Research Article

Iterative Solution for Systems of a Class of Abstract Operator Equations in Banach Spaces and Application

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In this paper, by using the partial order method, the existence and uniqueness of a solution for systems of a class of abstract operator equations in Banach spaces are discussed. The result obtained in this paper improves and unifies many recent results. Two applications to the system of nonlinear differential equations and the systems of nonlinear differential equations in Banach spaces are given, and the unique solution and interactive sequences which converge the unique solution and the error estimation are obtained.

1. Introduction

Guo and Lakshmikantham [1] introduced the definition of the mixed monotone operator and the coupled fixed point, and there are many good results (see [2–23]). Recently, from paper [6], using the monotone iterative techniques, the iterative unique solution of the following nonlinear mixed monotone Fredholm-type integral equations in Banach spaces

\[ u(t) = \int_{I} H(t, s, u(s))ds, \quad I = [a, b], \]

where \( I = [a, b] \) and \( H \in C([I \times I \times E, E]) \).

In this paper, the following nonlinear abstract operator equations in Banach spaces \( E \) are discussed:

\[
\begin{align*}
    u &= A(u, v), \\
    v &= B(v, u),
\end{align*}
\]

where \( A, B : D \times D \to E \) and \( D \) is a partial interval in \( E \) which is denoted as the following:

\[ D = [u_0, v_0] \equiv \{ u \in E | u_0 \leq u \leq v_0 \}. \]

For convenience, the following assumptions are made:

\( (H_1) \) There exist positive bounded operators \( T_i : E \to E (i = 1, 2) \) which satisfy:

\[ (I + T_1 - T_2)x \geq \theta \implies x \in P, \]

and for any \( u, v \in D (i = 1, 2), u_1 \leq u_2, v_1 \leq v_2, \) the following is obtained:

\[
\begin{align*}
    B(v_2, u_1) - B(v_1, u_2) &\leq -T_1(v_2 - v_1) - T_2(u_2 - u_1), \\
    A(v_2, u_1) - A(v_1, u_2) &\leq -T_1(v_2 - v_1) - T_2(u_2 - u_1).
\end{align*}
\]

\( (H_2) \) There exists a positive bounded operator \( L : E \to E \) and for any \( u, v \in D, u \leq v \), the following is obtained:

\[
\begin{align*}
    B(v_2, u_1) - B(v_1, u_2) &\leq -T_1(v_2 - v_1) - T_2(u_2 - u_1), \\
    A(v_2, u_1) - A(v_1, u_2) &\leq -T_1(v_2 - v_1) - T_2(u_2 - u_1),
\end{align*}
\]

\( (H_3) \) \( LT_2 T_1 = T_1 T_2 \) in which the spectral radius satisfies

\[ -(T_1 + T_2)(v - u) \leq B(v, u) - A(u, v) \leq L(v - u). \]

\( (H_4) \)

\[ LT_2 = T_2 L, LT_1 = T_1 L, T_1 T_2 = T_2 T_1 \] in which the spectral radius satisfies...
The zero element of normal if there exists a constant

$$r(L) + r(T_1) + r(T_2) < \inf \{ |\lambda| : \lambda \in \sigma(I + T_1 - T_2) \}.$$  

$$r(T_1 - T_2) < 1.$$  

In this paper, firstly, by using the partial order method, the existence and uniqueness of a solution for systems of a class of abstract operator equations in Banach spaces are discussed. And next, two applications to the system of nonlinear integral equations and the system of nonlinear differential equations in Banach spaces are given, and the unique solution and interactive sequences which converge a unique solution and the error estimation are obtained.

2. The Interactive Solution of Abstract Operator Equations

Let $P$ be a cone in $E$, i.e., a closed convex subset, such that $\lambda P \subset P$ for any $\lambda \geq 0$ and $P \cap \{-P\} = \emptyset$. A partial order $\leq$ in $P$ is defined as $x \leq y \Leftrightarrow y - x \in P$. A cone $P$ is said to be normal if there exists a constant $N > 0$ which satisfies $x, y \in E, \theta \leq x \leq y$, implying $\|x\| \leq N\|y\|$, where $\theta$ denotes the zero element of $E$. And, the smallest number $N$ is called as the normal constant of $P$ and denoted as $N_p$. The cone $P$ is normal iff every ordered interval $[x, y] = \{z \in E : x \leq z \leq y\}$ is bounded.

The following theorem is the main result in this section.

**Theorem 1.** Let $P$ be a cone in $E, u_0, v_0 \in E, u_0 \leq v_0$. Suppose that $A, B: D \times D \to E$ satisfies conditions $(H_1) - (H_4)$.

(i) There exists a unique solution of equation (2) $(u^*, u^*)$ in $D \times D$, and for any solutions of equation (2) $(u, u) \in D \times D$, one has $u = u^*$.

(ii) For any initial value $x_0, y_0 \in D, x_0 \leq y_0$, the following iterative sequences are constructed:

$$\begin{align*}
  x_n &= (I + T_1 - T_2)^{-1} \left[ A(x_{n-1}, y_{n-1}) + T_1 x_{n-1} - T_2 y_{n-1} \right], \\
  y_n &= (I + T_1 - T_2)^{-1} \left[ B(y_{n-1}, x_{n-1}) + T_1 y_{n-1} - T_2 x_{n-1} \right],
\end{align*}$$  

(7)

which satisfy $\|x_n - u^*\| \to 0, \|y_n - u^*\| \to 0 (n \to \infty)$, and for any $\delta$,

$$r(L) + r(T_1) + r(T_2) < \inf \{ |\lambda| : \lambda \in \sigma(I + T_1 - T_2) \} < \delta < 1,$$  

(8)

depends on $\|x_0\|, \|y_0\|$ and satisfies $n \geq n_0$ the following is obtained:

$$\begin{align*}
  \|x_n - u^*\| &\leq 2N_r \delta^n \|v_0 - u_0\|, \\
  \|y_n - u^*\| &\leq 2N_r \delta^n \|v_0 - u_0\|.
\end{align*}$$  

(9)

Proof. By $r(T_1 - T_2) < 1$, it is known that the operator $(I + T_1 - T_2)$ is reversible. And, from condition $(H_1)$, $(I + T_1 - T_2)^{-1}$ is the positive operator. Let

$$\begin{align*}
  F(u, v) &= (I + T_1 - T_2)^{-1} [A(u, v) + T_1 u - T_2 v], \\
  G(v, u) &= (I + T_1 - T_2)^{-1} [B(v, u) + T_1 v - T_2 u].
\end{align*}$$  

(10)

Then, equation (7) can be substituted by the following:

$$\begin{align*}
  x_n &= F(x_{n-1}, y_{n-1}), \\
  y_n &= G(y_{n-1}, x_{n-1}).
\end{align*}$$  

(11)

By conditions $(H_1) - (H_2)$, it is easy to obtain that operators $F$ and $G$ satisfy the following:

(1) $u_0 \leq F(u_0, v_0) \leq G(v_0, u_0) \leq v_0$

(2) $F, G: D \times D \to E$ are the mixed monotone operator

(3) $\theta \leq G(v, u) - F (u, v) \leq H(v - u), u_0 \leq u \leq v \leq v_0,$

where $H = (L + T_1 + T_2)(I + T_1 - T_2)^{-1}$

Letting $u_n = F(u_{n-1}, v_{n-1})$ and $v_n = G(v_{n-1}, u_{n-1})$ $(n = 1, 2, \ldots)$, the following two results are obtained by mathematical induction:

$$\begin{align*}
  u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0, \\
  u_n \leq v_n - u_0 \leq H^n (v_0 - u_0), \quad n = 1, 2, \ldots.
\end{align*}$$  

(12)

(13)

In fact, from (1) and (3), one has

$$\begin{align*}
  u_0 &\leq u_1 \leq v_1 \leq v_0, \\
  u_1 &\leq \cdots \leq u_n \leq \cdots \leq v_n, \\
  0 &\leq v_1 - u_1 \leq H(v_0 - u_0).
\end{align*}$$  

(14)

Suppose that for $n = k$, one has (12) and (13). Then, as $n = k + 1$, by (2) and (3), the following is obtained:

$$\begin{align*}
  u_{k+1} &= F(u_k, v_k) \leq x_{k+1} = F(x_k, y_k) \leq G(y_k, x_k) = y_{k+1}, \\
  \theta &\leq y_{k+1} - u_{k+1} = G(v_k, u_k) - F(u_k, v_k) \leq H(v_k - u_k) \leq H^{k+1} (v_k - u_k).
\end{align*}$$  

(15)

Then, it is known that

$$\begin{align*}
  u_k &\leq u_{k+1} \leq x_{k+1} \leq y_{k+1} \leq v_k, \\
  \theta &\leq y_{k+1} - u_{k+1} \leq H^{k+1} (v_k - u_k).
\end{align*}$$  

(16)

Then, for any natural number $n$, (12) and (13) are obtained by mathematical induction.

Next, it is proved that $\{x_n\}$ is Cauchy sequences. From condition $(H_4)$, it is known that

$$\begin{align*}
  \left( L + T_1 + T_2, (I + T_1 - T_2)^{-1} \right) \left( L + T_1 + T_2, (I + T_1 - T_2)^{-1} \right),
\end{align*}$$  

(17)

then by ([14], V 3.9), $r(H) \leq [r(L) + r(T_1) + r(T_2)]

\{1 + \{T_1 \{\{\{1\}\}\} \} \} \cdot \{1 + \{T_1 \{\{\{1\}\}\} \} \} \rightarrow \{1 + \{T_1 \{\{\{1\}\}\} \} \}$.

Thus, for any $\delta$: $(r(L) + r(T_1) + r(T_2))/(\inf \{ |\lambda| : \lambda \in \sigma(I + T_1 - T_2) \}) < \delta < 1$, the following is obtained:
\[ \lim \|H^n\|^{1/n} = r(H) \leq [r(L) + r(T_1) + r(T_2)]r \]
\[ \cdot \left(1 + T_1 - T_2 \right)^{-1} \]
\[ = \frac{[r(L) + r(T_1) + r(T_2)]}{\inf\{\|\lambda\|: \lambda \in \sigma(1 + T_1 - T_2)\}} < \delta < 1. \]

Then, there exists a natural number \( n_0 \) which satisfies
\[ \|H^n\| \leq \delta^n, \ \forall n \geq n_0. \] (19)

And, by (12) and (13), it is obtained that
\[ \theta \leq u_n \leq u_{n+1} \leq x_{n+1} \leq y_{n+1} \leq v_{n+1} \leq v_n, \]
\[ \theta \leq u_n \leq x_n \leq y_n \leq v_n, \quad p = 1, 2, \ldots. \] (20)

So, by (13), it is known that
\[ \theta \leq x_{n+1} - u_n \leq v_n - u_n \leq H^n(v_0 - u_0), \]
\[ \theta \leq x_n - u_n \leq H^n(v_0 - u_0). \] (21)

Then, by the normality of \( P \) and (19), it is known that
\[ \|x_{n+1} - u_n\| \leq \|x_n - u_n\| \leq \|H^n(v_0 - u_0)\|, \]
\[ \|x_n - u_n\| \leq \|H^n(v_0 - u_0)\|, \quad n \geq n_0, p = 1, 2, \ldots. \] (22)

Thus, the following is obtained:
\[ \|x_{n+1} - x_n\| \leq 2\|H^n(v_0 - u_0)\|, \quad n \geq n_0, p = 1, 2, \ldots, \] (24)
i.e., \([x_n]\) is Cauchy sequences. So, there exists \( u^* \in D \) (\( D \) is bounded), such that \( \lim_{n \to \infty} x_n = u^* \).

And, by \( \theta \leq y_n - x_n \leq v_n - u_n \leq H^n(v_0 - u_0) \), the normality of \( P \), and (19), one obtains
\[ \|y_n - x_n\| \leq \|H^n(v_0 - u_0)\|, \quad n \geq n_0. \] (25)

Therefore
\[ \lim_{n \to \infty} y_n = u^* = \lim_{n \to \infty} x_n, \quad x_n \leq u^* \leq y_n, n = 1, 2, \ldots. \] (26)

Thus, \( \|x_n - u_n\| \leq \|H^n(v_0 - u_0)\|, \quad \|y_n - x_n\| \leq \|H^n(v_0 - u_0)\|, \quad \|v_n - x_n\| \leq \|H^n(v_0 - u_0)\|, \quad \|v_n - u_n\| \leq \|H^n(v_0 - u_0)\|. \)

And
\[ \lim_{n \to \infty} u_n = u^* = \lim_{n \to \infty} v_n, \] (27)
\[ u_n \leq u^* \leq v_n, \quad n = 1, 2, \ldots, \] (28)
so by (2), (3), and (11), it is also obtained that
\[ u_n = F(u_{n-1}, v_{n-1}) \leq F(u^*, u^*) \leq G(u^*, u^*) \leq G(v_{n-1}, u_{n-1}) = v_n. \] (29)

Letting \( n \to \infty \) and by (27), \( F(u^*, u^*) = G(u^*, u^*) = u^*. \)

Then, by the definition of \( F \) and \( G \), one obtains
\[ u^* = A(u^*, u^*), u^* = B(u^*, u^*), \quad \text{i.e.,} \quad (u^*, u^*) \text{ is a solution of equation (2)}. \]

Lastly, it is proven that the solution is unique. Supposing that \((u, v) \in D \times D\) also satisfies equation (2), then by (11) and mathematical induction, the following is obtained:
\[ u_n \leq u \leq v_n, \quad (n = 1, 2, \ldots). \] (30)

Thus, \( u = u^* \).

And, letting \( \rho \to \infty \) in (24), as \( n \geq n_0 \), the following is obtained:
\[ \|x_n - u^*\| \leq 2N\rho \delta^n \|v_0 - u_0\|. \] (31)

Similarly, as \( n \geq n_0 \), the following is obtained:
\[ \|y_n - u^*\| \leq 2N\rho \delta^n \|v_0 - u_0\|. \] (32)

The proof is complete.

Remark 1. In Theorem 1, it is only supposed that operators \( A \) and \( B \) satisfy the partial conditional, and the unique solution and interactive sequences which converge a unique solution are obtained.

3. The Application of Nonlinear Integral Equations

In this section, the following nonlinear integral equations are considered:
\[ \left\{ \begin{array}{l}
u(t) = f_1(t, u(t), v(t)) + \int_0^t g_1(s, u(s), v(s))ds, \\
v(t) = f_2(t, v(t), u(t)) + \int_0^t g_2(s, u(s), v(s))ds, 
\end{array} \right. \] (33)
where \( f_i \in [I \times R_+ \times R_+], (i = 1, 2, \text{ and } I = [0, +\infty]) \), and \( E \) is a real Banach space with norm \( \| \cdot \| \).

In this section, the iterative solution of a nonlinear integral equation (33) is discussed. For convenience, the following assumptions are made:

(L1) For the nonnegative bounded continuous function \( a(t), b(t) \), and nonnegative integrable \( c(t), d(t) \), one has
\[ f_2(t, u, v) \leq a(t)u + b(t), \quad g_2(t, u, v) \leq c(t)u + d(t), \] (34)

(L2) There exists a constant \( M > 0 \), for any \( u, v \in E, u \leq v \), which satisfies
\[ f_i(t, u, v) - f_i(t, u, u) \geq -M(v - u), \quad \text{and} \quad g_i(t, u, v) - g_i(t, u, u) \geq 0, \quad (i = 1, 2). \] (35)

(L3) For any \( u, v \in E, u \leq v \), the following is satisfied:
\[-M(v-u) \leq f_2(t,v,u) - f_1(t,u,v) \leq c(t)(v-u),
\]
\[0 \leq g_2(t,v,u) - g_1(t,u,v) \leq a(t)(v-u). \tag{36}\]

\[(L_1) \max_{t \in I} a(t) < 1.\]

In this section, the following main theorem is obtained.

**Theorem 2.** Let \( P \) be a normal cone in \( E \). Suppose conditions \((L_1) - (L_4)\) hold. Then, there exists a unique solution of equation \((2)\) \((u^*,u^*) \in (E \times E)\), and there are iterative sequences converging to the unique solution, and corresponding error estimates are given.

**Proof.** Let \( E = C[I,R] \). Then, \( P_c = \{ x \in C[I,R] | x(t) \geq 0, \forall t \in I \} \) is a cone. Thus, by the normal of \( P, P_c \) is also normal.

The following operator is considered:

\[
\begin{align*}
A &= F_1 + G_1, \\
B &= F_2 + G_2,
\end{align*}
\tag{37}
\]

where for any \( u,v \in P_c, t \in I, \)

\[
F_1(u,v) = f_1(t,u(t),v(t)),
\]

\[
G_1(u,v) = \int_0^t g_1(s,u(s),v(s))ds,
\]

\[
F_2(v,u) = f_2(t,u(t),u(t)),
\]

\[
G_2(v,u) = \int_0^t g_2(s,v(s),u(s))ds.
\]

Then, \( A, B : P_c \times P_c \rightarrow E \). It is easy to know that \((u^*,u^*) \in P_c \times P_c \) is a solution of \((33)\) if and only if \((u^*,u^*) \) is a solution of the following integral equations:

\[
\begin{align*}
u &= A(u,v), \\
v &= B(v,u). \tag{39}
\end{align*}
\]

Next, from conditions \((L_4) - (L_4)\), it is obtained that the operators \( A \) and \( B \) satisfy the whole condition of Theorem 1.

In fact, \( \forall u_1, u_2, v_1, v_2 \in P_c, u_1 \leq u_2, v_1 \leq v_2 \): (i) Let

\[
B(v,u) - A(u,v) = f_2(t,v(t),u(t)) - f_1(t,v(t),u(t)) + \int_0^t [g_2(s,v(s),u(s)) - g_1(s,v(s),u(s))]ds
\]

\[
\geq - M(v-u) + \int_0^t [g_2(s,v(s),u(s)) - g_1(s,v(s),u(s))]ds \geq - M(v-u),
\]

\[
B(v,u) - A(u,v) = f_2(t,v(t),u(t)) - f_1(t,v(t),u(t)) + \int_0^t [g_2(s,v(s),u(s)) - g_1(s,v(s),u(s))]ds
\]

\[
\leq a(t)(v-u) + \int_0^t c(s)(v-u)ds = L(v-u). \tag{44}\]

\[
Lu = a(t)u + \int_0^t c(s)u(s)ds,
\]

\[
h = b(t) + \int_0^t d(s)ds, \quad t \in I,
\]

\[
L_1u = a(t)u,
\]

\[
L_2u = \int_0^t c(s)u(s)ds.
\]

Then, \( L_1L_2 = L_2L_1 \) and \( r(L_1) = \max_{t \in I} a(t), r(L_2) = 0 \). Thus,

\[
r(L) = r(L_1 + L_2) \leq r(L_1) + r(L_2) = \max_{t \in I} a(t) < 1. \tag{41}\]

Therefore, for the equation \((I - L)u = h\), there exists a unique solution \( v_0 = (I - L)^{-1}h = \sum_{n=0}^{\infty} L^n h \in P \). Then, by \( (L_1) \), for any \( t \in I \), the following is obtained:

\[
B(v_0, \theta) = F_2(v_0, \theta) + G_2(v_0, \theta) = f_2(t,v_0(t),\theta) + \int_0^t g_2(s,v_0(s),\theta)ds \leq a(t)v_0 + \int_0^t c(s)v_0(s)ds + b(t) + \int_0^t d(s)ds = Lv_0 + h = v_0. \tag{42}\]

Obviously, \( \theta \leq f_1(t,\theta, v_0(t)) + \int_0^t g_1(s,\theta, v_0(s))ds = A(\theta,v_0) \).

(ii) By \( (L_2) \), the following is obtained:

\[
B(v_2,u_1) - B(v_1,u_2) = f_2(t,v_2(t),u_1(t)) - f_2(t,v_1(t),u_2(t)) + \int_0^t [g_2(s,v_2(s),u_1(s)) - g_2(s,v_1(s),u_2(s))]ds \geq f_2(t,v_2(t),u_1(t)) - f_2(t,v_1(t),u_2(t)) \geq - M(v_2 - v_1). \tag{43}\]

Similarly, \( A(v_2,u_1) - A(v_1,u_2) \geq - M(v_2 - v_1) \).

(iii) From \((L_3)\) and \((L_4)\), the following is obtained:
Then, by (41), it is known that
\[-M(v - u) \leq B(v, u) - A(u, v) \leq L(v - u), \quad r(L) < 1.\]
(45)

Therefore, from (i), (ii), and (iii), letting \( T_2 = M, I, T_2 = 0 \) in Theorem 1, it is easy to know that the condition \((H_4)\) holds.

Finally, for any initial value \( x_0, y_0 \in [\theta, v_0], x_0 \leq y_0 \), by constructing the iterative sequences
\[
\begin{aligned}
x_n(t) &= f_1(t, x_{n-1}(t), y_{n-1}(t)) + \int_0^t g_1(s, x_{n-1}(s), y_{n-1}(s))ds, \\
y_n(t) &= f_2(t, y_{n-1}(t), x_{n-1}(t)) + \int_0^t g_2(s, y_{n-1}(s), x_{n-1}(s))ds,
\end{aligned}
\]
(46)

one has \( \|x_n - u^*\| \longrightarrow 0, \|y_n - u^*\| \longrightarrow 0 \) \( (n \longrightarrow \infty) \), and for any \( a \in (0, 1) \), there exists a natural number \( n_0 \) which satisfies as \( n \geq n_0 \), the following is obtained:
\[
\begin{aligned}
\|x_n - u^*\| &\leq 2N_p a^n \|v_0 - u_0\|, \\
\|y_n - u^*\| &\leq 2N_p a^n \|v_0 - u_0\|.
\end{aligned}
\]
(47)

This completes the proof of Theorem 2.

4. The Application of Nonlinear Differential Equations

In this section, the following nonlinear initial value problems of the differential equation are considered:
\[
\begin{aligned}
u'(t) &= f_1(t, u, v) + \int_0^t g_1(s, u, v)ds, \quad u(0) = u_0, \\
v'(t) &= f_2(t, u, v) + \int_0^t g_2(s, u, v)ds, \quad v(0) = v_0,
\end{aligned}
\]
(48)

where \( f_i, g_i \in C[\theta \times R, \times R], i = 1, 2, I = [0, T], \) and \( E \) is a real Banach space with norm \( \|\cdot\| \).

For convenience, the following assumptions are made:

\( (C_1) \) There exists the nonnegative bounded integrable functions \( a(t), b(t), c(t), d(t) \) which satisfy
\[
\begin{aligned}
f_2(t, u, \theta) &\leq a(t)u + b(t), \\
g_2(t, u, \theta) &\leq c(t)u + d(t).
\end{aligned}
\]
(49)

\( (C_2) \) There exists constant \( M > 0 \), for any \( u, v \in E, u \leq v \), which satisfies
\[
\begin{aligned}
f_i(t, v, u) - f_i(t, u, v) &\geq -M (v - u), \\
g_i(t, v, u) - g_i(t, u, v) &\geq 0, \quad (i = 1, 2).
\end{aligned}
\]
(50)

\( (C_3) \) For any \( u, v \in E, u \leq v \), the following is satisfied:
\[
\begin{aligned}
-M(v - u) &\leq f_2(t, v, u) - f_1(t, u, v) \leq c(t)(v - u), \\
0 &\leq g_2(t, v, u) - g_1(t, u, v) \leq a(t)(v - u).
\end{aligned}
\]
(51)

Then, the following theorem is obtained.

**Theorem 3.** Let \( P \) be a normal cone in \( E \). Suppose that conditions \((C_1) - (C_4)\) hold. Then, there exists a unique solution of equation (48) \((u^*, u^*)\), and there are iterative sequences converging to the unique solution, and corresponding error estimates are given.

**Proof.** Firstly, differential equation (48) is turned into integral equations. For any fixed \( \eta \in C^1[\theta, E] \), the following one-order linear differential initial value problems in Banach spaces are investigated:

\[
\begin{aligned}
u'(t) &= f_1(t, \eta, \eta) - M(u - \eta) + \int_0^t K(t, s)g_1(s, \eta, \eta)ds, \quad u(0) = u_0, \\
u'(t) &= f_2(t, \eta, \eta) - M(u - \eta) + \int_0^t K(t, s)g_2(s, \eta, \eta)ds, \quad u(0) = u_0.
\end{aligned}
\]
(52)
It is easy to know that \((u, u) \in C^1[I, E] \times C^1[I, E]\) is a solution of (52) if and only if \((u, u)\) is a solution of the following integral equations:

\[
\begin{align*}
    u(t) &= e^{-Mt} \left[ u_0 + \int_0^T g_1(r, \eta(r), \eta(r))dr \int_0^t K(s, r)ds \right] \\
    &\quad + e^{-Mt} \int_0^T e^{Mt} \left[ f_1(s, \eta(s), \eta(s)) + \eta \eta(s) \right] ds,
\end{align*}
\]

\[
\begin{align*}
    u(t) &= e^{-Mt} \left[ u_0 + \int_0^T g_2(r, \eta(r), \eta(r))dr \int_0^t K(s, r)ds \right] \\
    &\quad + e^{-Mt} \int_0^T e^{Mt} \left[ f_2(s, \eta(s), \eta(s)) + \eta \eta(s) \right] ds.
\end{align*}
\]

Next, the operator \(A, B: C^1[I, E] \times C^1[I, E] \rightarrow C^1[I, E]\) is defined as the following:

\[
\begin{align*}
    A(\eta, \eta) &= e^{-Mt} \left[ u_0 + \int_0^T g_1(r, \eta(r), \eta(r))dr \int_0^t K(s, r)ds \right] \\
    &\quad + e^{-Mt} \int_0^T e^{Mt} \left[ f_1(s, \eta(s), \eta(s)) + \eta \eta(s) \right] ds,
\end{align*}
\]

\[
\begin{align*}
    B(\eta, \eta) &= e^{-Mt} \left[ u_0 + \int_0^T g_2(r, \eta(r), \eta(r))dr \int_0^t K(s, r)ds \right] \\
    &\quad + e^{-Mt} \int_0^T e^{Mt} \left[ f_2(s, \eta(s), \eta(s)) + \eta \eta(s) \right] ds.
\end{align*}
\]

Obviously, \((\eta, \eta)\) is a solution of (48) if and only if

\[
\begin{align*}
    \eta &= A(\eta, \eta), \\
    \eta &= B(\eta, \eta).
\end{align*}
\]

Next, similar to the proof of Theorem 2, it is tested whether the operators \(A\) and \(B\) satisfy the whole condition of Theorem 1 from conditions \((C_1) - (C_3)\). Therefore, the result of Theorem 3 is obtained from Theorem 1.

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this paper.

**Authors’ Contributions**

The author read and approved the final manuscript.

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