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Research Article

A New Approach to the Robust Control Design of Fuzzy Automated Highway Systems

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In this paper, a new approach to the decentralized control design for vehicle platooning for uncertain automated highway systems is proposed. The uncertainty in the system, which is nonlinear and (possibly) fast time-varying, is bounded. The bound is assumed to be within a prescribed fuzzy set. A creative transformation is made to the system, which converts a local problem to a global problem. Based on the fuzzy description of the uncertainty bound and the transformation, a class of decentralized control is proposed in which each vehicle only needs the knowledge of its preceding vehicle in the platoon. No acceleration feedback or the information of the leading vehicle is required. Both the vehicle platooning system and the control are deterministic, hence not *if-then* fuzzy rule-based. The performance of the resulting controlled system is twofold. First, the collision avoidance performance is guaranteed under any safe initial conditions regardless of the value of the uncertainty. Second, the minimization of a fuzzy-based performance index is guaranteed based on an optimal choice of a control design parameter. Numerical simulations are conducted to validate the efficiency of the proposed algorithm.

1. Introduction

The Automated Highway System (AHS) is a proposed intelligent transportation system (ITS) technology designed to increase capacity and safety with the increasingly severe traffic congestion since the last two decades. The basic idea is by grouping vehicles into platoons at closer spacing under automatic control, which cannot be achieved by human drivers alone, to increase the capacity on highways [1, 2]. With the distances between vehicles becoming smaller, safety (or collision avoidance) is the first and foremost concern, which must be guaranteed for the AHS. From the control design point of view, stability is often the primary system performance. However, a platoon of automated vehicles is an interconnected system in which stability of each component system *per se* is not sufficient to guarantee the boundedness of the spacing errors for all the vehicles [3], which is directly related to safety.

For this reason, the alternative string stability has been proposed, which implies the boundedness of all the states of the interconnected system if the initial states of the vehicles are within some neighborhood of the equilibrium states [4]. Efforts are then devoted to linearize the system through control with required platoon information (acceleration feedback and/or information of the leading vehicle) and to use transfer function analysis to investigate string stability [5–8]. Many important contributions have been made (for a survey, see, for example, [9–12]). However, as it was pointed out in [13], the string stability alone does not provide a warranty for global collision avoidance of the platoon. Rather, the initial conditions should be within a neighborhood around the equilibrium position in order to prevent vehicle collision [14, 15], hence a *local* performance. Earlier research studies regarding *local* collision avoidance can be found in [16–18]. The restriction on the initial conditions reduces the range of applications. This promotes a need to further design a controller for the platoon which guarantees *global* collision avoidance under any *safe* initial condition (that is, as long as the collision does not occur initially).

One salient feature of the paper is that the uncertainty in the system, which may come from parameter variations (the mass variation and the movement of the passengers and/or the stowage) and aerodynamics (the unmeasurable side wind), as well as external disturbances (the rolling resistance and slope of road), is described using fuzzy set theory [19–21], instead of fuzzy *if-then* inference rules [22, 23]. Neither the platoon model nor the controller is fuzzy *if-then* rule-based, hence not the Takagi–Sugeno–Kang (TSK) fuzzy system [24, 25] nor the Mamdani-type fuzzy controller [26]. This distinguishes the current work from some others in fuzzy systems and control research. For example, the uncertainty bound may be *close to* 2. Here, "*close to*" is a linguistic variable which is associated with a fuzzy set. Earlier works related to fuzzy *if-then* rule-based approach for platooning can be found in [27, 28].

The main contributions of this paper are fourfold. First, a creative state transformation is made on the error dynamics of each vehicle in the platoon. This converts a *local* problem to a *global* problem. Second, fuzzy set instead of *fuzzy if-then rules* is used to describe the uncertainty bound of the system. A class of decentralized robust controllers is proposed based on the fuzzy description of the uncertainty bound. Each vehicle in the platoon only needs the information of its preceding vehicle for the control. No acceleration feedback or information of the following or leading vehicle or other vehicle is required. The control is proven to render each transformed error dynamics globally practically stable regardless of the uncertainty. Third, it is shown that the global practical stability leads to global collision avoidance and the spacing error will converge to a small region. Fourth, an optimal design problem associated with the fuzzy-based performance index and the control cost is formulated, which enables us to choose an optimal gain for a control parameter. It is proven that the global solution to the optimal problem exists and is unique. Furthermore, the closed-form (that is, the analytic) expression of the optimal solution is shown. This completely solves the problem.

2. Vehicle Platooning Model

2.1. Platoon Configuration. Consider a platoon of automated vehicles traveling on a straight line of a highway and closely following one another, as shown in Figure 1. The position of the leading vehicle and the i^{th} following vehicle in the platoon is denoted by $x_L \in \mathbf{R}$ and $x_i \in \mathbf{R}$ with respect to an inertial frame, respectively. The actual space Δ_i between the i^{th} vehicle and its preceding one is given by

$$\Delta_i(t) = x_{i-1}(t) - x_i(t) - l_{i-1}, \tag{1}$$

where l_{i-1} is the length of the $(i-1)^{\text{th}}$ vehicle in the platoon. Henceforth, subscripts L and i are used to label the leader and the i^{th} follower in the platoon, respectively.

Suppose the desired space between the i^{th} and $(i-1)^{\text{th}}$ vehicle is denoted by a constant scalar Δ_i^d. Then, the space error is given by

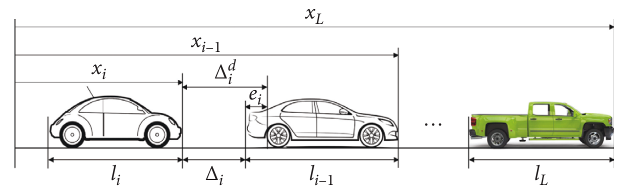

Figure 1: The position history of the four vehicles in the platoon under robust controller with zero initial conditions.

$$e_i(t) = \Delta_i^d - \Delta_i(t) = \Delta_i^d + x_i(t) - x_{i-1}(t) + l_{i-1}. \tag{2}$$

Notice that $\Delta_i(t_s) = 0$ means a collision happens at time t_s. A definition related to the collision avoidance performance is given as follows.

Definition 1 (collision avoidance). A platoon of controlled vehicles is collision avoidable if given any *safe* initial condition (that is, initially noncolliding or $e_i(t_0) < \Delta_i^d$):

$$e_i(t) < \Delta_i^d, \tag{3}$$

for all $t > t_0$.

Remark. It is worth pointing out here that collision avoidance (3) is the foremost performance to be guaranteed for automated vehicles. From the practical design point of view, one should first guarantee that the platoon is collision avoidable.

2.2. Platoon Error Dynamics. The longitudinal equation of motion for the ith vehicle is determined by the Newton's second law as

$$\begin{aligned} M_i(x_i(t), \sigma_i(t), t)\ddot{x}_i(t) = {} & u_i(t) - c_i(x(t), \dot{x}_i(t), \sigma_i(t), t) \\ & \times \dot{x}_i(t)\left|\dot{x}_i(t)\right| - F_i(x_i(t), \dot{x}_i(t), \sigma_i(t), t), \end{aligned} \tag{4}$$

where $t \in \mathbf{R}$ is the time, $\sigma_i \in \Sigma_i \subset \mathbf{R}^p$ is the uncertain parameter representing all the uncertainty in the vehicle, and $u_i \in \mathbf{R}$ is the control input which is from the vehicle propulsive/braking effort. Moreover, we assume $M_i(\cdot)$, $c_i(\cdot)$, and $F_i(\cdot)$ are continuous, where $M_i(x_i, \sigma_i, t) \in \mathbf{R}$ represents the vehicle mass, $-c_i(x, \dot{x}_i, \sigma_i, t)\dot{x}_i|\dot{x}_i| \in \mathbf{R}$ represents the aerodynamic drag force, and $-F_i(x_i, \dot{x}_i, \sigma_i, t) \in \mathbf{R}$ represents the rolling resistance force and other external disturbances acting on the vehicle.

Differentiating (2) with respect to time, we get the space error velocity and space error dynamics as follows:

$$\dot{e}_i(t) = \dot{x}_i(t) - \dot{x}_{i-1}(t), \tag{5}$$

$$e_i(t) = x_i(t) - x_{i-1}(t). \tag{6}$$

With (4) in (6) yields (henceforth, arguments of functions are sometimes omitted when no confusion is likely to arise),

$$\ddot{e}_i = M_i^{-1}\left(u_i - c_i\dot{x}_i|\dot{x}_i| - F_i\right) - M_{i-1}^{-1}\left(u_{i-1} - c_{i-1}\dot{x}_{i-1}|\dot{x}_{i-1}| - F_{i-1}\right). \tag{7}$$

The control objective for vehicle platooning is stated as follows: design the control u_i in (4) for the ith vehicle in the platoon with available information (will be elaborated later) such that the resulting controlled platoon is collision avoidable under arbitrary safe initial conditions.

2.3. Bounding and Structure Conditions. We decompose M_i, c_i, and F_i in (4) as follows:

$$\begin{aligned} M_i(x_i,\sigma_i,t) &= \overline{M}_i(x_i,t) + \Delta M_i(x_i,\sigma_i,t),\\ c_i(x_i,\dot{x}_i,\sigma_i,t) &= \overline{c}_i(x_i,\dot{x}_i,t) + \Delta c_i(x_i,\dot{x}_i,\sigma_i,t),\\ F_i(x_i,\dot{x}_i,\sigma_i,t) &= \overline{F}_i(x_i,\dot{x}_i,t) + \Delta F_i(x_i,\dot{x}_i,\sigma_i,t), \end{aligned} \tag{8}$$

where $\overline{M}_i$, $\overline{c}_i$, and $\overline{F}_i$ denote the "nominal" portions with $\overline{M}_i > 0$ (this is always feasible since the nominal portion is the designer's discretion), while ΔM_i, Δc_i, and Δg_i are the uncertain portions. The functions $\overline{M}_i(\cdot)$, $\Delta M_i(\cdot)$, $\overline{c}_i(\cdot)$, $\Delta c_i(\cdot)$, $\overline{F}_i(\cdot)$, and $\Delta F_i(\cdot)$ are all continuous.

Denote

$$D_i(x_i,t) := \overline{M}_i^{-1}(x_i,t), \tag{9}$$

$$\Delta D_i(x_i,\sigma_i,t) := M_i^{-1}(x_i,\sigma_i,t) - \overline{M}_i^{-1}(x_i,t), \tag{10}$$

$$E_i(x_i,\sigma_i,t) := \overline{M}_i(x_i,t)M_i^{-1}(x_i,\sigma_i,t) - 1. \tag{11}$$

Hence, $\Delta D_i(x_i,\sigma_i,t) = D_i(x_i,t)E_i(x_i,\sigma_i,t)$.

Assumption 1 (i) Let the initial state of the error dynamics of the ith vehicle in the platoon, which is uncertain, be represented by $q_{i0} = [e_{i0}\ \dot{e}_{i0}]^T$. For each entry of q_{i0}, namely, q_{i0j}, $j = 1, 2$, there exists a fuzzy set X_{i0j} in a universe of discourse $\Xi_{ij} \in \mathbf{R}$ characterized by a membership function $\mu_{\Xi_{ij}}: \Xi_{ij} \longrightarrow [0,1]$. That is,

$$X_{i0j} = \left\{\left(q_{i0j}, \mu_{\Xi_{ij}}\left(q_{i0j}\right)\right) \mid q_{i0j} \in \Xi_{ij}\right\}. \tag{12}$$

where Ξ_{ij} is known and compact. (ii) For each entry of the vector σ_i, namely, σ_{ij}, $j = 1, 2, \cdots, p$, the function $\sigma_{ij}(\cdot)$ is Lebesgue measurable. (iii) For each σ_{ij}, there exists a fuzzy set S_{ij} in a universe of discourse $\Sigma_{ij} \in \mathbf{R}$ characterized by a membership function $\mu_{ij}: \Sigma_{ij} \longrightarrow [0,1]$. That is,

$$S_{ij} = \left\{\left(\sigma_{ij}, \mu_{ij}\left(\sigma_{ij}\right)\right) \mid \sigma_{ij} \in \Sigma_{ij}\right\}. \tag{13}$$

where Σ_{ij} is known and compact.

Assumption 2

(1) There exists a (possibly unknown) parameter $\rho_{E_i} \in \mathbf{R}$ such that for all $(x_i,t) \in \mathbf{R}\times\mathbf{R}$, $\sigma_i \in \Sigma_i$,

$$\min_{\sigma_i \in \Sigma_i} E_i(x_i,\sigma_i,t) \geq \rho_{E_i} > -1. \tag{14}$$

(2) The unknown parameter ρ_{E_i} belongs to a known fuzzy number.

Remark. In the special case that $M_i = \overline{M}_i$ (i.e., no uncertainty in the i^{th} vehicle mass), we get $E_i = 0$. Hence, one can choose $\rho_{E_i} = 0$. The assumption imposes the effect of uncertainty on the possible deviation of M_i from $\overline{M}_i$ to be within a unidirectional threshold.

2.4. State Transformation. We propose the following transformation for the space error e_i:

$$z_{i1} = \ln\left(\frac{\Delta_i^d - e_i}{\Delta_i^d}\right), \tag{15}$$

which means that

$$e_i = \Delta_i^d - \Delta_i^d \exp(z_{i1}). \tag{16}$$

Differentiating (15) with respect to time, we get

$$\dot{z}_{i1} = -\frac{1}{\Delta_i^d}\exp(-z_{i1})\dot{e}_i, \tag{17}$$

$$\ddot{z}_{i1} = -\frac{1}{\Delta_i^d}\exp(-z_{i1})(-\dot{z}_{i1})\dot{e}_i - \frac{1}{\Delta_i^d}\exp(-z_{i1})\ddot{e}_i. \tag{18}$$

Let

$$z_{i2} := z_{i1} - \frac{1}{\Delta_i^d}\exp(-z_{i1})\dot{e}_i, \tag{19}$$

yields

$$\begin{aligned} \dot{z}_{i1} &= -z_{i1} + z_{i2},\\ \dot{z}_{i2} &= \dot{z}_{i1} - \frac{1}{\Delta_i^d}\exp(-z_{i1})(-\dot{z}_{i1})\dot{e}_i - \frac{1}{\Delta_i^d}\exp(-z_{i1})\ddot{e}_i. \end{aligned} \tag{20}$$

With (7) in (20), the transformed space error dynamics is in the form of

$$\begin{aligned} \dot{z}_{i1} &= -z_{i1} + z_{i2},\\ \dot{z}_{i2} &= -z_{i1} + z_{i2} - (z_{i1} - z_{i2})^2, \end{aligned} \tag{21}$$

$$\begin{aligned} &-\frac{1}{\Delta_i^d}\exp(-z_{i1})\left[M_i^{-1}\left(u_i - c_i\dot{x}_i|\dot{x}_i| - F_i\right)\right.\\ &\left.- M_{i-1}^{-1}\left(u_{i-1} - c_{i-1}\dot{x}_{i-1}|\dot{x}_{i-1}| - F_{i-1}\right)\right]. \end{aligned} \tag{22}$$

Remark. From (15), we have for all $-\infty < e_i < \Delta_i^d$, the corresponding z_{i1} is $-\infty < z_{i1} < \infty$. Furthermore, $z_{i1} = 0 \Longleftrightarrow e_i = 0$. This is in fact the main reason for the transformation from $(e_i, \dot{e}_i)$ to (z_{i1}, z_{i2}). If both $z_{i1}(t)$ and $z_{i2}(t)$ are bounded, then space error $e_i(t) \in (-\infty, \Delta_i^d)$, hence no collision.

Remark. The state z_{i2} given by (19) is selected to make the transformed space error dynamics in a lower triangular form so that the backstepping method [29] can be adopted for the control design, which will be explained in the next section.

3. Robust Control Design and Performance Analysis

We now propose a robust controller for the transformed system to realize the performance of uniform boundedness and uniform ultimate boundedness. Then, we show that the controlled platoon will have guaranteed collision avoidance in the error space.

3.1. Robust Control Design

Let

$$p_{i1} = \bar{c}_i\dot{x}_i|\dot{x}_i| + \bar{F}_i + D_i^{-1}D_{i-1}\left(u_{i-1} - \bar{c}_{i-1}\dot{x}_{i-1}|\dot{x}_{i-1}| - \bar{F}_{i-1}\right), \tag{23}$$

$$p_{i2} = -D_i^{-1}\Delta_i^d \exp(z_{i1})\left[-2z_{i2} + (z_{i1} - z_{i2})^2\right]. \tag{24}$$

Assumption 3

(1) There exists a possibly unknown parameter α_i and a known function $\Pi_i(\cdot)$: $\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R} \longrightarrow \mathbf{R}_+$ such that for all $(x_i, x_{i-1}, \dot{x}_i, \dot{x}_{i-1}, u_{i-1}, t) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}$, $\sigma_i \in \Sigma_i$,

$$\begin{aligned}&\left(1+\rho_{E_i}\right)^{-1}\max_{\sigma_i\in\Sigma_i}\Big\|\Delta D_i\left(-c_i\dot{x}_i|\dot{x}_i| - F_i + p_{i1} + p_{i2}\right) - \Delta D_{i-1}\\&\times\left(u_{i-1} - c_{i-1}\dot{x}_{i-1}|\dot{x}_{i-1}| - F_{i-1}\right) + D_i\left(-\Delta c_i\dot{x}_i|\dot{x}_i| - \Delta F_i\right)\\&- D_{i-1}\left(-\Delta c_{i-1}\dot{x}_{i-1}|\dot{x}_{i-1}| - \Delta F_{i-1}\right)\Big\|\\&\le \alpha_i\Pi_i(x_i, x_{i-1}, \dot{x}_i, \dot{x}_{i-1}, u_{i-1}, t).\end{aligned} \tag{25}$$

(2) The unknown parameter α_i belongs to a known fuzzy number.

Remark. The fuzzy numbers ρ_{E_i} and α_i are relevant to σ_i. Their associated membership functions can be determined via $\mu_i(\cdot)$'s, the fuzzy arithmetic and the decomposition theorem [30]. The extreme value of the fuzzy number ρ_{E_i} can be evaluated since the universes of discourse Σ_i's are known.

Let

$$\mu_i := -\frac{1}{\Delta_i^d}z_{i2}\exp(-z_{i1})\Pi_i. \tag{26}$$

We propose the following control:

$$u_i = p_{i1} + p_{i2} + p_{i3}, \tag{27}$$

where

$$\begin{aligned}p_{i3} &= -\gamma_i D_i^{-1}\mu_i\Pi_i,\\ \gamma_i &> 0.\end{aligned} \tag{28}$$

Theorem 1. *Let $\delta_i := [z_{i1}\ z_{i2}]^T \in \mathbf{R}^2$. Subject to Assumptions 1–3, the control (27) renders the transformed system (22) and (21) the following performance:*

(i) Uniform boundedness: for any $r_i > 0$, there is a $d_i(r_i) < \infty$ such that if $\|\delta_i(t_0)\| \le r_i$, then $\|\delta_i(t)\| \le d_i(r_i)$ for all $t > t_0$

(ii) Uniform ultimate boundedness: for any $r_i > 0$ with $\|\delta_i(t_0)\| \le r_i$, there exists a $\underline{d}_i > 0$ such that $\|\delta_i(t)\| \le \bar{d}_i$ for any $\bar{d}_i > \underline{d}_i$ as $t \ge t_0 + T_i(\bar{d}_i, r_i)$, where $T_i(\bar{d}_i, r_i) < \infty$

Proof. Consider the Lyapunov function candidate:

$$V_i = \frac{1}{2}\left(z_{i1}^2 + z_{i2}^2\right). \tag{29}$$

For a given uncertainty $\sigma_i(\cdot)$, the derivative of V_i along the trajectory of the controlled system is to be evaluated. First, we have

$$\dot{V}_i = z_{i1}\dot{z}_{i1} + z_{i2}\dot{z}_{i2}. \tag{30}$$

Next, in view of (21), the first term on the RHS of (30) is given by

$$z_{i1}\dot{z}_{i1} = -z_{i1}^2 + z_{i1}z_{i2}. \tag{31}$$

The second term on the RHS of (30) is

$$\begin{aligned}z_{i2}\dot{z}_{i2} &= z_{i2}(-z_{i1} + z_{i2}) - z_{i2}(z_{i1} - z_{i2})^2\\&\quad - \frac{1}{\Delta_i^d}z_{i2}\exp(-z_{i1})\left[M_i^{-1}\left(u_i - c_i\dot{x}_i|\dot{x}_i| - F_i\right)\right.\\&\quad\left. - M_{i-1}^{-1}\left(u_{i-1} - c_{i-1}\dot{x}_{i-1}|\dot{x}_{i-1}| - F_{i-1}\right)\right].\end{aligned} \tag{32}$$

By the third term on the RHS of (32), after decomposing M_i^{-1}, c_i, and F_i by (9)–(11) and noting that $M_i^{-1} = D_i + \Delta D_i$, we get

$$\begin{aligned}&-\frac{1}{\Delta_i^d}z_{i2}\exp(-z_{i1})\left[M_i^{-1}\left(u_i - c_i\dot{x}_i|\dot{x}_i| - F_i\right) - M_{i-1}^{-1}(u_{i-1}\right.\\&\left.- c_{i-1}\dot{x}_{i-1}|\dot{x}_{i-1}| - F_{i-1})\right]\\&= -\frac{1}{\Delta_i^d}z_{i2}\exp(-z_{i1})\{(D_i + \Delta D_i)[p_{i1} + p_{i2} + p_{i3}\\&- (\bar{c}_i + \Delta c_i)\dot{x}_i|\dot{x}_i| - (\bar{F}_i + \Delta F_i)] - (D_{i-1} + \Delta D_{i-1})\\&\times\left[u_{i-1} - (\bar{c}_{i-1} + \Delta c_{i-1})\dot{x}_{i-1}|\dot{x}_{i-1}| - (\bar{F}_{i-1} + \Delta F_{i-1})\right]\}\\&= -\frac{1}{\Delta_i^d}z_{i2}\exp(-z_{i1})\{\left[D_i\left(p_{i1} - \bar{c}_i\dot{x}_i|\dot{x}_i| - \bar{F}_i\right) - D_{i-1}(u_{i-1}\right.\\&\left.- \bar{c}_{i-1}\dot{x}_{i-1}|\dot{x}_{i-1}| - \bar{F}_{i-1})\right] + D_ip_{i2} + [\Delta D_i(p_{i1} + p_{i2}\\&- c_i\dot{x}_i|\dot{x}_i| - F_i) - \Delta D_{i-1}\left(u_{i-1} - c_{i-1}\dot{x}_{i-1}|\dot{x}_{i-1}| - F_{i-1}\right)\\&+ D_i\left(-\Delta c_i\dot{x}_i|\dot{x}_i| - \Delta F_i\right) - D_{i-1}(-\Delta c_{i-1}\dot{x}_{i-1}|\dot{x}_{i-1}|\\&- \Delta F_{i-1})] + (D_i + \Delta D_i)p_{i3}\}.\end{aligned} \tag{33}$$

By (23), it can be shown that

$$D_i\left(p_{i1} - \overline{c}_i\dot{x}_i|\dot{x}_i| - \overline{F}_i\right) - D_{i-1}\left(u_{i-1} - \overline{c}_{i-1}\dot{x}_{i-1}|\dot{x}_{i-1}| - \overline{F}_{i-1}\right) = 0. \tag{34}$$

Next, by (24),

$$\begin{aligned} &-\frac{1}{\Delta_i^d}z_{i2}\exp(-z_{i1})D_i p_{i2} \\ &= \frac{1}{\Delta_i^d}z_{i2}\exp(-z_{i1})D_i D_i^{-1}\Delta_i^d\exp(z_{i1})\left[-2z_{i2} + (z_{i1} - z_{i2})^2\right] \\ &= -2z_{i2}^2 + z_{i2}(z_{i1} - z_{i2})^2. \end{aligned} \tag{35}$$

By Assumption 3, we obtain

$$\begin{aligned} &-\frac{1}{\Delta_i^d}z_{i2}\exp(-z_{i1})\left\{\left[\Delta D_i\left(p_{i1} + p_{i2} - c_i\dot{x}_i|\dot{x}_i| - F_i\right)\right.\right. \\ &- \Delta D_{i-1}\left(u_{i-1} - c_{i-1}\dot{x}_{i-1}|\dot{x}_{i-1}| - F_{i-1}\right) + D_i\left(-\Delta c_i\dot{x}_i|\dot{x}_i|\right. \\ &\left.\left.\left. - \Delta F_i\right) - D_{i-1}\left(-\Delta c_{i-1}\dot{x}_{i-1}|\dot{x}_{i-1}| - \Delta F_{i-1}\right)\right]\right\} \\ &\le \left(1 + \rho_{E_i}\right)\frac{1}{\Delta_i^d}|z_{i2}|\exp(-z_{i1})\alpha_i\Pi_i \\ &\le \left(1 + \rho_{E_i}\right)\alpha_i\|\mu_i\|. \end{aligned} \tag{36}$$

By (28), Assumption 2, and (26),

$$\begin{aligned} &-\frac{1}{\Delta_i^d}z_{i2}\exp(-z_{i1})(D_i + \Delta D_i)p_{i3} \\ &\le -\frac{1}{\Delta_i^d}z_{i2}\exp(-z_{i1})(D_i + D_iE_i)\left(-\gamma_i D_i^{-1}\mu_i\Pi_i\right) \\ &= -\gamma_i D_i D_i^{-1}\left(1 + \rho_{E_i}\right)\left(-\frac{1}{\Delta_i^d}z_{i2}\exp(-z_{i1})\Pi_i\right)\mu_i \\ &= -\gamma_i\left(1 + \rho_{E_i}\right)\|\mu_i\|^2. \end{aligned} \tag{37}$$

With (33)–(37) in (32), we have

$$\begin{aligned} z_{i2}\dot{z}_{i2} &\le z_{i2}(-z_{i1} + z_{i2}) - z_{i2}(z_{i1} - z_{i2})^2 - 2z_{i2}^2 + z_{i2} \\ &\quad\cdot(z_{i1} - z_{i2})^2 + \left(1 + \rho_{E_i}\right)\alpha_i\|\mu_i\| - \gamma_i\left(1 + \rho_{E_i}\right)\|\mu_i\|^2 \\ &= -z_{i1}z_{i2} - z_{i2}^2 + \left(1 + \rho_{E_i}\right)\alpha_i\|\mu_i\| - \gamma_i\left(1 + \rho_{E_i}\right)\|\mu_i\|^2. \end{aligned} \tag{38}$$

With (31) and (38) in (30), we obtain

$$\dot{V}_i \le -z_{i1}^2 - z_{i2}^2 + \left(1 + \rho_{E_i}\right)\alpha_i\|\mu_i\| - \gamma_i\left(1 + \rho_{E_i}\right)\|\mu_i\|^2. \tag{39}$$

Since

$$\left(1 + \rho_{E_i}\right)\alpha_i\|\mu_i\| - \gamma_i\left(1 + \rho_{E_i}\right)\|\mu_i\|^2 \le \left(1 + \rho_{E_i}\right)\frac{\alpha_i^2}{4\gamma_i}, \tag{40}$$

we have (noting that $\|\delta\|_i^2 = \|z_{i1}\|^2 + \|z_{i2}\|^2$)

$$\dot{V}_i \le -\|\delta_i\|^2 + \frac{\varepsilon_i}{\gamma_i}, \tag{41}$$

where $\varepsilon_i := (1 + \rho_{E_i})\alpha_i^2/4$. This means that $\dot{V}_i$ is negative definite for all $\|\delta_i\|$ such that

$$\|\delta_i\| > \sqrt{\frac{\varepsilon_i}{\gamma_i}}. \tag{42}$$

Since all universes of discourse Σ_{ij}'s are compact (hence, closed and bounded), ε_i is bounded. Noticing that γ_i is crisp, we conclude that $\dot{V}_i$ is a negative definite for sufficiently large $\|\delta_i\|$. Therefore, upon invoking the standard arguments as in [31], we conclude the solution to the transformed error dynamics of the i^{th} vehicle is uniform boundedness with

$$\begin{aligned} d_i(r_i) &= \begin{cases} R_i \text{if } r_i \le R_i, \\ r_i \text{ if } r_i > R_i, \end{cases} \\ R_i &= \sqrt{\frac{\varepsilon_i}{\gamma_i}}. \end{aligned} \tag{43}$$

Furthermore, uniform ultimate boundedness also follows with

$$\begin{aligned} \underline{d}_i &= R_i, \\ T_i\left(\overline{d}_i, r_i\right) &= \begin{cases} 0, & \text{if } r_i \le \overline{d}_i, \\ \dfrac{r_i^2 - \overline{d}_i^2}{\overline{d}_i^2 - R_i^2}, & \text{otherwise}, \end{cases} \end{aligned} \tag{44}$$

Q.E.D. □

Remark. Notice the control given by (23) only needs the information of the preceding vehicle. No acceleration feedback or information of the leader or other followers are required. One may argue that the control given by (23) needs the information of $\overline{M}_{i-1}, \overline{c}_{i-1}, \overline{F}_{i-1}$, and u_{i-1} besides x_{i-1} and $\dot{x}_{i-1}$ of the preceding vehicle, which increases the burden of communication. However, a straightforward modification to the control will show that without the information of $\overline{M}_{i-1}, \overline{c}_{i-1}$, $\overline{F}_{i-1}$, and u_{i-1} (noting that $M_{i-1}^{-1}(u_{i-1} - c_{i-1}\dot{x}_{i-1}|\dot{x}_{i-1}| - F_{i-1})$ is the acceleration of the $i-1^{\text{th}}$ vehicle), the control still works as long as this unknown acceleration is bounded by a known function $\widetilde{\Pi}_i(x_i, x_{i-1}, \dot{x}_i, \dot{x}_{i-1}, t)$.

Remark. In the special case when the system is without uncertainty, that is $\Delta D_i \equiv 0$, $\Delta c_i \equiv 0$, and $\Delta F_i \equiv 0$, we may choose $\Pi_i = 0$ and hence $p_{i3} = 0$. The control $u_i = p_{i1} + p_{i2}$ will render $\dot{V}_i \le -\|\delta_i\|^2$. Therefore, we conclude that $\delta_i \longrightarrow 0$ as $t \longrightarrow \infty$. Notice that $z_{i1} = 0 \Longleftrightarrow e_i = 0$. Hence, $e_i \longrightarrow 0$ as $t \longrightarrow \infty$. Furthermore, if we choose $p_{i2} = 0$, then $\dot{V}_i = 0$. This means if $e_i = 0$ initially, then $e_i = 0$ for all $t \ge t_0$.

Remark. Notice that the third term in the control scheme (27) is for compensating the uncertainty and γ_i is the control gain. From (43) and (44), it can be concluded that the gain γ_i can be used to manipulate the size of the uniform ultimate boundedness region. In particular, the larger the gain, the smaller the size. This stands for a trade-off between the performance and the cost. As a result, it may be interested in seeking an optimal choice of γ_i for a compromise, which will be elaborated later.

3.2. Guaranteed Collision Avoidance

Theorem 2. *Consider a platoon of vehicles with the i^{th} vehicle described by (4). Suppose that Assumptions 1–3 are met. The control (27) for the i^{th} following vehicle guarantees the collision avoidance of the platoon under arbitrary safe initial condition.*

Proof. For the ith follower in the platoon, if an initial condition $[e_i(t_0)\,\dot{e}_i(t_0)]^T$ and a desired space Δ_i^d are specified, the corresponding initial condition in the transformed state of the same vehicle is given by

$$z_{i1}(t_0) = \ln\frac{\Delta_i^d - e_i(t_0)}{\Delta_i^d}, \tag{45}$$

$$z_{i2}(t_0) = z_{i1}(t_0) - \frac{1}{\Delta_i^d}\exp(-z_{i1}(t_0))\dot{e}_i(t_0). \tag{46}$$

Suppose control (27) is proposed, and hence the transformed state of the i^{th} follower is uniform bounded. By the definition of uniform boundedness in Theorem 1 with (43), we conclude that for any $r_i > 0$, there is a $d_i(r_i) < \infty$ such that if

$$\|\delta_i(t_0)\| = \sqrt{z_{i1}^2(t_0) + z_{i2}^2(t_0)} \le r_i, \tag{47}$$

then

$$\delta_i(t) \le d_i(r_i), \tag{48}$$

where

$$d_i(r_i) := \begin{cases} \sqrt{\dfrac{\varepsilon_i}{\gamma_i}}, & \text{if } r_i \le \sqrt{\dfrac{\varepsilon_i}{\gamma_i}}, \\[2ex] r_i, & \text{if } r_i > \sqrt{\dfrac{\varepsilon_i}{\gamma_i}}, \end{cases} \tag{49}$$

for all $t \ge t_0$. From (48), we have

$$\sqrt{z_{i1}^2(t) + z_{i2}^2(t)} \le d_i(r_i), \tag{50}$$

which means that in the worst case

$$|z_{i1}(t)| = \left|\ln\frac{\Delta_i^d - e_i(t)}{\Delta_i^d}\right| \le d_i(r_i). \tag{51}$$

From (51), we get

$$\frac{\Delta_i^d - e_i(t)}{\Delta_i^d} \le \exp(d_i(r_i)), \tag{52}$$

$$\frac{\Delta_i^d - e_i(t)}{\Delta_i^d} \ge \exp(-d_i(r_i)). \tag{53}$$

Therefore,

$$e_i(t) \ge [1 - \exp(d_i(r_i))]\Delta_i^d, \tag{54}$$

$$e_i(t) \le [1 - \exp(-d_i(r_i))]\Delta_i^d. \tag{55}$$

Since $0 < d_i(r_i) < \infty$, from (54), we get

$$e_i(t) > -\infty, \tag{56}$$

and from (55), we have

$$e_i(t) < \Delta_i^d, \tag{57}$$

for all $t > t_0$. Consequently, by Definition 1, we conclude that the platoon is collision avoidable. Q.E.D. □

4. Optimal Robust Control

4.1. Fuzzy-Based Performance Index. In the previous analysis, we know that the system performance can be guaranteed by a deterministic control design. The size of the uniform ultimate boundedness region decreases as γ_i increases. Furthermore, the size approaches zero when γ_i approaches infinity. Therefore, both the performance and control effort are affected by the control gain γ_i. From the practical design point of view, one may be interested in seeking an optimal choice of γ_i for a compromise among various conflicting criteria.

We first explore more on the deterministic performance of the uncertain system. From (41), we know that

$$\begin{aligned} \dot{V}_i(t) &\le -\|\delta_i\|^2 + \frac{\varepsilon_i}{\gamma_i}, \\ &\le -2V_i(t) + \frac{\varepsilon_i}{\gamma_i}, \end{aligned} \tag{58}$$

where $V_i(t_0) = (1/2)(z_{i1}^2(t_0) + z_{i2}^2(t_0))$. This is a *differential inequality* [32], whose analysis can be made according to [33]. The following is needed for our analysis of (58).

Definition 2 (see [32]). If $\omega(\psi, t)$ is a scalar function of the scalars ψ and t in some open connected set D, we say a function $\psi(t)$, $t_0 \le t \le \bar{t}$ and $\bar{t} \ge t_0$ is a solution of the differential inequality

$$\dot{\psi}(t) \le \omega(\psi(t), t), \tag{59}$$

on $[t_0, \bar{t}]$ if $\psi(t)$ is continuous on $[t_0, \bar{t}]$ and its derivative on $[t_0, \bar{t}]$ satisfies (59).

Theorem 3 (see [32]). *Let $\omega(\phi, t)$ be continuous on an open connected set $D \in \mathbf{R}^2$ such that the initial value problem for the scalar equation*

$$\dot{\phi}(t) = \omega(\phi(t), t), \quad \phi(t_0) = \phi_0, \tag{60}$$

has a unique solution. If $\phi(t)$ is a solution of (60) on $t_0 \le t \le \bar{t}$ and $\psi(t)$ is a solution of (59) on $t_0 \le t \le \bar{t}$ with $\psi(t_0) \le \phi(t_0)$, then $\psi(t) \le \phi(t)$ for $t_0 \le t \le \bar{t}$.

Instead of exploring the solution of the differential inequality, which is often nonunique and not available, the theorem suggests that it may be feasible to study the upper bound of the solution. The reasoning is, however, based on that the solution of (60) is unique.

Theorem 4 (see [33]). *Consider the differential inequality (59) and the differential equation (60). Suppose that for some constant $L > 0$, the function $\omega(\cdot)$ satisfies the Lipschitz condition*

$$|\omega(\nu_1, t) - \omega(\nu_2, t)| \le |\nu_1 - \nu_2|, \tag{61}$$

for all points $(\nu_1, t), (\nu_2, t) \in \mathcal{D}$. Then, any function $\psi(t)$ that satisfies the differential inequality (59) for $t_0 \le t < \bar{t}$ satisfies also the inequality

$$\psi(t) \le \phi(t), \tag{62}$$

for $t_0 \le t \le \bar{t}$.

We consider the differential equation

$$\begin{aligned} \dot{r}_i(t) &= -2r_i(t) + \frac{\varepsilon_i}{\gamma_i}, \\ r_i(t_0) &= V_{i0} = V_i(t_0). \end{aligned} \tag{63}$$

The RHS satisfies the global Lipschitz condition with $L = 2$. We proceed with solving the differential equation (63). This results in

$$r_i(t) = \left(V_{i0} - \frac{\varepsilon_i}{2\gamma_i}\right) \exp[-2(t - t_0)] + \frac{\varepsilon_i}{2\gamma_i}. \tag{64}$$

Therefore,

$$V_i(t) \le r_i(t), \tag{65}$$

or

$$V_i(t) \le \left(V_{i0} - \frac{\varepsilon_i}{2\gamma_i}\right) \exp[-2(t - t_0)] + \frac{\varepsilon_i}{2\gamma_i}, \tag{66}$$

for all $t \ge t_0$. By the same argument, we also have, for any t_s and any $\tau \ge t_s$,

$$V_i(\tau) \le \left(V_{is} - \frac{\varepsilon_i}{2\gamma_i}\right) \exp[-2(\tau - t_s)] + \frac{\varepsilon_i}{2\gamma_i}, \tag{67}$$

where $V_{is} = V_i(t_s) = (1/2)(z_{i1}^2(t_s) + z_{i2}^2(t_s))$. The time t_s is when the control scheme (27) starts to be executed. It does not need to be t_0.

Since $V_i(\tau) = (1/2)\|\delta_i(\tau)\|^2$, the RHS of (67) provides an upper bound of $(1/2)\|\delta_i(\tau)\|^2$. This in turn leads to an upper bound of $\|\delta_i(\tau)\|$. For each $\tau \ge t_s$, let

$$\eta_i(\gamma_i, \tau, t_s) = \left(V_{is} - \frac{\varepsilon_i}{2\gamma_i}\right) \exp[-2(\tau - t_s)], \tag{68}$$

$$\eta_{i\infty}(\gamma_i) = \frac{\varepsilon_i}{2\gamma_i}. \tag{69}$$

Notice that for each $\gamma_i, t_s, \eta_i(\gamma_i, \tau, t_s) \longrightarrow 0$ as $\tau \longrightarrow \infty$.

Definition 3. Consider a fuzzy set:

$$\mathcal{N} = \{(\nu, \mu_N(\nu)) | \nu \in N\}. \tag{70}$$

For any function $f: N \longrightarrow \mathbf{R}$, the D-operation $D[f(\nu)]$ is given by

$$D[f(\nu)] = \frac{\int_N f(\nu)\mu_N(\nu) d\nu}{\int_N \mu_N(\nu) d\nu}. \tag{71}$$

Remark. In a sense, the D-operation $D[f(\nu)]$ takes an average value of $f(\nu)$ over $\mu_N(\nu)$. In the special case that $f(\nu) = \nu$, this is reduced to the well-known center-of-gravity defuzzification method (see, e.g., [34]). If $\mathcal{N}$ is crisp (i.e., $\mu_N(\nu) = 1$ for all $\nu \in N$), then $D[f(\nu)] = f(\nu)$.

Lemma 1. *For any crisp constant $a \in \mathbf{R}$,*

$$D[af(\nu)] = aD[f(\nu)]. \tag{72}$$

Proof. By Definition 3,

$$\begin{aligned} D[af(\nu)] &= \frac{\int_N af(\nu)\mu_N(\nu) d\nu}{\int_N \mu_N(\nu) d\nu} \\ &= a\frac{\int_N f(\nu)\mu_N(\nu) d\nu}{\int_N \mu_N(\nu) d\nu} = aD[f(\nu)], \end{aligned} \tag{73}$$

Q.E.D.

We now propose the following performance index. For any t_s, let

$$\begin{aligned} J_i(\gamma_i, t_s) &:= D\left[\int_{t_s}^{\infty} \eta_i^2(\gamma_i, \tau, t_s) d\tau\right] + \Gamma_{i1} D\left[\eta_{i\infty}^2(\gamma_i)\right] + \Gamma_{i1}\gamma_i^2 \\ &=: J_{i1}(\gamma_i, t_s) + \Gamma_{i1} J_{i2}(\gamma_i) + \Gamma_{i2} J_{i3}(\gamma_i), \end{aligned} \tag{74}$$

where $\Gamma_{i1} > 0, \Gamma_{i2} > 0$. The performance index consists of three parts. The first part $J_{i1}(\gamma_i, t_s)$ may be interpreted as the value of the overall transient performance (via the integration) from time t_s. The second part $J_{i2}(\gamma_i)$ may be interpreted as the value of the steady state performance. The third part $J_{i3}(\gamma_i)$ may be interpreted as the value of the control effort. Both Γ_{i1} and Γ_{i2} are weighting factors. The weighting of J_1 is normalized to be unity.

For any t_s, given Δ_i, Γ_{i1}, and Γ_{i2}, our design problem is to choose $\gamma_i > 0$ in (28) such that the performance index $J_i(\gamma_i, t_s)$ is minimized. □

Remark. A standard LQG (i.e., linear-quadratic- Gaussian) problem in stochastic control is to minimize a performance index which is the average (via the expectation value operation in probability) of the overall state and control accumulation. The current approach may be viewed, loosely speaking, as a parallel, though not equivalent, in fuzzy dynamical systems. However, one cannot be too careful in distinguishing the differences. For example, the Gaussian probability distribution implies that the uncertainty is unbounded (although a higher bound is predicted by a lower probability). In the current consideration, the uncertainty bound is always finite. Also, the standard LQG does not take parameter uncertainty into account.

One can show that

$$\begin{aligned}\int_{t_s}^{\infty} \eta_i^2(\gamma_i, \tau, t_s)\mathrm{d}\tau &= \left(V_{is} - \frac{\varepsilon_i}{2\gamma_i}\right)^2 \int_{t_s}^{\infty} \exp[-4(\tau - t_s)]\mathrm{d}\tau \\ &= \left(V_{is} - \frac{\varepsilon_i}{2\gamma_i}\right)^2 \left(-\frac{1}{4}\right)\exp[-4(\tau - t_s)]\big|_{t_s}^{\infty} \\ &= \frac{1}{4}\left(V_{is} - \frac{\varepsilon_i}{2\gamma_i}\right)^2.\end{aligned} \tag{75}$$

Taking the D-operation yields,

$$\begin{aligned}D\left[\int_{t_s}^{\infty} \eta_i^2(\gamma_i, \tau, t_s)\mathrm{d}\tau\right] &= D\left[\frac{1}{4}\left(V_{is} - \frac{\varepsilon_i}{2\gamma_i}\right)^2\right] \\ &= \frac{1}{4}D\left[V_{is}^2\right] - \frac{1}{4\gamma_i}D\left[V_{is}\varepsilon_i\right] + \frac{1}{16\gamma_i^2}D\left[\varepsilon_i^2\right].\end{aligned} \tag{76}$$

The last equality is due to Lemma 1. Next, we analyze the cost $J_{i2}(\gamma_i)$. Again, by Lemma 1,

$$D\left[\eta_{i\infty}^2(\gamma_i)\right] = D\left[\frac{\varepsilon_i^2}{4\gamma_i^2}\right] = \frac{1}{4\gamma_i^2}D\left[\varepsilon_i^2\right]. \tag{77}$$

With (76) and (77) in (74), we obtain

$$\begin{aligned}J_i(\gamma_i, t_s) &= \frac{1}{4}D\left[V_{is}^2\right] - \frac{1}{4\gamma_i}D\left[V_{is}\varepsilon_i\right] + \frac{1}{16\gamma_i^2}D\left[\varepsilon_i^2\right] \\ &\quad + \Gamma_{i1}\frac{1}{4\gamma_i^2}D\left[\varepsilon_i^2\right] + \Gamma_{i2}\gamma_i^2 \\ &=: \lambda_1 - \frac{\lambda_2}{\gamma_i} + \frac{\lambda_3}{\gamma_i^2} + \Gamma_{i1}\frac{\lambda_4}{\gamma_i^2} + \Gamma_{i2}\gamma_i^2,\end{aligned} \tag{78}$$

where $\lambda_1 = (1/4)D[V_{is}^2]$, $\lambda_2 = (1/4)D[V_{is}\varepsilon_i]$, $\lambda_3 = (1/16)D[\varepsilon_i^2]$, and $\lambda_4 = (1/4)D[\varepsilon_i^2]$.

4.2. Formulation of the Optimal Design Problem. The optimal design problem is then equivalent to the following constrained optimization problem. For any t_s,

$$\begin{aligned}&\min_{\gamma_i} J_i(\gamma_i, t_s), \\ &\text{subject to} \quad \gamma_i > 0.\end{aligned} \tag{79}$$

For any t_s, taking the first order derivative of J_i with respect to γ_i,

$$\begin{aligned}\frac{\partial J_i}{\partial \gamma_i} &= \frac{\lambda_2}{\gamma_i^2} - 2\frac{\lambda_3}{\gamma_i^3} - 2\Gamma_{i1}\frac{\lambda_4}{\gamma_i^3} + 2\Gamma_{i2}\gamma_i^4 \\ &= \frac{1}{\gamma_i^3}\left(\lambda_2\gamma_i - 2\lambda_3 - 2\Gamma_{i1}\lambda_4 + 2\Gamma_{i2}\gamma_i^4\right).\end{aligned} \tag{80}$$

That $\partial J_i/\partial \gamma_i = 0$ leads to

$$\lambda_2\gamma_i - 2\lambda_3 - 2\Gamma_{i1}\lambda_4 + 2\Gamma_{i2}\gamma_i^4 = 0, \tag{81}$$

or

$$\lambda_2\gamma_i + 2\Gamma_{i2}\gamma_i^4 = 2(\lambda_3 + \Gamma_{i1}\lambda_4), \tag{82}$$

which is a quartic equation.

Theorem 5. *Suppose $D[\varepsilon_i] \neq 0$. For any t_s, given λ_1, λ_2, λ_3, and λ_4, the solution $\gamma_i > 0$ to (82) always exists and is unique, which globally minimizes the performance index (79).*

Proof. Let $f(\gamma_i) := \lambda_2\gamma_i + 2\Gamma_{i2}\gamma_i^4$. Then, $f(0) = 0$ and $f(\cdot)$ is continuous in γ_i. In view of $\lambda_2 > 0$ and $\Gamma_{i2} > 0$, we conclude that $f(\cdot)$ is strictly increasing in γ_i. Since $D[\varepsilon_i] \neq 0$, we have $D[\varepsilon_i] > 0$, $D[\varepsilon_i^2] > 0$, $\lambda_3 > 0$, $\lambda_4 > 0$, and, therefore, $2(\lambda_3 + \Gamma_{i1}\lambda_4) > 0$. Consequently, the solution $\gamma_i > 0$ to (82) always exists and is unique. For the unique solution $\gamma_i > 0$ that solves (82),

$$\begin{aligned}\frac{\partial^2 J_i}{\partial \gamma_i^2} &= -\frac{3}{\gamma_i^4}\left(\lambda_2\gamma_i^2\lambda_3 - 2\Gamma_{i1}\lambda_4 + 2\Gamma_{i2}\gamma_i^4\right) + \frac{1}{\gamma_i^3}\left(\lambda_2 + 8\Gamma_{i2}\gamma_i^3\right) \\ &= \frac{1}{\gamma_i^3}\left(\lambda_2 + 8\Gamma_{i2}\gamma_i^3\right) > 0.\end{aligned} \tag{83}$$

Therefore, the positive solution $\gamma_i > 0$ of the quartic equation (82) solves the constrained minimization problem (68). Q.E.D. □

Remark. In the special case that the fuzzy sets S_i are crisp, $D[V_{is}] = V_{is}$, $D[\varepsilon_i] = \varepsilon_i$, etc. The current setting still applies. The optimal design can also be found by solving (82), the solution of which will be given in the next section.

4.3. Solution to the Optimal Problem. The solutions of the quartic equation (82) depend on the following *cubic resolvent*:

$$z^3 + (-4\bar{r})z - \bar{q}^2 = 0, \tag{84}$$

where

$$\bar{r} = -\frac{1}{\Gamma_{i2}}(\lambda_3 + \Gamma_{i1}\lambda_4), \qquad \bar{q} = \frac{\lambda_2}{2\Gamma_{i2}}. \tag{85}$$

Let $\hat{p}_1 = -4\bar{r}$ and $\hat{p}_2 = -\bar{q}^2$. The discriminant $\hat{D}$ of the cubic resolvent is

$$\hat{D} = \left(\frac{\hat{p}_1}{3}\right)^3 + \left(\frac{\hat{p}_2}{2}\right)^2. \tag{86}$$

Since $\Gamma_{i2} > 0$, $\lambda_3 > 0$, $\Gamma_{i1} > 0$, and $\lambda_4 > 0$, we have $\bar{r} < 0$, thus $\hat{p}_1 > 0$, so that $\hat{D} > 0$. The solutions of the cubic resolvent are given by

$$z_1 = \hat{u} + \hat{v}, \tag{87}$$

$$z_2 = -\frac{\hat{u} + \hat{v}}{2} + (\hat{u} - \hat{v})i\sqrt{\frac{3}{2}}, \tag{88}$$

$$z_3 = -\frac{\hat{u} + \hat{v}}{2} - (\hat{u} - \hat{v})i\sqrt{\frac{3}{2}}, \tag{89}$$

where

$$\hat{u} = \left(-\frac{\hat{p}_2}{2} + \sqrt{\hat{D}}\right)^{1/3}, \qquad \hat{v} = \left(-\frac{\hat{p}_2}{2} - \sqrt{\hat{D}}\right)^{1/3}. \tag{90}$$

The cubic resolvent has one real solution and two complex conjugate solutions. This in turn implies that the quartic solution has two real solutions and one pair of complex conjugate solutions. The maximum real solution, which is positive and is therefore the optimal solution to the constrained optimization problem, of the quartic equation is given by

$$\gamma_{i\text{opt}} = \frac{1}{2}\left(\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}\right). \tag{91}$$

With z_1, z_2, and z_3 into (91), the positive solution of the quartic equation is given by

$$\gamma_{i\text{opt}} = \frac{1}{2}\left(\sqrt{\hat{u} + \hat{v}} + \sqrt{7\hat{u}^2 + 7\hat{v}^2 - 10\hat{u}\hat{v}}\cos\frac{\psi}{2}\right), \tag{92}$$

where

$$\psi = \tan^{-1}\frac{\sqrt{3/2}\,(\hat{u} - \hat{v})}{-(1/2)(\hat{u} + \hat{v})}. \tag{93}$$

With (82), the cost J in (78) can be rewritten as

$$\begin{aligned} J_i &= \lambda_1 - \frac{\lambda_2}{\gamma_i} + \frac{\lambda_3}{\gamma_i^2} + \Gamma_{i1}\frac{\lambda_4}{\gamma_i^2} + \Gamma_{i2}\gamma_i^2 \\ &= \lambda_1 - \frac{1}{\gamma_i^2}\left(\lambda_2\gamma_i + 2\Gamma_{i2}\gamma_i^4\right) + \frac{\lambda_3}{\gamma_i^2} + \Gamma_{i1}\frac{\lambda_4}{\gamma_i^2} + 3\Gamma_{i2}\gamma_i^2 \\ &= \lambda_1 - \frac{1}{\gamma_i^2}\left[2\left(\lambda_3 + \Gamma_{i1}\lambda_4\right)\right] + \frac{\lambda_3}{\gamma_i^2} + \Gamma_{i1}\frac{\lambda_4}{\gamma_i^2} + 3\Gamma_{i2}\gamma_i^2 \\ &= \lambda_1 - \frac{\lambda_3}{\gamma_i^2} - \Gamma_{i1}\frac{\lambda_4}{\gamma_i^2} + 3\Gamma_{i2}\gamma_i^2 \\ &= \lambda_1 - \frac{1}{\gamma_i^2}\left(\lambda_3 + \Gamma_{i1}\lambda_4 + 3\Gamma_{i2}\gamma_i^4\right). \end{aligned} \tag{94}$$

With (92), the minimum cost is given by

$$\begin{aligned} J_{i\min} = \lambda_1 - \frac{4}{\left(\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}\right)^2} \times \Big[\lambda_3 + \Gamma_{i1}\lambda_4 - \frac{3}{16}\Gamma_{i2} \\ \cdot \left(\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}\right)^4\Big]. \end{aligned} \tag{95}$$

Remark. Combining the results of Sections 2–4, the robust control scheme (27) using the optimal design of $\gamma_i > 0$ renders the solution of the transformed error dynamic system uniformly bounded and uniformly ultimately bounded. In addition, the performance index J_i given by (78) is globally minimized. Furthermore, the closed-form (i.e., analytic) solution is shown. The optimization problem is completely solved.

The optimal design procedure is summarized as follows:

Step 1: for given Δ_i^d, the control terms p_1 in (23) and p_2 in (24) are obtained.

Step 2: obtain the fuzzy descriptions of α_i and ρ_{E_i} and the bounding function $\Pi(\cdot)$. The control p_3 is given by (28) with γ_i undetermined.

Step 3: calculate $\lambda_{1,2,3,4}$ based on the α-cuts of the membership functions, the fuzzy arithmetic, and the decomposition theorem.

Step 4: for given $\Gamma_{i1} > 0$ and $\Gamma_{i2} > 0$, calculate the optimal gain $\gamma_{i\text{opt}}$ in (92). The resulting minimal performance index can be obtained by (95).

Step 5: the robust control scheme is given by (27).

5. Simulation Results

In this section, numerical simulations will be conducted to examine the behavior of a vehicle platoon traveling on a highway under the robust control law (27) developed in previous sections. Consider there are totally four vehicles in a platoon, that is, one leader with three followers. The following parameters are chosen for numerical simulations. The nominal vehicle masses (kg) $\overline{M}_L = \overline{M}_1 = \overline{M}_2 = \overline{M}_3 = 1000$, the nominal aerodynamic coefficients $\overline{c}_L = \overline{c}_1 = \overline{c}_2 = \overline{c}_3 = 0.3$, the nominal resistance forces (N) $\overline{F}_L = \overline{F}_1 = \overline{F}_2 = \overline{F}_3 = 100$, and the vehicle lengths $l_L = l_1 = l_2 = l_3 = 5$ m. The uncertainties in each vehicle which are (possibly) time-varying are given by $\Delta M_L = 35\sin 0.1t$, $\Delta M_1 = 32\cos 0.5t$, $\Delta M_2 = 40\cos t$, $\Delta M_3 = -25\cos 0.1t$, $\Delta c_L = 0.02$, $\Delta c_1 = 0.01$, $\Delta c_2 = -0.03$, $\Delta c_3 = -0.02$, $\Delta F_L = 180\sin 0.5t$, $\Delta F_1 = 200\sin t$, $\Delta F_2 = 190\sin(t-\pi/6)$, and $\Delta F_3 = 160\sin(t-\pi/6)$. Notice that all uncertainties are bounded. Suppose the parameters ρ_{E_1}, ρ_{E_2}, and ρ_{E_3} in Assumption 2 are all "close to 0", where the associated membership functions are given by (all triangular)

$$\mu_{\rho_{E_{1,2,3}}}(\nu) = \begin{cases} \dfrac{10}{3}\nu + 1, & -0.3 \le \nu \le 0, \\ \\ -\dfrac{10}{3}\nu + 1, & 0 \le \nu \le 0.3, \end{cases} \quad (96)$$

and the known functions in Assumption 3 are given by

$$\Pi_{1,2,3} = \alpha_{1,2,3}\left(0.1\dot{e}_i^2 + 0.1e_i^2 + 0.6\right), \quad (97)$$

with the uncertain parameters $\alpha_{1,2,3}$ all "close to 1," where the associated membership functions are given by (all triangular)

$$\mu_{\alpha_{1,2,3}}(\nu) = \begin{cases} 2\nu - 1, & \dfrac{1}{2} \le \nu \le 1, \\ \\ -2\nu + 3, & 1 \le \nu \le \dfrac{3}{2}. \end{cases} \quad (98)$$

Hence, Assumptions 2 and 3 are met.

The leading vehicle is under the control as follows:

$$\overline{u}_L := \overline{c}_L \dot{x}_L \left|\dot{x}_L\right| + \overline{F}_L, \quad (99)$$

$$u_L = \begin{cases} \overline{u}_L + 2500\sin 0.2\pi(t-5), & \text{if } 5 < t \le 10, \\ \overline{u}_L - 1500\sin 0.2\pi(t-15), & \text{if } 15 < t \le 20, \\ \overline{u}_L, & \text{otherwise}. \end{cases} \quad (100)$$

For comparison purpose, we consider two types of control for the followers. The first is the robust controller (27):

$$u_i = p_{i1} + p_{i2} + p_{i3}, \quad i = 1, 2, 3. \quad (101)$$

The second is the PD controller:

$$u_i = -k_{i1}e_i - k_{i2}\dot{e}_i, \quad i = 1, 2, 3, \quad (102)$$

where k_{i1} and k_{i2} are the proportional and derivative control gains.

Suppose the desired space is given by $\Delta_1^d = \Delta_2^d = \Delta_3^d = 5$ m. For the PD controller, we choose $k_{i1} = 220$ and $k_{i2} = 500$, $i = 1, 2, 3$. Two cases of initial conditions are considered (notice that all initial conditions are crisp): first, the zero initial condition case with initial positions $x(t_s) = x(t_0) = x(0) = [100\ 90\ 80\ 70]^T$ and initial velocities $\dot{x}(t_s) = \dot{x}(t_0) = \dot{x}(0) = [10\ 10\ 10\ 10]^T$. We find that the position errors $e = [0\ 0\ 0]^T$ and velocity errors $\dot{e} = [0\ 0\ 0]^T$. Recalling that $V_{is} = (1/2)(z_{i1}^2(t_s) + z_{i2}^2(t_s))$, we have $V_{is} = 0$, $i = 1, 2, 3, 4$, By using the fuzzy arithmetic and the decomposition theorem, we obtain $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0.0039$, and $\lambda_4 = 0.0156$, and the quartic equation (82) is given by

$$2\Gamma_{i2}\gamma_i^4 = 2\left(0.0039 + 0.0156\Gamma_{i2}\right). \quad (103)$$

We choose five sets of weighting factors Γ_1 and Γ_2 for demonstration. Their values and the corresponding $\gamma_{i\mathrm{opt}}$ and $J_{\min}$ are summarized in Table 1.

Next, the critical case with initial positions $x(t_s) = x(t_0) = x(0) = [100\ 94\ 88\ 82]^T$ and initial velocities $\dot{x}(t_s) = \dot{x}(t_0) = \dot{x}(0) = [10\ 12\ 14\ 16]^T$. Similarly, it can be found that $e(t_s) = [4\ 4\ 4]^T$ and $\dot{e}(t_s) = [2\ 2\ 2]^T$ (these mean the vehicles are only 1m apart and they are approaching initially). Recalling that $V_{is} = (1/2)(z_{i1}^2(t_s) + z_{i2}^2(t_s))$, we have $V_{is} = 1.3747$, $i = 1, 2, 3, 4$. By using the fuzzy arithmetic and the decomposition theorem, we obtain $\lambda_1 = 0.4725$, $\lambda_2 = 0.0859$, $\lambda_3 = 0.0039$, and $\lambda_4 = 0.0156$. By following (84) to (95), the quartic equation (82) is given by

$$0.0859\gamma_i + 2\Gamma_{i2}\gamma_i^4 = 2\left(0.0039 + 0.0156\Gamma_{i2}\right). \quad (104)$$

Again, five sets of weighting factors Γ_{i1} and Γ_{i2} are chosen for demonstration. Their values and the corresponding $\gamma_{i\mathrm{opt}}$ and $J_{i\min}$ are summarized in Table 2.

With these parameters obtained, numerical simulations are conducted. The final results are shown in Figures 2–13. Figures 2–9 show the performance of the platoon under zero initial conditions. Figures 2–5 show the position and velocity histories along x-direction of each vehicle in the platoon under the PD controller and robust controller ($\gamma_{i\mathrm{opt}} = 0.3738$, when using $\Gamma_{i1} = \Gamma_{i2} = 1$) with zero initial condition, respectively. All trajectories are smooth and no abrupt changes occurred. Figures 6 and 7 show the space error histories for the vehicles under the PD controller and robust controller, respectively. It is shown that the space errors of the followers under PD control oscillate and the maximum magnitude increases. The collisions occur ($\Delta_{2,3} > 5$ m) for the second and third vehicles at around $t = 18$. This is not surprising since this control does not guarantee stability. On the other hand, the space errors of the followers under robust control always stay in a region smaller than 0.3 m and hence no collisions.

It is interesting to see that in Figure 7, the third vehicle is the one with least upper bound of the space error, which means the space error is well suppressed under the robust controller. Figures 8 and 9 show the control effort history

Table 1: Weighting factors/optimal gain/minimum cost.

$(\Gamma_{i1}, \Gamma_{i2})$	Γ_{i1}/Γ_{i2}	γ_{iopt}	J_{imin}
(100, 1)	100	1.1187	2.5031
(10, 1)	10	0.6326	0.8004
(1, 1)	1	0.3738	0.2795
(1, 10)	0.1	0.2102	0.8839
(1, 100)	0.01	0.1182	2.7951

Table 2: Weighting factors/optimal gain/minimum cost.

$(\Gamma_{i1}, \Gamma_{i2})$	Γ_{i1}/Γ_{i2}	γ_{iopt}	$J_{i\min}$
(100, 1)	100	2.5118	6.9956
(10, 1)	10	0.8280	1.2878
(1, 1)	1	0.3627	0.5156
(1, 10)	0.1	0.1133	1.3640
(1, 100)	0.01	0.0357	13.522

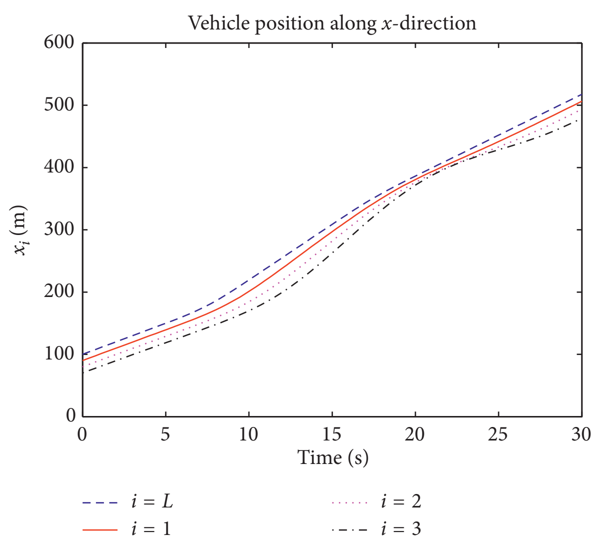

Figure 2: The position history of the four vehicles in the platoon under the PD controller with zero initial conditions.

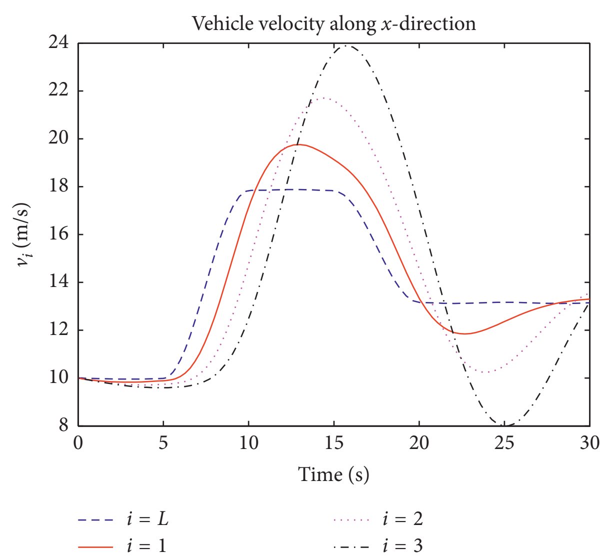

Figure 4: The velocity history of the four vehicles in the platoon under the PD controller with zero initial conditions.

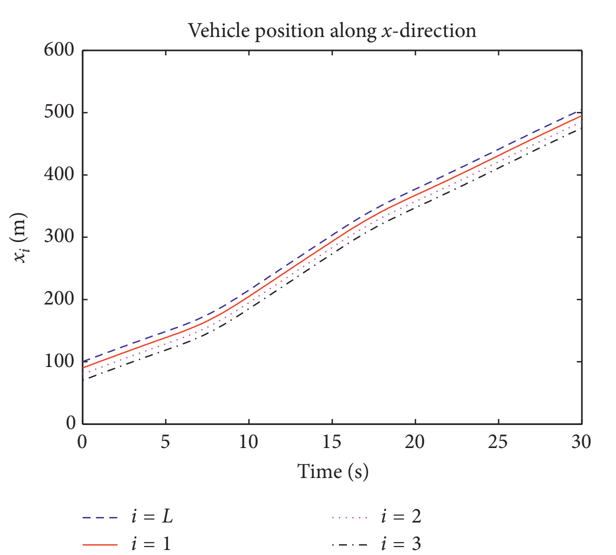

Figure 3: The position history of the four vehicles in the platoon under robust controller with zero initial conditions.

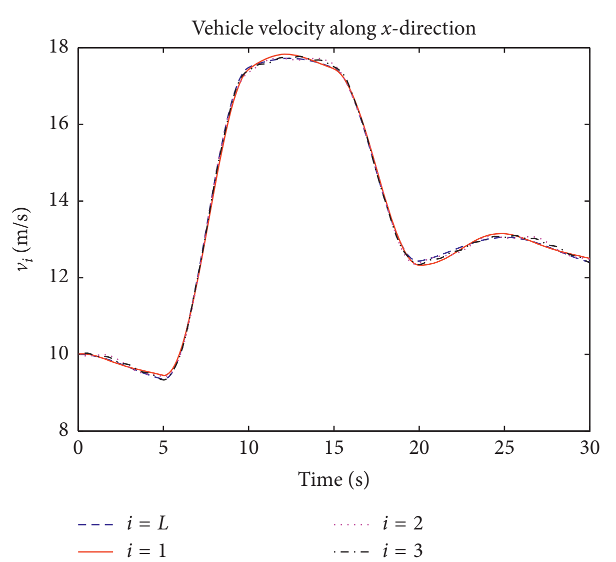

Figure 5: The velocity history of the four vehicles in the platoon under the robust controller with zero initial conditions.

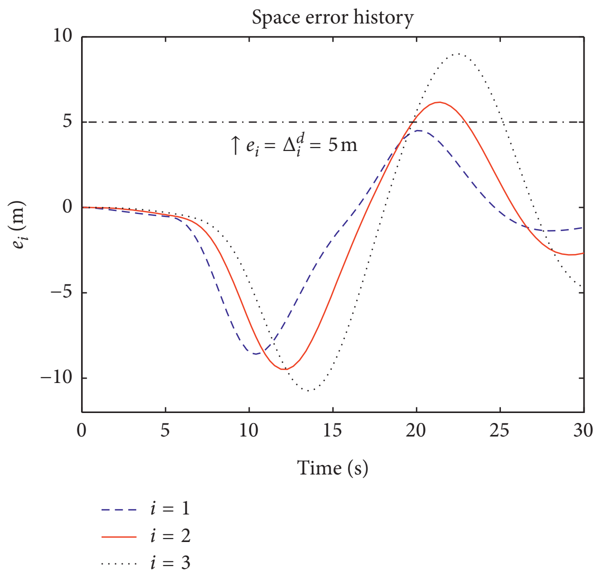

Figure 6: The space error history of the four vehicles in the platoon under the PD controller with zero initial conditions.

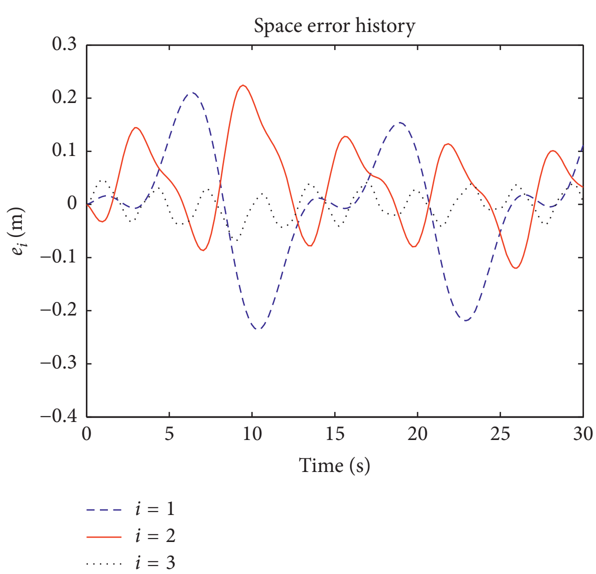

Figure 7: The space error history of the four vehicles in the platoon under the robust controller with zero initial conditions.

under two different controllers. It appears that the magnitude of the robust controller is smaller than that of the PD controller.

Figures 10–13 show the performance of the platoon under critical initial conditions. Figure 10 shows the space error histories for each vehicle under the robust controller. Despite initially being very close to each other, which means a very dangerous proximity, the space error $e_i(t)$ for $i = 1, 2, 3$, under robust control is always under 5m (i.e., $e_i(t) < \Delta_i^d$) and hence no collision. Moreover, it shows that the space error of each follower enters a small region (less than 0.3) in less than 5 s and stays there,

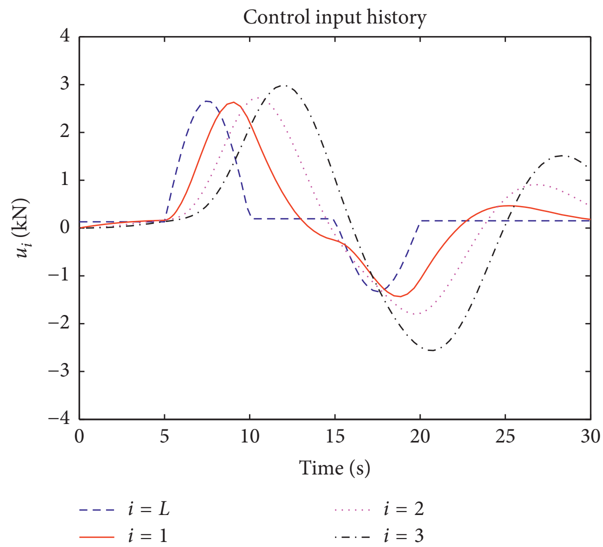

Figure 8: The control force history of the four vehicles in the platoon under the PD controller with zero initial conditions.

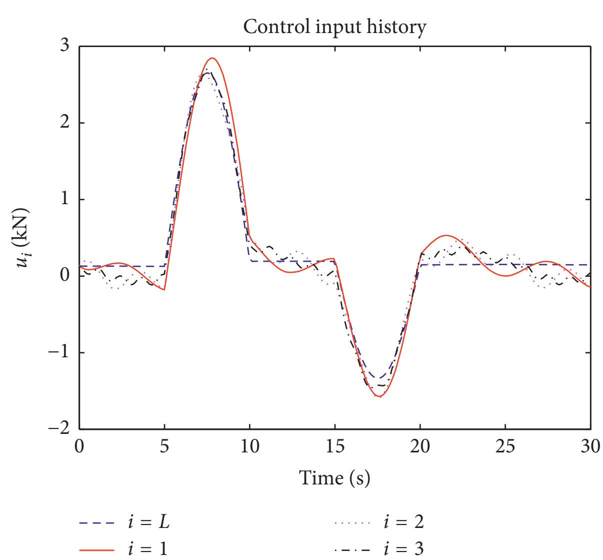

Figure 9: The control force history of the four vehicles in the platoon under the robust controller with zero initial conditions.

thereafter, regardless of the acceleration changes of the leader at the time periods $5 < t \le 10$ and $15 < t \le 20$. Figure 11 shows the control effort history under critical initial conditions. It can be shown that after the initial position and velocity error is suppressed (after 5 s), the effort of the robust controller is smaller than that of the PD controller. For comparison, Figures 12 and 13 show the space error history and its corresponding control effort of the first follower under five $\gamma_{i\text{opt}}$, respectively. It appears that a higher optimal gain $\gamma_{i\text{opt}}$ renders a faster settling of the space error e_i at the beginning. This is because more weighting was assigned to the transient-state performance J_{11}.

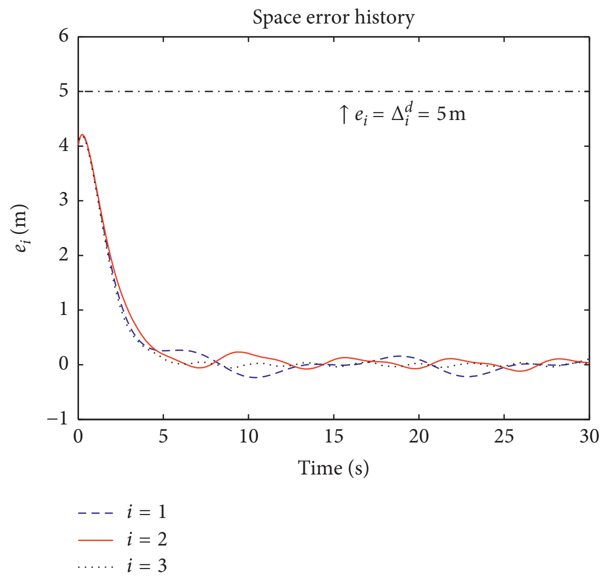

Figure 10: The space error history of the four vehicles in the platoon under the PD controller with critical initial conditions.

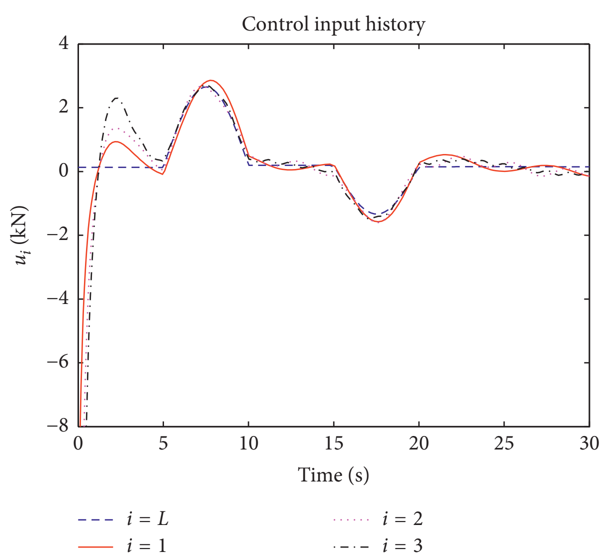

Figure 11: The control force history of the four vehicles in the platoon under the robust controller with critical initial conditions.

6. Conclusion

A new approach is proposed to the decentralized control design for vehicle platooning for uncertain automated highway systems. The uncertainty in the system is nonlinear and (possibly) fast time-varying. The only information about the uncertainty is that it is assumed to be within a prescribed fuzzy set. Based on a creative transformation, a class of decentralized control is proposed for the vehicle platoon. Each vehicle in the platoon only needs the knowledge of the preceding vehicle. No acceleration feedback or the information of the lead vehicle is required. The resulting controlled platoon is global collision avoidance. The resulting performance of the controlled platoon is two folds: one deterministic and one fuzzy. The deterministic performance

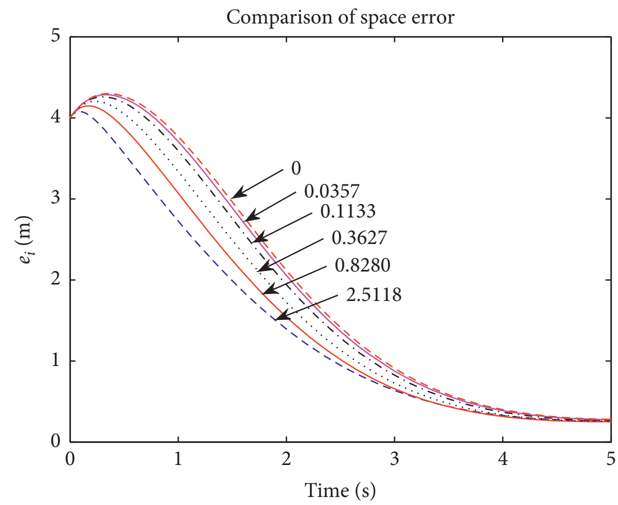

Figure 12: Comparison of the space error of the first follower under different $\gamma_{1\min}$.

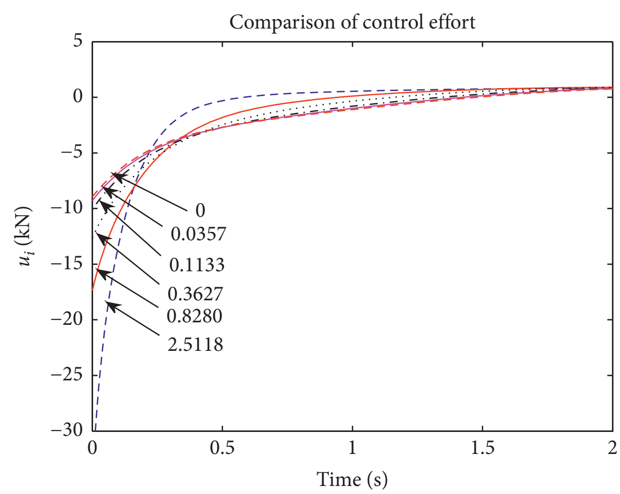

Figure 13: Comparison of the control effort of the first follower under different $\gamma_{1\min}$.

assures that under the worst case we still have collision avoidance under arbitrary safe initial conditions; this assures the bottom line, while the fuzzy information allows us further consider optimization problem. A control design parameter is selected to minimize a fuzzy-based performance index. Both the analytic forms (i.e., closed forms) of the control design parameter and the resulting minimum performance index are obtained. The optimization problem is completely solved. The simulation results compare the performance of the system under the proposed control with PD control. It shows that even under a very critical initial state, where collision appears to be imminent, the robust control was able to pull out and avoid collision.

Data Availability

The simulation data used to support the findings of this study are included within the article and can be made freely available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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