

## Research Article

# Uniform Continuity of Fractal Interpolation Function

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In order to research analysis properties of fractal interpolation function generated by the iterated function system defined by affine transformation, the continuity of fractal interpolation function is proved by the continuous definition of function and the uniform continuity of fractal interpolation function is proved by the definition of uniform continuity and compactness theorem of sequence of numbers or finite covering theorem in this paper. The result shows that the fractal interpolation function is uniformly continuous in a closed interval which is from the abscissa of the first interpolation point to that of the last one.

## 1. Introduction

In 1960s, fractal geometry was regarded as a new interdisciplinary firstly discovered by American mathematician Mandelbrot [1–5]. On the one hand, because most extremely irregular graphics in nature and very irregular social phenomenon are researched in the fractal geometry field, fractal geometry is called natural geometry. Therefore, fractal geometry is applied in almost all fields, such as mathematics, physics, chemistry, engineering, social science, and art [6–9]. On the other hand, for research on the natural properties of fractal itself, many methods used in researching fractals have been found by experts from 1960s to now, for example, fractal dimension method [10], multifractal spectrum method [11], adaptive fuzzy output-feedback method of nonlinear system [12], and nonlinear iterated method [13]. Especially, the fractal interpolation function method has been paid more and more attention by mathematicians. The theory of fractal interpolation function generated by the iterated function system defined by affine transformation was firstly proposed by Barnsley [14–16] and Massopust [17, 18]. They found that any part of a fractal graphic is similar to the whole, so they used mathematical language to express the similar iterated process. That is to say, first, the iterated function system consisting of affine transformation is defined and it is proved that the iterated function system

has a unique attractor that is the fixed point. Second, according to the theory of iterated function system and self-similar theory, complex fractal graphic, the graphic of fractal interpolation curve (refer with Figure 1), or fractal interpolation surface (refer with Figure 2) can be generated by computer program. Finally, the dimension theory and integrability of fractal interpolation function have been studied by Barnsley and Massopust.

Based on the research of fractal interpolation function above, the continuity and uniform continuity of the fractal interpolation function-generated iterated function system-defined affine mapping are proved in the paper.

## 2. Main Concepts and Lemmas

*Definition 1* (see [19, 20]). Let  $f$  be a function defined on interval  $I$ . If  $\forall \bar{x} \in I$  and  $\forall \varepsilon > 0$ , there is a  $\delta > 0$ , so that for any  $x \in I$  and  $|x - \bar{x}| < \delta \implies |f(x) - f(\bar{x})| < \varepsilon$ , we call that  $f$  is continuous on the point  $\bar{x}$  and the  $f$  is called continuous function on  $I$ .

*Definition 2* (see [19, 20]). Let  $f$  be a function defined on interval  $I$ . If  $\forall \varepsilon > 0$  and  $\exists \delta > 0$ , so that for any  $x', x'' \in I$ ,  $|x' - x''| < \delta \implies |f(x') - f(x'')| < \varepsilon$ , the  $f$  is called uniformly continuous function on  $I$ .

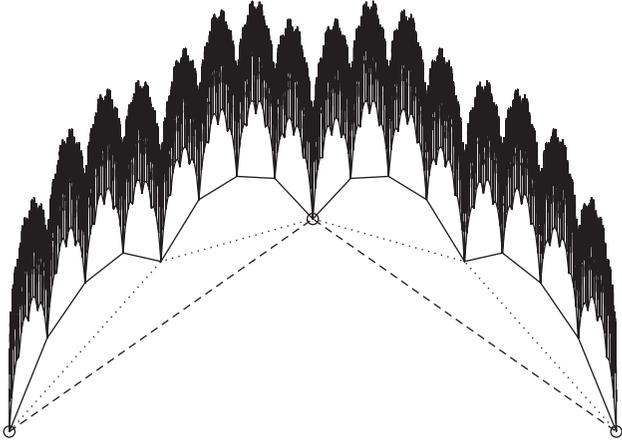


FIGURE 1: Fractal interpolation curve.

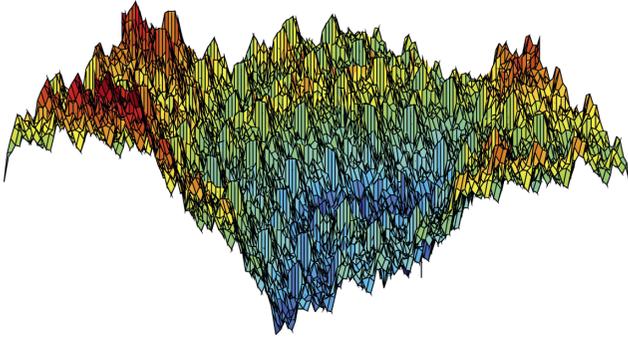


FIGURE 2: Fractal interpolation surface.

**Definition 3** (see [19]). Let  $S$  be a point subset on real number set  $R$ . Let  $H$  be an open-interval set; in other words, each element in  $H$  is the open-interval form of  $(\alpha, \beta)$ .  $H$  is called an open covering of  $S$  or  $H$  covers  $S$ , if any point in  $S$  is contained in at least an open interval in  $H$ . Furthermore, the number of open intervals in  $H$  is infinite (or finite) and we call that  $H$  is an infinite open covering of  $S$  (or a finite open covering of  $S$ ).

**Definition 4** (see [14–16]). A (hyperbolic) iterated function system consists of a complete metric space  $(X, d)$  together with a finite set of contraction mappings  $w_i: X \rightarrow X$ , with respective contractivity mapping factors  $s_i$ , for  $i = 1, 2, \dots, n$ . The abbreviation “IFS” is used for “iterated function system.” The notation for the IFS just announced is  $\{X; w_i, i = 1, 2, \dots, n\}$  and contractivity factor is  $s = \max\{s_i; i = 1, 2, \dots, n\}$ .

**Definition 5** (see [14]). Let  $\{(x_i, y_i) \in R^2; i = 0, 1, 2, \dots, n\}$  be a set of points, where  $x_0 < x_1 < x_2 < \dots < x_n$ . An interpolation function corresponding to this set of data is a function  $f: [x_0, x_n] \rightarrow R$  such that

$$f(x_i) = y_i, \quad i = 1, 2, \dots, n. \quad (1)$$

The points  $(x_i, y_i)$  are called the interpolation points. It is called that the function of  $f$  interpolates the data and that the graph of  $f$  passes through the interpolation points.

**Lemma 1** (see [19, 20]). *If a sequence  $\{x_n\}_{n=1}^{\infty}$  is bounded, the sequence  $\{x_n\}$  has a convergent subsequence  $\{x_{n_k}\}_{k=1}^{\infty}$ .*

**Lemma 2** (see [19, 20]). *Let  $H$  be a infinite open covering of closed interval  $[a, b]$ ; then, there exist finite open intervals selected from  $H$  to cover  $[a, b]$ .*

**Lemma 3** (see [14–18]). *Let  $n$  be a positive integer greater than 1. Let  $\{R^2; w_i, i = 1, 2, \dots, n\}$  denote the IFS defined above, associated with the data set*

$$\{(x_i, y_i) \in R^2; i = 0, 1, 2, \dots, n\}. \quad (2)$$

Let the vertical scaling factor  $d_i$  obey  $0 \leq d_i < 1$  for  $i = 1, 2, \dots, n$ . Then, there is a metric  $d$  on  $R^2$ , equivalent to the Euclidean metric, such that the IFS is hyperbolic with respect to  $d$ . In particular, there is a unique nonempty compact set  $G \subset R^2$ , such that

$$G = \bigcup_{i=1}^n w_i(G). \quad (3)$$

In particular, an IFS of the form  $\{R^2; w_i, i = 1, 2, \dots, n\}$  is considered, where the mapping is an affine transformation of the special structure

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}, \quad (4)$$

The transformations are constrained by the data according to

$$\begin{aligned} w_i \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} &= \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix}, \\ w_i \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad \text{for } i = 1, 2, \dots, n, \end{aligned} \quad (5)$$

$a_i, e_i, c_i$ , and  $f_i$  can be solved from equations (4)–(5) in terms of the data and vertical scaling factor  $d_i$ :

$$a_i = \frac{x_i - x_{i-1}}{x_n - x_0}, \quad (6)$$

$$e_i = \frac{x_n x_{i-1} - x_0 x_i}{x_n - x_0}, \quad (7)$$

$$c_i = \frac{y_i - y_{i-1} - d_i(y_n - y_0)}{x_n - x_0}, \quad (8)$$

$$f_i = \frac{x_n y_{i-1} - x_0 y_i}{x_n - x_0} - \frac{d_i(x_n y_0 - x_0 y_n)}{x_n - x_0}. \quad (9)$$

**Lemma 4** (see [14–16]). *Suppose  $F$  is a set of continuous functions which satisfy  $f: [x_0, x_n] \rightarrow R$  and  $f(x_0) = y_0$*

and  $f(x_n) = y_n$ . The metric is defined by the following formula:

$$d(f, g) = \max|f(x) - g(x)|, x \in [x_0, x_n], \quad \forall f, g \in F. \quad (10)$$

Then,  $(F, d)$  is a complete metric space. Let the real numbers  $a_i, c_i, e_i$ , and  $f_i$ , be defined by equations (5)–(9). Define a mapping  $T: F \rightarrow F$  by

$$(Tf)(x) = c_i L_i^{-1}(x) + d_i f(L_i^{-1}(x)) + f_i, x \in [x_{i-1}, x_i], \quad i = 1, 2, \dots, n, \quad (11)$$

where  $L_i: [x_0, x_n] \rightarrow [x_{i-1}, x_i]$  is the invertible transformation:

$$L_i(x) = a_i x + e_i, \quad (12)$$

$$L_i^{-1}(x) = \frac{x - e_i}{a_i}, L_i^{-1}(x_{i-1}) = x_0, L_i^{-1}(x_i) = x_n. \quad (13)$$

Then,  $Tf$  is continuous on the interval  $[x_{i-1}, x_i]$  and  $T$  is a contraction mapping on  $(F, d)$ , so  $T$  possesses a unique fixed point in  $F$ . That is, there exists a function  $f \in F$  such that

$$Tf = f, \quad \forall f \in F. \quad (14)$$

The function  $f$  is called fractal interpolation function. The abbreviation “FIF” is used for “fractal interpolation function.”

### 3. The Continuity and the Uniform Continuity of Fractal Interpolation Function

**Theorem 1.** Let function  $f$  be a FIF generated by IFS mentioned in Definition 4 and defined by affine mapping referred from equations (3)–(14) and Lemmas 3 and 4. Then, the FIF is continuous on the closed interval on the closed interval  $[x_0, x_n]$ .

*Proof.*  $\forall \bar{x} \in [x_0, x_n]$ , according to Lemmas 3 and 4 and equations (11) to (14),  $\forall \varepsilon > 0$ , and

$$\begin{aligned} \|f(x) - f(\bar{x})\| &= |Tf(x) - Tf(\bar{x})| \\ &= |c_i L_i^{-1}(x) + d_i f(L_i^{-1}(x)) + f_i - c_i L_i^{-1}(\bar{x}) - d_i f(L_i^{-1}(\bar{x})) - f_i| \\ &= |c_i((L_i^{-1}(x) - L_i^{-1}(\bar{x})) + d_i(f(L_i^{-1}(x)) - f(L_i^{-1}(\bar{x})))| \\ &= \left| c_i \left( \frac{x - e_i}{a_i} - \frac{\bar{x} - e_i}{a_i} \right) + d_i \left( f \left( \frac{x - e_i}{a_i} \right) - f \left( \frac{\bar{x} - e_i}{a_i} \right) \right) \right| \\ &= \left| \frac{c_i}{a_i} (x - \bar{x}) + d_i \left( c_i x + d_i \left( \frac{x - e_i}{a_i} \right) + f_i - c_i \bar{x} - d_i \left( \frac{\bar{x} - e_i}{a_i} \right) - f_i \right) \right| \\ &= \left| \left( \frac{c_i + d_i^2}{a_i} + c_i d_i \right) (x - \bar{x}) \right| < \varepsilon. \end{aligned} \quad (15)$$

Select

$$p = \max_{1 \leq i \leq n} \left\{ \left| \frac{c_i + d_i^2}{a_i} + c_i d_i \right| \right\}. \quad (16)$$

Then, we select

$$\delta = \frac{\varepsilon}{p}, \quad (17)$$

so  $\forall \varepsilon > 0$ , for any  $x \in [x_0, x_n]$  and  $|x - \bar{x}| < \delta$ , such that

$$|f(x) - f(\bar{x})| < \varepsilon. \quad (18)$$

Therefore, according to Definition 1, fractal interpolation function  $f$  is continuous on the point  $\bar{x}$ . Furthermore, because  $\bar{x}$  is arbitrarily selected from  $[x_0, x_n]$  and  $f$  is a continuous function on  $[x_0, x_n]$ .

The proof above illustrates that fractal interpolation function  $f$  is continuous on the closed interval  $[x_0, x_n]$ .  $\square$

**Theorem 2.** Let function  $f$  be a FIF generated by IFS and defined by affine transformation from equations (4)–(14) on the closed interval  $[x_0, x_n]$ . Then,  $f$  is uniformly continuous on the closed interval  $[x_0, x_n]$ .

*Proof.* The contradiction method will be used in the proof. Suppose that FIF is not uniformly continuous on the closed interval  $[x_0, x_n]$ . According to Definition 2,

$$\exists \varepsilon_0 > 0, \quad \forall \delta > 0, \exists x'_m, x''_m \in [x_0, x_n], \quad (19)$$

so that

$$\lim_{m \rightarrow \infty} |x'_m - x''_m| = 0, \quad (20)$$

that is,

$$|x'_m - x''_m| < \delta, \quad (21)$$

but

$$|f(x'_m) - f(x''_m)| \geq \varepsilon_0. \quad (22)$$

Because  $x'_m \in [x_0, x_n]$ , the number sequence  $\{x'_m\}$  is a bounded number sequence. According to Lemma 1,  $\{x'_m\}$  has a convergent subsequence  $\{x'_{m_k}\}$ . Let  $\{x'_{m_k}\} \rightarrow \bar{x}$ ; then,

$$\begin{aligned} \lim_{k \rightarrow \infty} x''_{m_k} &= \lim_{k \rightarrow \infty} (x''_{m_k} - x'_{m_k} + x'_{m_k}) \\ &= \lim_{k \rightarrow \infty} x''_{m_k} = \bar{x}, \end{aligned} \quad (23)$$

from  $x_0 \leq x'_{m_k} \leq x_n \implies x_0 \leq \bar{x} \leq x_n$ . Because  $f$  is a continuous function on the closed interval  $[x_0, x_n]$ ,

$$\lim_{x \rightarrow \bar{x}} f(x) = f(\bar{x}). \quad (24)$$

From the relationship between limit of function and that of number sequence,

$$\begin{aligned} \lim_{k \rightarrow \infty} |f(x'_{m_k}) - f(x''_{m_k})| &\geq \varepsilon_0, \\ |f(\bar{x}) - f(\bar{x})| &\geq \varepsilon_0, \end{aligned} \quad (25)$$

that is,

$$0 \geq \varepsilon_0, \quad (26)$$

which is contrary to  $\varepsilon_0 > 0$ . In other words, at first, the contrary hypothesis to the conclusion of Theorem 2 makes  $\varepsilon_0 > 0$  and  $\varepsilon_0 \leq 0$  contradictory. So,  $f$  is uniform continuous function on the interval  $[x_0, x_n]$ .

For the proof of Theorem 2, we have the following second method to prove Theorem 2.  $\square$

*Proof.* Because fractal interpolation function  $f$  is continuous on the interval  $[x_0, x_n]$ ,  $\forall \bar{x} \in [x_0, x_n]$ ,  $\forall \varepsilon > 0$ ,  $\exists \delta_{\bar{x}} > 0$ , so that  $\forall x \in \bigcup(\bar{x}; t\delta_{\bar{x}})$  and

$$|f(x) - f(\bar{x})| < \varepsilon/2. \quad (27)$$

Let

$$H = \bigcup \left( \bar{x}; t \frac{\delta_{\bar{x}}}{2} \right) | \bar{x} \in [x_0, x_n]. \quad (28)$$

Obviously,  $H$  is an open covering of  $[x_0, x_n]$ . According to Lemma 2 above, there exists

$$H^* = \left\{ \bigcup \left( \bar{x}_i; \frac{\delta_i}{2} \right) | t \text{ niq} = h1, 2, x \dots 7, CN \right\}, \quad (29)$$

to cover  $[x_0, x_n]$ . Suppose that

$$\delta = \min_{1 \leq i \leq N} \left\{ \frac{\delta_i}{2} \right\}, \quad (30)$$

$\forall x', x'' \in [x_0, x_n]$ , and  $|x' - x''| < \delta$ ; then,  $\exists \bigcup(\bar{x}_i; \delta_i/2) \in H^*$ , such that

$$x' \in \bigcup \left( \bar{x}_i; \frac{\delta_i}{2} \right). \quad (31)$$

On the one hand,

$$|x' - \bar{x}_i| < \frac{\delta_i}{2} < \delta_i \implies |f(x') - f(\bar{x}_i)| < \frac{\varepsilon}{2}. \quad (32)$$

On the other hand,

$$\begin{aligned} |x'' - \bar{x}_i| &= |x'' - x' + x' - \bar{x}_i| < \delta + \frac{\delta_i}{2} < \delta_i \\ \implies |f(x'') - f(\bar{x}_i)| &< \frac{\varepsilon}{2}. \end{aligned} \quad (33)$$

Therefore,

$$\begin{aligned} |f(x') - f(x'')| &= |f(x') - f(\bar{x}_i) \\ &+ f(\bar{x}_i) - f(x'')| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \quad (34)$$

According to Definition 2, fractal interpolation function is a uniformly continuous function proved by finite covering Lemma 2.

In other words, from the two proof methods of Theorem 2, fractal interpolation function  $f$  is uniform continuous on the interval  $[x_0, x_n]$ .  $\square$

## 4. Conclusion

From the above related theory of fractal interpolation function and function continuity, the following conclusion can be acquired through the above proof. Fractal interpolation function is not only continuous but also uniformly continuous on the closed interval  $[x_0, x_n]$ .

## Data Availability

The data of this paper are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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