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Research Article

Reliability Analysis of *k*-out-of-*n* Systems of Components with Potentially Brittle Behavior by Universal Generating Function and Linear Programming

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Due to initial cracks, careless construction, and extreme load conditions, components with brittle behavior may exist in a structural system. The presence of brittle behavior of components usually is accompanied by a low strength. However, existing methods for calculating the reliability of structures of components with brittle behavior are rather complicated or impossible. By means of decomposing the entire system into a set of subsystems, this paper proposed a method to estimate the bounds on failure probability of *k*-out-of-*n* system of components with potentially brittle behavior by using universal generating function (UGF) and linear programming (LP). Based on the individual component state probabilities and joint probabilities of the states of a small number of components, the proposed method can provide the bounds for the failure probability of a system with a large number of components. The accuracy and efficiency of the proposed method are investigated using numerical examples.

1. Introduction

A *k*-out-of-*n* system can be defined as a system with *n* components which functions if and only if *k* or more of the components function. The *k*-out-of-*n* system is one of the most popular and widely used systems in practice. Both series and parallel systems are special cases of the *k*-out-of-*n* system. A series system is equivalent to an *n*-out-of-*n* system, while a parallel system is equivalent to a 1-out-of-*n* system. A series of effective reliability methods of systems have been developed, including the first-order reliability method (FORM) [1], second-order reliability method (SORM) [2], logical differential calculus [3–5], Monte Carlo (MC) simulations [6], and so on [7–12]. However, most methods have assumed that there are ideal structures with perfectly ductile components. The ductile components have perfectly elastic-plastic behavior, i.e., the ductile component retains its load-bearing capacity after ductile failure. However, there could be components with brittle behavior in reality. The perfectly brittle failure of components indicates that deformation is zero after the peak capacity has been reached. The intermediate semibrittle behavior is also possible.

Bennett introduced a method of analysis for determining the reliability of frames with a few brittle components [13]. Based on the assumption of monotonic loading and the failure graph of frame structure, an accurate estimate of failure probability can be obtained. Heidweiller and Vrouwenvelder proposed a method of analysis for calculating the reliability of structures with potentially brittle components [14]. Important fundamental assumptions of the method are that the probability of the presence of a brittle component is relatively low and that only one component has brittle failure in the structure. A few research studies for evaluating the reliability of structures with brittle components have been done [15, 16]. It has been noted that most methods of the above research are rather complicated.

Because of the complexity and the incomplete probability information of structure, the exact evaluation of structural failure probability is often time-consuming or unavailable. In order to reduce the computational complexity, methods of computing bounds on structural failure probability have been proposed [17, 18]. Based on the individual component failure probabilities and the joint failure probabilities of components, the linear programming (LP) bounds methods for determining the upper and lower bounds on structural failure probability by using LP have been proposed [19, 20]. In order to overcome the size problem of the LP bounds method, the relaxed linear programming (RLP) bounds method has been proposed. It employs the universal generating function (UGF) to reduce the number of design variables from 2^n in LP bounds method to $n^2 - n + 2$ [21, 22]. However, the above methods of computing bounds are used for the structural system of components with ductile behavior, not for the structural system especially for the k-out-of-n system of components with potentially brittle behavior.

In order to accurately evaluate the failure probability of k-out-of-n system, a method will be proposed in this paper for components with potentially brittle behavior. Based on the individual component failure probabilities and the joint failure probabilities of a few components, the proposed method can obtain narrow bounds by using UGF and LP. The accuracy and applicability of proposed method will be demonstrated using numerical examples.

2. Fundamental Assumptions

2.1. Overview. Because of the complexity of properties of actual material, it is impossible to divide the behavior of a component by drawing a sharp demarcation. For the sake of simplicity, three basic behaviors of a component have been distinguished artificially in this paper as shown in Figure 1 [23].

The perfectly brittle behavior of a component means that there is no further capacity of displacement when the peak capacity has been reached. The component would completely lose its capacity after brittle failure. A component can be called perfectly ductile if it has sufficient displacement capacity. The component could maintain its load-bearing capacity after ductile failure. The intermediate mode can be called semibrittle, and the component would maintain partial load-bearing capacity after failure. For the sake of simplicity, this paper will focus on the structure of components with perfectly ductile behavior and perfectly brittle behavior, and the influence of the difference in the behavior of a component, either brittle or ductile, is only the statistical characteristics of the strength of a component depending on whether it is brittle or ductile.

2.2. Fundamental Assumptions. Most components in the structure and the governing components related to the failure should be ductile in reality. The presence of brittle behavior of a component often results from the initial cracks, the lack of care in construction, and the extreme loading conditions. On the basis of these considerations, the following fundamental assumptions have been arrived:

(1) The brittle behavior and ductile behavior of a component cannot exist at the same time

(2) The presence of brittle or ductile behavior of a component is independent of each other

(3) The probability of the presence of brittle behavior of a component is low

(4) The presence of brittle behavior of a component is independent of that of another

(5) The correlation coefficients among components have no change when the brittle behavior appears

For an ideal brittle component, its failure causes an immediate or a somewhat delayed redistribution of internal forces. Sometimes, dynamic effects need to be considered. Thus, based on the above fundamental assumptions, the concept that a component with potentially brittle behavior, which means an imperfectly ductile component could exhibit brittle behavior with some probability, has been introduced. For the sake of simplicity, both for brittle failure of an imperfectly ductile component (a component with potentially brittle behavior) and for subsequent failure of structure, it considers only just a few obvious dominant mechanisms. The redistribution of internal forces and the dynamic effects are ignored in this paper. For an imperfectly ductile component, the correlation coefficient with other components is independent of whether there is ductile behavior or brittle behavior. The method for evaluating the reliability of k-out-of-n system of components with potentially brittle behavior will be introduced in the next section.

3. Methodology

3.1. Reliability Analysis Procedure. Considering a structure with n components, the behavior of most components is ductile, and the other is potentially brittle. Since the ductile components should be the governing components that relate to the failure in reality, the main mode of structural failure is the plastic failure. According to the presence of brittle behavior of components, the failure of a structure can be divided into two categories, i.e., ductile failure mode or brittle-ductile failure mode, as shown in Figure 2. If there is no brittle behavior of components in a structure, the structural failure can be considered as the ductile failure mode. Otherwise, the structural failure should be considered as the brittle-ductile failure mode.

According to the fundamental assumptions in Section 2, the brittle behavior and ductile behavior of a component are mutually exclusive events. Thus, the ductile failure mode and the brittle-ductile failure mode of structure are mutually exclusive events. The failure probability of series structure can be expressed as

$$P_S = P_b \times P_{S_b} + (1 - P_b) \times P_{S_d}, \tag{1}$$

where S represents the structural system and P_S is its failure probability; S_b is the category of subsystems with brittle-ductile failure mode, P_{s_b} is the corresponding failure

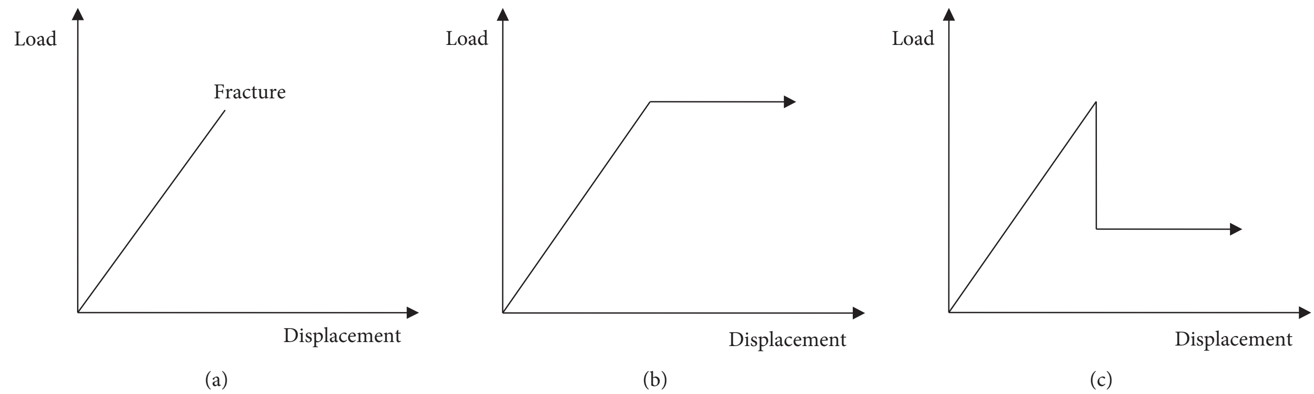


Figure 1: Force-displacement relationships for three types of a component behavior. (a) Perfectly brittle behavior. (b) Perfectly ductile behavior. (c) Semibrittle behavior.

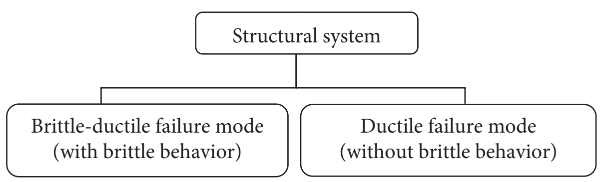


Figure 2: Schematic diagram of structural failure analysis.

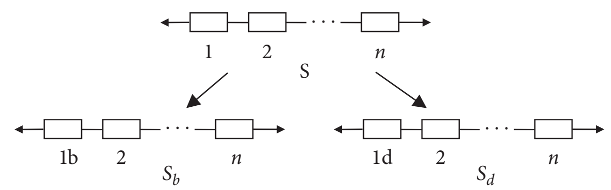


Figure 3: Schematic diagram of subsystems of series structure (component 1 with potential brittle behavior).

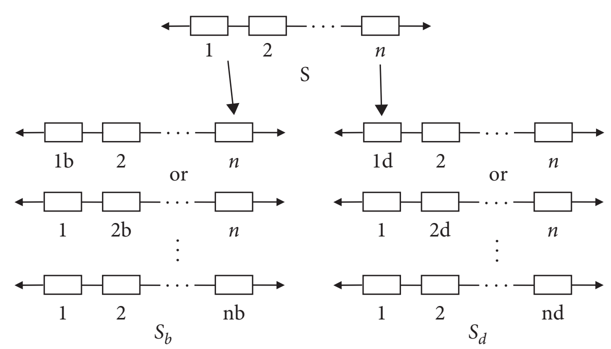


Figure 4: Schematic diagram of subsystems of series structure (one component with potential brittle behavior).

probability, and P_b is the probability of presence of brittle-ductile failure mode; and S_d is the category of subsystem with ductile failure mode, P_{s_d} is its failure probability, and $(1 - P_b)$ is the probability of presence of ductile failure mode.

3.1.1. One Component with Potentially Brittle Behavior. When a structure only includes one known component with potentially brittle behavior, the structural system can be divided into two categories of subsystems (S_b and S_d). For example, if component 1 in a series structure is a component with potentially brittle behavior, this series structure can be divided into two subsystems (S_b and S_d) as shown in Figure 3.

If the structure only includes one known component with potentially brittle behavior, the reason of presence of brittle-ductile failure mode is the presence of brittle behavior of component i. Then, P_b (the probability of presence of brittle-ductile failure mode) can be expressed as P_{b_i}(the probability of presence of brittle behavior of component i). Therefore, $1 - P_b$ (the probability of presence of ductile failure mode) can be expressed as $1 - P_{b_i}$ (the probability of presence of ductile behavior of component i). The above structural failure probability can be expressed as

$$P_S = P_{b_i} \times P_{S_b} + \left(1 - P_{b_i}\right) \times P_{S_d}. \qquad (2)$$

Taking the series system as the example, when the series number of component with potentially brittle behavior is unknown, the structure can be divided into one of combination of two categories of subsystems (S_b and S_d) as shown in Figure 4. The maximum failure probability of combination in Figure 4 is the upper bound of failure probability of this series system. The combination of subsystems (S_b and S_d) of other k-out-of-n systems is similar to the series system. The maximum failure probability of combination would be taken as the structural failure probability in this paper. Then, the failure probability can be expressed as

$$P_S = \max\left[p_{b_i} \times P_{S_b} + \left(1 - p_{b_i}\right) \times P_{S_d}\right]; \quad i \in (1, n). \qquad (3)$$

3.1.2. Two Components with Potentially Brittle Behavior. When a structure only includes two known components with potentially brittle behavior, the structure can be divided into three categories of subsystems (S_{b1}, S_{b2}, and S_d). For example, if components 1 and 2 in a parallel structure are components with potentially brittle behavior, this parallel structure can be divided into three subsystems (S_{b1}, S_{b2}, and S_d) as shown in Figure 5. S_{b1} is the category of subsystems

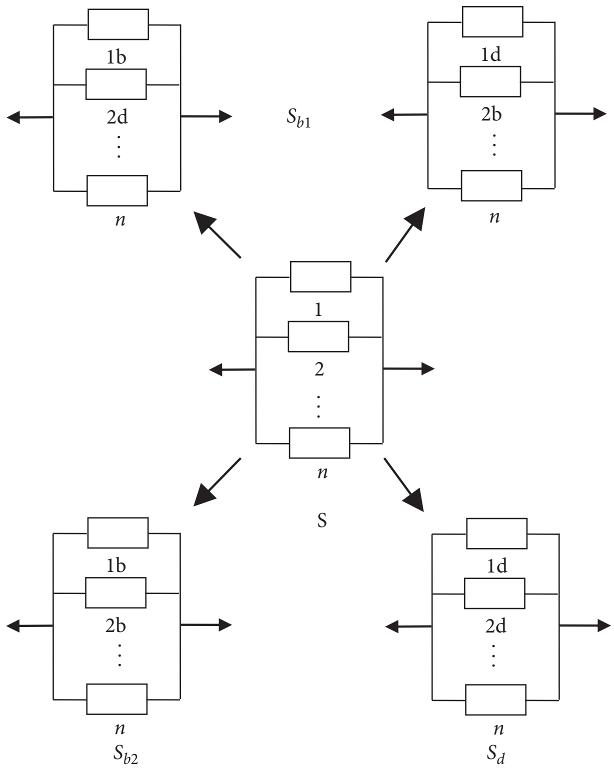


Figure 5: Schematic diagram of subsystems of parallel structure (two components with potential brittle behavior).

that only one component with brittle behavior, S_{b2} is the category of subsystems that two components with brittle behavior, and S_d is the category of subsystems that components only with ductile behavior. The structural failure probability can be expressed as

$$P_S = \left[P_{b_i}\left(1 - P_{b_j}\right) + P_{b_j}\left(1 - P_{b_i}\right)\right]P_{S_{b1}} + P_{b_{i,j}}P_{S_{b2}} + \left(1 - P_{b_i}\right)\left(1 - P_{b_j}\right)P_{S_d}, \quad i \neq j;\ i \in (1, n);\ j \in (1, n), \tag{4}$$

where $P_{S_{b1}}$ is the failure probability of category of subsystems (S_{b1}) that only one component with brittle behavior; $P_{S_{b2}}$ is the failure probability of category of subsystems (S_{b2}) that only two component with brittle behavior, $P_{b_{i,j}}$ denotes $P_{b_i} \times P_{b_j}$; P_{S_d} is the failure probability of category of subsystem (S_d) that components only with ductile behavior.

It has been noted that S_{bi} is not a single subsystem, but a category of subsystems. For example, S_{b1} in equation (4) is the subsystems that only components *i* or components *j* with brittle behavior. Therefore, the probability of presence of S_{b1} can be expressed as $P_{b_i}(1 - P_{b_j}) + P_{b_j}(1 - P_{b_i})$. S_{b2} in equation (4) is the subsystems that both components *i* and *j* with brittle behavior. According to fundamental assumptions in Section 2.2, the probability of presence of S_{b2} can be expressed as $P_{b_i} \times P_{b_j}$. S_d in equation (4) is the subsystem that both components *i* and *j* with ductile behavior. Therefore, the probability of presence of S_d can be expressed as $(1 - P_{b_i})(1 - P_{b_j})$.

Similarly, when the series number of components with potentially brittle behavior is unknown, the structure can be divided into one of combination of three categories of subsystems (S_{b1}, S_{b2}, and S_d). The maximum failure probability of combination would be taken as the failure probability of series structure in this paper. Then, the failure probability can be expressed as

$$P_S = \max\left\{\left[P_{b_i}\left(1 - P_{b_j}\right) + P_{b_j}\left(1 - P_{b_i}\right)\right]P_{S_{b1}} + P_{b_{i,j}}P_{S_{b2}} + \left(1 - P_{b_i}\right)\left(1 - P_{b_j}\right)P_{S_d}\right\}, \quad i \neq j;\ i \in (1, n);\ j \in (1, n). \tag{5}$$

3.1.3. n_1 Components with Potentially Brittle Behavior. Similarly, when a structure includes n_1 known components with potentially brittle behavior, its failure probability can be expressed as

$$\begin{aligned} P_s = & \left[P_{b_i}\prod_{m\in A;m\neq i}\left(1 - p_{b_m}\right) + P_{b_j}\prod_{m\in A;m\neq j}\left(1 - P_{b_m}\right) + \cdots + P_{b_r}\prod_{m\in A;m\neq r}\left(1 - P_{b_m}\right)\right]P_{S_{b1}} \\ & \cdot\left[P_{b_{i,j}}\prod_{m\in A;m\neq i\neq j}\left(1 - p_{b_m}\right) + P_{b_{i,k}}\prod_{m\in A;m\neq i\neq k}\left(1 - P_{b_m}\right) + \cdots + P_{b_{q,r}}\prod_{m\in A;m\neq q\neq r}\left(1 - P_{b_m}\right)\right]P_{S_{b2}} \\ & + \left[P_{b_{i,j,k}}\prod_{m\in A;m\neq i\neq j\neq k}\left(1 - p_{b_m}\right) + \cdots + P_{b_{p,q,r}}\prod_{m\in A;m\neq p\neq q\neq r}\left(1 - P_{b_m}\right)\right]P_{S_{b3}} + \cdots \\ & + P_{b_{i,j,k,\cdots,p,q,r}}P_{S_{bn_1}} + \prod_{m\in A}\left(1 - P_{b_m}\right)P_{S_d}, \\ & i\neq j\neq k\neq\cdots\neq p\neq q\neq r; i\in(1,n), j\in(1,n), k\in(1,n), \cdots, q\in(1,n), r\in(1,n); \\ & \cdot\{i, j, k, \cdots, p, q, r\}\in A; n_1\leq n, \end{aligned} \tag{6}$$

where n is the number of components and n_1 is the number of components with potentially brittle behavior; $i, j, k, \ldots, p, q, r$, are the series numbers of components with potentially brittle behavior; A is the set of them; P_{b_i} is the probability of presence of brittle behavior of component i; $1 - P_{b_i}$ is the probability of presence of ductile behavior of component i; $P_{b_{i,j}}$ denotes $P_{b_i} \times P_{b_j}$; $P_{b_i}\prod_{m\in A;m\neq i}(1-P_{b_m}) + P_{b_j}\prod_{m\in A;m\neq j}(1-P_{b_m}) + \cdots + P_{b_r}\prod_{m\in A;m\neq r}(1-P_{b_m})$ is the probability of presence of category of subsystems (S_{b1}) only one component with brittle behavior; and $P_{S_{b1}}$ is the failure probability of category of subsystems (S_{b1}); other symbols have similar meanings.

When the series number of components with potentially brittle behavior is unknown, the structure can be divided into one of combination of (n_1+1) categories of subsystems (S_{b1}, S_{b2}, ..., S_{bn1}, and S_d). The maximum failure probability of combination would be taken as the structural failure probability in this paper. Then, the failure probability can be expressed as

$$\begin{aligned} P_s = \max\Bigg\{ & \left[P_{b_i}\prod_{m\in A;m\neq i}\left(1-P_{b_m}\right) + P_{b_j}\prod_{m\in A;m\neq j}\left(1-P_{b_m}\right) + \cdots + P_{b_r}\prod_{m\in A;m\neq r}\left(1-P_{b_m}\right)\right]P_{S_{b1}} \\ & + \left[P_{b_{i,j}}\prod_{m\in A;m\neq i\neq j}\left(1-P_{b_m}\right) + P_{b_{i,k}}\prod_{m\in A;m\neq i\neq k}\left(1-P_{b_m}\right) + \cdots + P_{b_{q,r}}\prod_{m\in A;m\neq q\neq r}\left(1-P_{b_m}\right)\right]P_{S_{b2}} \\ & + \left[P_{b_{i,j,k}}\prod_{m\in A;m\neq i\neq j\neq k}\left(1-P_{b_m}\right) + \cdots + P_{b_{p,q,r}}\prod_{m\in A;m\neq p\neq q\neq r}\left(1-P_{b_m}\right)\right]P_{S_{b3}} + \cdots \\ & + P_{b_{i,j,k,\cdots,p,q,r}}P_{S_{bn_1}} + \prod_{m\in A}\left(1-P_{b_m}\right)P_{S_d}\Bigg\}, \\ & i\neq j\neq k\neq \cdots \neq p\neq q\neq r; i\in(1,n), j\in(1,n), k\in(1,n), \cdots, q\in(1,n), r\in(1,n); \\ & \cdot\{i,j,k,\cdots,p,q,r\}\in A; n_1\leq n. \end{aligned} \tag{7}$$

3.2. Reliability Bounds. Usually, the structural failure probability can be expressed as

$$P_S = \int\cdots\int_{g_x(X)\leq 0} f_X(x_1, x_2, x_3, \ldots x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n, \tag{8}$$

where $g_x(X)$ is the limit state function and $f_X(x)$ is the joint probability density function for the n-dimensional vector X of basic variables. Since the structural state consists of component states, the basic variables X can also be expressed as the variables of component states in equation (8). However, even for 2 possible states components (safe state and failure state), when the components states are not independent, this is extremely difficult for calculating multiple integral.

3.2.1. Revised RLP Bounds Method for K-out-of-n Systems. Chang and Mori proposed a RLP bounds method to calculate the bounds on failure probability of a structure only with 2 possible state components (ductile safe state and ductile failure state) [21, 22]. A conceptually simplified UGF of the system can be expressed as

$$\begin{aligned} U(z) = {} & p_1 z^0 + p_2 z^{x_1} + p_3 z^{x_2} + \ldots + p_{n+1} z^{x_n} \\ & + p_{n+2} z^{2x_1} + p_{n+3} z^{2x_2} + \ldots + p_{2n+1} z^{2x_n} \\ & + p_{2n+2} z^{3x_1} + p_{2n+3} z^{3x_2} + \ldots + p_{3n+1} z^{3x_n} + \ldots \\ & + p_{(n-2)n+2} z^{(n-1)x_1} + p_{(n-2)n+3} z^{(n-1)x_2} + \ldots \\ & + p_{(n-1)n+1} z^{(n-1)x_n} + p_{n^2-n+2} z^{nx}, \end{aligned} \tag{9}$$

where z^x is called the z-transform of x [24–26], 0 of z^0 encodes the subset of the state in which no component survives, and p_1 is the probability corresponding to the state encoded by the 0 of z^0; x_1 of z^{x_1} encodes the subset of the state in which only component 1 survives, and p_2 is the probability corresponding to the state encoded by the x_1 of z^{x_1}; $2x_1$ of z^{2x_1} encodes the portion of subset of state for which only two components including component 1 survive, p_{n+2} is the probability corresponding to the state encoded by the $2x_1$ of z^{2x_1}, and so on.

The lower bound and the upper bound of the system failure probability are obtained as the minimum and the maximum of the objective function of the LP, respectively. The equation of LP appropriate for this analysis has the following form:

$$\begin{aligned} & \text{minimize (maximize)} && C^T p \\ & \text{subject to} && A_{1_p} = B_1 \\ & && A_{2_p} \geq B_2, \end{aligned} \tag{10}$$

where $\mathbf{p} = \{p_1, p_2, \ldots, p_{n^2-n+2}\}$ is the vector of design variables and represents the probabilities of the basic events in equation (10); $\mathbf{C}$ is a matrix that relates the system failure event with the component failure events; $\mathbf{C}^T\mathbf{p}$ is the linear objective function; and $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{B_1}$, and $\mathbf{B_2}$ are the coefficient matrices and vectors, respectively, which represent the information given in terms of joint failure probabilities of k components. The above probabilities ($\mathbf{B_1}$ and $\mathbf{B_2}$) associating with the related matrices ($\mathbf{A_1}$ and $\mathbf{A_2}$) consist of the constraints of the LP problem.

Taking a system with 3 components as an example, the UGF of the system can be expressed as

$$U(z) = p_1 z^0 + p_2 z^{x_1} + p_3 z^{x_2} + p_4 z^{x_3} + p_5 z^{2x_1} + p_6 z^{2x_2} + p_7 z^{2x_3} + p_8 z^{3x}, \tag{11}$$

where p_1 is the probability corresponding to the state encoded by 0 of z^0 (no component survives); p_2 is the probability corresponding to the state encoded by x_1 of z^{x_1} (only component 1 survives); p_5 is the probability corresponding to the state encoded by $2x_1$ of z^{2x_1}(only two components including component 1 survive), and so on.

The bounds of failure probability of components 1, 2, and 3 can be expressed as

$$\begin{aligned} P(F_1) &> p_1 + p_3 + p_4 \\ &< p_1 + p_3 + p_4 + p_6 + p_7, \\ P(F_2) &> p_1 + p_2 + p_4 \\ &< p_1 + p_2 + p_4 + p_5 + p_7, \\ P(F_3) &> p_1 + p_2 + p_3 \\ &< p_1 + p_2 + p_3 + p_5 + p_6. \end{aligned} \tag{12}$$

$p_1 + p_3 + p_4$ corresponds to the probabilities of states that no component survives and only one component except component 1 survives. $p_5 + p_6 + p_7$ corresponds to the probabilities of states that only two components survives. It is obvious that $P(F_1)$, the failure probability of component 1, is greater than $p_1 + p_3 + p_4$ and smaller than $p_1 + p_3 + p_4 + p_5 + p_6 + p_7$. Because p_5 is the probability corresponding to the state that only two components including component 1 survive, it can be excluded from the inequality. The other inequalities can be derived similarly.

The sum of failure probability of components 1, 2, and 3 can also be expressed as

$$P(F_1) + P(F_2) + P(F_3) = 3p_1 + 2(p_2 + p_3 + p_4) + p_5 + p_6 + p_7. \tag{13}$$

Note that $\mathbf{p} = \{p_1, p_2, \ldots, p_8\}$ is the vector of design variables, and it can be obtained by using LP. If one knows $P(F_1) = 0.01$, $P(F_2) = 0.02$, and $P(F_3) = 0.03$, $\mathbf{A_1}$, $\mathbf{B_1}$, $\mathbf{A_2}$, and $\mathbf{B_2}$ in equation (10) can be expressed as

$$\begin{aligned} A_1 &= [3\ 2\ 2\ 2\ 1\ 1\ 1\ 0], \\ B_1 &= [0.06], \\ A_2 &= \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 0.01 \\ -0.01 \\ 0.02 \\ -0.02 \\ 0.03 \\ -0.03 \end{bmatrix}. \end{aligned} \tag{14}$$

By defining the matrix C^T as [1 1 . . . 1 0] or [1 0 . . . 0], the original RLP bounds method can solve problems involving only a pure series system or a pure parallel system. When C^T is expressed as [1 1 . . . 1 0], $\mathbf{C}^T\mathbf{p}$ means the objective function for the failure of a series system that as least any one of components failed; when C^T is expressed as [1 0 . . . 0], $\mathbf{C}^T\mathbf{p}$ means the objective function for the failure of a parallel system that every one of components failed. Obviously, one main drawback of the RLP method is that it is only available to a pure series system or a pure parallel system.

It can be found that the objective function $\mathbf{C}^T\mathbf{p}$ could have different means when matrix C^T is changing. The objective function $\mathbf{C}^T\mathbf{p}$ for the survival probability of k-out-of-n systems can be obtained when C^T is expressed as

$$C^T = \left[0\ \underbrace{\underbrace{0 \cdots 0}_{n}\ \underbrace{0 \cdots 0}_{n}\ \cdots}_{(k-1)n}\ \underbrace{\underbrace{1 \cdots 1}_{n}\ \underbrace{1 \cdots 1}_{n}\ \cdots}_{(n-k)n}\ 1 \right]. \tag{15}$$

Then, a revised RLP bounds method for k-out-of-n systems is proposed by changing matrix C^T as equation (15). For example, for the above system with 3 components, the objective function for the survival probability of 1-out-of-n systems can be obtained when C^T is expressed as

$$C^T = [0\ 1\ 1\ 1\ 1\ 1\ 1\ 1]. \tag{16}$$

The objective function for the survival probability of 2-out-of-n systems can be obtained when C^T is expressed as

$$C^T = [0\ 0\ 0\ 0\ 1\ 1\ 1\ 1]. \tag{17}$$

It should be noted that the above objective function $\mathbf{C}^T\mathbf{p}_m$ for k-out-of-n systems is the survival probability of k-out-of-n systems, and its failure probability can be obtained by subtracting the survival probability from 1.

3.2.2. Bounds of System of Components with Potentially Brittle Behavior. For a component with potentially brittle behavior, it has 4 possible states, i.e., brittle safe state, brittle failure state, ductile safe state, and ductile failure state. The calculation of failure probability of a structure of components with potentially brittle behavior is very difficult. According to equation (7), the failure probability of such a structure can be expressed as the sum of failure probability of a series of subsystems, S_{b1}, S_{b2}, . . ., S_{bn1}, and S_d. A component in any one of above subsystems has either brittle behavior or ductile behavior, which means each component has only 2 possible state components, i.e., either brittle safe state and brittle failure state or ductile safe state and ductile failure state.

Obviously, the state of each subsystem in equation (7), S_{b1}, S_{b2}, . . ., S_{bn1}, and S_d, can be expressed as its component states. Each subsystem can be expressed a conceptually simplified UGF by using equation (9), and its failure probability can be obtained by the revised RLP bounds method. Thus, by using the revised RLP bounds method, the failure probability of a structure of components with potentially brittle behavior can be expressed as the sum of failure probability of a series of subsystems, S_{b1}, S_{b2}, . . ., S_{bn1}, and S_d, as shown in Figure 6.

4. Numerical Example

4.1. Roof Truss with 47 Components. Consider a symmetrical roof truss of factory building with 47 components as a large series structure as shown in Figure 7. The left symmetrical distribution of internal force is shown in Figure 8. For the sake of simplicity, suppose the intensity of the load is deterministic, and the component strengths, i.e., the random variables X_i, $i = 1, 2, \ldots, 47$, are jointly normally distributed. Also, the reliability index of components with ductile behavior is assumed to be the same, and the reliability index of components with brittle behavior is also assumed to be the same as shown in Table 1. Suppose n_1 could be 1, 2 and 3, and its series number is unknown. Then, the failure probability of this truss can be obtained by equation (7).

4.1.1. One Component with Potentially Brittle Behavior ($n_1 = 1$). The probability of presence of brittle behavior of each component, i.e. P_{bi}, $i \in (1, n)$, has been supposed to be identical. When n_1 equals to 1, the relationship between structural failure probability and P_{bi} is shown in Table 2 and Figure 9. Obviously, with the increase of P_{bi}, the structural failure probability increases; along with the increase of correlation relationship among components, its bounds become wider. There are two reasons for the above wider bounds. One is that the bounds of failure probability by LP will become wider with the increase of the correlation coefficient. The other is related to the assumption that the failure probability can be estimated by the maximum of the combination considered such as equation (3). However, even if the bound of failure probability becomes wider with the increase of correlation coefficient, the bound of failure probability is relatively small. The bounds are still acceptable. Taking $P_{bi} = 0.05$, $i \in (1, n)$ as the example, one can find that with the increase of correlation coefficient among components, the failure probability decreases as shown in Figure 10. Also, the structural failure probability is estimated by the MC simulations with 10^7 samples. The result of proposed method is consistent with that of MC simulations.

4.1.2. Comparing the Effects of Different n_1. The structural failure probability obtained by the proposed method is shown in Table 3 and Figure 11. Similarly, the figure of relationship between structural failure probability and P_{bi} when n_1 equals to 2 or 3 is similar to its figure when n_1 equals to 1, and it has the same regular pattern as shown in Section 4.1.1 for structural failure probability, correlation coefficient, and P_{bi}. Also, the structural failure probability is estimated by the MC simulations as shown in Figure 12. MC simulations are conducted with 10^7 samples. Similar to the results of n_1 equals to 1, the result of proposed method is consistent with that of MC simulations.

4.2. Parallel System with 20 Components. Consider a parallel system with 20 components. The correlation coefficient is considered identical, and the reliability index of components is shown in Table 4. Suppose n_1 could be 1, 2, and 3, and its series number is unknown. Then, the failure probability of this parallel system can be obtained by equation (7).

4.2.1. One Component with Potentially Brittle Behavior ($n_1 = 1$). Similarly, the probability of presence of brittle behavior of each component, i.e. P_{bi}, $i \in (1, n)$, has been supposed to be identical in this example. When n_1 equals to 1, the structural failure probability is obtained by the proposed method. The relationship between structural failure probability and P_{bi} is shown in Table 5 and Figure 13; then, taking $P_{bi} = 0.05$, $i \in (1, n)$ as the example, the relationship between correlation coefficient and structural failure probability is shown in Figure 13. From Figure 13, Figure 14, and Table 5, one can find that it has the same regular pattern for structural failure probability, correlation coefficient, and P_{bi} as shown in Section 4.1.1. Also, the structural failure probability is estimated by the MC simulations with 10^7 samples. The result of proposed method is consistent with that of MC simulations.

4.2.2. Comparing the Effects of Different n_1. The structural failure probability obtained by the proposed method is shown in Table 6 and Figure 15. Similarly, the figure of relationship between structural failure probability and P_{bi} when n_1 equals to 2 or 3 is similar to its figure when n_1 equals to 1, and it has the same regular pattern as shown in Section 4.1.1 for structural failure probability, correlation coefficient, and P_{bi}. Also, the structural failure probability is estimated by the MC simulations with 10^7 samples as shown in Figure 16. Similar to the results of n_1 equals to 1, the result of RLP bounds method is consistent with that of MC simulations.

4.3. k-out-of-n System. For comparison purposes, the parallel structural system in Section 4.2 is considered as a k-out-of-n system in this Section. Since a parallel system is equivalent to a 1-out-of-n system, the 3-out-of-20 system is considered in this example.

4.3.1. One Component with Potentially Brittle Behavior ($n_1 = 1$). The relationship between structural failure probability and P_{bi} is shown in Table 7 and Figure 17, and the

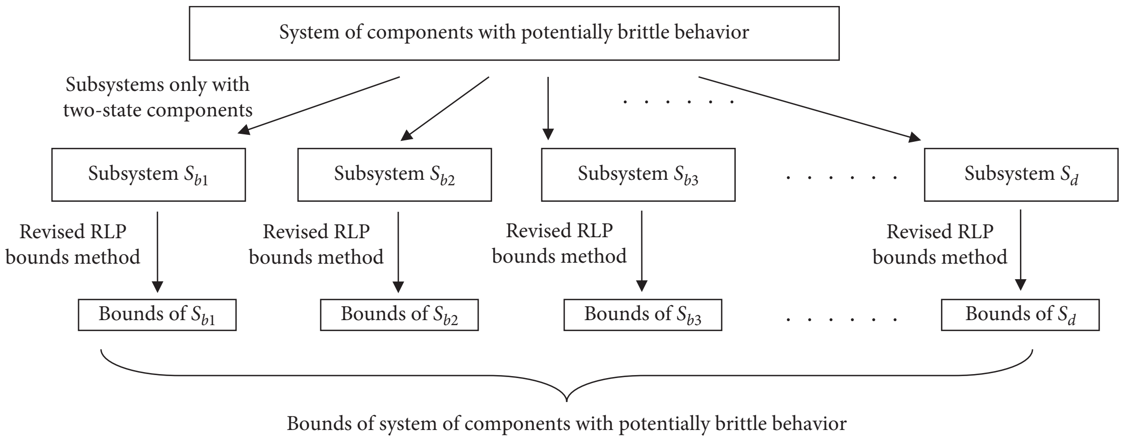


Figure 6: Schematic diagram of bounds of system of components with potentially brittle behavior.

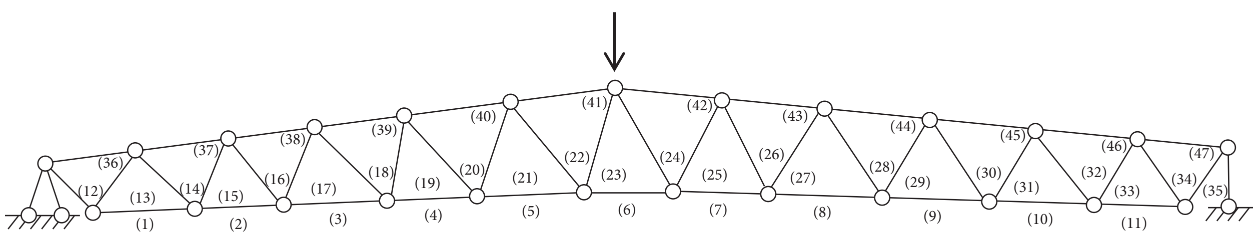


Figure 7: Roof truss of factory building as a large series structure.

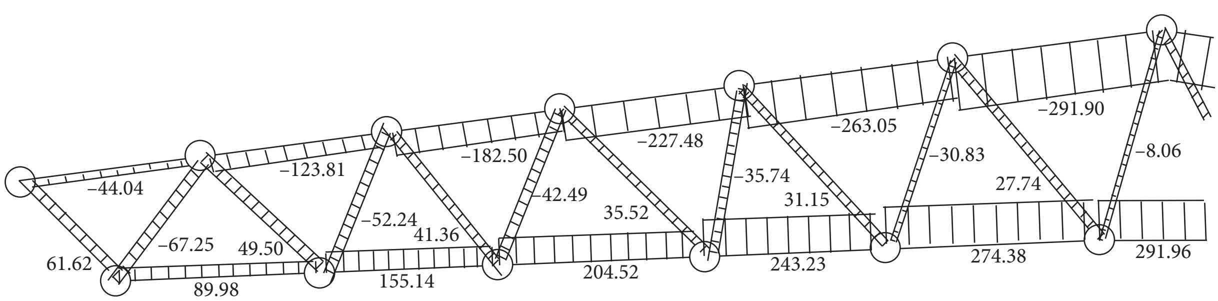


Figure 8: Left symmetrical distribution of internal force of roof truss.

Table 1: Basic parameters of components with potentially brittle behavior.

Series number of components	Behavior of component	Reliability index
$i = 1, 2, \ldots, 47$	Ductile	3.42
	Brittle	2.60

Table 2: Failure probability of roof truss of components with potentially brittle behavior.

P_{bi} $i \in (1, n)$	Bounds by RLP bounds method ($\times10^{-2}$)					
	$\rho = 0.1$		$\rho = 0.5$		$\rho = 0.9$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
0.05	1.451	1.454	0.641	1.198	0.119	0.598
0.1	1.472	1.477	0.652	1.216	0.137	0.615
0.2	1.514	1.524	0.674	1.252	0.174	0.650
0.3	1.556	1.571	0.696	1.288	0.211	0.685

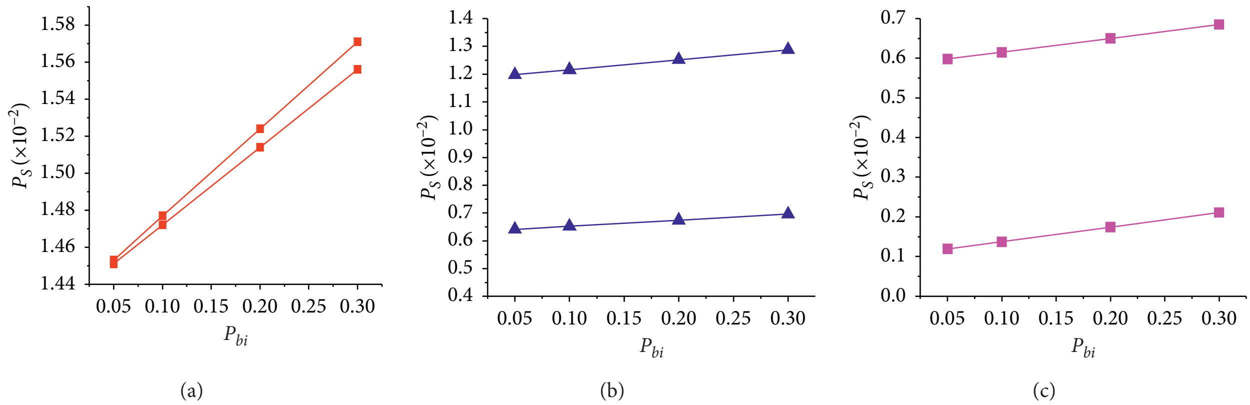


FIGURE 9: Relationship between structural failure probability and P_{bi}. (a) $\rho = 0.1$. (b) $\rho = 0.5$. (c) $\rho = 0.9$.

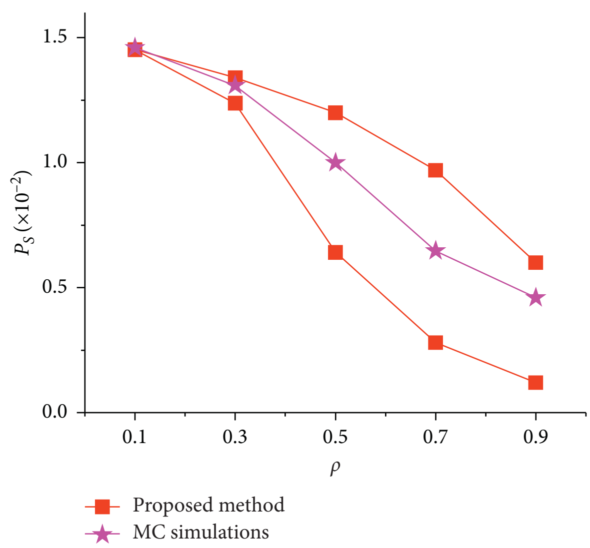


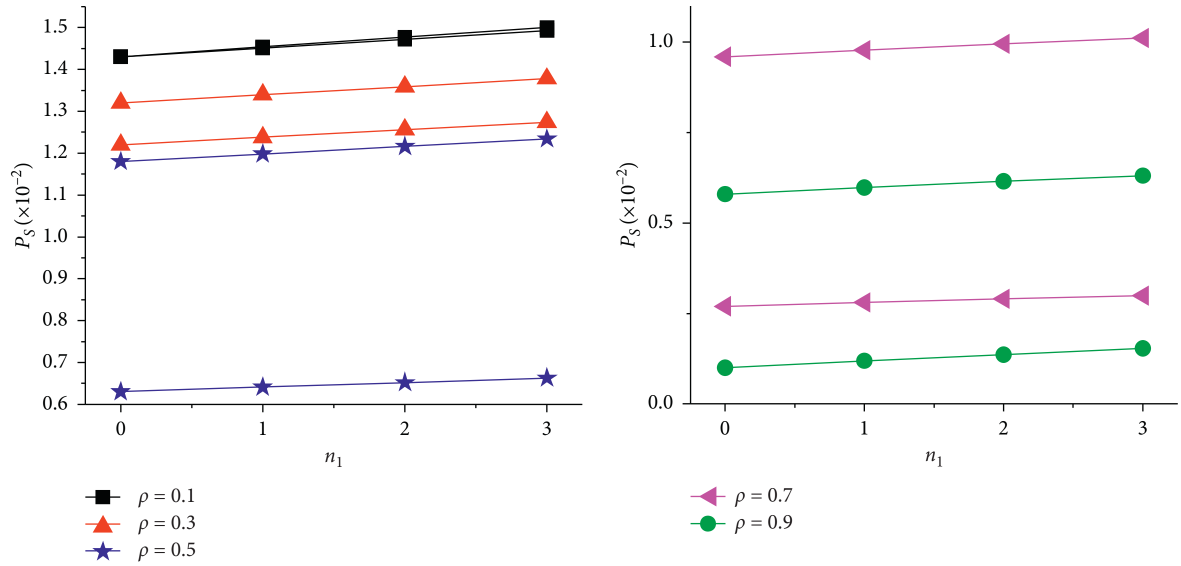
FIGURE 10: Relationship between structural failure probability and correlation coefficient.

TABLE 3: Failure probability for different ρ and n_1.

ρ	Bounds on failure probability ($\times 10^{-2}$)							
	$n_1 = 0$		$n_1 = 1$		$n_1 = 2$		$n_1 = 3$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
0.1	1.430	1.430	1.451	1.454	1.472	1.477	1.493	1.500
0.3	1.220	1.320	1.238	1.340	1.256	1.359	1.274	1.378
0.5	0.630	1.180	0.641	1.198	0.652	1.216	0.663	1.234
0.7	0.270	0.960	0.280	0.978	0.290	0.995	0.299	1.012
0.9	0.100	0.580	0.119	0.598	0.136	0.615	0.153	0.631

relationship between correlation coefficient and structural failure probability is shown in Figure 18. From Figure 17, Figure 18, and Table 7, one can find that it has the same regular pattern for structural failure probability, correlation coefficient, and P_{bi} as shown in Section 4.2.1. It should be noted that the failure probability increases with the increase of k in k-out-of-n system. Also, the structural failure probability is estimated by the MC simulations with 10^7 samples. The result of proposed method is consistent with that of MC simulations.

4.3.2. Comparing the Effects of Different n_1. The structural failure probability obtained by the proposed method is



(a) (b)

Figure 11: Relationship between structural failure probability and n_1. (a) $\rho = 0.1$, $\rho = 0.3$, and $\rho = 0.5$. (b) $\rho = 0.7$ and $\rho = 0.9$.

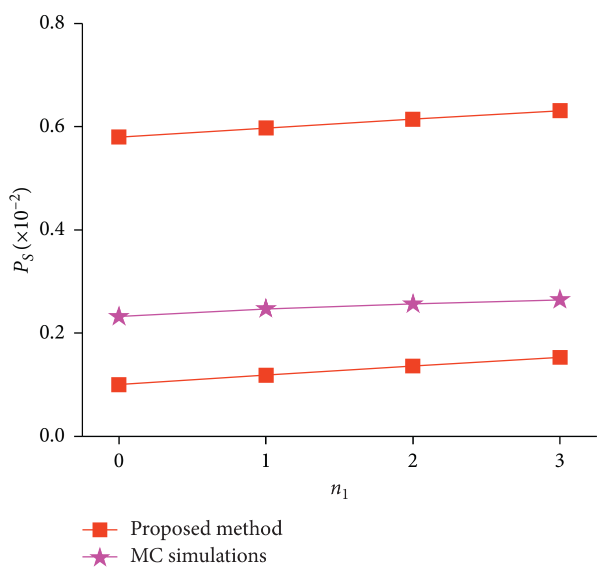


Figure 12: Relationship between structural failure probability and n_1 for different methods ($\rho = 0.9$).

Table 4: Basic parameters of components with potentially brittle behavior.

Series number of components	Behavior of component	Reliability index
$i = 1, 2, \ldots, 10$	Ductile Brittle	2.0 1.5

Table 5: Failure probability of parallel system of components with potentially brittle behavior.

P_{bi} $i \in (1, n)$	Bounds by RLP bounds method (×10^{-3})					
	$\rho = 0.1$		$\rho = 0.5$		$\rho = 0.9$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
0.05	0	0.050	0	1.105	0	9.515
0.1	0	0.050	0	1.110	0	9.530
0.2	0	0.050	0	1.120	0	9.560
0.3	0	0.050	0	1.130	0	9.590

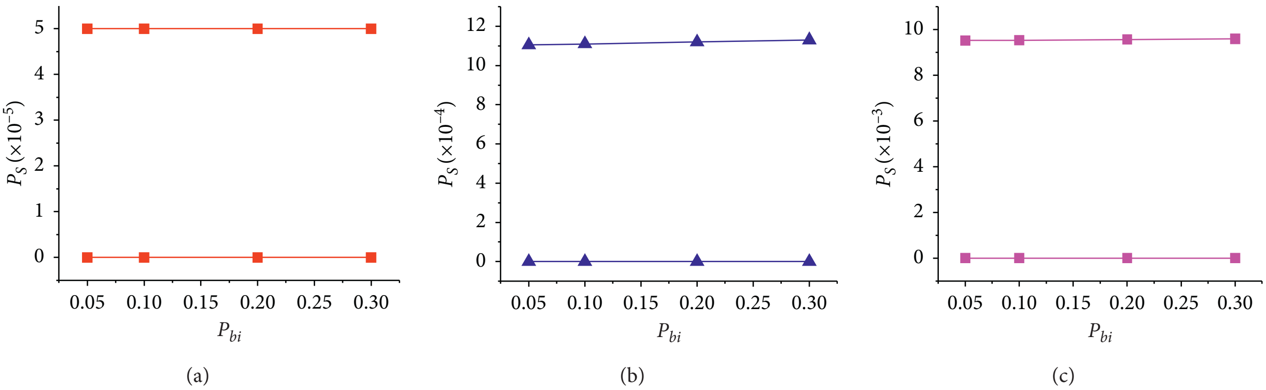


Figure 13: Relationship between structural failure probability and P_{bi}. (a) $\rho = 0.1$. (b) $\rho = 0.5$. (c) $\rho = 0.9$.

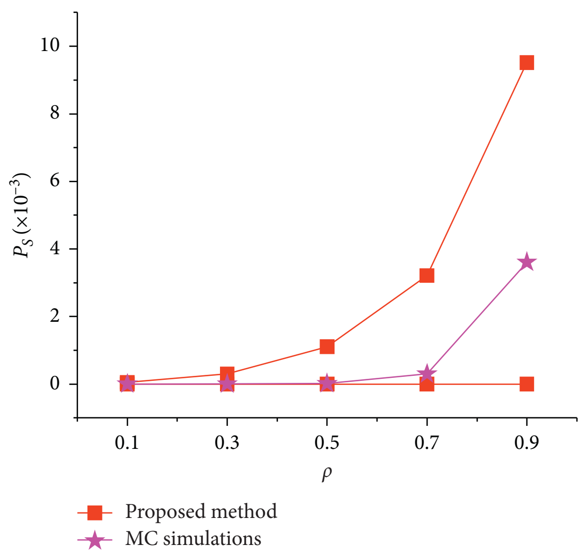


Figure 14: Relationship between structural failure probability and correlation coefficient.

Table 6: Failure probability for different ρ and n_1.

	Bounds on failure probability (×10^{-3})							
ρ	$n_1 = 0$		$n_1 = 1$		$n_1 = 2$		$n_1 = 3$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
0.1	0	0.050	0	0.050	0	0.050	0	0.050
0.3	0	0.295	0	0.298	0	0.301	0	0.304
0.5	0	1.100	0	1.105	0	1.110	0	1.115
0.7	0	3.200	0	3.210	0	3.220	0	3.231
0.9	0	9.500	0	9.515	0	9.531	0	9.547

shown in Table 8 and Figure 19. Similarly, the figure of relationship between structural failure probability and P_{bi} when n_1 equals to 2 or 3 is similar to its figure when n_1 equals to 1, and it has the same regular pattern as shown in Section 4.2.2 for structural failure probability, correlation coefficient, and P_{bi}. Also, the structural failure probability is estimated

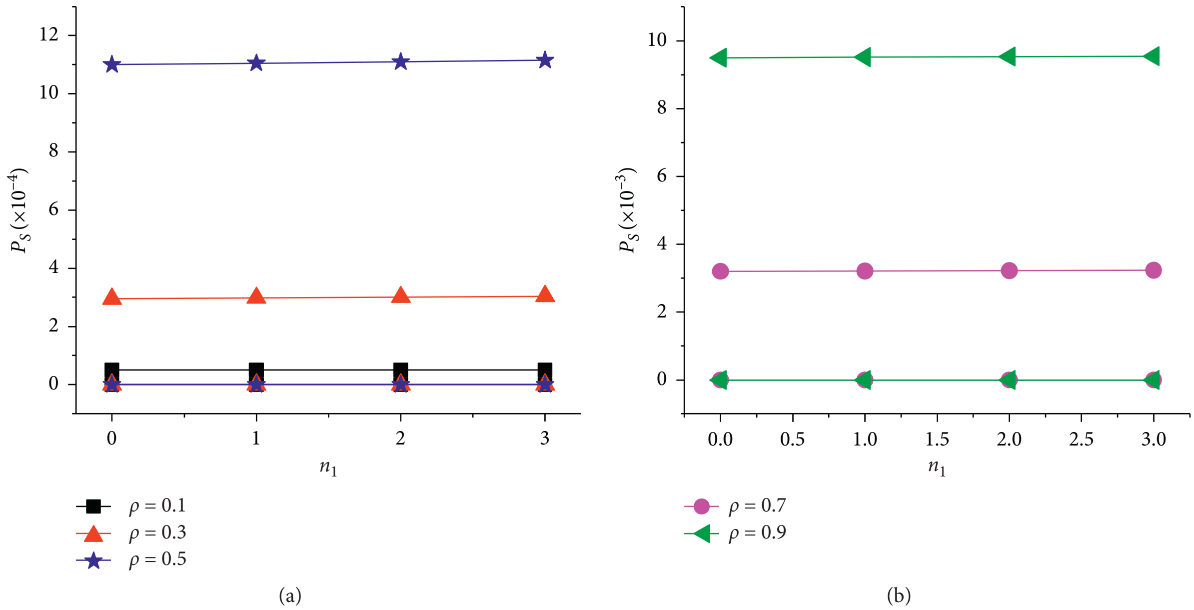


FIGURE 15: Relationship between structural failure probability and n_1. (a) $\rho = 0.1$, $\rho = 0.3$, and $\rho = 0.5$. (b) $\rho = 0.7$ and $\rho = 0.9$.

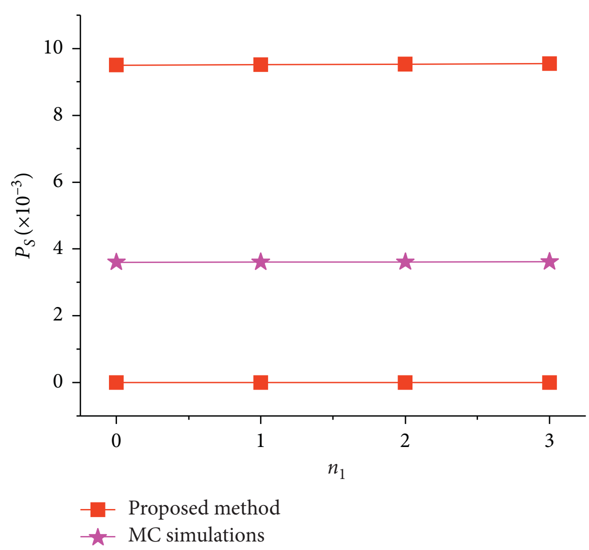


FIGURE 16: Relationship between structural failure probability and n_1 for different methods ($\rho = 0.9$).

TABLE 7: Failure probability of parallel system of components with potentially brittle behavior.

P_{bi} $i \in (1, n)$	Bounds by RLP bounds method ($\times 10^{-3}$)					
	$\rho = 0.1$		$\rho = 0.5$		$\rho = 0.9$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
0.05	0	0.100	0	1.605	0	14.430
0.1	0	0.100	0	1.610	0	14.460
0.2	0	0.100	0	1.620	0	14.520
0.3	0	0.100	0	1.630	0	14.580

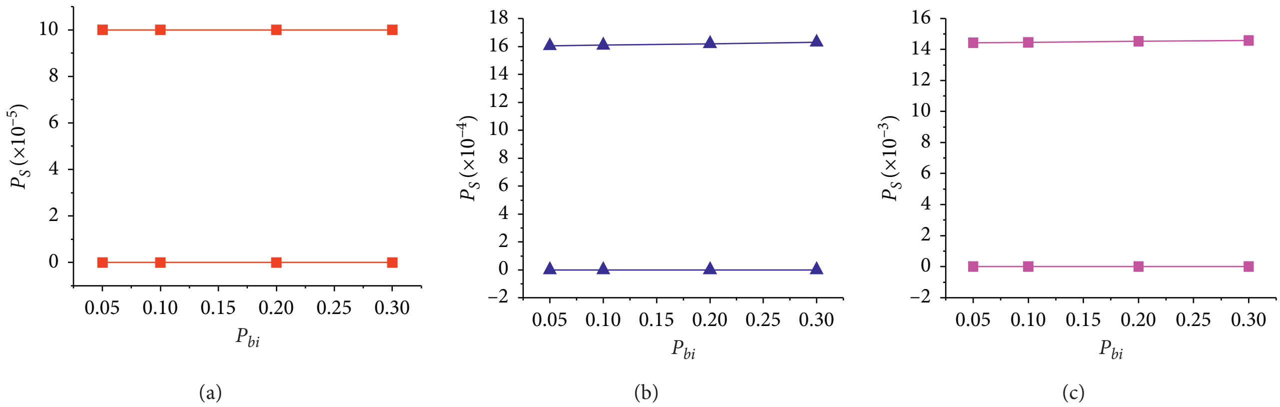


Figure 17: Relationship between structural failure probability and P_{bi}. (a) $\rho = 0.1$. (b) $\rho = 0.5$. (c) $\rho = 0.9$.

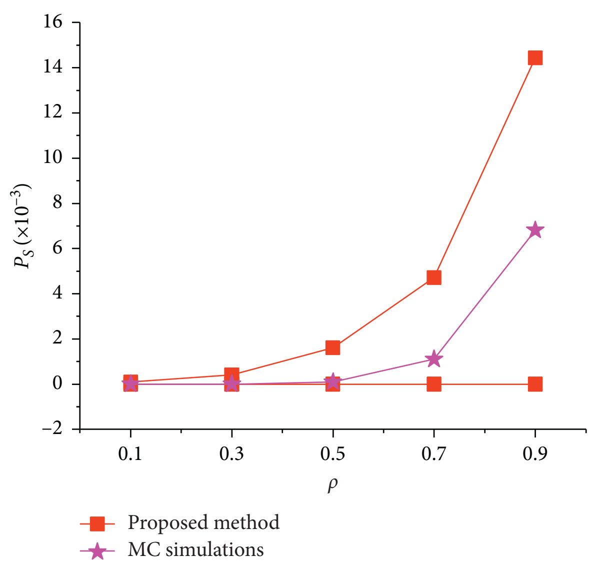


Figure 18: Relationship between structural failure probability and correlation coefficient.

Table 8: Failure probability for different ρ and n_1.

ρ	Bounds on failure probability ($\times 10^{-3}$)							
	$n_1 = 0$		$n_1 = 1$		$n_1 = 2$		$n_1 = 3$	
	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
0.1	0	0.100	0	0.100	0	0.100	0	0.100
0.3	0	0.400	0	0.405	0	0.410	0	0.414
0.5	0	1.600	0	1.605	0	1.610	0	1.616
0.7	0	4.700	0	4.715	0	4.730	0	4.746
0.9	0	14.400	0	14.430	0	14.460	0	14.491

by the MC simulations with 10^7 samples as shown in Figure 20. Similar to the results of n_1 equals to 1, the result of RLP bounds method is consistent with that of MC simulations.

Note that the lower bounds of the failure probability of the system are considered as a parallel system and k-out-of-n system is not always equals to zero. With the increase of k, the failure probability of k-out-of-n system will increase. Then, the lower bound of failure probability could be not zero. Also, even for the parallel system, if the system failure probability is not small, the lower bound of failure probability is not zero.

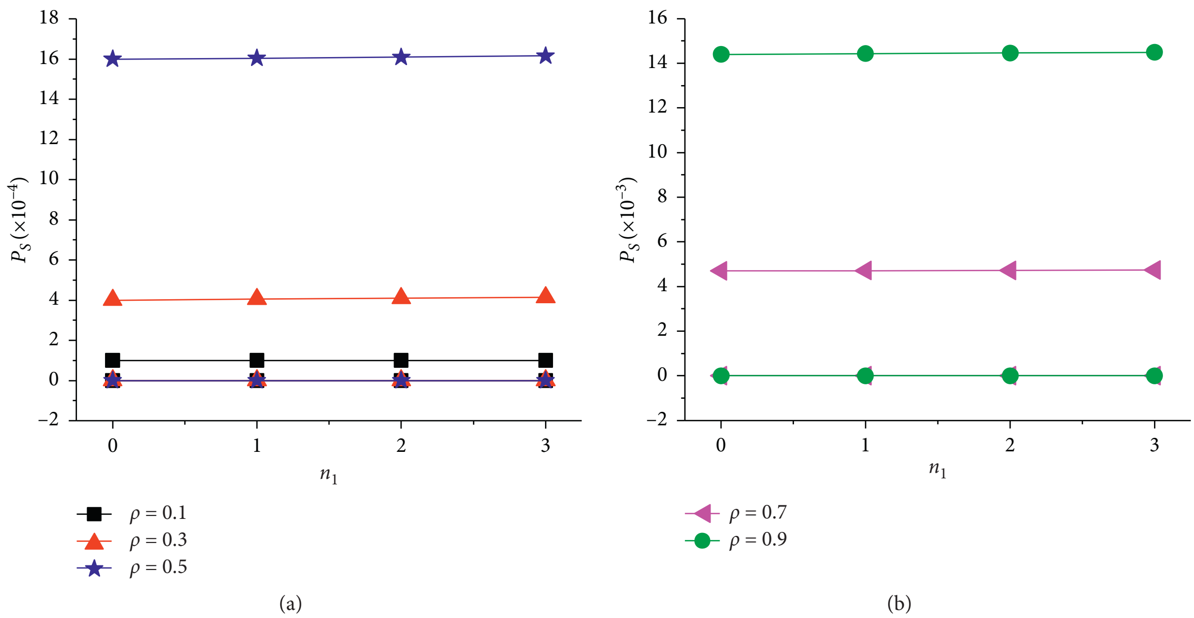


FIGURE 19: Relationship between structural failure probability and n_1. (a) $\rho = 0.1$, $\rho = 0.3$, and $\rho = 0.5$. (b) $\rho = 0.7$ and $\rho = 0.9$.

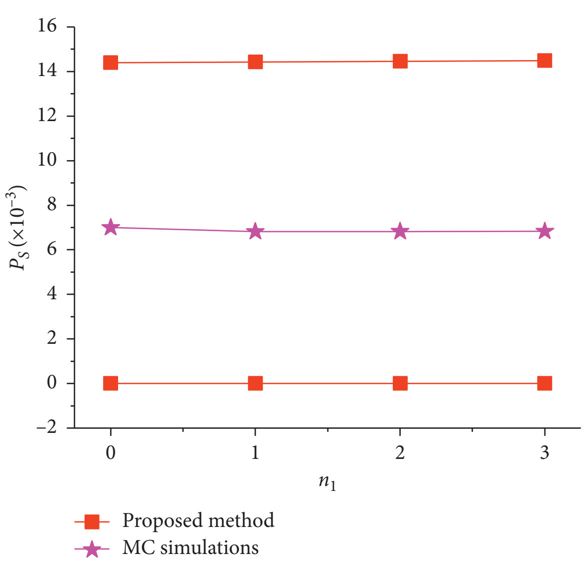


FIGURE 20: Relationship between structural failure probability and n_1 for different methods ($\rho = 0.9$).

5. Conclusion

This paper proposed a method to estimate the failure probability of k-out-of-n system of components with potentially brittle behavior. By decomposing the entire system that a component has 4 possible states into subsystems that a component has only 2 possible states, i.e., either brittle safe state and brittle failure state or ductile safe state and ductile failure state, the bounds on failure probability can be obtained by using the UGF and LP. Based on the individual component failure probabilities and the joint failure probabilities of a few components, narrow bounds can be obtained.

The accuracy and applicability of the proposed method are investigated along with the MC simulations using numerical examples. The proposed method can be used to estimate a structure with a large number of components and provides a result comparable to that of MC simulations. Not like MC simulations, it should be noted that the proposed method only based individual component failure probabilities and the joint failure probabilities, the incomplete sets of joint failure probabilities can also be handled.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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