**Research Article**

**Tempered Mittag–Leffler Stability of Tempered Fractional Dynamical Systems**

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Due to finite lifespan of the particles or boundedness of the physical space, tempered fractional calculus seems to be a more reasonable physical choice. Stability is a central issue for the tempered fractional system. This paper focuses on the tempered Mittag–Leffler stability for tempered fractional systems, being much different from the ones for pure fractional case. Some new lemmas for tempered fractional Caputo or Riemann–Liouville derivatives are established. Besides, tempered fractional comparison principle and extended Lyapunov direct method are used to construct stability for tempered fractional system. Finally, two examples are presented to illustrate the effectiveness of theoretical results.

1. Introduction

Fractional derivatives were first proposed by Leibnitz soon after the more familiar classic integer order derivatives. In recent decades, the study of fractional differential systems has attracted wide attention. Compared with the classical calculus, fractional calculus can better characterize memory and hereditary properties of processes and materials. They are now used to model the dynamical evolution in the fields of physics, chemistry, biology, and so on. Fractional calculus can be most easily understood in terms of probability. The relationships among random walks, Brownian motion, and diffusion processes were given in [1]. It is more reasonable to replace classic derivatives by fractional analogues in the diffusion equation [2].

Fractional calculus involves the operation of convolution with a power law function. Multiplying by an exponential factor results in tempered fractional derivatives and integrals [3], this exponential tempering has many merits both in mathematical and practical. A truncated Lévy flight was investigated to capture the natural cutoff in real physical systems [4]. Without a sharp cutoff, the tempered Lévy flight was studied as a smoother alternative [5]. Cartea and del-Castillo-Negrete [6] explored the tempered fractional diffusion equation by the tempered Lévy flight. In finance, the tempered stable process models describe price fluctuations with semi-heavy tails [7–10]. Tempered fractional time derivatives can be also found in geophysics [11–13], Brownian motion [14], and so on.

As in classical calculus, stability analysis is still one of the most important tasks in fractional differential system [15–20]. It is a basic feature in fractional physical and biological systems, such as Duffing oscillator [21], neural networks [22–24], and predator-prey models [25]. At present, Lyapunov method has been applied to analyze Mittag–Leffler stability of different fractional systems [26–30]. Li et al. [26, 27] obtained a series of conclusions on the Mittag–Leffler stable for nonlinear fractional equations. In [28], Mittag–Leffler stability of multiple equilibrium points of fractional recurrent neural networks was considered. In [29], a convex and positive definite function was used to analyze Mittag–Leffler stable for fractional systems. In [30], the authors presented the Lyapunov stability analysis for fractional nonlinear systems with impulses.

As far as we know, no paper has discussed stability analysis for tempered fractional system. Motivated by this, we think it is very necessary and meaningful to study Mittag–Leffler stability of tempered fractional dynamical systems both in theoretical research and practical application. Because tempered fractional operators combine with nonlocal, weak singularity, and exponential factors [31–33], it has many differences to fractional case in stability analysis. In this paper, tempered Mittag–Leffler...
stability is first proposed. It is a more appropriate concept for tempered fractional system. Tempered comparison principle and some inequalities are given for tempered fractional calculus or systems. Then, sufficient conditions for tempered Mittag–Leffler stability are provided and verified by the Lyapunov method. Finally, the theoretical results are applied to some examples.

This paper is organized as follows. In Section 2, some necessary definitions and lemmas are prepared. Section 3 mainly discusses the sufficient criterions ensuring Mittag–Leffler stability of the tempered fractional systems. In Section 4, two examples are presented to show the effectiveness of theoretical results. We conclude the paper with some discussions in Section 5.

2. Preliminaries

Tempered fractional calculus plays an important role in different fields [34, 35]. In practical application, different tempered fractional derivatives are proposed, such as Caputo, Riemann–Liouville, and Riesz. Some definitions and lemmas are stated below, which will be used later.

Definition 1 (see [13]). The tempered fractional integral of order $\alpha > 0$ and tempered parameter $\lambda \geq 0$ is defined as

$$\int_{\tau}^{t} a_{I}^{\alpha, \lambda} \{x(s)\} ds = \frac{1}{\Gamma(\alpha)} \int_{\tau}^{t} e^{\lambda (t-s)} (t-s)^{\alpha-1} x(s) ds,$$

where $\Gamma$ presents the Euler gamma function.

Definition 2 (see [3]). The tempered fractional Caputo derivative of tempered parameter $\lambda \geq 0$ is defined as

$$\int_{\tau}^{t} a_{D}^{\alpha, \lambda} \{x(t)\} = e^{-\lambda t} \int_{\tau}^{t} \frac{1}{\Gamma(n-\alpha)} \left( \begin{array}{c} \int_{\tau}^{s} \cdots \int_{\tau}^{s} \cdots \int_{\tau}^{s} \cdots \int_{\tau}^{s} \end{array} \right) \frac{ds}{(t-s)^{n-\alpha+1}} ds,$$

where $n-1 \leq \alpha < n, n \in \mathbb{N}$, and $a_{D}^{\alpha, \lambda}$ is the Caputo fractional derivative.

Definition 3 (see [9]). The tempered Riemann–Liouville derivative of tempered parameter $\lambda \geq 0$ is defined as

$$\int_{\tau}^{t} a_{R}^{\alpha, \lambda} \{x(t)\} = e^{-\lambda t} \int_{\tau}^{t} \frac{\mathcal{D}_{\lambda}^{\alpha} \{x(t)\}}{\Gamma(n-\alpha)} \frac{ds}{(t-s)^{n-\alpha+1}},$$

where $n-1 \leq \alpha < n, n \in \mathbb{N}$, and $a_{D}^{\alpha, \lambda}$ is the Riemann–Liouville fractional derivative operator.

In order to study the stability of tempered fractional systems, several lemmas are needed.

Lemma 1 (see [36]). Let $0 < \alpha < 1, \lambda \geq 0$ and $t \geq \alpha$, then

$$C_{a}^{\alpha, \lambda} \{x(t)\} = a_{D}^{\alpha, \lambda} \{x(t)\} = e^{-\lambda t} \left( \frac{-e^{-\lambda (t-a)} - e^{-\lambda t}}{1 - \alpha} \right) \left[ e^{\lambda t} x(t) \right]_{t=a}.$$

Lemma 2 (see [36]). Let $0 < \alpha < 1, \lambda \geq 0$, then

(i) $a_{D}^{\alpha, \lambda} \{C_{D}^{\alpha, \lambda} \{x(t)\}\} = x(t) - e^{\lambda t} (t-a)^{-\alpha} \{x(a)\}$,

(ii) $a_{D}^{\alpha, \lambda} \{I_{C}^{\alpha, \lambda} \{x(t)\}\} = x(t) - (e^{\lambda t} (t-a)^{-\alpha} / \Gamma(\alpha)) \left[ a_{D}^{\alpha, \lambda} \{e^{\lambda t} u(t)\} \right]_{t=a}$ and $a_{D}^{\alpha, \lambda} \{I_{C}^{\alpha, \lambda} \{x(t)\}\} = x(t) - (e^{\lambda t} (t-a)^{-\alpha} / \Gamma(\alpha)) \left[ a_{D}^{\alpha, \lambda} \{e^{\lambda t} u(t)\} \right]_{t=a}$.

Lemma 3 (see [36]). The Laplace transform of tempered fractional integral and Caputo derivative (2) are given as

(i) $\mathcal{L} \{a_{I}^{\alpha, \lambda} \{x(t)\}\} = \lambda + a_{I}^{\alpha, \lambda} \{x(s)\}$,

(ii) $\mathcal{L} \{a_{D}^{\alpha, \lambda} \{x(t)\}\} = (s + \lambda)^{\alpha} \{x(s)\} - \sum_{k=0}^{\alpha-1} (s + \lambda)^{\alpha-1-k} |(\mathcal{D}_{\lambda}^{\alpha} \{x(t)\}|_{t=0})$, where $X(s) = \mathcal{L} \{x(t)\}$ denotes the Laplace transform of $x(t)$

3. Main Results

In this section, tempered fractional comparison principles, some inequalities, and tempered Mittag–Leffler stability are derived.

3.1. Tempered Fractional Comparison Principles

In this section, we establish tempered fractional comparison principles.

Lemma 4. Assume that $C_{a}^{\alpha, \lambda} \{x(t)\} \geq C_{a}^{\alpha, \lambda} \{y(t)\}, x(0) = y(0), \alpha \in (0, 1)$ and $\lambda \geq 0$, then $x(t) \geq y(t)$.

Proof. Following from $C_{a}^{\alpha, \lambda} \{x(t)\} \geq C_{a}^{\alpha, \lambda} \{y(t)\}$, there exists a function $m(t) \geq 0$ such that

$$C_{a}^{\alpha, \lambda} \{x(t)\} = m(t) + C_{a}^{\alpha, \lambda} \{y(t)\}.$$

By Lemma 3, equation (5) yields

$$(s + \lambda)^{\alpha} \{x(s)\} - (s + \lambda)^{\alpha-1} x(0) = M(s) + (s + \lambda)^{\alpha} Y(s) - (s + \lambda)^{\alpha-1} Y(0).$$

According to $x(0) = y(0)$, we have

$$(s + \lambda)^{\alpha} \{X(s)\} = M(s) + (s + \lambda)^{\alpha} \{Y(s)\},$$

Thus,

$$X(s) = Y(s) + (s + \lambda)^{-\alpha} \{M(s)\}.\tag{6}$$

Taking the inverse Laplace transform on (6), solution of system (5) can be written as

$$x(t) = y(t) + \mathcal{L}^{-1} \{m(t)\} = y(t) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} e^{-\lambda (t-r)} \{t-r \}^{\alpha-1} m(r)dr.$$

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According to \( m(t) \geq 0 \), therefore we obtain \( x(t) \geq y(t) \).

**Lemma 5.** Assume that \( \alpha D_t^{\alpha, x}y(t) \geq \beta D_t^{\alpha, x}x(t) \), \( x(0) = y(0) \), and \( \alpha \in (0, 1) \), then \( x(t) \geq y(t) \).

**Proof.** From Lemma 1 and \( \alpha D_t^{\alpha, x}x(t) \geq \beta D_t^{\alpha, x}y(t) \), we derive

\[
\frac{\alpha}{\beta} D_t^{\alpha, x}(x(t)) + \frac{e^{-\lambda t}}{\Gamma(1 - \alpha)} x(0) \geq \frac{\alpha}{\beta} D_t^{\alpha, x}(y(t)) + \frac{e^{-\lambda t}}{\Gamma(1 - \alpha)} y(0) .
\]

(10)

That is \( \frac{\alpha}{\beta} D_t^{\alpha, x}(x(t)) \geq \frac{\alpha}{\beta} D_t^{\alpha, x}(y(t)) \). From Lemma 4, we obtain \( x(t) \geq y(t) \).

3.2. Some Inequalities. In this section, we construct some inequalities for tempered fractional derivatives or systems.

From Lemma 1, we could easily obtain the following lemma.

**Lemma 6.** The relationship between \( \alpha D_t^{\alpha, x}x(t) \) and \( \alpha D_t^{\alpha, x}y(t) \) is as follows:

\[
\begin{cases}
\frac{\alpha}{\beta} D_t^{\alpha, x}(x(t)) \leq \frac{\alpha}{\beta} D_t^{\alpha, x}(y(t)), & \text{if } x(0) \geq 0, \\
\frac{\alpha}{\beta} D_t^{\alpha, x}(x(t)) \geq \frac{\alpha}{\beta} D_t^{\alpha, x}(y(t)), & \text{if } x(0) \leq 0,
\end{cases}
\]

where \( \alpha \in (0, 1) \).

**Lemma 7.** If \( x(t) \in C^1([0, \infty), \mathbb{R}) \) is a continuously differentiable function, the following inequality holds:

\[
\frac{\alpha}{\beta} D_t^{\alpha, x}(x(t)) \leq \text{sgn}(x(t)) \frac{\alpha}{\beta} D_t^{\alpha, x}(x(t)), \quad 0 < \alpha < 1, \lambda \geq 0,
\]

(12)

where \( x(t^*) = \lim_{t \to t^*} x(t) \).

**Proof.** We take \( y(t) = e^{\lambda t}x(t) \) into Theorem 2 in [22]

\[
\frac{\alpha}{\beta} D_t^{\alpha, x}(y(t)) \leq \text{sgn}(y(t)) \frac{\alpha}{\beta} D_t^{\alpha, y}(y(t)), \quad 0 < \alpha < 1,
\]

for the Caputo fractional derivative. That is,

\[
\frac{\alpha}{\beta} D_t^{\alpha, x}(e^{\lambda t} x(t^*)) \leq \text{sgn}(x(t)) \frac{\alpha}{\beta} D_t^{\alpha, x}(e^{\lambda t} x(t)).
\]

(14)

Multiplying both sides of equation (14) by \( e^{-\lambda t} \), it gives

\[
e^{-\lambda t} \frac{\alpha}{\beta} D_t^{\alpha, x}(e^{\lambda t} x(t^*)) \leq \text{sgn}(x(t)) e^{-\lambda t} \frac{\alpha}{\beta} D_t^{\alpha, x}(e^{\lambda t} x(t)).
\]

(15)

Using Definition 2, we obtain (12).

Consider the following tempered fractional system

\[ t_1 \alpha D_t^{\alpha, x}(x(t)) = f(t, x(t)), \quad 0 < t < T, \]

(16)

subjects to the proper initial conditions, where \( \alpha \in (0, 1) \), \( \lambda \geq 0 \), \( D \) denotes either \( \alpha D_t^{\alpha, x} \) or \( \alpha D_t^{\alpha, x} \), \( f: [t_0, \infty) \times \mathbb{R} \to \mathbb{R}^n \) is piecewise continuous in \( t \) and locally Lipschitz in \( x \), and \( \mathbb{D} \subset \mathbb{R}^n \) is a domain contain the origin.

**Theorem 1.** For the real-valued continuous function \( f(t, x) \) in (16), we have

\[
\| t_1 \alpha D_t^{\alpha, x}(f(t, x)) \| \leq \| t_1 \alpha D_t^{\alpha, x}(f(t, x)) \|, \quad \alpha > 0, \lambda > 0, \text{ and } \| \| \text{ denotes an arbitrary norm.}
\]

**Proof.** It follows from (1) that

\[
\| t_1 \alpha D_t^{\alpha, x}(f(t, x)) \| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)} (t-s)^{\alpha-1} f(s, x(s)) ds \right|
\]

\[
\leq \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)} (t-s)^{\alpha-1} \| f(s, x(s)) \| ds
\]

\[
= \frac{\alpha}{\beta} D_t^{\alpha, x}(f(t, x(t))).
\]

(18)

**Theorem 2.** If \( x = 0 \) is an equilibrium point of system (16) with \( t_1 \alpha D_t^{\alpha, x} \) to system (16), it follows from Lemma 2 and Lipschitz condition in \( f(t, x) \) that

\[
\| x(t) \| - e^{-\lambda t} \| x(0) \| \leq \| x(t) - e^{-\lambda t} x(0) \|
\]

\[
= \| t_1 \alpha D_t^{\alpha, x}(f(t, x(t))) \| - \| t_1 \alpha D_t^{\alpha, x}(f(t, x(t))) \|
\]

\[
\leq t_1 \alpha D_t^{\alpha, x}(f(t, x(t))) \| \leq t_1 \alpha D_t^{\alpha, x}(f(t, x(t))) \|.
\]

(20)

There exists a function \( M(t) \geq 0 \) such that

\[
\| x(t) \| - e^{-\lambda t} \| x(0) \| = t_1 \alpha D_t^{\alpha, x}(f(t, x(t))) - M(t).
\]

(21)

Combining with Lemma 3 and Laplace transform to (21), we obtain

\[
\| \mathcal{L}(x(t)) \| \leq \frac{\| (s + \lambda)^{\alpha-1} x(0) \| - (s + \lambda)^{\alpha} M(s)}{(s + \lambda)^{\alpha} - 1},
\]

where \( \mathcal{L}(\| x(t) \|) \). Taking the inverse Laplace transform to (22) gives

\[
\| x(t) \| = \| x(0) \| e^{-\lambda t} E_{\alpha, \lambda}(lt^\alpha) - M(t) \star \left[ e^{-\lambda t} E_{\alpha, \lambda}(lt^\alpha) \right],
\]

(23)

where \( \star \) denotes the convolution operator. Obviously, \( e^{-\lambda t+1} E_{\alpha, \lambda}(lt^\alpha) \geq 0 \), then inequality \( \| x(t) \| \leq \| x(0) \| e^{-\lambda t} E_{\alpha, \lambda}(lt^\alpha) \) is obtained.

3.3. Tempered Mittag-Leffler Stability. In this section, some sufficient conditions are established for the tempered Mittag-Leffler stability of system (16).

**Definition 4.** If and only if \( f(t, x) = t_1 \alpha D_t^{\alpha, x}x \), then \( x \in \mathbb{R}^n \) is an equilibrium point of tempered fractional system (16).
Theorem 3. Assume $x = 0$ be an equilibrium point for (16) and domain $\mathcal{D} \subset \mathbb{R}^n$ contains the origin. Let $V(t, x(t)) : [0, +\infty) \times \mathcal{D} \rightarrow \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to $x$, such that

$$
\alpha_1 \|x(t)\|^\alpha \leq V(t, x(t)) \leq \alpha_2 \|x(t)\|^\beta,
$$

Proof. It follows from equations (25) and (26) that

$$
\frac{C}{\alpha_1} D_t^{\beta,\lambda} V(t, x(t)) \leq -\frac{\alpha_2}{\alpha_2} V(t, x(t)).
$$

Taking the Laplace transform to (28) gives

$$
(s + \lambda)^{\beta} V(s) - (s + \lambda)^{\beta-1} V(0) + M(s) = -\frac{\alpha_2}{\alpha_2} V(s),
$$

where nonnegative constant $V(0) = V(x(0))$ and $V(s) = \mathcal{L}[V(t, x(t))]$. We rewrite this in the form

$$
V(s) = \frac{V(0)}{(s + \lambda)^{\beta-1} - M(s)}.
$$

Applying the inverse Laplace transform to (30), unique solution of (28) is

$$
V(t) = V(0)e^{-\lambda t} E_\beta\left(-\frac{\alpha_3}{\alpha_2}\right) - M(t) + \left[e^{-\lambda t} E_\beta\left(-\frac{\alpha_3}{\alpha_2}\right) - \frac{\alpha_2}{\alpha_2} V(0)\right].
$$

Because $t^{\beta-1}$ and $E_{\beta, \phi}(-x/\alpha_2)t^\beta$ are nonnegative functions, we obtain

$$
V(t) \leq V(0)e^{-\lambda t} E_\beta\left(-\frac{\alpha_3}{\alpha_2}\right).
$$

Substituting (32) into (25) satisfies

$$
\|x(t)\| \leq \left[\frac{V(0)}{\alpha_1} e^{-\lambda t} E_\beta\left(-\frac{\alpha_3}{\alpha_2}\right)\right]^\alpha.
$$

and $x(0) = 0$ if and only if $(V(0)/\alpha_1)e^{-\lambda t} = 0$.

Because $V(t, x)$ is local Lipschitz condition with respect to $x$ and $x(0) = 0$ if and only if $(V(0)/\alpha_1)e^{-\lambda t}$ satisfies local Lipschitz condition about $x(0)$. So, system (16) is tempered Mittag--Leffler stable. \(\square\)

Theorem 4. Assume all conditions in Theorem 3 are satisfied except replacing $\frac{C}{\alpha_1} D_t^{\beta,\lambda}$ by $\frac{C}{\alpha_1} D_t^{\beta,\lambda}$, then we have

$$
\|x(t)\| \leq \left[\frac{V(0)}{\alpha_1} e^{-\lambda t} E_\beta\left(-\frac{\alpha_3}{\alpha_2}\right)\right]^\alpha.
$$

A similar proof method in Theorem 3 shows result (35). \(\square\)

Theorem 5. For the tempered fractional system (10), where $D_t^{\alpha,\lambda} = D_t^{\alpha,\lambda}$, $f(t, x)$ is Lipschitz on $x$ with constant $l > 0$, and $f(t, 0) = 0$, if there exists a Lyapunov candidate $V(t, x)$ yielding

$$
\alpha_1 \|x(t)\|^\alpha \leq V(t, x(t)) \leq \alpha_2 \|x(t)\|^\beta,
$$

$$
\frac{D_t^{\alpha,\lambda} V(t, x(t))}{\Gamma(\alpha)} \leq -\frac{\alpha_3}{\alpha_2} \|x(t)\|^\beta,
$$

where $\alpha_3 > \lambda \alpha_2$, $\alpha_1$, $\alpha_2$, $\alpha_3$ are given positive constants and $V(t, x(t)) = (dV(t, x(t))/dt)$, then

$$
\|x(t)\| \leq \left[\frac{V(0)}{\alpha_1} e^{-\lambda t} E_\beta\left(-\frac{\alpha_3 - \lambda \alpha_2}{\alpha_2}\right)\right]^\alpha.
$$

A similar proof method in Theorem 3 shows result (35). \(\square\)
Using (36) and (37) and Lipschitz condition on \( f(t,x) \), we obtain
\[
\begin{align*}
\kappa_t^{1-\alpha} V(t,x) &\leq \lambda \alpha_0 \kappa_t^{\alpha} \| x(t) \| - \alpha_3 \kappa_t^{\alpha - 1} \| x(t) \| \\
&= -(\alpha_3 - \lambda \alpha_2) \kappa_t^{\alpha} \| x(t) \| \\
&\leq -(\alpha_3 - \lambda \alpha_2) \Gamma^{1-\alpha} \| f(t,x(t)) \|. 
\end{align*}
\] (40)

We can use Lemmas 7 and 2 to write
\[
\begin{align*}
\kappa_t^{1-\alpha} V(t,x) &\leq -(\alpha_3 - \lambda \alpha_2) \Gamma^{1-\alpha} \| f(t,x(t)) \| \\
&= -(\alpha_3 - \lambda \alpha_2) \Gamma^{1-\alpha} \| x(t) \|, 
\end{align*}
\] (41)

where \( \alpha_t^{-1}(e^{\alpha} u(t)) \big|_{t=0} = 0 \). By the same proof in Theorem 3, conclusion (38) holds.

4. Applications

In this section, we will give three examples to demonstrate theoretical analysis. The Adams–Bashforth–Moulton method [37] is employed for solving tempered fractional differential equations in the simulations.

Example 1. Consider the tempered fractional Riemann–Liouville system:
\[
\kappa_t^{\alpha} x(t) = -|x(t)|, \quad 0 < \alpha < 1, \quad \lambda \geq 0, 
\] (42)

where \( x(0) > 0 \). The Lyapunov function candidate is chosen as \( V(t,x) = |x| \). From Lemma 1, we obtain
\[
\begin{align*}
\kappa_t^{\alpha} V &= \kappa_t^{\alpha} |x| \leq \kappa_t^{\alpha} |x| = -|V|.
\end{align*}
\] (43)

By Theorem 3, we have
\[
|x(t)| \leq |x(0)| e^{-\lambda t} E_\alpha(-t^\alpha). 
\] (44)

Then, \( x = 0 \) is tempered Mittag–Leffler stable. To verify the result, we choose parameters as \( \alpha = 0.95 \) and \( x(0) = 0 \) and the tempered parameters as \( \lambda = 2, 4, 6, 8 \), respectively. The time evolution of the system states (42) is shown in Figure 1. It is presented that the system (42) converges to the equilibrium point \( x = 0 \). The larger the tempered parameter \( \lambda \) is, the faster the convergence speed becomes.

Example 2. Consider the tempered fractional Caputo Hopfield neural networks:
\[
\kappa_t^{\alpha} x_i(t) = -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(t)) + I_i, 
\] (45)

where \( 0 < \alpha < 1, \lambda \geq 0, i = 1, 2, \ldots, n \) and \( n \) is number of units. \( x_i(t) \) is the \( i \)th state, \( f_j \) is the \( j \)th activation function, \( b_{ij} \) is constant connection weight of the \( j \)th neuron on the \( i \)th neuron, \( a_i > 0 \) denotes the resting rate when the \( i \)th neuron disconnected, and \( I_i \) is the external inputs. Under the conditions

\[
|f_j(x)| \leq l_j |x|, \quad l_j > 0, \quad j = 1, 2, \ldots, n, 
\] (46)

\[
c_i = a_i - \sum_{j=1}^n |b_{ij}| l_j > 0, \quad i = 1, 2, \ldots, n, 
\] (47)

system (45) is globally tempered Mittag–Leffler stable. Let \( x(t) = (x_1, x_2, \ldots, x_n)^T \) be any solution of system (45). We choose Lyapunov function as follows:
\[
V(t,x(t)) = \sum_{i=1}^n |x_i(t)|. 
\] (48)

By inequalities (46) and (47) and Lemma 7, tempered fractional Caputo derivative on \( V(t,x(t)) \) can be written as
\[
\begin{align*}
\kappa_t^{\alpha} V(t,x(t)) &= \sum_{i=1}^n \kappa_t^{\alpha} x_i(t) \leq \sum_{i=1}^n \sum_{j=1}^n \text{sgn}(x_i(t)) \kappa_t^{\alpha} x_j(t) \\
&\leq -a_i \sum_{i=1}^n |x_i(t)| + \sum_{j=1}^n |b_{ij}| l_j |x_j(t)| \\
&= \sum_{i=1}^n \left[ -a_i + \sum_{j=1}^n |b_{ij}| l_j \right] |x_i(t)| \\
&\leq -c \| x(t) \|, 
\end{align*}
\] (49)

where \( c = \min\{c_1,c_2,\ldots,c_n\} \). From (49) and Theorem 3, system (45) is globally tempered Mittag–Leffler stable.

To illustrate the effectiveness of Example 2, in system (45), we let \( x = (x_1, x_2, x_3)^T, \alpha = 0.98, x_1(0) = 5, x_2(0) = 0, x_3(0) = 3, f_j(x_j) = \tanh(x_j) \), and \( c_j = 6 \) for \( j = 1, 2, 3 \) and
\[
A = (a_{ij})_{3 \times 3} = \begin{bmatrix} 2 & -1.2 & 0 \\ 1.8 & 1.71 & 1.15 \\ -4.75 & 0 & 1.1 \end{bmatrix}. 
\] (50)

It is obvious that condition (47) is satisfied. Let tempered parameters \( \lambda = 0, 2, 4, 6 \), respectively. As shown in Figure 2,
the equilibrium point \( x = 0 \) is tempered Mittag–Leffler stable and the solution of system (45) converges to \( x = 0 \).

Example 3. Consider the following tempered fractional system:

\[
\begin{align*}
\mathcal{C}_0 D^\alpha_t x_1 &= -2x_1 + \frac{\sin(x_3)}{1 + t^2} x_1, \\
\mathcal{C}_0 D^\alpha_t x_2 &= -2x_2 + \cos(x_1) x_2, \\
\mathcal{C}_0 D^\alpha_t x_3 &= x_3,
\end{align*}
\]

where \( 0 < \alpha < 1, \lambda \geq 0 \) and \( x(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^3 \).

Let the Lyapunov function \( V(t, x) = |x_1| + |x_2| \). By Lemma 7, we obtain

\[
\mathcal{C}_0 D^\alpha_t V(t, x) \leq \text{sgn}(x_1(t)) \mathcal{C}_0 D^\alpha_t x_1(t) + \text{sgn}(x_2(t)) \mathcal{C}_0 D^\alpha_t x_2(t)
\]

\[
= \text{sgn}(x_1(t)) \left(-2x_1(t) + \frac{\sin(x_3(t))}{1 + t^2} x_1(t)\right)
\]

\[
+ \text{sgn}(x_2(t)) \left(-2x_2(t) + \cos(x_1(t)) x_2(t)\right)
\]

\[
\leq -2|x_1(t)| + \frac{\sin(x_3(t))}{1 + t^2} |x_1(t)| - 2|x_2(t)|
\]

\[
+ \cos(x_1(t)) |x_2(t)|
\]

\[
\leq -\left(|x_1(t)| + |x_2(t)|\right).
\]

(52)
Then, the conditions of Theorem 3 are satisfied. Hence, $x = 0$ is globally tempered Mittag–Leffler stable with respect to $(x_1, x_2)$. Take $\alpha = 0.9, x_1 (0) = 10, x_2 (0) = -5, \text{ and } x_3 (0) = 5$. The numerical simulation is shown as Figure 3 with different tempered parameters $\lambda = 0, 0.5, 1, 1.5$. It is obvious $x_1 (t)$ and $x_2 (t)$ converge to 0. When tempered parameter of the system increase, the part solution of system converges faster.

### 5. Conclusions

In this paper, we present some stability results for the tempered fractional systems. Based on the Laplace transform, we obtain the comparison principle for the tempered fractional systems. Some theorems about tempered Mittag–Leffler stability are derived, which enrich the knowledge of the system theory and the tempered fractional calculus and are helpful in characterizing the tempered fractional system models. Furthermore, we will study stability of tempered fractional systems with time-varying delays in future work.

### Data Availability

The authors affirm that all data necessary for confirming the conclusions of the article are present in the article.

### Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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