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Research Article

Optimal Investment Strategy under the CEV Model with Stochastic Interest Rate

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Interest rate is an important macrofactor that affects asset prices in the financial market. As the interest rate in the real market has the property of fluctuation, it might lead to a great bias in asset allocation if we only view the interest rate as a constant in portfolio management. In this paper, we mainly study an optimal investment strategy problem by employing a constant elasticity of variance (CEV) process and stochastic interest rate. The assets of investment for individuals are supposed to be composed of one risk-free asset and one risky asset. The interest rate for risk-free asset is assumed to follow the Cox–Ingersoll–Ross (CIR) process, and the price of risky asset follows the CEV process. The objective is to maximize the expected utility of terminal wealth. By applying the dual method, Legendre transformation, and asymptotic expansion approach, we successfully obtain an asymptotic solution for the optimal investment strategy under constant absolute risk aversion (CARA) utility function. In the end, some numerical examples are provided to support our theoretical results and to illustrate the effect of stochastic interest rates and some other model parameters on the optimal investment strategy.

1. Introduction

The optimal investment strategy problems, as a critical part of portfolio management with behaviors, are studied by [1, 2], in which the stochastic control method is used and some analytical solutions are provided. In the model of [2], it supposes that individuals allocate their wealths between one risky asset and one risk-free asset and further search an optimal consumption rate to maximize the total expectation on the discounted utility of the consumption. Inspired by the work of [2], many scholars have made plentiful research studies on the extensions and applications for the optimal investment problems. For instances, [3] consider one more factor, borrowing constraints, and they analyze an investment decision for a single agent by using the dynamic programming principle (DPP) method to derive an analytical solution for the objective function with CRRA utility. Vila and Zariphopoulou [4] study a similar problem, but they develop a new method to approach the viscosity solution of Hamilton–Jacobi–Bellman (HJB) equation. Thus, they provide a wider way to analyze investment strategy problems than before. In [5–7], the stochastic interest rate is involved and the investment strategy with infinite time horizon is discussed by applying a method with lower and upper solutions. Moreover, the existence of the solution of HJB equation is also verified. [8–10] consider the effect factor of transaction costs in investment problem. Zhang and Rong [11] consider the optimal investment strategies for defined contribution (DC) pension with an affine interest rate. By applying the HJB equation, dual theory, and Legendre transformation, they find the explicit solutions for the CRRA and CARA utility functions, respectively. Zhang [12] considers a continuous-time dynamic mean-variance portfolio selection problem of DC pension funds with stochastic salary. An explicit solution with a closed form for the optimal investment portfolio as well as the efficient frontier is obtained. In all the abovementioned models, it is assumed that stock prices follow a geometric Brownian motion

(GBM), which implies that the volatility coefficient of risky assets is just a constant in the model.

Volatility is a very important factor that affects the portfolio selection. In the real financial market, volatility always fluctuates with the spread of many kinds of information, especially in the bear market. Therefore, if the volatility is regarded as a constant in the process of risky assets, then it would not accurately reflect the fluctuation of risky asset in time, which may lead to the underestimation of the risk of risky assets. Thereby, we must consider volatility as a stochastic process and we employ a constant elasticity of variance (CEV) process to cover the dynamic volatility. But in the models of [11, 12], volatility is viewed as a constant, which is a special case in our model. Therefore, the results of optimal portfolio selection of our model cover those of the models in [11, 12]. As a diffusion process for European option pricing, a CEV process is firstly proposed in [13] and it is a natural extension of the geometric Brownian motion. The advantage of the CEV process covers that the volatility of such a model has correlation with risky asset prices and can explain volatility smile efficiently. Many literatures provide detailed analyses on the option pricing formula under the CEV model (see [14–17], etc.). In recent years, the CEV process has been employed to pension plans and also used to search optimal investment strategy under different utility functions (see [18–25]).

Unfortunately, in these works no one considers to employ the CEV process and stochastic interest rate to search optimal investment strategy. In other words, only the constant interest rate is involved in these models. One of important reasons is that the CEV process combining with the stochastic interest rate will make it very difficult to obtain the analytic solution for the optimal investment strategy. However, in real market the interest rate is not a constant, but a process with fluctuation. The volatility of interest rate is also an important source of market risks. In addition, the interest rate is a critical macrofactor influencing the prices of various financial assets. Thereby, if we just simplify the interest rate as a constant in the optimal investment problem, it might underestimate some risks in the portfolio management and lead to a big error in the asset allocation. Consequently, it is necessary for us to involve the stochastic interest rate in the model of investment problems.

In this paper, we assume that individuals are allowed to invest one risk-free asset and one risky asset, such as a stock. Considering that the interest rate has the property of mean reverting and positive values, we assume that the interest rate follows Cox–Ingersoll–Ross (CIR) process. Moreover, the price of risky asset is supposed to follow the CEV process. Our optimal objective function is to maximize the expected utility of terminal wealth. Unfortunately, it is very hard to directly derive the analytic solution of such an objective function. To reduce the difficulty, we employ the dual method to simplify the original problem. Meanwhile, Legendre transformation is also applied as a critical technique in the dual method. Even the dual method is used, and the equation after Legendre transformation is still a nonlinear problem. Fortunately, we find an asymptotic technique to approach an asymptotic solution of the dual problem. This is the innovation of our paper and the main difference between our paper and [11], in which asymptotic technique is not mentioned. This is the main contribution of our paper. In the end, some numerical examples are illustrated to support our theoretical results and to test the effect of stochastic interest rates on the optimal investment strategy.

The rest of this paper is organized as follows. In Section 2, we introduce some continuous process for interest rate and risky asset. Furthermore, an optimal investment problem is put forward. In Section 3, we derive a HJB equation by applying dynamic programming principle, and then transform it into its dual one by using Legendre transformation. In Section 4, we obtain the asymptotic solution for the optimal investment strategy with CARA utility. In Section 5, we provide some numerical examples to support our theoretical results.

2. Model Formulations

In this paper, we consider a simple portfolio, which is just composed of two financial assets. One is a risk-free asset and the other one is a risky asset. We denote the price of the risk-free asset at time t by $S_0(t)$, whose dynamics is expressed as

$$\mathrm{d}S_0(t) = r(t)S_0(t)\mathrm{d}t, \quad \text{with } S(0) = S_0 > 0, \tag{1}$$

where $r(t)$ is an interest rate, which follows the CIR process:

$$\mathrm{d}r(t) = (k_1 - k_2 r(t))\mathrm{d}t + \delta\sqrt{r(t)}\mathrm{d}W_1(t), \quad \text{with } r(0) = r_0 > 0, \tag{2}$$

where k_1, k_2, and δ are positive constants, and the Feller condition is assumed to be satisfied, i.e., $2k_1 > \delta^2$ holds.

The price $S(t)$ of the risky asset is supposed to follow the CEV process as follows:

$$\mathrm{d}S(t) = S(t)\cdot\left(\mu\mathrm{d}t + \sigma S^{\theta}(t)\mathrm{d}W_2(t)\right), \tag{3}$$

where μ is an expected instantaneous rate of return, σ is the instantaneous volatility, and θ is the elasticity parameter. Both $W_1(t)$ and $W_2(t)$ in (2) are standard Brownian motions. Furthermore, we assume that the Brownian motion $W_1(t)$ is correlated with $W_2(t)$ and that the instantaneous correlation coefficient is ρ, i.e., $\mathrm{d}W_1(t)\mathrm{d}W_2(t) = \rho\mathrm{d}t$.

Let $V(t)$ denote the individual wealth at time t, $\pi(t)$ be the proportion of the wealth invested in risky asset. Then, the proportion of wealth invested in risk-free asset is $1 - \pi(t)$. SSuppose that the events of short selling and borrowing at the risk-free interest rate r are allowed. The wealth process $V(t)$ satisfies the following stochastic process:

$$\begin{cases} dV(t) = [V(t)(\pi(t)(\mu - r(t)) + r(t))]\mathrm{d}t + \pi(t)\sigma S^{\theta}(t) \\ V(t)\mathrm{d}W_2(t), \\ V(0) = V_0 > 0, \end{cases} \tag{4}$$

where V_0 denotes an initial wealth.

Definition 1. A strategy $\pi(t)$ is said to be admissible if the following conditions are satisfied:

(i) $\pi(t)$ is $\mathcal{F}_t$-measurable and satisfies $\int_0^T \pi(t)\mathrm{d}t < \infty$, $T > 0$, almost surely

(ii) $\mathbb{E}\left(\int_0^T (\pi(t)\sigma)^2 \mathrm{d}t \right) < \infty$

(iii) The stochastic differential equation (2) has a unique strong solution corresponding to any $\pi(t)$

Assume that the set of all admissible strategies is denoted by $\mathcal{A} = \{\pi(t), 0 \le t \le T\}$. The aim of individuals is to maximize the expected utility of terminal wealth in the finite horizon as follows:

$$\sup_{\pi(t)\in\mathcal{A}} \mathbb{E}[\mathrm{U}(\mathrm{V}(\mathrm{T}))], \tag{5}$$

where $U(\cdot)$ is a utility function. In this paper, the constant absolute risk aversion (CARA) utility is considered and it has the expression as

$$U(x) = -\frac{e^{-qx}}{q}. \tag{6}$$

3. Optimization Process

In this section, we mainly discuss how to find a solution for the optimal objective function in Section 2. Firstly, we define a value function $K(t, s, r, v)$ as follows:

$$K(t,s,r,v) \coloneqq \sup_{\pi(t)\in\mathcal{A}} \mathbb{E}[U(V(T)) \mid S(t) = s, V(t) = v, r(t) = r], \quad 0 \le t \le T, \tag{7}$$

with boundary condition $K(T, s, r, v) = U(v)$. The HJB equation of the optimal problem (7) can be given by

$$\begin{aligned} &K_t + \mu s K_s + \frac{1}{2}\sigma^2 s^{2\theta+2} K_{ss} + rvK_v + (k_1 - k_2 r)K_r \\ &+ \frac{1}{2} r\delta^2 K_{rr} + \rho\sigma\delta\sqrt{r} s^{\theta+1} K_{sr} + \sup_{\pi}\left\{ \frac{1}{2}\pi^2\sigma^2 s^{2\theta} v^2 K_{vv} \right. \\ &\left. + \pi\left[(\mu - r)vK_v + s^{2\theta+1}\sigma^2 v K_{vs} + \rho s^{\theta} v\sigma\delta\sqrt{r}\, K_{vr} \right] \right\} = 0, \end{aligned} \tag{8}$$

where $K_t, K_s, K_v, K_r, K_{rr}, K_{vv}, K_{ss}, K_{vr}, K_{vs}$, and K_{rs} denote different partial derivatives with respect to parameters t, s, v, and r.

The first order maximal condition of (8) is

$$\pi^*(t) = -\frac{(\mu - r)K_v + \sigma^2 s^{2\theta+1} v K_{vs} + \rho\sigma v\delta\sqrt{r} s^{\theta} K_{vr}}{v^2\sigma^2 s^{2\theta} K_{vv}}. \tag{9}$$

Putting (9) into (8), it follows that

$$\begin{aligned} &K_t + \mu s K_s + \frac{1}{2}\sigma^2 s^{2\theta+2} K_{ss} + rvK_v + (k_1 - k_2 r)K_r + \frac{1}{2}\delta^2 r K_{rr} \\ &+ \rho\sigma\delta\sqrt{r} s^{\theta+1} K_{sr} - \frac{\left[(\mu - r)K_v + \sigma^2 s^{2\theta+1} K_{vs} + \rho\sigma\delta\sqrt{r}\, s^{\theta} K_{vr} \right]^2}{2\sigma^2 s^{2\theta} K_{vv}} = 0. \end{aligned} \tag{10}$$

For problem (10), it is very hard to obtain its analytic solution directly. Following the work of [26], by applying Legendre transformation and dual method, firstly, we can search a dual problem of (10). Now, we define Legendre transformation for K as

$$\widehat{K}(t,s,r,z) \coloneqq \sup_{v>0}\{K(t,s,r,v) - zv \mid 0 < t < T\}, \tag{11}$$

where $z > 0$ denotes the dual variable of v. The optimal value of v is denoted by $g(t, s, r, z)$ as

$$g(t,s,r,z) \coloneqq \inf_{v>0}\left\{ v \mid K(t,s,r,v) \ge zv + \widehat{K}(t,s,r,z), \quad 0 < t < T \right\}, \tag{12}$$

and this leads to

$$\widehat{K}(t,s,r,z) = K(t,s,r,g) - zg, \quad g(t,s,r,z) = v. \tag{13}$$

Consequently, the relationship of the derivatives for the value function K and the dual function $\widehat{K}$ can be given as follows:

$$\begin{aligned} &K_v = z,\ K_t = \widehat{K}_t,\ K_r = \widehat{K}_r,\ K_s = \widehat{K}_s, \\ &g = -\widehat{K}_z,\ K_{vv} = -\frac{1}{\widehat{K}_{zz}},\ K_{ss} = \widehat{K}_{ss} - \frac{\widehat{K}_{sz}^2}{\widehat{K}_{zz}}, \\ &K_{rr} = \widehat{K}_{rr} - \frac{\widehat{K}_{rz}^2}{\widehat{K}_{zz}},\ K_{vs} = -\frac{\widehat{K}_{sz}}{\widehat{K}_{zz}}, \\ &K_{vr} = -\frac{\widehat{K}_{rz}}{\widehat{K}_{zz}},\ K_{sr} = \widehat{K}_{sr} - \frac{\widehat{K}_{sz}\widehat{K}_{rz}}{\widehat{K}_{zz}}. \end{aligned} \tag{14}$$

In this paper, we use the function g as the dual function of K. Then, it would provide an easy way to compute optimal strategy.

At the terminal time T, $K(T, s, x, v) = U(v)$. Similarly, we define

$$\begin{aligned} &g(T,s,r,z) \coloneqq \inf_{v>0}\left\{ v \mid U(v) \ge zv + \widehat{K}(T,s,r,z) \right\}, \\ &\widehat{K}(t,s,r,z) \coloneqq \sup_{v>0}\{U(v) - zv \mid v \rangle 0\}. \end{aligned} \tag{15}$$

Then, it follows that

$$g(T,s,r,z) = \left(U'\right)^{-1}(z). \tag{16}$$

Substituting (14) into (10), we can obtain the following equation of the function $\widehat{K}$:

$$\begin{aligned} &\widehat{K}_t + \mu s\widehat{K}_s + \frac{1}{2}\sigma^2 s^{2\theta+2}\widehat{K}_{ss} + rgz + (k_1 - k_2 r)\widehat{K}_r \\ &+ \frac{1}{2}\delta^2 r\widehat{K}_{rr} - \frac{1}{2}\delta^2 r\left(1 - \rho^2\right)\frac{\widehat{K}_{rz}^2}{\widehat{K}_{zz}} + \rho\sigma\delta\sqrt{r} s^{\theta+1}\widehat{K}_{sr} \\ &- (\mu - r)sz\widehat{K}_{sz} + \frac{(\mu - r)^2 z^2}{2\sigma^2 s^{2\theta}}\widehat{K}_{zz} - \frac{(\mu - r)\rho\delta\sqrt{r} z}{\sigma s^{\theta}}\widehat{K}_{rz} = 0. \end{aligned} \tag{17}$$

For equality $g(t,s,r,z) = -\widehat{K}_z$, differentiating on its both sides with respect to the variables $t, s, r,$ and z, we achieve the first- and second-order partial derivatives as follows:

$$\widehat{K}_{tz} = -g_t,\ \widehat{K}_{sz} = -g_s,\ \widehat{K}_{ssz} = -g_{ss},\ \widehat{K}_{zz} = -g_z,$$

$$\widehat{K}_{srz} = -g_{sr},\ \widehat{K}_{zzz} = -g_{zz},\ \widehat{K}_{szz} = -g_{sz},\ \widehat{K}_{rzz} = -g_{rz},$$

$$\widehat{K}_{rz} = -g_r,\ \widehat{K}_{rrz} = -g_{rr},\ \frac{\widehat{K}_{rz}^2}{\widehat{K}_{zz}} = -\frac{g_r^2}{g_z}. \tag{18}$$

Similarly, differentiating both sides of equation (17) with respect to z and then using the relationships in (18), we can have that

$$\begin{aligned} &g_t + rsg_s + \frac{1}{2}\sigma^2 s^{2\theta+2} g_{ss} - rg + \left(\frac{(\mu-r)^2}{\sigma^2 s^{2\theta}} - r\right) z g_z \\ &+ \frac{(\mu-r)^2}{2\sigma^2 s^{2\theta}} z^2 g_{zz} - (\mu-r) sz g_{sz} + \rho\sigma\delta\sqrt{r}\, s^{\theta+1} g_{sr} \\ &+ (k_1 - k_2 r) g_r + \frac{1}{2}\delta^2 r g_{rr} - \frac{\rho\delta\sqrt{r}\,(\mu-r)}{\sigma s^{\theta}} (g_r + z g_{rz}) \\ &- \frac{1}{2} r\delta^2 \left(1-\rho^2\right)\left(\frac{2 g_r g_{rz}}{g_z} - \frac{g_r^2 g_{zz}}{g_z^2}\right) = 0, \end{aligned} \tag{19}$$

with the boundary condition

$$g(T,s,r,z) = \left(U'\right)^{-1}(z). \tag{20}$$

Note that from (14) and (18), it follows that

$$\begin{aligned} &K_v = z,\ K_{vs} = -\frac{\widehat{K}_{sz}}{\widehat{K}_{zz}},\ K_{vr} = -\frac{\widehat{K}_{rz}}{\widehat{K}_{zz}}, \\ &\widehat{K}_{zz} = -g_z,\ \widehat{K}_{sz} = -g_s,\ \widehat{K}_{rz} = -g_r,\ g = v. \end{aligned} \tag{21}$$

Thus, by applying the equalities above, the optimal strategy (9) can be written as

$$\begin{aligned} \pi_t^* &= -\frac{(\mu-r)K_v + \sigma^2 s^{2\theta+1} K_{vs} + \rho\sigma\delta\sqrt{r}\, s^{\theta} K_{vr}}{v\sigma^2 s^{2\theta} K_{vv}} \\ &= -\frac{(\mu-r)z + \sigma^2 s^{2\theta+1}\left(-\widehat{K}_{sz}/\widehat{K}_{zz}\right) + \rho\sigma\delta\sqrt{r}\, s^{\theta}\left(-\widehat{K}_{rz}/\widehat{K}_{zz}\right)}{v\sigma^2 s^{2\theta}\left(-1/\widehat{K}_{zz}\right)} \\ &= \frac{(\mu-r)z\widehat{K}_{zz} - \sigma^2 s^{2\theta+1}\widehat{K}_{sz} - \rho\sigma\delta\sqrt{r}\, s^{\theta}\widehat{K}_{rz}}{v\sigma^2 s^{2\theta}} \\ &= \frac{-(\mu-r)z g_z + \sigma^2 s^{2\theta+1} g_s + \rho\sigma\delta\sqrt{r}\, s^{\theta} g_r}{g\sigma^2 s^{2\theta}}. \end{aligned} \tag{22}$$

Remark 1. Notice that equation (10) of K has been transformed into (19) for the dual function g, and there exists a relationship of

$$\frac{\partial}{\partial z}\left(\frac{g_r^2}{g_z}\right) = \frac{2 g_r g_{rz}}{g_z} - \frac{g_r^2 g_{zz}}{g_z^2}. \tag{23}$$

Now, the problem is to solve (19) for the dual function g. For (19), we can conjecture the solution via variable transformation technique. Furthermore, it combines with (22), the optimal strategy can be obtained.

4. Asymptotic Solution

In this section, by applying variable transformation and asymptotic expansion approach, we try to obtain an asymptotic solution for (19). In this paper, the CARA utility function described by (6) is considered.

From (16) and (6), we have that

$$g(T,s,r,z) = -\frac{1}{q}\ln z. \tag{24}$$

Similar to [18], the form of a solution to (19) can be presented by

$$\begin{cases} g(t,s,r,z) = -\dfrac{1}{q}\left[h(t)(\ln z + m(t,y,r))\right] + n(t), \\ y = s^{-2\theta}, \end{cases} \tag{25}$$

with the boundary conditions, $h(T) = 1$, $n(T) = 0$, and $m(T,y,x) = 0$. Then, it follows that

$$\begin{aligned} &g_t = -\frac{1}{q}\left[h_t(\ln z + m(t,y,r)) + h(t) m_t\right] + n_t, \\ &g_s = \frac{2\theta}{q} s^{-2\theta-1} h(t) m_y,\ g_z = -\frac{h(t)}{qz},\ g_r = -\frac{h(t) m_r}{q}, \\ &g_{rr} = -\frac{h(t) m_{rr}}{q},\ g_{zz} = \frac{h(t)}{qz^2},\ g_{sz} = g_{rz} = 0, \\ &g_{ss} = -\frac{h(t)}{q}\left(4\theta^2 s^{-4\theta-2} m_{yy} + 2\theta(2\theta+1) s^{-2\theta-2} m_y\right), \\ &g_{sr} = \frac{2\theta}{q} s^{-2\theta-1} h(t) m_{yr}. \end{aligned} \tag{26}$$

Inputting the derivatives above in (19), the following equation can be obtained:

$$\begin{aligned} &[h_t - r h(t)]\ln z + [r n(t) - n_t] q + h(t)\Big[\left(L_0 + L_1 + L_2\right) m \\ &- \frac{1}{2}\delta^2 r\left(1-\rho^2\right)(m_r)^2\Big] = 0, \end{aligned} \tag{27}$$

where

$$L_0 m = (k_1 - k_2 r) m_r + \frac{1}{2}\delta^2 r m_{rr}, \tag{28}$$

$$L_1 m = m_t + \theta\left[(2\theta+1)\sigma^2 - 2ry\right] m_y + 2\theta^2\sigma^2 y m_{yy} + \frac{(\mu - r)^2}{2\sigma^2} y - r, \tag{29}$$

$$L_2 m = -2\rho\sigma\theta\delta\sqrt{r}\sqrt{y} m_{yr} - \frac{\rho\delta\sqrt{r}(\mu - r)}{\sigma}\sqrt{y} m_r. \tag{30}$$

Furthermore, we decompose (27) into the following three equations by setting the coefficients of terms $\ln z$, q and $h(t)$ to be zeros:

$$h_t - rh(t) = 0, \tag{31}$$

$$rn(t) - n_t = 0, \tag{32}$$

$$(L_0 + L_1 + L_2) m - \frac{1}{2} r\delta^2 \left(1 - \rho^2\right)(m_r)^2 = 0. \tag{33}$$

Considering that $h(T) = 1$ and $n(T) = 0$, the solution of (31) and (32) can be given by

$$\begin{aligned} h(t) &= e^{r(t-T)}, \\ n(t) &= 0. \end{aligned} \tag{34}$$

But equation (33) is a nonlinear partial differential equation, and it is very difficult to obtain the analytical solution. Fortunately, the asymptotic expansion approach is an effective method to solve such nonlinear problems. For instance, in [27, 28], it assumes that the volatility follows a slow-fluctuating process and the asymptotic formulas for option pricing under different stochastic volatility models are derived. Similarly, we try to find an asymptotic solution of (33) by a following slow-fluctuating process $r^\epsilon(t)$ to replace (2), in which $0 < \epsilon \ll 1$ is a small positive parameter:

$$\mathrm{d}r^\epsilon(t) = \epsilon(k_1 - k_2 r(t))\mathrm{d}t + \sqrt{\epsilon}\sqrt{r(t)}\mathrm{d}W_1(t). \tag{35}$$

Input (35) into (33), and replace $k_1 - k_2 r$ and $\sqrt{r}$ with $\epsilon(k_1 - k_2 r)$ and $\sqrt{\epsilon}\sqrt{r}$, respectively. Then, m^ϵ for (33) can be written as follows:

$$(\epsilon L_0 + L_1 + \sqrt{\epsilon} L_2) m^\epsilon - \frac{1}{2}\epsilon r\left(1 - \rho^2\right)(m_r^\epsilon)^2 = 0. \tag{36}$$

A solution to (36) is assumed to follow the form as

$$m^\epsilon(t, y, r) = m^{(0)}(t, y, r) + \sqrt{\epsilon} m^{(1)}(t, y, r) + \epsilon m^{(2)}(t, y, r). \tag{37}$$

Substituting (37) into (36), we obtain that

$$L_1 m^{(0)} + \sqrt{\epsilon}\left(L_1 m^{(1)} + L_2 m^{(0)}\right) + \epsilon\left[L_0 m^{(0)} + L_1 m^{(2)} + L_2 m^{(1)} - \frac{1}{2} r\left(1 - \rho^2\right)\left(m_r^{(0)}\right)^2\right] = 0. \tag{38}$$

Collecting the coefficients of terms with the same order in (38), three equations can be presented as follows. For the term of ϵ^0, it follows that

$$L_1 m^{(0)} = 0, \quad \text{with } m^{(0)}(T, y, r) = 0. \tag{39}$$

For the term of $\sqrt{\epsilon}$, from its coefficient we have that

$$L_1 m^{(1)} + L_2 m^{(0)} = 0, \quad \text{with } m^{(1)}(T, y, r) = 0. \tag{40}$$

For the term of ϵ, we can obtain the equation as

$$\begin{aligned} &L_0 m^{(0)} + L_1 m^{(2)} + L_2 m^{(1)} - \frac{1}{2} r\left(1 - \rho^2\right)\left(m_r^{(0)}\right)^2 = 0, \\ &\text{with } m^{(2)}(T, y, r) = 0. \end{aligned} \tag{41}$$

In the following lemmas, we will try to solve three equations in (39)–(41), respectively.

Lemma 1. *The solution for equation (39) can be given by*

$$m^{(0)}(t, y, r) = A(t, r) + B(t, r) y, \tag{42}$$

where

$$\begin{aligned} A(t, r) &= \left[\frac{(2\theta+1)(\mu - r)^2}{4r} - r\right](T - t) - \frac{2\theta + 1}{8r^2\theta}(\mu - r)^2 \cdot \left[1 - e^{2r\theta(t-T)}\right], \\ B(t, r) &= \sigma^{-2}\frac{(\mu - r)^2}{4r\theta}\left[1 - e^{2r\theta(t-T)}\right]. \end{aligned} \tag{43}$$

Proof. From (29), equation (39) can be rewritten as

$$m_t^{(0)} + \theta\left[(2\theta+1)\sigma^2 - 2ry\right] m_y^{(0)} + 2\theta^2\sigma^2 y m_{yy}^{(0)} + \frac{(\mu - r)^2}{2\sigma^2} y - r = 0, \tag{44}$$

with the boundary condition $m^{(0)}(T, y, r) = 0$. We try to use the following form to solve (44):

$$m^{(0)}(t, y, r) = A(t, r) + B(t, r) y, \tag{45}$$

with boundary conditions, $A(T, r) = B(T, r) = 0$.

Inserting (45) into (44), we have the following equation:

$$A_t + \theta(2\theta+1)\sigma^2 B(t, r) - r + y\left[B_t - 2r\theta B(t, r) + \frac{(\mu - r)^2}{2\sigma^2}\right] = 0. \tag{46}$$

By the merger of similar items y^0 and y, equality (46) can be decomposed into two equations:

$$B_t - 2r\theta B(t, r) + \frac{(\mu - r)^2}{2\sigma^2} = 0, \tag{47}$$

$$A_t + \theta(2\theta+1)\sigma^2 B(t, r) - r = 0. \tag{48}$$

By solving (47) and (48), we can obtain the solutions of them as follows:

$$A(t,r) = \left[\frac{(2\theta+1)(\mu-r)^2}{4r} - r\right](T-t) - \frac{2\theta+1}{8r^2\theta}(\mu-r)^2 \cdot \left[1 - e^{2r\theta(t-T)}\right],$$
$$B(t,r) = \sigma^{-2}\frac{(\mu-r)^2}{4r\theta}\left[1 - e^{2r\theta(t-T)}\right]. \tag{49}$$

Therefore, (42) and (44) are verified. □

Lemma 2. *The solution for equation (40) can be given by*

$$m^{(1)}(t,y,r) = c(t,r) + d(t,r)y^{1/2} + f(t,r)y + w(t,r)y^{3/2}, \tag{50}$$

where

$$f(t,r) = \frac{(\mu-r)^2}{4r\theta\sigma^2}\left(1 - e^{2r\theta(t-T)}\right), \tag{51}$$

$$\begin{aligned} c(t,r) &= \theta(2\theta+1)\sigma^2\int_t^T f(s,r)\mathrm{d}s - r(T-t),\\ w(t,r) &= -\frac{\rho\delta\sqrt{r}(\mu-r)}{\sigma}e^{3r\theta(t-T)}\int_t^T B_r e^{3r\theta(T-s)}\mathrm{d}s,\\ d(t,r) &= \frac{3}{2}\theta(3\theta+1)\sigma^2 e^{r\theta(t-T)}\int_t^T w(s,r)e^{r\theta(T-s)}\mathrm{d}s\\ &\quad - \frac{\rho\delta\sqrt{r}(\mu-r)}{\sigma}e^{r\theta(t-T)}\int_t^T A_r e^{r\theta(T-s)}\mathrm{d}s\\ &\quad - 2\delta\sqrt{r}\rho\theta\sigma e^{r\theta(t-T)}\int_t^T B_r e^{r\theta(T-s)}\mathrm{d}s. \end{aligned} \tag{52}$$

Proof. Equality (40) can be rewritten as follows:

$$m_t^{(1)} + \theta\left[(2\theta+1)\sigma^2 - 2ry\right]m_y^{(1)} + 2\theta^2\sigma^2 y m_{yy}^{(1)} + \frac{(\mu-r)^2}{2\sigma^2}y - r - 2\rho\sigma\theta\delta\sqrt{r}\sqrt{y}m_{yr}^{(0)} - \frac{\rho\delta\sqrt{r}(\mu-r)}{\sigma}\sqrt{y}m_r^{(0)} = 0. \tag{53}$$

For (53), we try to use the following form to obtain its solution:

$$m^{(1)}(t,y,r) = c(t,r) + d(t,r)y^{1/2} + f(t,r)y + w(t,r)y^{3/2}, \tag{54}$$

with boundary conditions, $c(T,r) = d(T,r) = f(T,r) = w(T,r) = 0$.

From (42), we have that

$$m_r^{(0)} = A_r + B_r y, \quad m_{yr}^{(0)} = B_r. \tag{55}$$

Inputting (54) and (55) into (53), it follows that

$$\begin{aligned} &c_t + \mathrm{d}_t y^{1/2} + f_t y + w_t y^{3/2} + \theta\left[(2\theta+1)\sigma^2 - 2ry\right]\left(\frac{1}{2}\mathrm{d}(t)y^{-1/2} + f(t,r) + \frac{3}{2}w(t,r)y^{1/2}\right)\\ &\quad + 2\theta^2\sigma^2 y\left[-\frac{1}{4}y^{-3/2}d(t,r) + \frac{3}{2}w(t,r)y^{-1/2}\right] + \frac{(\mu-r)^2}{2\sigma^2}y - r\\ &\quad - 2\rho\sigma\theta\delta\sqrt{r}\sqrt{y}B_r - \frac{\rho(\mu-r)\delta\sqrt{r}}{\sigma}\sqrt{y}\left(A_r + B_r y\right) = 0. \end{aligned} \tag{56}$$

Collecting (56) by the same orders of $y^{-1/2}$, $y^{1/2}$, y, and $y^{3/2}$, we can achieve the following equations:

$$\begin{aligned} f_t - 2\theta r f(t,r) + \frac{(\mu-r)^2}{2\sigma^2} &= 0,\\ c_t + \theta(2\theta+1)\sigma^2 f(t,r) - r &= 0,\\ w_t - 3\theta r w(t,r) + \frac{\rho\delta\delta\sqrt{r}(\mu-r)}{\sigma}B_r &= 0,\\ d_t - r\theta d(t,r) + \frac{3}{2}\theta(3\theta+1)\sigma^2 w(t,r) - \frac{\sqrt{r}(\mu-r)\rho\delta A_r}{\sigma} - 2\sqrt{r}\theta\rho\delta\sigma B_r &= 0. \end{aligned} \tag{57}$$

Then, the solutions of equations 58–61 can be obtained as follows, respectively:

$$\begin{aligned}
f(t,r) &= \frac{(\mu-r)^2}{4r\theta\sigma^2}\left(1-e^{2r\theta(t-T)}\right),\\
c(t,r) &= \theta(2\theta+1)\sigma^2\int_t^T f(s,r)\mathrm{d}s - r(T-t),\\
w(t,r) &= -\frac{\rho\delta\sqrt{r}\,(\mu-r)}{\sigma}e^{3r\theta(t-T)}\int_t^T B_r e^{3r\theta(T-s)}\mathrm{d}s,\\
d(t,r) &= \frac{3}{2}\theta(3\theta+1)\sigma^2 e^{r\theta(t-T)}\int_t^T w(s,r)e^{r\theta(T-s)}\mathrm{d}s\\
&\quad -\frac{\rho\delta\sqrt{r}\,(\mu-r)}{\sigma}e^{r\theta(t-T)}\int_t^T A_r e^{r\theta(T-s)}\mathrm{d}s\\
&\quad -2\sqrt{r}\rho\theta\sigma\delta e^{r\theta(t-T)}\int_t^T B_r e^{r\theta(T-s)}\mathrm{d}s.
\end{aligned} \tag{58}$$

This lemma is verified. □

Lemma 3. *The solution for (41) can be given by*

$$\begin{aligned}
m^{(2)}(t,y,r) &= C(t,r) + D(t,r)y^{1/2} + F(t,r)y + W(t,r)y^{3/2}\\
&\quad + Q(t,r)y^2,
\end{aligned} \tag{59}$$

where

$$\begin{aligned}
Q(t,r) &= -\frac{r}{2}\left(1-\rho^2\right)\delta^2 e^{4r\theta(t-T)}\int_t^T B_r^2 e^{4r\theta(T-s)}\mathrm{d}s\\
&\quad -\frac{\rho\delta\sqrt{r}\,(\mu-r)}{\sigma}e^{4r\theta(t-T)}\int_t^T w_r e^{4r\theta(T-s)}\mathrm{d}s,
\end{aligned} \tag{60}$$

$$W(t,r) = -\frac{\rho\delta\sqrt{r}\,(\mu-r)}{\sigma}e^{3r\theta(t-T)}\int_t^T f_r e^{3r\theta(T-s)}\mathrm{d}s, \tag{61}$$

$$\begin{aligned}
F(t,r) &= 2\theta(4\theta+1)\sigma^2 e^{2r\theta(t-T)}\int_t^T Q(s,r)e^{2r\theta(T-s)}\mathrm{d}s\\
&\quad + (k_1-k_2r)e^{2r\theta(t-T)}\int_t^T B_r e^{2r\theta(T-s)}\mathrm{d}s\\
&\quad - r\left(1-\rho^2\right)\delta^2 e^{2r\theta(t-T)}\int_t^T A_r B_r e^{2r\theta(T-s)}\mathrm{d}s\\
&\quad -\frac{\rho\delta\sqrt{r}\,(\mu-r)}{\sigma}e^{2r\theta(t-T)}\int_t^T d_r e^{2r\theta(T-s)}\mathrm{d}s\\
&\quad -3\rho\sigma\theta\delta\sqrt{r}e^{2r\theta(t-T)}\int_t^T w_r e^{2r\theta(T-s)}\mathrm{d}s\\
&\quad +\frac{1}{2}r\delta^2 e^{2r\theta(t-T)}\int_t^T B_{rr}e^{2r\theta(T-s)}\mathrm{d}s\\
&\quad +\frac{(\mu-r)^2}{4r\theta\sigma^2}\left(1-e^{2r\theta(t-T)}\right),
\end{aligned} \tag{62}$$

$$\begin{aligned}
D(t,r) &= \frac{3}{2}\theta(3\theta+1)\sigma^2 e^{r\theta(t-T)}\int_t^T W(s,r)e^{r\theta(T-s)}\mathrm{d}s\\
&\quad -\frac{\rho\delta\sqrt{r}\,(\mu-r)}{\sigma}e^{r\theta(t-T)}\int_t^T c_r e^{r\theta(T-s)}\mathrm{d}s\\
&\quad -2\sqrt{r}\theta\rho\sigma\delta e^{r\theta(t-T)}\int_t^T c_r f^{r\theta(T-s)}\mathrm{d}s,\\
C(t,r) &= \theta(2\theta+1)\sigma^2\int_t^T F(s,r)\mathrm{d}s + \left(k_1-k_2r\right)\int_t^T A_r\mathrm{d}s\\
&\quad -\frac{1}{2}r\left(1-\rho^2\right)\delta^2\int_t^T A_r^2\mathrm{d}s - \sqrt{r}\theta\rho\sigma\delta\int_t^T d_r\mathrm{d}s\\
&\quad +\frac{1}{2}r\delta^2\int_t^T A_{rr}\mathrm{d}s - r(T-t).
\end{aligned} \tag{63}$$

Proof. Based on the solutions of (39) and (40), (41) can be rewritten as

$$\begin{aligned}
&m_t^{(2)} + \theta\left[(2\theta+1)\sigma^2 - 2ry\right]m_y^{(2)} + 2\theta^2\sigma^2 y m_{yy}^{(2)} + \frac{(\mu-r)^2}{2\sigma^2}y - r\\
&\quad + (k_1-k_2r)m_r^{(0)} + \frac{1}{2}\delta^2 r m_{rr}^{(0)} - 2\rho\sigma\theta\delta\sqrt{r}\sqrt{y}m_{yr}^{(1)}\\
&\quad -\frac{\rho\delta\sqrt{r}\,(\mu-r)}{\sigma}\sqrt{y}m_r^{(1)} - \frac{1}{2}\delta^2 r\left(1-\rho^2\right)\left(m_r^{(0)}\right)^2 = 0.
\end{aligned} \tag{64}$$

We try the following form to solve the solution of (64):

$$\begin{aligned}
m^{(2)}(t,y,r) &= C(t,r) + D(t,r)y^{1/2} + F(t,r)y + W(t,r)y^{3/2}\\
&\quad + Q(t,r)y^2,
\end{aligned} \tag{65}$$

with boundary conditions

$$C(T,r) = D(T,r) = F(T,r) = W(T,r) = Q(T,r) = 0. \tag{66}$$

Considering that

$$m_r^{(0)} = A_r + B_r y, \quad m_{rr}^{(0)} = A_{rr} + B_{rr}y, \tag{67}$$

$$\begin{aligned}
m_r^{(1)} &= c_r + d_r y^{1/2} + f_r y + w_r y^{3/2}, \quad m_{yr}^{(1)} = \frac{1}{2}d_r y^{-1/2}\\
&\quad + f_r + \frac{3}{2}w_r y^{1/2}.
\end{aligned} \tag{68}$$

Inputting (69), (71), and (72) into (78), it follows that

$$
\begin{aligned}
&C_t + D_t y^{1/2} + F_t y + W_t y^{1/2} + Q_t y^2 + \theta\left[(2\theta+1)\sigma^2 - 2ry\right] \\
&\cdot\left(\frac{1}{2}D(t,r)y^{-1/2} + F(t,r) + \frac{3}{2}W(t,r)y^{1/2} + 2Q(t,r)y\right) \\
&+ 2\theta^2\sigma^2 y\left(-\frac{1}{4}D(t,r)y^{-3/2} - \frac{3}{4}W(t,r)y^{-1/2} + 2Q(t,r)\right) \\
&+ \left(k_1 - k_2 r\right)\left(A_r + B_r y\right) + \frac{1}{2}\delta^2 r\left(A_{rr} + B_{rr} y\right) \\
&- 2\rho\sigma\theta\delta\sqrt{r}\sqrt{y}\left(\frac{1}{2}d_r y^{-1/2} + f_r + \frac{3}{2}w_r y^{1/2}\right) \\
&- \frac{\rho\delta\sqrt{r}(\mu - r)}{\sigma}\sqrt{y}\left(c_r + d_r y^{1/2} + f_r y + w_r y^{3/2}\right) \\
&- \frac{1}{2}\delta^2 r\left(1-\rho^2\right)\left(A_r + B_r y\right)^2 + \frac{(\mu - r)^2}{2\sigma^2}y - r = 0.
\end{aligned}
\tag{69}
$$

Collecting the same orders of $y^{-1/2}$, $y^{1/2}$, y, $y^{3/2}$, and y^2 in (73), we obtain five equations as follows:

$$
Q_t - 4r\theta Q(t,r) - \frac{r}{2}\left(1-\rho^2\right)\delta^2 B_r^2 - \frac{\rho\delta\sqrt{r}(\mu - r)}{\sigma}w_r = 0, \tag{70}
$$

$$
W_t - 3r\theta W(t,r) - \frac{\rho\delta\sqrt{r}(\mu - r)}{\sigma}f_r = 0, \tag{71}
$$

$$
\begin{aligned}
&F_t - 2r\theta F(t,r) + 2\theta(4\theta+1)\sigma^2 Q(t,r) \\
&+ \left(k_1 - k_2 r\right)B_r - r\left(1-\rho^2\right)\delta^2 A_r B_r - \frac{\rho\delta\sqrt{r}(\mu - r)}{\sigma}d_r \\
&- 3\rho\sigma\theta\delta\sqrt{r}w_r + \frac{1}{2}r\delta^2 B_{rr} + \frac{(\mu - r)^2}{2\sigma^2} = 0,
\end{aligned}
\tag{72}
$$

$$
\begin{aligned}
&D_t - r\theta\, D(t,r) + \frac{3}{2}\theta(3\theta+1)\sigma^2 W(t,r) \\
&- \frac{\rho\delta\sqrt{r}(\mu - r)}{\sigma}c_r - 2\sqrt{r}\theta\rho\sigma\delta f_r = 0,
\end{aligned}
\tag{73}
$$

$$
\begin{aligned}
&C_t + \theta(2\theta+1)\sigma^2 F(t,r) - r + \left(k_1 - k_2 r\right)A_r \\
&- \frac{1}{2}r\left(1-\rho^2\right)\delta^2 A_r^2 - \sqrt{r}\theta\rho\sigma\delta d_r + \frac{1}{2}r\delta^2 A_{rr} = 0.
\end{aligned}
\tag{74}
$$

Considering the boundary conditions in (70), the solution of (74)–(78) can be presented as (63)–(67), respectively. □

Theorem 1. *The optimal investment strategy for the risky asset is the following proportion:*

$$
\pi_t^* = \nu^{-1}R(t,y,r)\left(J_0(t,y,r) + \sqrt{\epsilon}\,J_1(t,y,r) + \epsilon J_2(t,y,r)\right), \quad y = s^{-2\theta}. \tag{75}
$$

where

$$
\begin{aligned}
R(t,y,r) &= \frac{\mu - r}{q\nu\sigma^2}\,y e^{r(t-T)}, \\
J_0(t,y,r) &= 1 + \frac{2\theta\sigma^2 m_y^{(0)} - \rho\sigma\delta\sqrt{r}s^\theta m_r^{(0)}}{\mu - r}, \\
J_1(t,y,r) &= \frac{2\theta\sigma^2 m_y^{(1)} - \rho\sigma\delta\sqrt{r}s^\theta m_r^{(1)}}{\mu - r}, \\
J_2(t,y,r) &= \frac{2\theta\sigma^2 m_y^{(2)} - \rho\sigma\delta\sqrt{r}s^\theta m_r^{(2)}}{\mu - r}.
\end{aligned}
\tag{76}
$$

Proof. From (26) and (31), the following equalities hold:

$$
\begin{aligned}
&zg_z = -\frac{h(t)}{q},\ g_s = \frac{2\theta}{q}s^{-2\theta-1}h(t)m_y, \\
&g_r = -\frac{h(t)m_r}{q},\ h(t) = e^{r(t-T)}, \quad g = \nu.
\end{aligned}
\tag{77}
$$

Combining (22) with the equalities above, we obtain

$$
\begin{aligned}
\pi_t^* &= \frac{-(\mu - r)zg_z + \sigma^2 s^{2\theta+1}g_s + \rho\sigma\delta\sqrt{r}s^\theta g_r}{g\sigma^2 s^{2\theta}} \\
&= \frac{-(\mu - r)h(t) + 2\theta\sigma^2 h(t)m_y - \rho\sigma\delta\sqrt{r}s^\theta h(t)m_r}{qg\sigma^2 s^{2\theta}} \\
&= \frac{(\mu - r)h(t)}{q\nu\sigma^2 s^{2\theta}}\left[1 + \frac{2\theta\sigma^2}{\mu - r}m_y - \frac{\rho\sigma\delta\sqrt{r}\,s^\theta}{\mu - r}m_r\right] \\
&= \frac{(\mu - r)e^{r(t-T)}}{q\nu\sigma^2 s^{2\theta}}\left[1 + \frac{2\theta\sigma^2}{\mu - r}m_y - \frac{\rho\sigma\delta\sqrt{r}\,s^\theta}{\mu - r}m_r\right] \\
&= \frac{(\mu - r)e^{r(t-T)}}{q\nu\sigma^2 s^{2\theta}}\left[1 + \frac{2\theta\sigma^2}{\mu - r}\left(m_y^{(0)} + \sqrt{\epsilon}\,m_y^{(1)} + \epsilon m_y^{(2)}\right)\right. \\
&\quad \left. - \frac{\rho\sigma\delta\sqrt{r}\,s^\theta}{\mu - r}\left(m_r^{(0)} + \sqrt{\epsilon}\,m_r^{(1)} + \epsilon m_r^{(2)}\right)\right] \\
&= \frac{y(\mu - r)e^{r(t-T)}}{q\nu\sigma^2}\left[1 + \frac{2\theta\sigma^2 m_y^{(0)} - \rho\sigma\delta\sqrt{r}\,s^\theta m_r^{(0)}}{\mu - r}\right. \\
&\quad + \sqrt{\epsilon}\,\frac{2\theta\sigma^2 m_y^{(1)} - \rho\sigma\delta\sqrt{r}\,s^\theta m_r^{(1)}}{\mu - r} \\
&\quad \left. + \epsilon\frac{2\theta\sigma^2 m_y^{(2)} - \rho\sigma\delta\sqrt{r}\,s^\theta m_r^{(2)}}{\mu - r}\right].
\end{aligned}
\tag{78}
$$

Let

$$R(t,y,r) = \frac{\mu - r}{q\nu\sigma^2} ye^{r(t-T)},$$

$$J_0(t,y,r) = 1 + \frac{2\theta\sigma^2 m_y^{(0)} - \rho\sigma\delta\sqrt{r}s^\theta m_r^{(0)}}{\mu - r},$$

$$J_1(t,y,r) = \frac{2\theta\sigma^2 m_y^{(1)} - \rho\sigma\delta\sqrt{r}s^\theta m_r^{(1)}}{\mu - r}, \quad (79)$$

$$J_2(t,y,r) = \frac{2\theta\sigma^2 m_y^{(2)} - \rho\sigma\delta\sqrt{r}s^\theta m_r^{(2)}}{\mu - r}.$$

Thereby, the optimal strategy (79) is obtained. □

Remark 2. The value ϵ reflects the fluctuating degree of interest rate $r(t)$. Especially, when $\epsilon = 0$, it means that interest rate of risk-free asset is a constant, and then the optimal investment strategy $\pi_t^* = \nu^{-1}R(t,y,r)J_0(t,y,r)$ is equal to the optimal strategy under the classical CEV model.

5. Numerical Analysis

In this section, we provide some numerical examples to illustrate the dynamic behavior of the optimal investment strategy with CARA utility at time points $t = 0, 1, 2$. In these examples, in convenience of comparison we assume that all of the expiration dates are $T = 3$. In the whole numerical analysis, unless otherwise stated, the basic parameters employed are as follows: $\epsilon = 0.1$, $r = 0.05$, $\theta = -1$, $\nu = 300$, $\sigma = 1$, $k_1 = 1.5$, $k_2 = 0.2$, $\rho = 0.5$, $q = 0.05$, $T = 3$, $s_0 = 3$, $r = 0.08$, $\delta = 1$, and $\mu = 0.3$.

In Figures 1–6, we present some numerical analysis for the optimal investment strategy by oscillating one certain parameter and keeping all other parameters. For example, in Figure 1, the optimal proportion π^* of investment on risky asset versus parameter μ is plotted when other parameters are fixed.

From Figure 1, it shows that with the increase of μ the optimal proportion π^* increases first; then, as μ continues to increase to a certain point, π^* begins to decline. The reason for this kind of phenomena is explained as the moving Merton strategy and correlation factor (see the details in [16]). As the time of investment, for example, $t = 0$, is farther to the expiration date $T = 3$, the corresponding optimal proportion would be bigger. It means that individuals would like to invest more money on risky asset when the maturity length of the investment is bigger. The reason is that, as $\mu > r$, the investment on risky asset can obtain more returns for longer maturity.

Figure 2 plots the effect of interest rate r on the value of π^*. From Figure 2, we find that the optimal proportion of investment on the risky asset is a convex function of interest rate. Another interesting phenomena is that, as the investment time t is closer to expiration time $T = 3$, the lines would become flatter. From Figure 3, it shows that the bigger the correlation coefficient between $W_1(t)$ and $W_2(t)$ is, the larger the proportion π^* of investment would be on risky asset. That is, the investment proportion π^* is an increasing function with respect to the correlation coefficient ρ. Similarly, we can see that the investment time t, for example, $t = 2$, is closer to expiration time $T = 3$, and the proportion

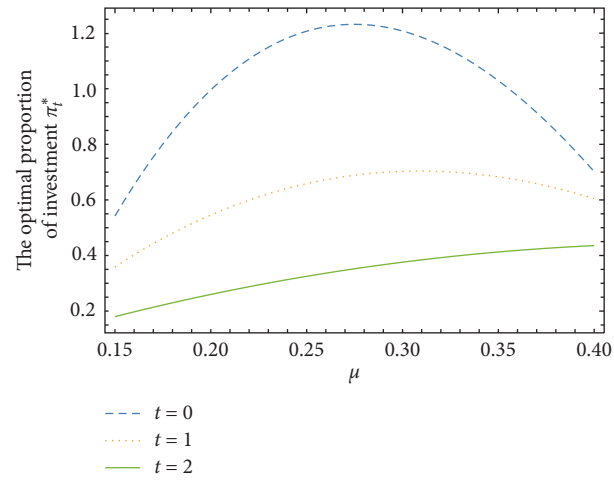

Figure 1: The impact of μ on π^*.

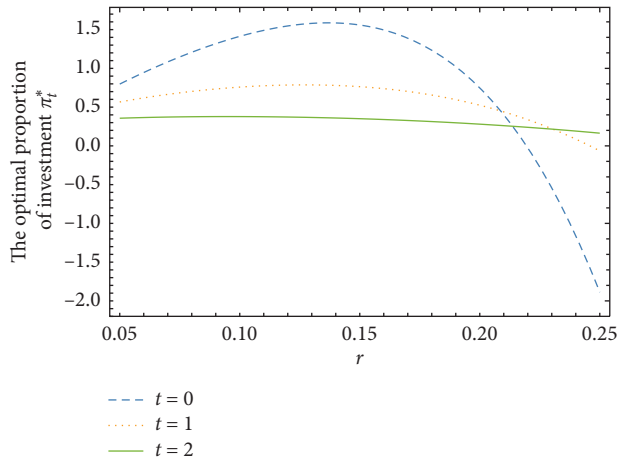

Figure 2: The impact of r on π^*.

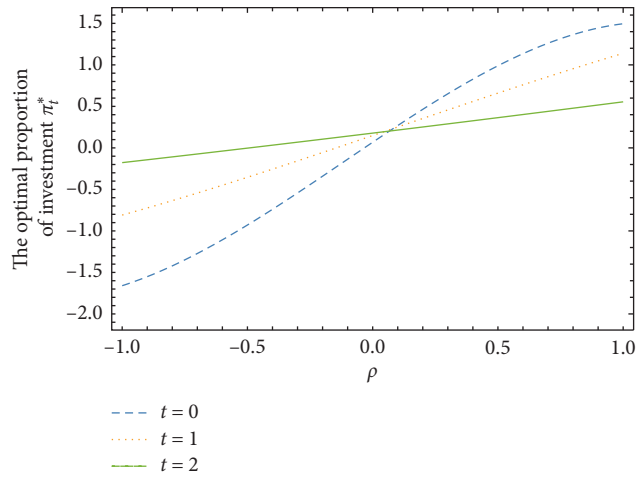

Figure 3: The impact of ρ on π^*.

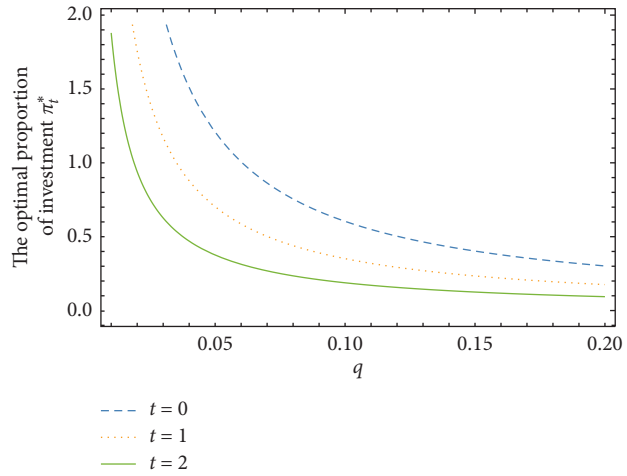

Figure 4: The impact of q on π^*.

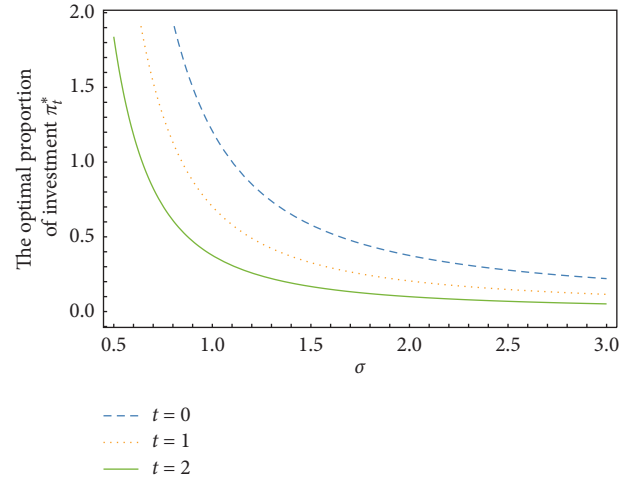

Figure 5: The impact of σ on π^*.

of investment π^* would be larger for $\rho < 0$. Conversely, for the case $\rho > 0$, the investment time t, for example, $t = 0$, is further from $T = 3$ and the proportion of investment would be larger, which implies that individuals would get more returns in a longer maturity.

In Figure 4, the parameter q denotes the risk aversion coefficient of investors. The bigger the value of q, the smaller the proportion of investment in risky assets. This is consistent with the fact that if an investor is a risk aversion person, then he/she would invest less wealth on the risky assets. In Figure 5, it shows that the optimal investment proportion π^* is a decreasing function of the value of σ. Here, σ denotes the risk coefficient of risky asset. As individuals are risk averters, the bigger the value of σ, the smaller the investment amount of individuals in the risky assets. Figure 6 shows that the optimal investment proportion is an increasing function of the parameter δ. That means, if the volatility δ of cash revenue increases, the amount invested in the risky asset also grows correspondingly. Another phenomena is that, as the investment time t is closer to expiration time $T = 3$, the parameter δ has smaller influence on the optimal investment strategy.

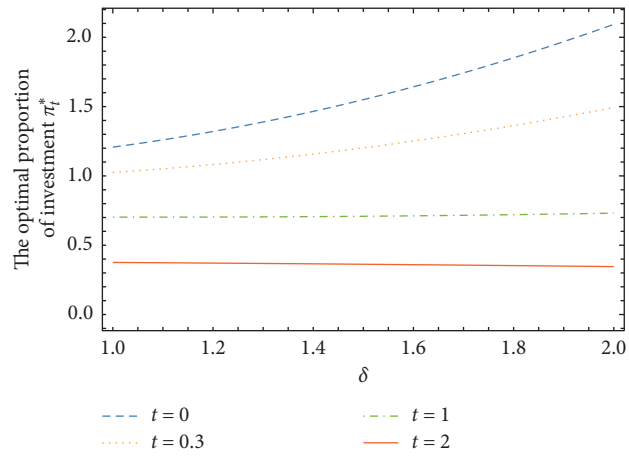

Figure 6: The impact of δ on π^*.

6. Conclusion

In this paper, we consider a CEV Model with a stochastic interest rate for the optimal investment problem. By applying the dual method, Legendre transformation, and asymptotic expansion approach, we derive an asymptotic strategy for CARA utility function. Finally, numerical examples are presented to illustrate the effect of parameters on the optimal investment strategy.

In the future work, we will consider more sophisticated situations for the optimal investment problems, such as involving transaction costs, stochastic affine interest rate, and the other uncertain factors, which would lead to a more complicated PDE to solve. We leave these points in the future research.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] R. C. Merton, "Lifetime portfolio selection under uncertainty: the continuous-time case," *The Review of Economics and Statistics*, vol. 51, no. 3, pp. 247–257, 1969.

[2] R. C. Merton, "Optimum consumption and portfolio rules in a continuous-time model," *Journal of Economic Theory*, vol. 3, no. 4, pp. 373–413, 1971.

[3] W. H. Fleming and T. Zariphopoulou, "An optimal investment/consumption model with borrowing," *Mathematics of Operations Research*, vol. 16, no. 4, pp. 802–822, 1991.

[4] J.-L. Vila and T. Zariphopoulou, "Optimal consumption and portfolio choice with borrowing constraints," *Journal of Economic Theory*, vol. 77, no. 2, pp. 402–431, 1997.

[5] C. Munk and C. Sørensen, "Optimal consumption and investment strategies with stochastic interest rates," *Journal of Banking & Finance*, vol. 28, no. 8, pp. 1987–2013, 2004.

[6] W. H. Fleming and T. Pang, "An application of stochastic control theory to financial economics," *SIAM Journal on Control and Optimization*, vol. 43, no. 2, pp. 502–531, 2004.

[7] R. Yao and H. H. Zhang, "Optimal consumption and portfolio choices with risky housing and borrowing constraints," *Review of Financial Studies*, vol. 18, no. 1, pp. 197–239, 2004.

[8] B. Dumas and E. Luciano, "An exact solution to a dynamic portfolio choice problem under transactions costs," *The Journal of Finance*, vol. 46, no. 2, pp. 577–595, 1991.

[9] S. E. Shreve and H. M. Soner, "Optimal investment and consumption with transaction costs," *The Annals of Applied Probability*, vol. 4, no. 3, pp. 609–692, 1994.

[10] M. Dai, L. Jiang, P. Li, and F. Yi, "Finite horizon optimal investment and consumption with transaction costs," *SIAM Journal on Control and Optimization*, vol. 48, no. 2, pp. 1134–1154, 2009.

[11] C. B. Zhang and X. M. Rong, "Optimal investment strategies for DC pension with stochastic salary under affine interest rate model," *Discrete Dynamics in Nature and Society*, vol. 10, 2013.

[12] C. B. Zhang, "Mean-Variance portfolio selection for defined-contribution pension funds with stochastic salary," *Discrete The Scientific World Journal*, vol. 10, 2014.

[13] J. C. Cox and S. A. Ross, "The valuation of options for alternative stochastic processes," *Journal of Financial Economics*, vol. 3, no. 1-2, pp. 145–166, 1976.

[14] Y. L. Hsu, T. I. Lin, and C. F. Lee, "Constant elasticity of variance (cev) option pricing model: integration and detailed derivation," *Mathematics and Computers in Simulation*, vol. 79, no. 1, pp. 60–71, 2008.

[15] R. R. Chen and C. F. Lee, "A constant elasticity of variance (CEV) family of stock price distributions in option pricing, review, and integration," *Handbook of Quantitative Finance and Risk Management*, vol. 10, 2010.

[16] R.-R. Chen, C.-F. Lee, and H.-H. Lee, "Empirical performance of the constant elasticity variance option pricing model," *Review of Pacific Basin Financial Markets and Policies*, vol. 12, no. 2, pp. 177–217, 2009.

[17] D. P. Nicholls and A. Sward, "A discontinuous galerkin method for pricing american options under the constant elasticity of variance model," *Communications in Computational Physics*, vol. 17, no. 3, pp. 761–778, 2015.

[18] J. Gao, "Optimal investment strategy for annuity contracts under the constant elasticity of variance (CEV) model," *Insurance: Mathematics and Economics*, vol. 45, no. 1, pp. 9–18, 2009.

[19] J. Gao, "An extended CEV model and the Legendre transform-dual-asymptotic solutions for annuity contracts," *Insurance: Mathematics and Economics*, vol. 46, no. 3, pp. 511–530, 2010.

[20] J. Xiao, Z. Hong, and C. Qin, "The constant elasticity of variance (CEV) model and the Legendre transform-dual solution for annuity contracts," *Insurance: Mathematics and Economics*, vol. 40, no. 2, pp. 302–310, 2007.

[21] X. Lin and Y. Li, "Optimal reinsurance and investment for a jump diffusion risk process under the CEV model," *North American Actuarial Journal*, vol. 15, no. 3, pp. 417–431, 2011.

[22] M. Gu, Y. Yang, S. Li, and J. Zhang, "Constant elasticity of variance model for proportional reinsurance and investment strategies," *Insurance: Mathematics and Economics*, vol. 46, no. 3, pp. 580–587, 2010.

[23] A. Gu, X. Guo, Z. Li, and Y. Zeng, "Optimal control of excess-of-loss reinsurance and investment for insurers under a CEV model," *Insurance: Mathematics and Economics*, vol. 51, no. 3, pp. 674–684, 2012.

[24] H. Zhao and X. Rong, "Portfolio selection problem with multiple risky assets under the constant elasticity of variance model," *Insurance: Mathematics and Economics*, vol. 50, no. 1, pp. 179–190, 2012.

[25] E. J. Jung and J. H. Kim, "Optimal investment strategies for the HARA utility under the constant elasticity of variance model," *Insurance: Mathematics and Economics*, vol. 51, no. 1, pp. 667–673, 2012.

[26] M. Jonsson and R. Sircar, "Optimal investment problems and volatility homogenization approximations," *NATO Science Series II*, vol. 51, 2002.

[27] A. Takahashi, "An asymptotic expansion approach to pricing contingent claims," *Asia-Pacific Financial Markets*, vol. 6, no. 1, pp. 115–151, 1999.

[28] M. Widdicks, P. Duck, A. Andricopoulos, and D. P. Newton, "The Black Scholes equation revisited: asymptotic expansions and singular perturbations," *Mathematical Finance*, vol. 15, no. 3, pp. 373–391, 2005.