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Research Article

Coupled System of Nonlinear Fractional Langevin Equations with Multipoint and Nonlocal Integral Boundary Conditions

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This research paper is about the existence and uniqueness of the coupled system of nonlinear fractional Langevin equations with multipoint and nonlocal integral boundary conditions. The Caputo fractional derivative is used to formulate the fractional differential equations, and the fractional integrals mentioned in the boundary conditions are due to Atangana–Baleanu and Katugampola. The existence of solution has been proven by two main fixed-point theorems: O'Regan's fixed-point theorem and Krasnoselskii's fixed-point theorem. By applying Banach's fixed-point theorem, we proved the uniqueness result for the concerned problem. This research paper highlights the examples related with theorems that have already been proven.

1. Introduction

Recently, many mathematical fields have been developed rapidly via fractional calculus. Different applications can be described by fractional equations involving fractional derivatives. Fractional calculus was an essential element in many recently published articles, such as a fractional biological population model, a fractional SISR-SI malaria disease model, a fractional Biswas–Milovic model, fractional wave equations, fractional reaction-diffusion equations, and nonlinear fractional shock wave equations. More recent published articles related with fractional calculus can be clearly found in [1–9]. Fractional differential equations have obtained a remarkable reputation among the mathematicians due to rapid development which is applicable in many fields such as mathematics, chemistry, and electronics. For more details, we refer to [10–16]. The coupled systems of fractional differential equations are mainly significant because such systems occur frequently in various scientific applications (see [17–19]).

The Langevin equations (first formulated by Langevin in 1908) have been done with accuracy in order to have a full description of evolution of physical phenomena in fluctuating environment [20]. There is a clear progress on fractional Langevin equations in physics (see [21, 22]). New results on Langevin equations under the variety of boundary value conditions have been published [23–26].

Different forms of fractional integral have been identified and employed in many different applications. Three of the fractional integrals will be used: Riemann–Liouville [10], Atangana and Baleanu [27, 28], and Ntouyas et al. [29, 30].

Recent paper [31] has discussed the existence and uniqueness of solutions obtained from boundary value conditions for nonlinear fractional differential equations for Riemann–Liouville type under the generalized nonlocal integral boundary condition. In addition, the authors in [32] have studied existence and uniqueness of the solution for a certain class of ordinary differential equations of Atangana–Baleanu fractional derivative.

In this paper, we modify the boundary value conditions of coupled systems of Langevin fractional differential equations of Caputo type into new boundary value conditions. So, we deal with the following coupled systems of nonlinear fractional Langevin equations of α and β fractional orders:

$$\begin{cases} {}^cD^{\beta_1}\left({}^cD^{\alpha_1}+\lambda_1\right)x_1(t)=f_1\left(t,x_1(t),x_2(t)\right), & t\in[0,1],\\ {}^cD^{\beta_2}\left({}^cD^{\alpha_2}+\lambda_2\right)x_2(t)=f_2\left(t,x_1(t),x_2(t)\right), & t\in[0,1],\end{cases} \tag{1}$$

supplemented by the following:

$$\begin{cases} x_1(0)=0, \quad {}^cD^{\alpha_1}x_1(0)=\Gamma(\alpha_1+1)^{\rho_1}I^{\gamma_1}x_1(\eta_1),\\ \sum_{j=1}^{m_1}a_{j_1}x_1\left(\xi_{j_1}\right)=\mu_1{}^{\mathrm{AB}}I^{\gamma_2}x_1(\eta_2),\\ x_2(0)=0, \quad {}^cD^{\alpha_1}x_2(0)=\Gamma(\alpha_2+1)^{\rho_2}I^{\gamma_3}x_2(\eta_3),\\ \sum_{j=1}^{m_2}a_{j_2}x_2\left(\xi_{j_2}\right)=\mu_2{}^{\mathrm{AB}}I^{\gamma_4}x_2(\eta_4),\end{cases} \tag{2}$$

where ${}^cD$ is the Caputo fractional derivative of order $0<\alpha_i\leq 1$ and $1<\beta_i\leq 2$ for $i=1,2$. ${}^{\mathrm{AB}}I$ and ${}^{\rho}I^{\gamma}$ are Atangana–Baleanu, and Katugampola fractional integrals, respectively. $\rho_i>0$ and λ_i, $\mu_i\in\mathbb{R}$ for $i=1,2$, $\gamma_k>0$ for $k=1,2,3,4$, $a_{j_i},\in\mathbb{R}$ for $j=1,2,\ldots,m_i$, and $i=1,2$. $0<\eta_k<\xi_{1_i}<\xi_{2_i}<\cdots<\xi_{m_i}<1$ for $i=1,2$ and $k=1,2,3,4,\ldots f_1,f_2\colon [0,1]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ are continuous functions.

From the definitions of fractional integrals mentioned in the next section, it is worth pointing out that the fractional integral of Katugampola is a generalization for Riemann–Liouville fractional integral ($\rho\longrightarrow 1$) and Hadamard fractional integral ($\rho\longrightarrow 0$). Also, the fractional integral of Atangana–Baleanu contains the Riemann–Liouville fractional integral and when $\gamma=0$, we recover the initial function and if $\gamma=1$, we obtain the ordinary integral. These motivate us to choose these fractional integrals in our boundary conditions. Furthermore, to the extent of our knowledge, this is the first paper that discusses the existence and uniqueness of the solutions to coupled systems of fractional Langevin equations involving the nonlocal integro-multipoint of Atangana–Baleanu type and the nonlocal integral of Katugampola type as boundary value conditions.

The research article has been organized as follows. In the second section, we introduce some main concepts and essential lemma. In the third one, the main results show the existence and uniqueness of solutions to (1) and (2) by O'Regan's fixed-point theorem, Krasnoselskii's fixed-point theorem, and Banach's fixed-point theorem, respectively. Under each one, examples have been considered in order to cover all theorems clearly.

2. Basic Concepts and Relevant Lemmas

We will deduce the main outcomes by the following preliminary concepts in fractional calculus.

Definition 1 (see [33]). For $n\in\mathbb{N}$, let $f\in C^n[0,\infty)$, then the Caputo fractional derivative of order β for a continuous function f is defined by

$${}^cD^{\beta}f(\varsigma)=\int_0^{\varsigma}\frac{(\varsigma-x)^{n-(\beta+1)}}{\Gamma(n-\beta)}f^{(n)}(x)\mathrm{d}x, \quad n-1<\beta\leq n, \tag{3}$$

provided the right-hand side exists.

Definition 2 (see [33]). The Riemann–Liouville fractional integral of order ω for a continuous function $f\colon(0,\infty)\longrightarrow\mathbb{R}$ is given by

$$I^{\omega}f(\tau)=\int_0^{\tau}\frac{(\tau-x)^{\omega-1}f(x)}{\Gamma(\omega)}\mathrm{d}x, \quad \omega>0,\tau>0, \tag{4}$$

provided the integral exists.

Lemma 1 (see [33]). *If ${}^cD^{\beta}f(x)$ is a continuous function on $[0,\infty)$, then*

$$I^{\beta c}D^{\beta}f(x)=f(x)+\sum_{i=1}^{n}c_{i-1}x^{i-1}, \quad n-1<\beta\leq n, \tag{5}$$

where $c_{i-1}\in\mathbb{R}$, $i=1,2,\ldots,n$.

Lemma 2 (see [33]). *Let $\gamma>0$. Then,*

$$I^{\gamma}x^{n}=\frac{\Gamma(n+1)}{\Gamma(n+\gamma+1)}x^{n+\gamma}, \quad n>-1. \tag{6}$$

Definition 3 (see [34]). The Katugampola fractional integral of order β for a function g defined on $(0,\infty)$ is given by the following formula:

$${}^{\rho}I^{\beta}g(\tau)=\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_0^{\tau}\left(\tau^{\rho}-x^{\rho}\right)^{\beta-1}x^{\rho-1}g(x)\mathrm{d}x, \quad \beta>0,\rho>0, \tag{7}$$

provided the right-hand side exists.

Definition 4 (see [35]). The AB "Atangana–Baleanu" fractional integral ${}^{\mathrm{AB}}I^{\gamma}$ is defined by

$$\begin{aligned}{}^{\mathrm{AB}}I^{\gamma}f(\theta)&=\frac{1-\gamma}{B(\gamma)}f(\theta)\\ &\quad+\frac{\gamma}{B(\gamma)}\int_0^{\theta}\frac{(\theta-s)^{\gamma-1}f(s)\mathrm{d}s}{\Gamma(\gamma)}, \quad \theta>0,\end{aligned} \tag{8}$$

provided the right-hand side exists whenever $\gamma\in(0,1)$ and $f\in L^1(0,\infty)$. $B(\gamma)$ is called a normalization function satisfying $B(0)=B(1)=1$.

Lemma 3. *For $h_1,h_2\in C([0,1],\mathbb{R})$, the coupled system*

$$\begin{cases} {}^cD^{\beta_1}\left({}^cD^{\alpha_1}+\lambda_1\right)x_1(t)=h_1(t), & t\in[0,1],\\ {}^cD^{\beta_2}\left({}^cD^{\alpha_2}+\lambda_2\right)x_2(t)=h_2(t), & t\in[0,1],\end{cases} \tag{9}$$

supplemented by

$$\begin{cases} x_1(0)=0,\ {}^cD^{\alpha_1}x_1(0)=\Gamma(\alpha_1+1)\displaystyle\int_0^{\eta_1}\frac{\rho_1^{1-\gamma_1}\left(\eta_1^{\rho_1}-s^{\rho_1}\right)^{\gamma_1-1}s^{\rho_1-1}x_1(s)\,\mathrm{d}s}{\Gamma(\gamma_1)}, \\ \displaystyle\sum_{j=1}^{m_1}a_{j_1}x_1\left(\xi_{j_1}\right)=\mu_1\left(\frac{1-\gamma_2}{B(\gamma_2)}x_1(\eta_2)+\frac{\gamma_2}{B(\gamma_2)}\int_0^{\eta_2}\frac{(\eta_2-s)^{\gamma_2-1}x_1(s)\,\mathrm{d}s}{\Gamma(\gamma_2)},\right) \\ x_2(0)=0,\ {}^cD^{\alpha_2}x_2(0)=\Gamma(\alpha_2+1)\displaystyle\int_0^{\eta_3}\frac{\rho_2^{1-\gamma_3}\left(\eta_3^{\rho_2}-s^{\rho_2}\right)^{\gamma_3-1}s^{\rho_2-1}x_2(s)\,\mathrm{d}s}{\Gamma(\gamma_3)}, \\ \displaystyle\sum_{j=1}^{m_2}a_{j_2}x_2\left(\xi_{j_2}\right)=\mu_2\left(\frac{1-\gamma_4}{B(\gamma_4)}x_2(\eta_4)+\frac{\gamma_4}{B(\gamma_4)}\int_0^{\eta_4}\frac{(\eta_4-s)^{\gamma_4-1}x_2(s)\,\mathrm{d}s}{\Gamma(\gamma_4)}\right), \end{cases} \tag{10}$$

has a solution given by

$$x_1(t)=\int_0^t\frac{(t-s)^{\alpha_1+\beta_1-1}h_1(s)\,\mathrm{d}s}{\Gamma(\alpha_1+\beta_1)}-\lambda_1\int_0^t\frac{(t-s)^{\alpha_1-1}x_1(s)\,\mathrm{d}s}{\Gamma(\alpha_1)}+S_1(t)\int_0^{\eta_1}\frac{\rho_1^{1-\gamma_1}\left(\eta_1^{\rho_1}-s^{\rho_1}\right)^{\gamma_1-1}s^{\rho_1-1}x_1(s)\,\mathrm{d}s}{\Gamma(\gamma_1)}+\frac{t^{\alpha_1+1}}{S(\alpha_1+1)}\Lambda_1(x_1), \tag{11}$$

$$x_2(t)=\int_0^t\frac{(t-s)^{\alpha_2+\beta_2-1}h_2(s)\,\mathrm{d}s}{\Gamma(\alpha_2+\beta_2)}-\lambda_2\int_0^t\frac{(t-s)^{\alpha_2-1}x_2(s)\,\mathrm{d}s}{\Gamma(\alpha_2)}+S_2(t)\int_0^{\eta_1}\frac{\rho_2^{1-\gamma_1}\left(\eta_3^{\rho_2}-s^{\rho_2}\right)^{\gamma_1-1}s^{\rho_2-1}x_2(s)\,\mathrm{d}s}{\Gamma(\gamma_3)}+\frac{t^{\alpha_2+1}}{S(\alpha_2+1)}\Lambda_2(x_2), \tag{12}$$

where

$$S_i(t)=t^{\alpha_i}-\frac{S(\alpha_i)}{S(\alpha_i+1)}t^{\alpha_i+1},$$
$$S(\alpha_i)=\sum_{j=1}^{m_i}a_{j_i}\xi_{j_i}^{\alpha_i}, \tag{13}$$
$$i=1,2,$$

and for $i=1,2$

$$\Lambda_i(x_i)=\mu_i\left(\frac{1-\gamma_{2i}}{B(\gamma_{2i})}x_i(\eta_{2i})+\frac{\gamma_{2i}}{B(\gamma_{2i})}\int_0^{\eta_{2i}}\frac{(\eta_{2i}-s)^{\gamma_{2i}-1}x_i(s)\mathrm{d}s}{\Gamma(\gamma_{2i})}\right)-\sum_{j=1}^{m_i}a_{j_i}\int_0^{\xi_{ji}}\frac{\left(\xi_{j_i}-s\right)^{\alpha_i+\beta_i-1}h_i(s)\mathrm{d}s}{\Gamma(\alpha_i+\beta_i)}$$
$$+\lambda_i\sum_{j=1}^{m_i}a_{j_i}\int_0^{\xi_{ji}}\frac{\left(\xi_{j_i}-s\right)^{\alpha_i-1}x_i(s)\mathrm{d}s}{\Gamma(\alpha_i)}. \tag{14}$$

Proof. Clearly, by direct computation, both (11) and (12) are solutions to (1). Conversely, by using Lemma 1, the general solution of (1) can be given as

$$x_i(t)=\int_0^t\frac{(t-s)^{\alpha_i+\beta_i-1}h_i(s)\,\mathrm{d}s}{\Gamma(\alpha_i+\beta_i)}+\lambda_i\int_0^t\frac{(t-s)^{\alpha_i-1}x_i(s)\,\mathrm{d}s}{\Gamma(\alpha_i)}$$
$$+c_{0i}\frac{t^{\alpha_i}}{\Gamma(\alpha_i+1)}+c_{1i}\frac{t^{\alpha_i+1}}{\Gamma(\alpha_i+2)}+c_{2i},\quad i=1,2. \tag{15}$$

For $i=1$, both first and second boundary conditions, we obtain $c_{21}=0$ and $c_{01}=\Gamma(\alpha_1+1)\int_0^{\eta_1}((\rho_1^{1-\gamma_1}(\eta_1^{\rho_1}-s^{\rho_1})^{\gamma_1-1}s^{\rho_1-1}x_1(s))/\Gamma(\gamma_1))ds$. Consequently, $c_{11}=(\Gamma(\alpha_1+1)/S(\alpha_1+1))\Lambda_1(x_1)$. By substituting c_{11} and c_{01} in (15), we get (11). Similarly, in case of $i=2$, the proof is done. □

Lemma 4. *$\forall t\in[0,1]$ and $i=1,2$, we have*

$$\widehat{S}_i=\max_{t\in[0,1]}|S_i(t)|=\max\left\{1-\frac{S(\alpha_i)}{S(\alpha_i+1)},\frac{\alpha_i^{\alpha_i}}{(\alpha_i+1)^{\alpha_i+1}}\left[\frac{S(\alpha_i+1)}{S(\alpha_i)}\right]^{\alpha_i}\right\}. \tag{16}$$

Proof. For each $i=1,2$, we have

$$S_i'(t)=t^{\alpha_i-1}\left(\alpha_i-(\alpha_i+1)\frac{S(\alpha_i)}{S(\alpha_i+1)}t\right). \tag{17}$$

If $S(\alpha_i)/S(\alpha_i+1)\le 0$, then $S_i'(t)>0$ for all $t\in[0,1]$ which means that $S_i(t)$ is increasing on $(0,1)$ and so

$$S_i(t)\le S_i(1)=1-\frac{S(\alpha_i)}{S(\alpha_i+1)}. \tag{18}$$

If $S(\alpha_i)/S(\alpha_i+1)>0$, then $S_i'(t)>0$ for all $t\in[0,t_0)$ and $S_i'(t)<0$ for all $t\in(t_0,1)$ where

$$t_0=\frac{\alpha_i}{\alpha_i+1}\frac{S(\alpha_i+1)}{S(\alpha_i)}. \tag{19}$$

These mean that $S_i(t)$ is increasing on $(0,t_0)$ and decreasing on $(t_0,1)$ and so

$$S_i(t)\le S_i(t_0)=\frac{\alpha_i^{\alpha_i}}{(\alpha_i+1)^{\alpha_i+1}}\left[\frac{S(\alpha_i+1)}{S(\alpha_i)}\right]^{\alpha_i}. \tag{20}$$

The proof is done. □

3. Main Results

For each $i=1,2$, consider $X_i=\{x_i(t)\mid x_i(t)\in C([0,1],\mathbb{R})\}$ is the Banach space of all continuous functions from $[0,1]$ to all real numbers introduced by the norm $\|x_i\|=\sup_{t\in[0,1]}|x_i(t)|$. Moreover, product space $(X_1\times X_2,\|(x_1,x_2)\|)$ is a Banach space equipped with $\|x_i\|=\|(x_1,x_2)\|=\|x_1\|+\|x_2\|$.

For convenience, we simplify the following expressions:

$$\Lambda_1=\frac{|\lambda_1|}{\Gamma(\alpha_1+1)}+\frac{\widehat{S}_1\eta_1^{\rho_1-\gamma_1}}{\Gamma(\gamma_1+1)}+\frac{1}{S(\alpha_1+1)}\left[|\mu_1|\left(\frac{1-\gamma_2}{B(\gamma_2)}+\frac{\eta_2^{\gamma_2}}{B(\gamma_2)\Gamma(\gamma_2)}\right)+\frac{|\lambda_1|S(\alpha_1)}{\Gamma(\alpha_1+1)}\right], \tag{21}$$

$$\Lambda_2=\frac{|\lambda_2|}{\Gamma(\alpha_2+1)}+\frac{\widehat{S}_2\eta_3^{\rho_2-\gamma_3}}{\Gamma(\gamma_3+1)}+\frac{1}{S(\alpha_2+1)}\left[|\mu_2|\left(\frac{1-\gamma_4}{B(\gamma_4)}+\frac{\eta_4^{\gamma_4}}{B(\gamma_4)\Gamma(\gamma_4)}\right)\frac{|\lambda_2|S(\alpha_2)}{\Gamma(\alpha_2+1)}\right], \tag{22}$$

$$\Lambda_3=\frac{1}{\Gamma(\alpha_1+\beta_1+1)}\left[1+\frac{S(\alpha_1+\beta_1)}{S(\alpha_1+1)}\right], \tag{23}$$

$$\Lambda_4=\frac{1}{\Gamma(\alpha_2+\beta_2+1)}\left[1+\frac{S(\alpha_2+\beta_2)}{S(\alpha_2+1)}\right]. \tag{24}$$

Define the operator $T\colon X_1\times X_2\longrightarrow X_1\times X_2$ by

$$T(x_1,x_2)(t)=(U(x_1,x_2)(t),V(x_1,x_2)(t)), \tag{25}$$

where

$$\begin{aligned}U(x_1,x_2)(t)&=\int_0^t\frac{(t-s)^{\alpha_1+\beta_1-1}f_1(s,x_1,x_2)\,ds}{\Gamma(\alpha_1+\beta_1)}-\lambda_1\int_0^t\frac{(t-s)^{\alpha_1-1}x_1(s)\,ds}{\Gamma(\alpha_1)}+S_1(t)\int_0^{\eta_1}\frac{\rho_1^{1-\gamma_1}(\eta_1^{\rho_1}-s^{\rho_1})^{\gamma_1-1}s^{\rho_1-1}x_1(s)\,ds}{\Gamma(\gamma_1)}\\&\quad+\frac{t^{\alpha_1+1}}{S(\alpha_1+1)}\Lambda_1(x_1),\\V(x_1,x_2)(t)&=\int_0^t\frac{(t-s)^{\alpha_2+\beta_2-1}f_2(s,x_1,x_2)\,ds}{\Gamma(\alpha_2+\beta_2)}-\lambda_2\int_0^t\frac{(t-s)^{\alpha_2-1}x_2(s)\,ds}{\Gamma(\alpha_2)}+S_2(t)\int_0^{\eta_3}\frac{\rho_2^{1-\gamma_3}(\eta_3^{\rho_2}-s^{\rho_2})^{\gamma_3-1}s^{\rho_2-1}x_2(s)\,ds}{\Gamma(\gamma_3)}\\&\quad+\frac{t^{\alpha_2+1}}{S(\alpha_2+1)}\Lambda_2(x_2),\end{aligned} \tag{26}$$

$$V(x_1,x_2)(t) = \int_0^t \frac{(t-s)^{\alpha_2+\beta_2-1} f_2(s,x_1,x_2)\,ds}{\Gamma(\alpha_2+\beta_2)} - \lambda_2 \int_0^t \frac{(t-s)^{\alpha_2-1} x_2(s)\,ds}{\Gamma(\alpha_2)} + S_2(t) \int_0^{\eta_3} \frac{\rho_2^{1-\gamma_3} (\eta_3^{\rho_2} - s^{\rho_2})^{\gamma_3-1} s^{\rho_2-1} x_2(s)\,ds}{\Gamma(\gamma_3)} + \frac{t^{\alpha_2+1}}{S(\alpha_2+1)} \Lambda_2(x_2). \tag{27}$$

By splitting both (26) and (27), we have

$$\begin{aligned}
U_1(x_1,x_2)(t) &= \int_0^t \frac{(t-s)^{\alpha_1+\beta_1-1} f_1(s,x_1,x_2)\,ds}{\Gamma(\alpha_1+\beta_1)} - \frac{t^{\alpha_1+1}}{S(\alpha_1+1)} \sum_{j=1}^{m_1} a_{j1} \int_0^{\xi_{j_1}} \frac{\left(\xi_{j_1}-s\right)^{\alpha_1+\beta_1-1} f_1(s,x_1,x_2)\,ds}{\Gamma(\alpha_1+\beta_1)}, \\
U_2(x_1)(t) &= -\lambda_1 \int_0^t \frac{(t-s)^{\alpha_1-1} x_1(s)\,ds}{\Gamma(\alpha_1)} + S_1(t) \int_0^{\eta_1} \frac{\rho_1^{1-\gamma_1} (\eta_1^{\rho_1} - s^{\rho_1})^{\gamma_1-1} s^{\rho_1-1} x_1(s)\,ds}{\Gamma(\gamma_1)} \\
&\quad + \frac{t^{\alpha_1+1}}{S(\alpha_1+1)} \left[\mu_1 \left(\frac{1-\gamma_2}{B(\gamma_2)} x_1(\eta_2) + \frac{\gamma_2}{B(\gamma_2)} \int_0^{\eta_2} \frac{(\eta_2-s)^{\gamma_2-1} x_1(s)\,ds}{\Gamma(\gamma_2)} \right) + \lambda_1 \sum_{j=1}^{m_1} a_{j1} \int_0^{\xi_{j_1}} \frac{\left(\xi_{j_1}-s\right)^{\alpha_1-1} x_1(s)\,ds}{\Gamma(\alpha_1)} \right], \\
V_1(x_1,x_2)(t) &= \int_0^t \frac{(t-s)^{\alpha_2+\beta_2-1} f_2(s,x_1,x_2)\,ds}{\Gamma(\alpha_2+\beta_2)} - \frac{t^{\alpha_2+1}}{S(\alpha_2+1)} \sum_{j=1}^{m_2} a_{j2} \int_0^{\xi_{j_2}} \frac{\left(\xi_{j_2}-s\right)^{\alpha_2+\beta_2-1} f_2(s,x_1,x_2)\,ds}{\Gamma(\alpha_2+\beta_2)}, \\
V_2(x_2)(t) &= -\lambda_2 \int_0^t \frac{(t-s)^{\alpha_2-1} x_2(s)\,ds}{\Gamma(\alpha_2)} + S_2(t) \int_0^{\eta_3} \frac{\rho_2^{1-\gamma_3} (\eta_3^{\rho_2} - s^{\rho_2})^{\gamma_3-1} s^{\rho_2-1} x_2(s)\,ds}{\Gamma(\gamma_3)} \\
&\quad + \frac{t^{\alpha_2+1}}{S(\alpha_2+1)} \left[\mu_2 \left(\frac{1-\gamma_4}{B(\gamma_4)} x_2(\eta_4) + \frac{\gamma_4}{B(\gamma_4)} \int_0^{\eta_2} \frac{(\eta_4-s)^{\gamma_4-1} x_2(s)\,ds}{\Gamma(\gamma_4)} \right) + \lambda_2 \sum_{j=1}^{m_2} a_{j2} \int_0^{\xi_{j_2}} \frac{\left(\xi_{j_2}-s\right)^{\alpha_2-1} x_2(s)\,ds}{\Gamma(\alpha_2)} \right].
\end{aligned} \tag{28}$$

Forthcoming theorems proof need more convenient dialog for the operator T. So, it is required to rewrite it as follows:

$$T(x_1,x_2)(t) = T_1(x_1,x_2)(t) + T_2(x_1,x_2)(t), \tag{29}$$

where

$$T_1(x_1,x_2)(t) = (U_1(x_1,x_2), V_1(x_1,x_2))(t), \tag{30}$$

$$T_2(x_1,x_2)(t) = (U_2(x_1), V_2(x_2))(t). \tag{31}$$

3.1. Existence via O'Regan's Theorem. The first main result counts on O'Regan's theorem which proves the existence of the solutions for (1) and (2).

Lemma 5 (see [36]). *(O'Regan's theorem)*

Let $0 \in \mathscr{V} \subset \mathscr{D}$ where $\mathscr{V}$ be an open subset of a closed and convex set $\mathscr{D}$ in Banach space X. Assume G: $\overline{\mathscr{V}} \longrightarrow \mathscr{D}$, where $G = G_1 + G_2$, $G(\overline{\mathscr{V}})$ is bounded, G_1: $\overline{\mathscr{V}} \longrightarrow X$ is completely continuous and G_2: $\overline{\mathscr{V}} \longrightarrow X$ is so called nonlinear contraction (i.e., there exists a nonnegative nondecreasing function ω: $[0,\infty) \longrightarrow [0,\infty)$ such that $\omega(y) < y, \forall\, y > 0$ and $\|_2(y_2) - G_2(y_1)\| \le \omega(\|y_2 - y_1\|), \forall y_2, y_1 \in \overline{\mathscr{V}}$). Then, either

- *(i) G has a fixed point $y \in \overline{\mathscr{V}}$ or*
- *(ii) There exists a point $y \in \partial\mathscr{V}$ and $\theta \in (0,1)$ with $y = \theta G(y)$.*

Theorem 1. *Let f_1, f_2: $[0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be continuous functions. Assume the following conditions hold:*

(E_1): there exist nonnegative functions $\Theta_1, \Theta_2 \in C([0,1], \mathbb{R})$ and nondecreasing functions φ_i, ϕ_i: $[0,\infty) \longrightarrow [0,\infty)$ for each $i = 1,2$ such that

$$\begin{aligned} \left|f_1(t,x_1,x_2)\right| &\leq \Theta_1(t)\left[\varphi_1\left(\|x_1\|\right) + \varphi_2\left(\|x_2\|\right)\right], \quad \text{for all}\,(t,x_1,x_2) \in [0,1]\times\mathbb{R}^2, \\ \left|f_2(t,x_1,x_2)\right| &\leq \Theta_2(t)\left[\phi_1\left(\|x_1\|\right) + \phi_2\left(\|x_2\|\right)\right], \quad \text{for all}\,(t,x_1,x_2) \in [0,1]\times\mathbb{R}^2. \end{aligned} \tag{32}$$

(E_2): there exists

$$\Delta = \max\{\Lambda_1, \Lambda_2\} < 1, \tag{33}$$

where Λ_i for each $i = 1,2$ are defined in (21) and (22), respectively.

Then, (1) and (2) have at least one solution on $[0,1]$.

Proof. Consider $V_r = \{(x_1,x_2) \in X_1 \times X_2: \|(x_1,x_2)\| \leq r\}$ with fix radius r as

$$r \geq \frac{\|\Theta_1\|[\varphi_1(\|r\|) + \varphi_2(\|r\|)]\Lambda_3 + \|\Theta_2\|[\varphi_1(\|r\|) + \varphi_2(\|r\|)]\Lambda_4}{1-\Delta}, \tag{34}$$

where Λ_i for each $i = 1,2,3,4$, are defined in (21)–(24), respectively.

From (29) which are defined in both (30) and (31) as two separate operators, first of all, we will show that T is uniformly bounded on $\overline{V_r}$. Indeed, $T_1(x_1,x_2)$ is bounded since $\forall(x_1,x_2) \in \overline{V_r}$, then we have

$$\begin{aligned} \left|U_1(x_1,x_2)(t)\right| &\leq \int_0^t \frac{(t-s)^{\alpha_1+\beta_1-1}\left|f_1(s,x_1,x_2)\right| \mathrm{d}s}{\Gamma(\alpha_1+\beta_1)} + \frac{t^{\alpha_1+1}}{S(\alpha_1+1)}\sum_{j=1}^{m_1}\left|a_{j1}\right|\int_0^{\xi_{j1}} \frac{\left(\xi_{j_1}-s\right)^{\alpha_1+\beta_1-1}\left|f_1(s,x_1,x_2)\right| \mathrm{d}s}{\Gamma(\alpha_1+\beta_1)} \\ &\leq \Theta_1(t)\left[\varphi_1\left(\|x_1\|\right) + \varphi_2\left(\|x_2\|\right)\right]\Lambda_3. \end{aligned} \tag{35}$$

By taking the norm over both sides, we obtain

$$\left\|U_1(x_1,x_2)\right\| \leq \|\Theta_1\|[\varphi_1(\|r\|) + \varphi_2(\|r\|)]\Lambda_3. \tag{36}$$

Similarly,

$$\left\|V_1(x_1,x_2)\right\| \leq \|\Theta_2\|[\phi_1(\|r\|) + \phi_2(\|r\|)]\Lambda_4. \tag{37}$$

Therefore, T_1 is uniformly bounded:

$$\begin{aligned} \left\|T_1(x_1,x_2)\right\| &\leq \Theta_1[\varphi_1(\|r\|) + \varphi_2(\|r\|)]\Lambda_3 \\ &\quad + \|\Theta_2\|[\phi_1(\|r\|) + \phi_2(\|r\|)]\Lambda_4. \end{aligned} \tag{38}$$

T_2 is bounded, i.e., $\forall(x_1,x_2) \in \overline{V_r}$, and we have

$$\begin{aligned} \left|U_2(x_1)\right| &\leq |\lambda_1|\int_0^t \frac{(t-s)^{\alpha_1-1}\left|x_1(s)\right|\mathrm{d}s}{\Gamma(\alpha_1)} + S_1(t)\int_0^{\eta_1} \frac{\rho_1^{1-\gamma_1}\left(\eta_1^{\rho_1}-s^{\rho_1}\right)^{\gamma_1-1}s^{\rho_1-1}\left|x_1(s)\right|\mathrm{d}s}{\Gamma(\gamma_1)} \\ &\quad + \frac{t^{\alpha_1+1}}{S(\alpha_1+1)}\left[|\mu_1|\left(\frac{1-\gamma_2}{B(\gamma_2)}\left|x_1(\eta_2)\right| + \frac{\gamma_2}{B(\gamma_2)}\int_0^{\eta_2}\frac{(\eta_2-s)^{\gamma_2-1}\left|x_1(s)\right|\mathrm{d}s}{\Gamma(\gamma_2)}\right)\right. \\ &\quad \left. + |\lambda_1|\sum_{j=1}^{m_1}\left|a_{j1}\right|\int_0^{\xi_{j1}}\frac{\left(\xi_{j_1}-s\right)^{\alpha_1-1}\left|x_1(s)\right|}{\Gamma(\alpha_1)}\mathrm{d}x\right] \\ &\leq \Lambda_1\|x_1\|. \end{aligned} \tag{39}$$

By taking the norm over both sides, it yields

$$\left\|U_2(x_1)\right\| \leq \Lambda_1\|x_1\|. \tag{40}$$

Similarly, we can deduce that

$$\left\|V_2(x_2)\right\| \leq \Lambda_2\|x_2\|, \tag{41}$$

which implies that

$$\left\|T_2(x_1,x_2)\right\| \leq \Lambda_1\|x_1\| + \Lambda_2\|x_2\| \leq \Delta r. \tag{42}$$

Thus,

$$\begin{aligned} \left\|T(x_1,x_2)\right\| &\leq \|\Theta_1\|[\varphi_1(||r\|) + \varphi_2(\|r\|)]\Lambda_3 + \|\Theta_2\|[\phi_1(\|r\|) \\ &\quad + \phi_2(\|r\|)]\Lambda_4 + \Delta r \leq r. \end{aligned} \tag{43}$$

Now, we shall show that T_1 is completely continuous, $\forall t_1, t_2 \in [0,1]$, then we have

$$\left|U_1(x_1,x_2)(t_2)-U_1(x_1,x_2)(t_1)\right|\le\left\|\Theta_1\right\|\left[\varphi_1(\|r\|)+\varphi_2(\|r\|)\right]\left|\int_0^{t_2}\frac{(t_2-s)^{\alpha_1+\beta_1-1}\mathrm{d}s}{\Gamma(\alpha_1+\beta_1)}-\int_0^{t_1}\frac{(t_1-s)^{\alpha_1+\beta_1-1}\mathrm{d}s}{\Gamma(\alpha_1+\beta_1)}\right.$$
$$\left.+\frac{\left(t_2^{\alpha_1+1}-t_1^{\alpha_1+1}\right)}{S(\alpha_1+1)}\sum_{j=1}^{m_1}a_{j_1}\int_0^{\xi_{j_1}}\frac{\left(\xi_{j_1}-s\right)^{\alpha_1+\beta_1-1}\mathrm{d}s}{\Gamma(\alpha_1+\beta_1)}\right|$$
$$\le\frac{\left\|\Theta_1\right\|\left[\varphi_1(\|r\|)+\varphi_2(\|r\|)\right]}{\Gamma(\alpha_1+\beta_1+1)}\left[\left(t_2^{\alpha_1+\beta_1}-t_1^{\alpha_1+\beta_1}\right)+\frac{S(\alpha_1+\beta_1)}{S(\alpha_1+1)}\left(t_2^{\alpha_1+1}-t_1^{\alpha_1+1}\right)\right]. \tag{44}$$

Similarly,

$$\left|V_1(x_1,x_2)(t_2)-V_1(x_1,x_2)(t_1)\right|\le\frac{\left\|\Theta_2\right\|\left[\phi_1(\|r\|)+\phi_2(\|r\|)\right]}{\Gamma(\alpha_2+\beta_2+1)}\left[\left(t_2^{\alpha_2+\beta_2}-t_1^{\alpha_2+\beta_2}\right)+\frac{S(\alpha_2+\beta_2)}{S(\alpha_2+1)}\left(t_2^{\alpha_2+1}-t_1^{\alpha_2+1}\right)\right], \tag{45}$$

which shows the independence of the pair (x_1,x_2) and $(t_2-t_1)\longrightarrow 0$. We conclude that T_1 is equicontinuous. By Arzelá–Ascoli theorem, $T_1(\overline{V_r})$ is relatively compact. Thus, T_1 is completely continuous.

We shall show that T_2 is a contraction mapping. Indeed, $\forall t\in[0,1]$ and for any $(x_1,x_1'),(x_2,x_2')\in X\times Y$, we have

$$\begin{aligned}\left\|U_2(x_2)-U_2(x_1)\right\|&\le\Lambda_1\left\|x_2-x_1\right\|,\\ \left\|V_2(x_2')-V_2(x_1')\right\|&\le\Lambda_2\left\|x_2'-x_1'\right\|.\end{aligned} \tag{46}$$

So, we can write

$$\begin{aligned}\left\|T_2(x_2,x_2')-T_2(x_1,x_1')\right\|&\le\Delta\left\|x_2-x_1\right\|+\Delta\left\|x_2'-x_1'\right\|\\ &=\Delta\left\|(x_2-x_1)+(x_2'-x_1')\right\|,\end{aligned} \tag{47}$$

which clarifies that T_2 is a contraction mapping via (3.11). The last step is to show the first case of Lemma 5. By the way of contradiction, so we suppose $\exists\theta\in(0,1)$ and $(x_1,x_2)\in\partial\overline{V}_r$ such that $(x_1,x_2)=\theta T(x_1,x_2)$. Then, we have $\|(x_1,x_2)\|=r$, where

$$\begin{aligned}\left\|x_1\right\|&\le\Theta_1\left\|\left[\varphi_1(\|r\|)+\varphi_2(\|r\|)\right]\Lambda_3+\Lambda_1\right\|x_1\|,\\ \left\|x_2\right\|&\le\Theta_2\left\|\left[\phi_1(\|r\|)+\phi_2(\|r\|)\right]\Lambda_4+\Lambda_2\right\|x_2\|,\end{aligned} \tag{48}$$

consequently,

$$\begin{aligned}\left\|x_1\right\|+\left\|x_2\right\|\le\Theta_1\left\|\left[\varphi_1(\|r\|)+\varphi_2(\|r\|)\right]\Lambda_3+\Theta_2\right\|\left[\phi_1(\|r\|)\right.\\ \left.+\phi_2(\|r\|)\right]\Lambda_4+\Delta r,\end{aligned} \tag{49}$$

equivalently,

$$r\le\frac{\left[\left\|\Theta_1\right\|\left[\varphi_1(\|r\|)+\varphi_2(\|r\|)\right]\Lambda_3+\left\|\Theta_2\right\|\left[\varphi_1(\|r\|)+\varphi_2(\|r\|)\right]\Lambda_4\right]}{1-\Delta}, \tag{50}$$

which clearly contradicts (E_2). Hence, the operator T has at least one fixed point $(x_1,x_2)\in\overline{V}_r$. This ensures that (1) and (2) has a solution on $[0,1]$. □

Example 1. Consider the following coupled system of the nonlinear fractional Langevin equation subject to both the nonlocal integro-multipoint of Atangana–Baleanu type and the nonlocal integral of Katugampola type as boundary value conditions.

$$\begin{cases} {}^{C}D^{3/2}\left({}^{C}D^{1/2}+\frac{1}{101}\right)x(t)=f(t,x,y), & 0<t<1,\\ {}^{C}D^{4/3}\left({}^{C}D^{1/3}+\frac{1}{110}\right)y(t)=g(t,x,y), & 0<t<1,\\ x(0)=0,\quad {}^{C}D^{1/2}x(0)=\Gamma\left(\frac{3}{2}\right)^{3}I^{1/4}x\left(\frac{1}{109}\right),\quad \frac{1}{4}x\left(\frac{1}{5}\right)+\frac{1}{6}x\left(\frac{2}{7}\right)+\frac{3}{4}x\left(\frac{5}{9}\right)=\frac{1}{105}I^{1/10}x\left(\frac{1}{119}\right), & \\ y(0)=0,\quad {}^{C}D^{1/3}y(0)=\Gamma\left(\frac{4}{3}\right)^{2}I^{5/9}y\left(\frac{1}{103}\right),\quad \frac{1}{7}y\left(\frac{1}{5}\right)+\frac{5}{8}y\left(\frac{2}{9}\right)+\frac{3}{4}y\left(\frac{4}{11}\right)=\frac{4}{115}{}^{AB}I^{3/11}y\left(\frac{1}{135}\right). & \end{cases} \tag{51}$$

Obviously,

$$\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{3}, \beta_1 = \frac{3}{2}, \beta_2 = \frac{4}{3}, \lambda_1 = \frac{1}{101}, \lambda_2 = \frac{1}{110}, \rho_1 = 3, \rho_2 = 2, \mu_1 = \frac{1}{105}, \mu_2 = \frac{1}{115}, \gamma_1 = \frac{1}{4}, \gamma_2 = \frac{1}{10},$$

$$\gamma_3 = \frac{5}{9}, \gamma_4 = \frac{3}{11}, \eta_1 = \frac{1}{109}, \eta_2 = \frac{1}{119}, \eta_3 = \frac{1}{103}, \eta_4 = \frac{1}{135}, a_1 = \frac{1}{4}, a_2 = \frac{1}{6}, a_3 = \frac{3}{4}, b_1 = \frac{1}{7}, b_2 = \frac{5}{8}, b_3 = \frac{3}{4}, \xi_{1_1} = \frac{1}{5},$$

$$\xi_{2_1} = \frac{2}{7}, \xi_{3_1} = \frac{5}{9}, \xi_{1_2} = \frac{1}{5}, \xi_{2_2} = \frac{2}{9}, \xi_{3_2} = \frac{4}{11}, B\left(\frac{1}{10}\right) = \frac{99}{100}, B\left(\frac{3}{11}\right)$$

$$f(t,x,y) = \frac{t+2}{5(t+1)}\left[\frac{|x(t)|^2}{3|x(t)|+2} + \frac{|y(t)|}{2|y(t)|+2}\right] \Longrightarrow |f(t,x,y)| \le \frac{t+2}{5(t+1)}\left[\frac{|x(t)|^2}{2} + \frac{|y(t)|}{2}\right], \tag{52}$$

$$g(t,x,y) = \frac{t+2}{4(t+3)}\left[\frac{|x(t)|}{2|x(t)|+3} + \frac{|y(t)|^2}{7|y(t)|+3}\right] \Longrightarrow |g(t,x,y)| = \frac{t+2}{4(t+3)}\left[\frac{|x(t)|}{3} + \frac{|y(t)|^2}{3}\right],$$

$$\Theta_1(t) = \frac{t+2}{5(t+1)}, \Theta_2(t) = \frac{t+2}{4(t+3)}, \varphi_1(|x|) = \frac{|x(t)|^2}{2}, \varphi_2(|y|) = \frac{|y(t)|}{2}, \phi_1(|x|) = \frac{|x(t)|}{3}, \phi_2(|y|) = \frac{|y(t)|^2}{3}.$$

From the shown data above, we have

$$\Lambda_1 = 0.0883, \Lambda_2 = 0.1502, \Lambda_3 = 0.8559, \quad \Lambda_4 = 1.1138, \Delta = 0.1502. \tag{53}$$

Clearly, by applying Theorem 1 we have

$$\sup_{r\in(0,\infty)} \frac{r}{\|\Theta_1\|[\varphi_1(\|r\|)+\varphi_2(\|r\|)]\Lambda_3 + \|\Theta_2\|[\varphi_1(\|r\|)+\varphi_2(\|r\|)]\Lambda_4} \sim 2.5292 \ge \frac{1}{1-\Delta} \sim 1.1767. \tag{54}$$

Hence, our example possess at least one solution on $[0,1]$.

3.2. Existence via Krasnoselskii's Theorem. The second outcome relies on Krasnoselskii's theorem in order to prove solution existence for equations (1) and (2).

Lemma 6 (see [37]) (Krasnoselskii's theorem). *Let Ω be a bounded, closed, convex, and nonempty subset of Banach space X. Let T_1 and T_2 be two operators on Ω as the following:*

(i) $T_1(x_1) + T_2(x_2) \in \Omega, \forall x_1, x_2 \in \Omega$,

(ii) T_1 is compact and continuous,

(iii) T_2 is a contraction mapping.

Then, there exists $x_3 \in \Omega$ satisfying $x_3 = T_1(x_3) + T_2(x_3)$.

Theorem 2. *Assume the following conditions hold*

$(A_1)|f_1(t,x_1,x_2)| \le \Phi(t)$, $\forall (t,x_1,x_2) \in [0,1]\times\mathbb{R}^2$ *and* $\Phi \in C([0,1],\mathbb{R}_+)$.

$(A_2)|f_2(t,x_1,x_2)| \le \Psi(t)$, $\forall (t,x_1,x_2) \in [0,1]\times\mathbb{R}^2$ *and* $\Psi \in C([0,1],\mathbb{R}_+)$.

Then, the boundary value problem (1) and (2) has at least one solution on $[0,1]$ if $\Delta < 1$ where Δ is defined in (33).

Proof. Set $\|\Phi\| = \sup_{t\in[0,1]}|\Phi(t)|$, $\|\Psi\| = \sup_{t\in[0,1]}|\Psi(t)|$ and

$$\frac{\|\Phi\|\Lambda_3 + \|\Psi\|\Lambda_4}{1-\Delta} < r. \tag{55}$$

Consider

$$B_r = \left\{(x_1,x_2) \in X_1 \times X_2 : \left\|(x_1,x_2)\right\| \le r\right\}, \tag{56}$$

is a subset of Banach space $X_1 \times X_2$. Our claim is to prove that for any point (x_1,x_2) of B_r, it implies $T_1(x_1,x_2) \in B_r$. Indeed, given (x_1,x_2) of B_r arbitrarily, we have

$$
\begin{aligned}
\left|U_1(x_1,x_2)(t)+U_2(x_1)(t)\right| \le{} & \int_0^t \frac{(t-s)^{\alpha_1+\beta_1-1}\left|f_1(s,x_1,x_2)\right|\mathrm{d}s}{\Gamma(\alpha_1+\beta_1)} + |\lambda_1|\int_0^t \frac{(t-s)^{\alpha_1-1}|x_1(s)|\mathrm{d}s}{\Gamma(\alpha_1)} \\
& + \frac{t^{\alpha_1+1}}{S(\alpha_1+1)}\sum_{j=1}^{m_1}|a_{j1}|\int_0^{\xi_{j1}} \frac{\left(\xi_{j_1}-s\right)^{\alpha_1+\beta_1-1}\left|f_1(s,x_1,x_2)\right|\mathrm{d}s}{\Gamma(\alpha_1+\beta_1)} + S_1(t)\int_0^{\eta_1} \frac{\rho_1^{1-\gamma_1}\left(\eta_1^{\rho_1}-s^{\rho_1}\right)^{\gamma_1-1}s^{\rho_1-1}|x_1(s)|\mathrm{d}s}{\Gamma(\gamma_1)} \\
& + \frac{t^{\alpha_1+1}}{S(\alpha_1+1)}\left[|\mu_1|\left(\frac{1-\gamma_2}{B(\gamma_2)}|x_1(\eta_2)| + \frac{\gamma_2}{B(\gamma_2)}\int_0^{\eta_2}\frac{(\eta_2-s)^{\gamma_2-1}|x_1(s)|\mathrm{d}s}{\Gamma(\gamma_2)}\right) + |\lambda_1|\sum_{j=1}^{m_1}|a_{j_1}|\int_0^{\xi_{j1}}\frac{\left(\xi_{j_1}-s\right)^{\alpha_1-1}|x_1(s)|\mathrm{d}s}{\Gamma(\alpha_1)}\right] \\
\le{} & \|\Phi\|\left[\int_0^t \frac{(t-s)^{\alpha_1+\beta_1-1}\mathrm{d}s}{\Gamma(\alpha_1+\beta_1)} + \frac{1}{S(\alpha_1+1)}\sum_{j=1}^{m_1}|a_{j1}|\int_0^{\xi_{j_1}}\frac{\left(\xi_{j_1}-s\right)^{\alpha_1+\beta_1-1}\mathrm{d}s}{\Gamma(\alpha_1+\beta_1)}\right] \\
& + \|x_1\|\left[|\lambda_1|\int_0^t\frac{(t-s)^{\alpha_1-1}\mathrm{d}s}{\Gamma(\alpha_1)} + \widehat{S}_1\int_0^{\eta_1}\frac{\rho_1^{1-\gamma_1}\left(\eta_1^{\rho_1}-s^{\rho_1}\right)^{\gamma_1-1}s^{\rho_1-1}\mathrm{d}s}{\Gamma(\gamma_1)}\right. \\
& \left. + \frac{1}{S(\alpha_1+1)}\left[|\mu_1|\left(\frac{1-\gamma_2}{B(\gamma_2)} + \frac{\gamma_2}{B(\gamma_2)}\int_0^{\eta_2}\frac{(\eta_2-s)^{\gamma_2-1}\mathrm{d}s}{\Gamma(\gamma_2)}\right)\right] + |\lambda_1|\sum_{j=1}^{m_1}|a_{j1}|\int_0^{\xi_{j_1}}\frac{\left(\xi_{j_1}-s\right)^{\alpha_1-1}\mathrm{d}s}{\Gamma(\alpha_1)}\right] \\
\le{} & \|\Phi\|\Lambda_3 + \|x_2\|\Lambda_1.
\end{aligned}
\tag{57}
$$

Similarly,

$$
\left|V_1(x_1,x_2)(t)+V_2(x_2)(t)\right| \le \|\Psi\|\Lambda_4 + \|y_2\|\Lambda_2. \tag{58}
$$

This shows that $\|T_1(x_1,x_2)+T_2(x_1,x_2)\| \le r$.

Hence, $T_1(x_1,x_2)+T_2(x_1,x_2) \in B_r$. T_2 satisfies the contraction principle as it has been shown in (47). f is a continuous function which helps us to say that T_1 is continuous. Moreover, T_1 is uniformly bounded on B_r

$$
\left\|T_1(x_1,x_2)\right\| \le \|\Phi\|\Lambda_3 + \|\Psi\|\Lambda_4. \tag{59}
$$

The last step in this proof is to show compactness of the operator T_1. Indeed, $\forall t_1,t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$
\begin{aligned}
\left|U_1(t_2)-U_1(t_1)\right| \le{} & \left|\int_0^{t_2}\frac{(t_2-s)^{\alpha_1+\beta_1-1}f_1(s,x_1,x_2)\,\mathrm{d}s}{\Gamma(\alpha_1+\beta_1)} - \int_0^{t_1}\frac{(t_1-s)^{\alpha_1+\beta_1-1}f_1(s,x_1,x_2)\,\mathrm{d}s}{\Gamma(\alpha_1+\beta_1)}\right| \\
& + \left|\frac{\left(t_2^{\alpha_1+1}-t_1^{\alpha_1+1}\right)}{S(\alpha_1+1)}\sum_{j=1}^{m_1}|a_{j1}|\int_0^{\xi_{j1}}\frac{\left(\xi_{j_1}-s\right)^{\alpha_1+\beta_1-1}f_1(s,x_1,x_2)\,\mathrm{d}s}{\Gamma(\alpha_1+\beta_1)}\right| \\
\le{} & \|\Phi\|\left[\frac{\left(t_2^{\alpha_1+\beta_1}-t_1^{\alpha_1+\beta_1}\right)}{\Gamma(\alpha_1+\beta_1+1)} + \frac{\left(t_2^{\alpha_1+1}-t_1^{\alpha_1+1}\right)}{\Gamma(\alpha_1+\beta_1+1)}\frac{S(\alpha_1+\beta_1)}{S(\alpha_1+1)}\right], \\
\left|V_1(t_2)+V_1(t_1)\right| \le{} & \left|\int_0^{t_2}\frac{(t_2-s)^{\alpha_2+\beta_2-1}f_2(s,x_1,x_2)\,\mathrm{d}s}{\Gamma(\alpha_2+\beta_2)} - \int_0^{t_1}\frac{(t_1-s)^{\alpha_2+\beta_2-1}f_2(s,x_1,x_2)\,\mathrm{d}s}{\Gamma(\alpha_2+\beta_2)}\right| \\
& + \left|\frac{\left(t_2^{\alpha_2+1}-t_1^{\alpha_2+1}\right)}{S(\alpha_2+1)}\sum_{j=1}^{m_2}|a_{j2}|\int_0^{\xi_{j2}}\frac{\left(\xi_{j_2}-s\right)^{\alpha_2+\beta_2-1}f_2(s,x_1,x_2)\,\mathrm{d}s}{\Gamma(\alpha_2+\beta_2)}\right| \\
\le{} & \|\Psi\|\left[\frac{\left(t_2^{\alpha_2+\beta_2}-t_1^{\alpha_2+\beta_2}\right)}{\Gamma(\alpha_2+\beta_2+1)} + \frac{\left(t_2^{\alpha_2+1}-t_1^{\alpha_2+1}\right)}{\Gamma(\alpha_2+\beta_2+1)}\frac{S(\alpha_2+\beta_2)}{S(\alpha_2+1)}\right],
\end{aligned}
\tag{60}
$$

which is not dependent on the pair (x_1,x_2) and the quantity $(t_2-t_1) \longrightarrow 0$ which ensures that T_1 is equicontinuous. So, T_1 is relatively compact on B_r. T_1 is a compact operator on B_r since T_1 satisfies the Arzelá–Ascoli theorem. In conclusion, all terms of Krasnoselskii's theorem have been applied perfectly. Hence, (1) and (2) possess at least one solution on the given period. □

Example 2. Consider the following coupled system of nonlinear fractional Langevin equation subject to both the nonlocal integro-multipoint of Atangana–Baleanu type and the nonlocal integral of Katugampola type as boundary value conditions.

$$\begin{cases} {}^{C}D^{5/3}\left({}^{C}D^{1/3}+\frac{1}{100}\right)x(t)=f(t,x,y)\,0<t<1, \\ {}^{C}D^{8/7}\left({}^{C}D^{1/7}+\frac{1}{108}\right)y(t)=g(t,x,y)\,0<t<1, \\ x(0)=0,\quad {}^{C}D^{1/3}x(0)=\Gamma\left(\frac{4}{3}\right)^{3}I^{3/10}x\left(\frac{1}{107}\right),\quad \frac{1}{3}x\left(\frac{1}{4}\right)+\frac{1}{4}x\left(\frac{3}{5}\right)+\frac{1}{4}x\left(\frac{5}{6}\right)=\frac{1}{102}{}^{AB}I^{1/10}x\left(\frac{1}{120}\right), \\ y(0)=0,\quad {}^{C}D^{1/7}y(0)=\Gamma\left(\frac{8}{7}\right)^{1/3}I^{5/12}y\left(\frac{1}{108}\right),\quad \frac{1}{8}y\left(\frac{1}{3}\right)+\frac{3}{8}y\left(\frac{2}{3}\right)+\frac{1}{4}y\left(\frac{4}{5}\right)=\frac{4}{110}{}^{AB}I^{1/12}y\left(\frac{1}{140}\right). \end{cases} \tag{61}$$

Clearly,

$$\begin{aligned} &\alpha_1=\frac{1}{3},\alpha_2=\frac{1}{7},\beta_1=\frac{5}{3},\beta_2=\frac{8}{7},\lambda_1=\frac{1}{100},\lambda_2=\frac{1}{108},\rho_1=3,\rho_2=\frac{1}{3},\mu_1=\frac{1}{102},\mu_2=\frac{1}{110},\gamma_1=\frac{3}{10},\gamma_2=\frac{1}{10},\\ &\gamma_3=\frac{5}{12},\gamma_4=\frac{1}{12},\eta_1=\frac{1}{107},\eta_2=\frac{1}{120},\eta_3=\frac{1}{108},\eta_4=\frac{1}{140},a_1=\frac{1}{3},a_2=\frac{1}{4},a_3=\frac{1}{4},b_1=\frac{1}{8},b_2=\frac{3}{8},b_3=\frac{1}{4},\xi_{1_1}=\frac{1}{4},\\ &\xi_{2_1}=\frac{3}{5},\xi_{3_1}=\frac{5}{6},\xi_{1_2}=\frac{1}{3},\xi_{2_2}=\frac{2}{3},\xi_{3_2}=\frac{4}{5},B\left(\frac{1}{10}\right)=\frac{999}{1000},B\left(\frac{1}{12}\right)=\frac{998}{1000},\\ &f(t,x,y)=\frac{(t+1)^2|x(t)|}{\left(20(t+2)^2+2\right)(13|x(t)|+1)}+\frac{|y(t)|\sin^4 y(t)}{\left(24(t+3)^2+1\right)(8|y(t)|+5)},\\ &g(t,x,y)=\frac{(t+1)^3|x(t)|}{\left(1+18(t+2)^3\right)(7|y(t)|+5)}+\frac{|y(t)|\cos^2 t^2\cos y(t)}{\left(1+14(t+2)^2\right)(12|y(t)|+2)},\\ &|f(t,x,y)|\le\frac{(t+1)^2}{260(t+2)^2+26}+\frac{1}{192(t+3)^2+8},|g(t,x,y)|\le\frac{(t+1)^3}{126(t+2)^3+7}+\frac{1}{168(t+2)^2+12},\\ &\Phi(t)=\frac{(t+1)^2}{260(t+2)^2+26}+\frac{1}{192(t+3)^2+8},\Psi(t)=\frac{(t+1)^3}{126(t+2)^3+7}+\frac{1}{168(t+2)^2+12}. \end{aligned} \tag{62}$$

After calculating, we find $\Lambda_1\approx 0.0644$, $\Lambda_2\approx 0.5194$,$\Lambda_3\approx 0.8791$, and $\Lambda_4\approx 1.6832$. According to Theorem 2, we see that $\Delta=0.5194$ which indicates that our example has at least one solution on $[0,1]$.

3.3. Uniqueness via Banach Fixed-Point Theorem. The last result in this paper is about uniqueness criteria for the solution of (1) and (2), which can be achieved by Banach's fixed-point theorem.

Theorem 3. *Assume that* $f_1,f_2\colon [0,1]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ *are continuous functions and there exist positive real constants* l_1,l_2,j_1*, and* j_2 *such that* $\forall\,(t,x_1,x_2)\in[0,1]\times\mathbb{R}^2$,

$(E_3)|f_1(t,x_1,x_1')-f(t,x_2,x_2')|\le l_1|x_2-x_1|+l_2|x_2'-x_1'|$ $(E_4)|f_2(t,x_1,x_1')-g(t,x_2,x_2')|\le j_1|x_2-x_1|+j_2\quad|x_2'-x_1'|$

Then (1) and (2) have a unique solution on $[0,1]$ *if* $N_1+N_2<1$ *where*

$$\begin{aligned} N_1&=\Lambda_3(l_1+l_2)+\Lambda_1,\\ N_2&=\Lambda_4(j_1+j_2)+\Lambda_2, \end{aligned} \tag{63}$$

and Λ_i *for each* $i=1,2,3,4$ *are defined in (21)–(24).*

Proof. Set $F_1=\sup_{t\in[0,1]}f_1(t,0,0)$ and $F_2=\sup_{t\in[0,1]}f_2(t,0,0)$ and $r_0>0$ such that

$$\frac{\Lambda_3 F_1 + \Lambda_4 F_2}{1-(N_1+N_2)} \le r_0. \tag{64}$$

First of all, we will show that $T(B_{r_0}) \subseteq B_{r_0}$ where

$$B_{r_0} = \{(x_1,x_2) \in X_1 \times X_2 : \|(x_1,x_2)\| \le r_0\}. \tag{65}$$

For all $(x_1,x_2) \in B_{r_0}$, we have

$$\begin{aligned}
|U(x_1,x_2)(t)| \le{}& \int_0^t \frac{(t-s)^{\alpha_1+\beta_1-1}\left[|f_1(s,x_1,x_2)-f_1(s,0,0)|+|f_1(s,0,0)|\right]ds}{\Gamma(\alpha_1+\beta_1)} \\
&+ \frac{1}{S(\alpha_1+1)} \sum_{j=1}^{m_1} |a_{j1}| \int_0^{\xi_{j1}} \frac{\left(\xi_{j_1}-s\right)^{\alpha_1+\beta_1-1}\left[|f_1(s,x_1,x_2)-f_1(s,0,0)|+|f_1(s,0,0)|\right]ds}{\Gamma(\alpha_1+\beta_1)} \\
&+ |\lambda_1| \int_0^t \frac{(t-s)^{\alpha_1-1}|x_1(s)|ds}{\Gamma(\alpha_1)} + \widehat{S}_1 \int_0^{\eta_1} \frac{\rho_1^{1-\gamma_1}(\eta_1^{\rho_1}-s^{\rho_1})^{\gamma_1-1}s^{\rho_1-1}|x_1(s)|ds}{\Gamma(\gamma_1)} \\
&+ \frac{1}{S(\alpha_1+1)}\left[|\mu_1|\left(\frac{1-\gamma_2}{B(\gamma_2)}|x_1(\eta_2)| + \frac{\gamma_2}{B(\gamma_2)}\int_0^{\eta_2}\frac{(\eta_2-s)^{\gamma_2-1}|x_1(s)|ds}{\Gamma(\gamma_2)}\right)\right. \\
&\left. + |\lambda_1| \sum_{j=1}^{m_1}|a_{j1}| \int_0^{\xi_{j1}} \frac{\left(\xi_{j_1}-s\right)^{\alpha_1-1}|x_1(s)|ds}{\Gamma(\alpha_1)}\right] \\
\le{}& \left(l_1\|x_1\| + l_2\|x_2\| + F_1\right)\left(\int_0^t \frac{(t-s)^{\alpha_1+\beta_1-1}ds}{\Gamma(\alpha_1+\beta_1)} + \frac{1}{S(\alpha_1+1)}\sum_{j=1}^{m_1}|a_{j1}|\int_0^{\xi_{j1}}\frac{\left(\xi_{j_1}-s\right)^{\alpha_1+\beta_1-1}ds}{\Gamma(\alpha_1+\beta_1)}\right) \\
&+ \|x_1\|\left\{|\lambda_1|\int_0^t \frac{(t-s)^{\alpha_1-1}ds}{\Gamma(\alpha_1)} + \widehat{S}_1\int_0^{\eta_1}\frac{\rho_1^{1-\gamma_1}(\eta_1^{\rho_1}-s^{\rho_1})^{\gamma_1-1}s^{\rho_1-1}ds}{\Gamma(\gamma_1)}\right. \\
&\left. + \frac{1}{S(\alpha_1+1)}\left[\mu_1\left(\frac{1-\gamma_2}{B(\gamma_2)} + \frac{\gamma_2}{B(\gamma_2)}\int_0^{\eta_2}\frac{(\eta_2-s)^{\gamma_2-1}ds}{\Gamma(\gamma_2)}\right) + |\lambda_1|\sum_{j=1}^{m_1}|a_{j1}|\int_0^{\xi_{j_1}}\frac{\left(\xi_{j_1}-s\right)^{\alpha_1-1}ds}{\Gamma(\alpha_1)}\right]\right\} \\
={}& \Lambda_3\left(l_1\|x_1\| + l_2\|x_2\| + F_1\right) + \Lambda_1\|x_1\| \\
={}& (\Lambda_3 l_1 + \Lambda_1)\|x_1\| + \Lambda_3 l_2\|x_2\| + \Lambda_3 F_1 \\
\le{}& (\Lambda_3 l_1 + \Lambda_1)r_0 + \Lambda_3 l_2 r_0 + \Lambda_3 F_1 = N_1 r_0 + \Lambda_3 F_1.
\end{aligned} \tag{66}$$

Likewise,

$$\begin{aligned}
|V(x_1,x_2)(t)| &\le (\Lambda_4 j_2 + \Lambda_2)r_0 + \Lambda_4 j_1 r_0 + \Lambda_4 F_2 \\
&= N_2 r_0 + \Lambda_4 F_2.
\end{aligned} \tag{67}$$

Therefore, we obtain

$$\begin{aligned}
\|T(x_1,x_2)\| = \|U(x_1,x_2)\| + \|V(x_1,x_2)\| &\le (N_1+N_2)r_0 \\
&+ \Lambda_3 F_1 + \Lambda_4 F_2 < r_0.
\end{aligned} \tag{68}$$

Next, we shall show that the operator T is a contraction operator on $[0,1]$. Indeed, for any distinct two pairs (x_1,x_2), $(x_1',x_2') \in X_1 \times X_2$, we see that

$$\begin{aligned}
\left|U(x_1,x_2)(t)-U(x_1',x_2')(t)\right| \le & \int_0^t \frac{(t-s)^{\alpha_1+\beta_1-1}\left|f_1(s,x_1,x_2)-f_1(s,x_1',x_2')\right|\mathrm{d}s}{\Gamma(\alpha_1+\beta_1)} \\
& + \frac{t^{\alpha_1+1}}{S(\alpha_1+1)}\sum_{j=1}^{m_1}\left|a_{j1}\right|\int_0^{\xi_{j_1}} \frac{\left(\xi_{j_1}-s\right)^{\alpha_1+\beta_1-1}\left|f_1(s,x_1,x_2)-f_1(s,x_1',x_2')\right|ds}{\Gamma(\alpha_1+\beta_1)} \\
& + \left|\lambda_1\right|\int_0^t \frac{(t-s)^{\alpha_1-1}\left|x_1(s)-x_1'(s)\right|\mathrm{d}s}{\Gamma(\alpha_1)} + S_1(t)\int_0^{\eta_1} \frac{\rho_1^{1-\gamma_1}\left(\eta_1^{\rho_1}-s^{\rho_1}\right)^{\gamma_1-1}s^{\rho_1-1}\left|x_1(s)-x_1'(s)\right|\mathrm{d}s}{\Gamma(\gamma_1)} \\
& + \frac{t^{\alpha_1+1}}{S(\alpha_1+1)}\left[\left|\mu_1\right|\left|\left(\frac{1-\gamma_2}{B(\gamma_2)}\left|x_1(\eta_2)-x_1'(\eta_2)\right| + \frac{\gamma_2}{B(\gamma_2)}\int_0^{\eta_2} \frac{(\eta_2-s)^{\gamma_2-1}\left|x_1(s)-x_1'(s)\right|\mathrm{d}s}{\Gamma(\gamma_2)}\right)\right.\right. \\
& \left. + \left|\lambda_1\right|\sum_{j=1}^{m_1}\left|a_{j1}\right|\int_0^{\xi_{j_1}} \frac{\left(\xi_{j_1}-s\right)^{\alpha_1-1}\left|x_1(s)-x_1'(s)\right|\mathrm{d}s}{\Gamma(\alpha_1)}\right], \\
\le & \Lambda_3\left(l_1\left\|x_1-x_1'\right\| + l_2\left\|x_2-x_2'\right\|\right) + \Lambda_1\left\|x_1-x_1'\right\| \\
= & \left(l_1\Lambda_3+\Lambda_1\right)\left\|x_1-x_1'\right\| + l_2\Lambda_3\left\|x_2-x_2'\right\|,
\end{aligned} \tag{69}$$

obviously, it gives us

$$\left\|U(x_1,x_2)(t)-U(x_1',x_2')\right\| \le N_1\left(\left\|x_1-x_1'\right\| + \left\|x_2-x_2'\right\|\right). \tag{70}$$

In a similar technique, we can also have

$$\left\|V(x_2,x_2')(t)-V(x_1,x_1')\right\| \le N_2\left(\left\|x_1-x_1'\right\| + \left\|x_2-x_2'\right\|\right). \tag{71}$$

From (70) and (71), we obtain

$$\left\|T(x_2,x_2')-T(x_1,x_1')\right\| \le (N_1+N_2)\left(\left\|x_1-x_1'\right\| + \left\|x_2-x_2'\right\|\right). \tag{72}$$

Since the sum of both N_1 and N_2 is strictly less than one, we can say that the operator T satisfies a contraction criteria. Banach's fixed-point theorem ensures that the operator T has a unique fixed point. Hence (1) and (2) have a unique solution on $[0,1]$. □

Example 3. Consider the previous example of the coupled system of nonlinear fractional Langevin equation subject to both the nonlocal integro-multipoint of Atangana–Baleanu type the and nonlocal integral of Katugampola type as boundary value conditions.

$$\begin{cases}
{}^{C}D^{5/3}\left({}^{C}D^{1/3}+\dfrac{1}{100}\right)x(t) = f(t,x,y), 0<t<1, \\
{}^{C}D^{8/7}\left({}^{C}D^{1/7}+\dfrac{1}{108}\right)y(t) = g(t,x,y), 0<t<1, \\
x(0)=0, \quad {}^{C}D^{1/3}x(0) = \Gamma\left(\dfrac{4}{3}\right)^{3} I^{3/10}x\left(\dfrac{1}{107}\right), \quad \dfrac{1}{3}x\left(\dfrac{1}{4}\right)+\dfrac{1}{4}x\left(\dfrac{3}{5}\right)+\dfrac{1}{4}x\left(\dfrac{5}{6}\right) = \dfrac{1}{102}{}^{AB}I^{1/10}x\left(\dfrac{1}{120}\right), \\
y(0)=0, \quad {}^{C}D^{1/7}y(0) = \Gamma\left(\dfrac{8}{7}\right)^{1/3} I^{5/12}y\left(\dfrac{1}{108}\right), \quad \dfrac{1}{8}y\left(\dfrac{1}{3}\right)+\dfrac{3}{8}y\left(\dfrac{2}{3}\right)+\dfrac{1}{4}y\left(\dfrac{4}{5}\right) = \dfrac{4}{110}{}^{AB}I^{1/12}y\left(\dfrac{1}{140}\right).
\end{cases} \tag{73}$$

Clearly,

$$\alpha_1 = \frac{1}{3}, \alpha_2 = \frac{1}{7}, \beta_1 = \frac{5}{3}, \beta_2 = \frac{8}{7}, \lambda_1 = \frac{1}{100}, \lambda_2 = \frac{1}{108}, \rho_1 = 3, \rho_2 = \frac{1}{3}, \mu_1 = \frac{1}{102}, \mu_2 = \frac{1}{110}, \gamma_1 = \frac{3}{10},$$

$$\gamma_2 = \frac{1}{10}, \gamma_3 = \frac{5}{12}, \gamma_4 = \frac{1}{12}, \eta_1 = \frac{1}{107}, \eta_2 = \frac{1}{120}, \eta_3 = \frac{1}{108}, \eta_4 = \frac{1}{140}, a_1 = \frac{1}{3}, a_2 = \frac{1}{4}, a_3 = \frac{1}{4},$$

$$b_1 = \frac{1}{8}, b_2 = \frac{3}{8}, b_3 = \frac{1}{4}, \xi_{11} = \frac{1}{4}, \xi_{2_1} = \frac{3}{5}, \xi_{3_1} = \frac{5}{6}, \xi_{1_2} = \frac{1}{3}, \xi_{2_2} = \frac{2}{3}, \xi_{3_2} = \frac{4}{5}, B\left(\frac{1}{10}\right) = \frac{999}{1000}, B\left(\frac{1}{12}\right) = \frac{998}{1000},$$

$$f(t,x,y) = \frac{(t+1)^2|x(t)|}{\left(20(t+2)^2+2\right)(13|x(t)|+1)} + \frac{|y(t)|\sin^4 y(t)}{\left(24(t+3)^2+1\right)(8|y(t)|+5)}, \tag{74}$$

$$g(t,x,y) = \frac{(t+1)^3|x(t)|}{\left(1+18(t+2)^3\right)(7|x(t)|+5)} + \frac{|y(t)|\cos^2 t^2 \cos y(t)}{\left(1+14(t+2)^2\right)(12|y(t)|+2)},$$

$$|f(t,x,y)| \le \frac{(t+1)^2}{260(t+2)^2+26} + \frac{1}{192(t+3)^2+8}, |g(t,x,y)| \le \frac{(t+1)^3}{126(t+2)^3+7} + \frac{1}{168(t+2)^2+12},$$

$$\Phi(t) = \frac{(t+1)^2}{260(t+2)^2+26} + \frac{1}{192(t+3)^2+8}, \Psi(t) = \frac{(t+1)^3}{126(t+2)^3+7} + \frac{1}{168(t+2)^2+12}.$$

Based on Theorem 3, we can rewrite $f(t,x,y)$ and $g(t,x,y)$ as follows:

$$\left|f(t,x_2,y_2) - f(t,x_1,y_1)\right| \le \frac{1}{1066}\left|x_2 - x_1\right| + \frac{1}{1736}\left|y_2 - y_1\right|,$$
$$\left|g(t,x_2,y_2) - g(t,x_1,y_1)\right| \le \frac{1}{1015}\left|x_2 - x_1\right| + \frac{1}{684}\left|y_2 - y_1\right|. \tag{75}$$

Obviously, $l_1 = 1/1066$, $l_2 = 1/1736$, $j_1 = 1/1015$, $j_2 = 1/684$, $N_1 \approx 0.0657$, and $N_2 \approx 0.5235$. We conclude that our example has a unique solution on $[0,1]$ since $N_1 + N_2 \approx 0.5892 < 1$.

4. Conclusion

In this research paper, we have proven the existence and uniqueness of solutions for the coupled system of nonlinear fractional Langevin equations with multipoint and nonlocal integral boundary value conditions by selecting $0 < \alpha < 1 < \beta < 2$. Boundary value conditions have been chosen as two types of fractional integrals as we have shown in (2) for which have never been used together before in any article as far as we know. Existence of solutions have been shown by Krasnoselskii's theorem and O'Regan's theorem, and uniqueness solutions have been investigated by Banach's fixed-point theorem. Examples have been supported in order to demonstrate all theorems very well. Results of this paper are not new in giving configuration, but also provide us new cases related with the choice of the parameters involving in the given problem. For example, the results associated with nonperiodic and nonlocal multipoint nonclassical integral boundary conditions follow by considering $\gamma > 1$ of this problem. In case of $a_{j_1} = a_{j_2} = 0$ for all $j = 1, 2, \cdots$, the results change with respect to the boundary conditions:

$$\begin{cases} x_1(0) = 0, \quad {}^cD^{\alpha_1} x_1(0) = \Gamma(\alpha_1 + 1)^{\rho_1} I^{\gamma_1} x_1(\eta_1), \mu_1 {}^{AB}I^{\gamma_2} x_1(\eta_2) = 0, \\ x_2(0) = 0, \quad {}^cD^{\alpha_2} x_2(0) = \Gamma(\alpha_2 + 1)^{\rho_2} I^{\gamma_3} x_2(\eta_3), \mu_2 {}^{AB}I^{\gamma_4} x_2(\eta_4) = 0. \end{cases} \tag{76}$$

Other results can be considered if we take $\mu_1 = \mu_2 = 0$, then the boundary conditions will be

$$\begin{cases} x_1(0) = 0, \quad {}^cD^{\alpha_1} x_1(0) = \Gamma(\alpha_1 + 1)^{\rho_1} I^{\gamma_1} x_1(\eta_1), \quad \sum_{j=1}^{m_1} a_{j_1} x_1(\xi_{j_1}) = 0, \\ x_2(0) = 0, \quad {}^cD^{\alpha_2} x_2(0) = \Gamma(\alpha_2 + 1)^{\rho_2} I^{\gamma_3} x_2(\eta_3), \quad \sum_{j=1}^{m_2} a_{j_2} x_2(\xi_{j_2}) = 0. \end{cases} \tag{77}$$

In the future, in the case of obtaining theorems related with the existence and uniqueness of solutions under certain boundary conditions, both how can they be used to prove existence and uniqueness of solutions to the given problem and what are the conditions have to be considered.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] J. Singh, D. Kumar, and Sushila, "On the solutions of fractional reaction-diffusion equations," *Le Matematiche*, vol. 68, no. 1, pp. 23–32, 2013.

[2] J. Singh, D. Kumar, Sushila, and S. Gupta, "Application of homotopy perturbation transform method to linear and nonlinear space-time fractional reaction-diffusion equations," *Journal of Mathematics and Computer Science*, vol. 5, no. 1, pp. 40–52, 2012.

[3] J. Singh, D. Kumar, and Sushila, "Application of homotopy analysis transform method to fractional biological population model," *Romanian Reports in Physics*, vol. 65, no. 1, pp. 63–75, 2013.

[4] D. Kumar, J. Singh, S. Kumar, Sushila, and B. P. Singh, "Numerical computation of nonlinear shock wave equation of fractional order," *Ain Shams Engineering Journal*, vol. 6, no. 2, pp. 605–611, 2015.

[5] J. Singh, D. Kumar, and D. Baleanu, "New aspects of fractional biswas-milovic model with mittag-leffler law," *Mathematical Modelling of Natural Phenomena*, vol. 14, no. 3, p. 303, 2019.

[6] A. Goswami, J. Singh, D. Kumar, and Sushila, "An efficient analytical approach for fractional equal width equations describing hydro-magnetic waves in cold plasma," *Physica A: Statistical Mechanics and Its Applications*, vol. 524, pp. 563–575, 2019.

[7] D. Kumar, J. Singh, M. Al Qurashi, and D. Baleanu, "A new fractional SIRS-SI malaria disease model with application of vaccines, anti-malarial drugs, and spraying," *Advances in Difference Equations*, vol. 2019, p. 278, 2019.

[8] J. Singh, D. Kumar, D. Baleanu, and S. Rathore, "On the local fractional wave equation in fractal strings," *Mathematical Methods in the Applied Sciences*, vol. 42, no. 5, pp. 1588–1595, 2019.

[9] D. Kumar, J. Singh, and D. Baleanu, "On the analysis of vibration equation involving a fractional derivative with Mittag-Leffler law," *Mathematical Methods in the Applied Sciences*, vol. 43, no. 1, pp. 443–457, 2019.

[10] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, USA, 1999.

[11] B. Ahmad and J. J. Nieto, "Boundary value problems for a class of sequential integro-differential equations of fractional order," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 149659, 8 pages, 2013, http://dx.doi.org/10.1155/2013/149659.

[12] J. Tariboon, S. K. Ntouyas, and P. Thiramanus, "Riemann-liouville fractional differential equations with hadamard fractional integral conditions," *International Journal of Applied Mathematics and Statistics*, vol. 54, no. 1, pp. 119–134, 2016.

[13] J. d. J. Rubio, "Robust feedback linearization for nonlinear processes control," *ISA Transactions*, vol. 74, pp. 155–164, 2018.

[14] J. J. Rubio, G. Ochoa, D. Mujica, and E. Garcia-Trinidad, "Structure regulator for the perturbations attenuation in a quadrotor," *IEEE Access*, vol. 7, no. 1, pp. 138244–138252, 2019.

[15] J. Kumar, V. Kumar, and K. Rana, "Design of robust fractional order fuzzy sliding mode PID controller for two link robotic manipulator system," *Journal of Intelligent & Fuzzy Systems*, vol. 35, no. 5, pp. 5301–5315, 2018.

[16] J. J. Rubio, A. Aguilar, J. A. Meda-Campana, G. Ochoa, R. Balcazar, and J. Lopez, "An electricity generator based on the interaction of static and dynamic magnets," *IEEE Transactions on Magnetics*, vol. 55, no. 8, p. 8204511, 2019.

[17] S. K. Ntouyas and M. Obaid, "A coupled system of fractional differential equations with non-local integral boundary conditions," *Advances in Difference Equations*, vol. 130, 2012.

[18] K. Shah, A. Ali, and R. A. Khan, "Degree theory and existence of positive solutions to coupled systems of multi-point boundary value problems," *Boundary Value Problems*, vol. 2016, no. 1, p. 43, 2016.

[19] K. Shah and R. A. Khan, "Existence and uniqueness of positive solutions to a coupled system of nonlinear fractional order differential equations with anti periodic boundary conditions," *Differential Equations & Applications*, vol. 7, no. 2, pp. 245–262, 2015.

[20] WT. Coffey, YP. Kalmykov, and JT. Waldron, *The Langevin Equation*, World Scientific, Singapore, 2nd edition, 2004.

[21] S. C. Lim, M. Li, and L. P. Teo, "Langevin equation with two fractional orders," *Physics Letters A*, vol. 372, no. 42, pp. 6309–6320, 2008.

[22] A. Lozinski, R. G. Owen, and T. N. Philips, "The Langevin and Fokker-Planck equations in polymer rheology," in *Handbook of Numerical Analysis*, vol. 16, pp. 211–303, Elsevier, Amsterdam, Netherlands, 2011.

[23] J. Tariboon, S. K. Ntouyas, and C. Thaiprayoon, "Nonlinear Langevin equation of hadamard-caputo type fractional derivatives with non-local fractional integral conditions," *Advances in Mathematical Physics*, vol. 2014, Article ID 372749, p. 15, 2014.

[24] A. Salem, F. Alzahrani, and L. Almaghamsi, "Fractional Langevin equations with nonlocal integral boundary conditions," *Mathematics*, vol. 7, no. 5, pp. 402–411, 2019.

[25] A. Salem and B. Alghamdi, "Multi-point and anti-periodic conditions for generalized Langevin equation with two fractional orders," *Fractal and Fractional*, vol. 3, no. 4, pp. 1–14, 2019.

[26] W. Sudsutad, S. K. Ntouyas, and J. Tariboon, "System of fractional Langevin equations of Riemann-Liouville and hadamard types," *Advances in Difference Equations*, vol. 2015, p. 235, 2015.

[27] A. Atangana and D. Baleanu, "New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model," *Thermal Science*, vol. 20, no. 2, pp. 763–769, 2016.

[28] D. Kumar, J. Singh, and D. Baleanu, "Analysis of regularized long-wave equation associated with a new fractional operator with Mittag-Leffler type kernel," *Physica A: Statistical Mechanics and its Applications*, vol. 492, pp. 155–167, 2018.

[29] S. K. Ntouyas, J. Tariboon, and C. Thaiprayoon, "On the nonlocal Katugampola fractional integral conditions for fractional Langevin equation," *Advances in Difference Equations*, vol. 2015, p. 374, 2015.

[30] A. B. Malinowska, T. Odzijewicz, and D. F. M. Torres, *Advanced Methods in Fractional Calculus of Variation*, Springer, Berlin, Germany, 2015.

[31] C. Thaiprayoon, S. K. Ntouyas, and J. Tariboon, "On systems of fractional Langevin equations of riemann-liouville type with generalization non-local fractional integral boundary conditions," *Journal of Computational Analysis and Applications*, vol. 27, pp. 723–737, 2019.

[32] F. Jarad, T. Abdeljawad, and Z. Hammouch, "On a class of ordinary differential equations in the frame of AtanganaBaleanu fractional derivative," *Chaos, Solitons and Fractals*, vol. 117, p. 1620, 2018.

[33] A. A. Kilbas, H. H. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, Netherlands, 2006.

[34] U. N. Katugampola, "A new approach to generalized fractional fractional derivatives," *Bulletin of Mathematical Analysis and Applications*, vol. 6, pp. 1–15, 2014.

[35] D. Baleanu and A. Fernandez, "On some new properties of fractional derivatives with Mittag-Leffler kernel," *Communications in Nonlinear Science and Numerical Simulation*, vol. 59, pp. 444–462, 2018.

[36] D. O'Regan, "Fixed-point theory for the sum of two operators," *Applied Mathematics Letters*, vol. 9, pp. 1–8, 1996.

[37] A. Krasnoselskii, "Two remarks on the method of successive approximations," *Uspekhi Matematicheskikh Nauk*, vol. 10, pp. 123–127, 1955.