \text{ if } \Delta < 1 \) where \( \Delta \) is defined in (33).

**Lemma 6** (see [37]) (Krasnoselskii’s theorem). Let \( \Omega \) be a bounded, closed, convex, and nonempty subset of Banach space \( X \). Let \( T_1 \) and \( T_2 \) be two operators on \( \Omega \) as the following:

(i) \( T_1(x_1) + T_2(x_2) \in \Omega, \forall x_1, x_2 \in \Omega \),

(ii) \( T_1 \) is compact and continuous,

(iii) \( T_2 \) is a contraction mapping.

Then, there exists \( x_3 \in \Omega \) satisfying \( x_3 = T_1(x_1) + T_2(x_2) \).

**Theorem 2.** Assume the following conditions hold

\[ (A_1) f_1(t, x_1, x_2) \leq \Phi(t), \quad \forall (t, x_1, x_2) \in [0, 1] \times \mathbb{R}^2 \]

and \( \Phi \in C([0, 1], \mathbb{R}_+) \).

**Proof.** Set \( \Phi = \sup_{t \in [0, 1]} |\Phi(t)| \), \( \Psi = \sup_{t \in [0, 1]} |\Psi(t)| \) and

\[ \frac{||\Phi||\Lambda_3 + ||\Psi||\Lambda_4}{1 - \Delta} < r, \quad (55) \]

Consider

\[ B_r = \{(x_1, x_2) \in X_1 \times X_2 : \|x_1, x_2\| \leq r\}, \quad (56) \]

is a subset of Banach space \( X_1 \times X_2 \). Our claim is to prove that for any point \( (x_1, x_2) \) of \( B_r \), it implies \( T_1(x_1, x_2) \in B_r \). Indeed, given \((x_1, x_2)\) of \( B_r \) arbitrarily, we have
$$\left| U_1(x_1, x_2)(t) + U_2(x_1)(t) \right| \leq \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1 + \beta_1)} f_1(s, x_1, x_2) \, ds + |\lambda_1| \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} x_1(s) \, ds$$
$$+ \frac{t^{\alpha_1+1}}{\Gamma(\alpha_1 + 1)} \sum_{j=1}^m |d_j| \int_0^t (\xi_j - s)^{\alpha_j+1} f_1(s, x_1, x_2) \, ds + S_1(t) \int_0^t \frac{\rho_1^{\gamma_j} (\eta_1^{\gamma_j} - \rho_0^{\gamma_j})^{\gamma_j+1} x_1(s) \, ds}{\Gamma(y_1)}$$
$$+ \frac{t^{\alpha_1+1}}{\Gamma(\alpha_1 + 1)} \left[ |\mu_1| \left( 1 - \frac{1}{B(y_2)} \right) x_1(\eta_2) + \frac{y_2}{B(y_2)} \int_0^{\eta_2} (\eta_2 - s)^{\gamma_j+1} x_1(s) \, ds \right] + |\lambda_1| \sum_{j=1}^m |d_j| \int_0^t (\xi_j - s)^{\alpha_j+1} x_1(s) \, ds$$
$$\leq \|\Phi\| \left[ \int_0^t \frac{(t-s)^{\alpha_1+1}}{\Gamma(\alpha_1 + 1)} + \frac{1}{\Gamma(\alpha_1 + 1)} \sum_{j=1}^m |d_j| \int_0^t (\xi_j - s)^{\alpha_j+1} \, ds \right]$$
$$+ \|x_1\| \left[ |\lambda_1| \left( 1 - \frac{1}{B(\gamma_2)} \right) + \frac{y_2}{B(\gamma_2)} \int_0^{\eta_2} (\eta_2 - s)^{\gamma_j+1} \, ds \right] + |\lambda_1| \sum_{j=1}^m |d_j| \int_0^t (\xi_j - s)^{\alpha_j+1} \, ds$$
$$\leq \|\Phi\| \|\Lambda_3\| + \|x_2\| \|\Lambda_1\|. \tag{57}$$

Similarly,
$$|V_1(x_1, x_2)(t) + V_2(x_1)(t)| \leq \|\Psi\| \|\Lambda_4\| + \|y_2\| \|\Lambda_2\|. \tag{58}$$

This shows that $\|T_1(x_1, x_2) + T_2(x_1, x_2)\| \leq r$. Hence, $T_1(x_1, x_2) + T_2(x_1, x_2) \in B_r$. $T_2$ satisfies the contraction principle as it has been shown in (47). $f$ is a continuous function which helps us to say that $T_1$ is continuous. Moreover, $T_1$ is uniformly bounded on $B_r$,
$$\|T_1(x_1, x_2)\| \leq \|\Phi\| \|\Lambda_3\| + \|\Psi\| \|\Lambda_4\|. \tag{59}$$

The last step in this proof is to show compactness of the operator $T_1$. Indeed, $\forall t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, we have

$$\left| U_1(t_2) - U_1(t_1) \right| \leq \int_0^{t_1} \frac{(t_1-s)^{\alpha_1+1}}{\Gamma(\alpha_1 + \beta_1)} f_1(s, x_1, x_2) \, ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha_1+1}}{\Gamma(\alpha_1 + \beta_1)} f_1(s, x_1, x_2) \, ds$$
$$+ \frac{t_2^{\alpha_1+1} - t_1^{\alpha_1+1}}{\Gamma(\alpha_1 + 1)} \sum_{j=1}^m |d_j| \int_0^t (\xi_j - s)^{\alpha_j+1} f_1(s, x_1, x_2) \, ds$$
$$\leq \|\Phi\| \left[ \frac{t_2^{\alpha_1+1} - t_1^{\alpha_1+1}}{\Gamma(\alpha_1 + 1)} \left( \frac{t_2^{\alpha_1+1} - t_1^{\alpha_1+1}}{\Gamma(\alpha_1 + 1)} \right) \right], \tag{60}$$

$$\left| V_1(t_2) + V_1(t_1) \right| \leq \int_0^{t_1} \frac{(t_1-s)^{\alpha_2+1}}{\Gamma(\alpha_2 + \beta_2)} f_2(s, x_1, x_2) \, ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha_2+1}}{\Gamma(\alpha_2 + \beta_2)} f_2(s, x_1, x_2) \, ds$$
$$+ \frac{t_2^{\alpha_2+1} - t_1^{\alpha_2+1}}{\Gamma(\alpha_2 + 1)} \sum_{j=1}^m |d_j| \int_0^t (\xi_j - s)^{\alpha_j+1} f_2(s, x_1, x_2) \, ds$$
$$\leq \|\Psi\| \left[ \frac{t_2^{\alpha_2+1} - t_1^{\alpha_2+1}}{\Gamma(\alpha_2 + 1)} \left( \frac{t_2^{\alpha_2+1} - t_1^{\alpha_2+1}}{\Gamma(\alpha_2 + 1)} \right) \right],$$

which is not dependent on the pair $(x_1, x_2)$ and the quantity $(t_2 - t_1) \to 0$ which ensures that $T_1$ is equicontinuous. So, $T_1$ is relatively compact on $B_r$. $T_1$ is a compact operator on $B_r$, since $T_1$ satisfies the Arzelà–Ascoli theorem. In conclusion, all terms of Krasnoselskii’s theorem have been applied perfectly. Hence, (1) and (2) possess at least one solution on the given period. \hfill \Box

**Example 2.** Consider the following coupled system of nonlinear fractional Langevin equation subject to both the
nonlocal integro-multipoint of Atangana–Baléanu type and the nonlocal integral of Katugampola type as boundary value conditions.

\[
\begin{aligned}
&cD^{5/3} \left[ cD^{1/3} + \frac{1}{100} \right] x(t) = f(t, x, y) \quad 0 < t < 1, \\
cD^{8/7} \left[ cD^{1/7} + \frac{1}{108} \right] y(t) = g(t, x, y) \quad 0 < t < 1, \\
x(0) = 0, \quad cD^{1/3} x(0) = \Gamma \left( \frac{4}{3} \right) \int_0^{1/10} x \left( \frac{1}{107} \right), \\
y(0) = 0, \quad cD^{1/7} y(0) = \Gamma \left( \frac{8}{7} \right) \int_{1/108}^{1/10} y \left( \frac{1}{108} \right), \\
\end{aligned}
\]

(61)

Clearly,

\[
\begin{align*}
\alpha_1 &= \frac{1}{3}, \quad \alpha_2 = \frac{1}{3}, \quad \beta_1 = \frac{5}{3}, \quad \beta_2 = \frac{7}{3}, \quad \lambda_1 = \frac{1}{100}, \quad \lambda_2 = \frac{1}{108}, \\
\rho_1 &= \frac{1}{1}, \quad \rho_2 = \frac{3}{3}, \quad \mu_1 = \frac{1}{102}, \quad \mu_2 = \frac{1}{10}, \quad \gamma_1 = \frac{3}{10}, \quad \gamma_2 = \frac{1}{10}, \\
\eta_3 &= \frac{5}{12}, \quad \eta_4 = \frac{1}{2}, \quad \eta_5 = \frac{1}{108}, \quad \eta_6 = \frac{1}{108}, \\
\eta_7 &= \frac{1}{140}, \quad \eta_8 = \frac{1}{3}, \quad \eta_9 = \frac{1}{4}, \quad \eta_10, \ 
\end{align*}
\]

\[
f(t, x, y) = \frac{(t + 1)^2 |x(t)|}{(20t + 2)(20t + 2) + 13|x(t)| + 1} + \frac{|y(t)\sin^3 y(t)|}{(24t + 3)(24t + 3) + 8|y(t)| + 5},
\]

(62)

\[
g(t, x, y) = \frac{(t + 1)^2 |x(t)|}{(1 + 14(t + 2)^2)(1 + 14(t + 2)^2)} + \frac{|y(t)\cos^2 t\cos y(t)|}{(1 + 14(t + 2)^2)(12|y(t)| + 2)},
\]

\[
|f(t, x, y)| \leq \frac{(t + 1)^2}{260(t + 2)^2 + 26} + \frac{1}{192(t + 3)^2 + 8} |g(t, x, y)| \leq \frac{(t + 1)^3}{126(t + 2)^3 + 7} + \frac{1}{168(t + 2)^2 + 12},
\]

\[
\Phi(t) = \frac{(t + 1)^2}{260(t + 2)^2 + 26} + \frac{1}{192(t + 3)^2 + 8} \Psi(t) = \frac{(t + 1)^3}{126(t + 2)^3 + 7} + \frac{1}{168(t + 2)^2 + 12},
\]

\[
(\mathcal{E}_3) |f_1(t, x_1, x'_1) - f(t, x_2, x'_2)| \leq l_1|x_2 - x_1| + l_2|x'_2 - x'_1| (\mathcal{E}_4) |f_2(t, x_1, x'_1) - g(t, x_2, x'_2)| \leq j_1|x_2 - x_1| + j_2|x'_2 - x'_1|
\]

Then (1) and (2) have a unique solution on \([0, 1]\) if \(N_1 + N_2 < 1\) where

\[
N_1 = \Lambda_3(l_1 + l_2) + \Lambda_1, \quad N_2 = \Lambda_4(j_1 + j_2) + \Lambda_2,
\]

and \(\Lambda_i\) for each \(i = 1, 2, 3, 4\) are defined in (21)-(24).

Proof. Set \(F_1 = \sup_{t \in [0, 1]} f_1(t, 0, 0)\) and \(F_2 = \sup_{t \in [0, 1]} f_2(t, 0, 0)\) and \(r_0 > 0\) such that

\[
(E_2) |f_1(t, x_1, x'_1) - f(t, x_2, x'_2)| \leq l_1|x_2 - x_1| + l_2|x'_2 - x'_1| (E_4) |f_2(t, x_1, x'_1) - g(t, x_2, x'_2)| \leq j_1|x_2 - x_1| + j_2|x'_2 - x'_1|
\]

3.3. Uniqueness via Banach Fixed-Point Theorem. The last result in this paper is about uniqueness criteria for the solution of (1) and (2), which can be achieved by Banach’s fixed-point theorem.

**Theorem 3.** Assume that \(f_1, f_2: [0, 1] \times \mathbb{R}^2 \to \mathbb{R}\) are continuous functions and there exist positive real constants \(l_1, l_2, j_1, j_2\) such that \(\forall (t, x_1, x_2) \in [0, 1] \times \mathbb{R}^2,\)
First of all, we will show that \( T(B_{r_0}) \subseteq B_{r_0} \), where

\[
T(x_1, x_2) = \left\{ \left(\frac{\Lambda_3 F_1 + \Lambda_4 F_2}{1 - (N_1 + N_2)} \right) \leq r_0, \quad B_{r_0} = \{ (x_1, x_2) \in X_1 \times X_2 : \| (x_1, x_2) \| \leq r_0 \} \right. 
\]

For all \( (x_1, x_2) \in B_{r_0} \), we have

\[
|U(x_1, x_2)(t)| \leq \int_0^t (t-s)^{\alpha_1-1} \frac{[f_1(s, x_1, x_2) - f_1(s, 0, 0)]}{\Gamma(\alpha_1 + \beta_1)} \, ds 
+ \frac{1}{S(\alpha_1 + 1)} \sum_{j=1}^{m_1} \left[ \int_0^{\xi_j} \left( \frac{(\xi_j - s)^{\alpha_1-1} x_j(s)}{\Gamma(\alpha_1)} \right) \, ds \right] 
+ |\lambda_1| \int_0^t \frac{(t-s)^{\alpha_1-1} x_1(s)}{\Gamma(\alpha_1)} \, ds 
+ \frac{1}{S(\alpha_1 + 1)} \sum_{j=1}^{m_1} \left[ \int_0^{\xi_j} \frac{(\xi_j - s)^{\alpha_1-1} x_j(s)}{\Gamma(\alpha_1)} \, ds \right] 
\]

\[
\leq (l_1 \| x_1 \| + l_2 \| x_2 \| + F_1) \left( \int_0^t \frac{(t-s)^{\alpha_1-1} ds}{\Gamma(\alpha_1 + \beta_1)} + \frac{1}{S(\alpha_1 + 1)} \sum_{j=1}^{m_1} \left[ \int_0^{\xi_j} \frac{(\xi_j - s)^{\alpha_1-1} ds}{\Gamma(\alpha_1 + \beta_1)} \right] \right) 
+ \left\{ |\lambda_1| \int_0^t \frac{(t-s)^{\alpha_1-1} ds}{\Gamma(\alpha_1)} + \sum_{j=1}^{m_1} \left[ \int_0^{\xi_j} \frac{(\xi_j - s)^{\alpha_1-1} ds}{\Gamma(\alpha_1 + \beta_1)} \right] \right\} 
\]

Likewise,

\[
|V(x_1, x_2)(t)| \leq \Lambda_3 l_1 \| x_1 \| + \Lambda_4 l_2 \| x_2 \| + F_1 + \Lambda_3 \| x_1 \| 
= (\Lambda_3 l_1 + \Lambda_4 l_2) \| x_1 \| + \Lambda_3 F_1 + \Lambda_4 F_2 
\]

Therefore, we obtain

\[
T(x_1, x_2) = \left\{ \left(\frac{\Lambda_3 F_1 + \Lambda_4 F_2}{1 - (N_1 + N_2)} \right) \leq r_0, \quad \int_0^t \frac{(t-s)^{\alpha_1-1} ds}{\Gamma(\alpha_1 + \beta_1)} \right. 
\]

Next, we shall show that the operator \( T \) is a contraction operator on \([0, 1]\). Indeed, for any distinct two pairs \( (x_1, x_2), (x'_1, x'_2) \in X_1 \times X_2 \), we see that

\[
\| T(x_1, x_2) - T(x'_1, x'_2) \| \leq \Lambda_3 l_1 \| x_1 - x'_1 \| + \Lambda_4 l_2 \| x_2 - x'_2 \| + \Lambda_3 F_1 + \Lambda_4 F_2 
\]

\[
= \Lambda_3 l_1 \| x_1 - x'_1 \| + \Lambda_4 l_2 \| x_2 - x'_2 \| + \Lambda_3 l_1 \| x_1 - x'_1 \| + \Lambda_4 l_2 \| x_2 - x'_2 \| + \Lambda_3 F_1 + \Lambda_4 F_2 
\]
obviously, it gives us
\[ \|U(x_1, x_2)(t) - U(x_1', x_2')(t)\| \leq N_1(\|x_1 - x_1'\| + \|x_2 - x_2'\|). \]  
(70)

In a similar technique, we can also have
\[ \|V(x_2, x_2')(t) - V(x_1, x_1')(t)\| \leq N_2(\|x_1 - x_1'\| + \|x_2 - x_2'\|). \]  
(71)

From (70) and (71), we obtain
\[ \|T(x_2, x_2') - T(x_1, x_1')\| \leq (N_1 + N_2)(\|x_1 - x_1'\| + \|x_2 - x_2'\|). \]  
(72)

Since the sum of both \(N_1\) and \(N_2\) is strictly less than one, we can say that the operator \(T\) satisfies a contraction criteria. Banach’s fixed-point theorem ensures that the operator \(T\) has a unique fixed point. Hence (1) and (2) have a unique solution on \([0, 1]\). □

**Example 3.** Consider the previous example of the coupled system of nonlinear fractional Langevin equation subject to both the nonlocal integro-multipoint of Atangana–Baleanu type and nonlocal integral of Katugampola type as boundary value conditions.

\[
\begin{align*}
\frac{C_{D_{1.5}^0}}{C_{D_{1.5}^0} + \frac{1}{100}} x(t) &= f(t, x, y), 0 < t < 1, \\
\frac{C_{D_{1.7}^0}}{C_{D_{1.7}^0} + \frac{1}{100}} y(t) &= g(t, x, y), 0 < t < 1, \\
x(0) &= 0, \quad \frac{C_{D_{1.5}^1}}{C_{D_{1.5}^1} + \frac{1}{100}} x(0) = \Gamma\left[\frac{4}{3}\right] \frac{1}{100} \left(\frac{1}{107}\right), \\
y(0) &= 0, \quad \frac{C_{D_{1.7}^1}}{C_{D_{1.7}^1} + \frac{1}{100}} y(0) = \Gamma\left[\frac{8}{7}\right] \frac{1}{108} \left(\frac{1}{107}\right).
\end{align*}
\]  
(73)
Clearly,

\begin{align*}
\alpha_1 &= \frac{1}{3}, \alpha_2 = \frac{1}{7}, \beta_1 = \frac{5}{3}, \beta_2 = \frac{8}{7}, \lambda_1 = \frac{1}{100}, \lambda_2 = \frac{1}{108}, \rho_1 = 3, \rho_2 = \frac{1}{3}, \mu_1 = \frac{1}{102}, \mu_2 = \frac{1}{110}, \gamma_1 = \frac{3}{10}, \\
\gamma_2 &= \frac{1}{12}, \gamma_3 = \frac{5}{12}, \gamma_4 = \frac{1}{12}, \eta_1 = \frac{1}{107}, \eta_2 = \frac{1}{120}, \eta_3 = \frac{1}{140}, a_1 = \frac{1}{3}, a_2 = \frac{1}{4}, a_3 = \frac{1}{4}, \\
b_1 &= \frac{3}{8}, b_2 = \frac{3}{8}, b_3 = \frac{1}{4}, \xi_1 = \frac{1}{4}, \xi_2 = \frac{3}{5}, \xi_3 = \frac{5}{6}, \xi_4 = \frac{2}{3}, \xi_5 = \frac{2}{3}, \xi_6 = \frac{4}{5}, B \left( \frac{1}{10} \right) = \frac{999}{1000}, B \left( \frac{1}{12} \right) = \frac{998}{1000}.
\end{align*}

\begin{equation}
 f(t, x, y) = \frac{(t + 1)^2|x(t)|}{(20(t + 2)^2 + 2)(13|x(t)| + 1)} + \frac{|y(t)|\sin^4 y(t)}{(24(t + 3)^2 + 1)(8|y(t)| + 5)},
\end{equation}

\begin{equation}
g(t, x, y) = \frac{(t + 1)^3|x(t)|}{(1 + 18(t + 2)^2)(7|x(t)| + 5)} + \frac{|y(t)|\cos^2 t \cos y(t)}{(1 + 14(t + 2)^2)(12|y(t)| + 2)},
\end{equation}

\begin{equation}
|f(t, x, y)| \leq \frac{(t + 1)^2}{260(t + 2)^2 + 26} + \frac{1}{192(t + 3)^2 + 8}, |g(t, x, y)| \leq \frac{(t + 1)^3}{126(t + 2)^3 + 7} + \frac{1}{168(t + 2)^2 + 12},
\end{equation}

\begin{equation}
\Phi(t) = \frac{(t + 1)^2}{260(t + 2)^2 + 26} + \frac{1}{192(t + 3)^2 + 8}, \Psi(t) = \frac{(t + 1)^3}{126(t + 2)^3 + 7} + \frac{1}{168(t + 2)^2 + 12}.
\end{equation}

Based on Theorem 3, we can rewrite \( f(t, x, y) \) and \( g(t, x, y) \) as follows:

\begin{equation}
|f(t, x_2, y_2) - f(t, x_1, y_1)| \leq \frac{1}{1066}|x_2 - x_1| + \frac{1}{1736}|y_2 - y_1|, \\
|g(t, x_2, y_2) - g(t, x_1, y_1)| \leq \frac{1}{1015}|x_2 - x_1| + \frac{1}{684}|y_2 - y_1|.
\end{equation}

Obviously, \( l_1 = 1/1066, l_2 = 1/1736, j_1 = 1/1015, j_2 = 1/684, N_1 = 0.0657, \) and \( N_2 = 0.5235. \) We conclude that our example has a unique solution on \([0, 1]\) since \( N_1 + N_2 \approx 0.5892 < 1. \)

4. Conclusion

In this research paper, we have proven the existence and uniqueness of solutions for the coupled system of nonlinear fractional Langevin equations with multipoint and nonlocal integral boundary value conditions by selecting \( 0 < \alpha < 1 < \beta < 2. \) Boundary value conditions have been chosen as two types of fractional integrals as we have shown in (2) for which have never been used together before in any article as far as we know. Existence of solutions have been shown by Krasnoselskii’s theorem and O’Regan’s theorem, and uniqueness solutions have been investigated by Banach’s fixed-point theorem. Examples have been supported in order to demonstrate all theorems very well. Results of this paper are not new in giving configuration, but also provide us new cases related with the choice of the parameters involving in the given problem. For example, the results associated with nonperiodic and nonlocal multipoint nonclassical integral boundary conditions follow by considering \( \gamma > 1 \) of this problem. In case of \( a_j = a_{j+1} = 0 \) for all \( j = 1, 2, \cdots, \) the results change with respect to the boundary conditions:

\begin{equation}
\begin{cases}
x_1(0) = 0, \quad D^\alpha_0 x_1(0) = \Gamma (\alpha_1 + 1) D^\alpha_0 x_1(\eta_1), \mu_1^{AB} D^\gamma_1 x_1(\eta_2) = 0, \\
x_2(0) = 0, \quad D^\alpha_0 x_2(0) = \Gamma (\alpha_2 + 1) D^\alpha_0 x_2(\eta_3), \mu_2^{AB} D^\gamma_1 x_2(\eta_4) = 0.
\end{cases}
\end{equation}
Other results can be considered if we take \( \mu_1 = \mu_2 = 0 \), then the boundary conditions will be

\[
\begin{align*}
x_1(0) &= 0, \quad cD^{\alpha_1}x_1(0) = \Gamma(\alpha_1 + 1)\rho_1 I^{\alpha_1}x_1(\eta_1), \\
x_2(0) &= 0, \quad cD^{\alpha_2}x_2(0) = \Gamma(\alpha_2 + 1)\rho_1 I^{\alpha_2}x_2(\eta_2),
\end{align*}
\]

In the future, in the case of obtaining theorems related with the existence and uniqueness of solutions under certain boundary conditions, both how can they be used to prove existence and uniqueness of solutions to the given problem and what are the conditions have to be considered.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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