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Research Article

LU Decomposition Scheme for Solving *m*-Polar Fuzzy System of Linear Equations

Ali N. A. Koam,[1] Muhammad Akram,[2] Ghulam Muhammad,[2] and Nawab Hussain[3]

[1]*Department of Mathematics, College of Science, Jazan University, New Campus, P.O. Box 2097, Jazan 45142, Saudi Arabia*
[2]*Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan*
[3]*Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Correspondence should be addressed to Muhammad Akram; m.akram@pucit.edu.pk

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This paper presents a new scheme for solving *m*-polar fuzzy system of linear equations (*m*-PFSLEs) by using *LU* decomposition method. We assume the coefficient matrix of the system is symmetric positive definite, and we discuss this point in detail with some numerical examples. Furthermore, we investigate the inconsistent $m \times nm$-polar fuzzy matrix equation (*m*-PFME) and find the least square solution (LSS) of this system by using generalized inverse matrix theory. Moreover, we discuss the strong solution of *m*-polar fuzzy LSS of the inconsistent *m*-PFME. In the end, we present a numerical example to illustrate our approach.

1. Introduction

Certain types of uncertainties arise in several areas of engineering and decision-making. To handle such uncertainties, probability theory, fuzzy set theory [1], and their related models have been proposed as suitable mathematical tools. It helps to define the problem in real form and the solution for these uncertain variables has been obtained. Many researchers [2–6] have studied their basic arithmetical operations and the methods of the fuzzy numbers. Goetschel and Voxman [7] proposed the concept of fuzzy calculus. They presented the parametric form of fuzzy number by using cut expansion and inserted the class of fuzzy numbers into a topological vector space. Moghadam et al. [8] described the concept of trapezoidal fuzzy numbers and also other affected investigations were shown in [9, 10]. The notion of *m*-polar fuzzy set was proposed by Chen et al. [11] as the generalization of bipolar fuzzy set [12]. Nowadays, analysts believe that the world is moving towards multipolarity. Therefore, it comes as no surprise that multipolarity in data and information plays a vital role in various fields of science and technology. The remarkable contribution on applications of *m*-polar fuzzy sets is presented in [13–15].

Linear system plays an important role in many fields of engineering and science. In the wide majority of problems, we often deal with approximate data. Some parameters are represented as a fuzzy number and more general *m*-polar fuzzy number rather than a crisp number. A numerical approach that would be suitable to handle and solve a *m*-polar fuzzy linear system is extremely essential. The notion of *m*-polar fuzzy linear system with crisp coefficient entries of matrix and the right-hand side is that parametric *m*-polar fuzzy number vectors appear in many domains of engineering sciences such as economics, statistics, technology, telecommunications, image processing, social sciences, and physics. Some applications of linear system in fuzzy environment were presented in [16, 17]. *LU* decomposition method is used to solve many different kinds of systems of linear equations in *m*-polar fuzzy environment. It is faster and more numerically stable than computing explicit inverses. *LU* decomposition method to solve *m*-polar fuzzy linear system is used in electrical engineering and circuit designing, and this system is used to solve complex circuits. This technique is also used in dynamics to solve Diffusion Load Balancing. However, there is a vast literature in mathematics to solve the fuzzy linear system. But we

introduce the new approach to solve linear system in m-polar fuzzy environment. First we study the basic literature to solve linear system in fuzzy environment. Friedman [18] presented the idea of a fuzzy linear system of equations having the crisp coefficient matrix and the right-hand side is parametric fuzzy number vector. Friedman also proposed an embedding scheme and replaced $n \times n$ original system of a fuzzy linear system with the extended $2n \times 2n$ system.

The iterative scheme to solve the linear system of equation in the form $\mathcal{X} = \mathcal{A}\mathcal{X} + \mathcal{U}$ was studied by Wang et al. [19]. Asady et al. [20] developed the general linear system and used an embedding technique to construct the different schemes in a fuzzy environment. Vroman et al. [21] used a parametric technique of fuzzy numbers to find the general solution. Sevastjanov and Dymova [22] introduced a practical approach for interval fuzzy systems. The numerical technique was also studied by Garg and Singh [23] to solve a fuzzy linear system by using the Gaussian fuzzy weight (membership) function. Behera and Chakraverty [24] introduced a new scheme for handling the real as well as the complex fuzzy linear system. Moreover, Abbasbandy et al. [25, 26] presented the steepest descent method and the LU decomposition method to solve the fuzzy system of linear equations. Allahviranloo et al. [27–30] developed some important numerical schemes for solving a fuzzy linear system of equations (FLSEs). Moreover, certain methods to solve fuzzy linear systems have been discussed in [31–34]. Akram et al. [35–39] studied certain schemes for solving the bipolar fuzzy linear system of equations. Akram et al. [40] discussed the solution of linear system in m-polar fuzzy environment. This paper presents a new scheme for solving m-polar fuzzy system of linear equations (m-PFSLEs) by using LU decomposition method. We assume a special case when the coefficient matrix of the system is symmetric and positive definite and then we discuss this point in detail with some numerical examples. Furthermore, we investigate the inconsistent $m \times nm$-polar fuzzy matrix equation (m-PFME) and find the least square solution (LSS) of this system by using generalized inverse matrix theory. Moreover, we discuss the strong solution of m-polar fuzzy LSS of the inconsistent m-PFME. In the end, we present a numerical example to illustrate our approach.

The rest of the paper is structured as follows. In Section 2, we present the solution of m-polar fuzzy linear system by using LU decomposition method. Section 3 presents the inconsistent m-polar fuzzy matrix equation with some examples and Section 4 develops to obtain the least square solution of m-polar fuzzy matrix equations. Some results are investigated by giving the true reasoning and the conclusion of this research work is in Section 5.

2. LU Decomposition Method for Solving m-PFSLEs

Definition 1 (see [11]). An m-polar fuzzy set on an underlying set Z is a mapping $M\colon Z \longrightarrow [0,1]^m$. The truthness degree of each element $z \in Z$ is defined as

$$M(z) = (\mathcal{P}_1 \circ M(z), \mathcal{P}_2 \circ M(z), \ldots, \mathcal{P}_m \circ M(z)), \quad (1)$$

where $\mathcal{P}_i \circ M\colon [0,1]^m \longrightarrow [0,1]$ is the i-th projection mapping.

Definition 2 (see [40]). An m-polar fuzzy number (m-PFN) in parametric form is an m-tuple $\langle [\underline{\mathcal{K}}^{(1)}(\delta_i), \overline{\mathcal{K}}^{(1)}(\delta_i)], [\underline{\mathcal{K}}^{(2)}(\delta_i), \overline{\mathcal{K}}^{(2)}(\delta_i)], \ldots, [\underline{\mathcal{K}}^{(m)}(\delta_i), \overline{\mathcal{K}}^{(m)}(\delta_i)]\rangle = \langle [\underline{\mathcal{K}}^{(i)}(\delta_i), \overline{\mathcal{K}}^{(i)}(\delta_i)]\rangle$ of functions $\underline{\mathcal{K}}^{(i)}(\delta_i), \overline{\mathcal{K}}^{(i)}(\delta_i), \delta_i \in [0,1]^m, i = 1, 2, \ldots, m$, which satisfy the following properties:

(i) $\underline{\mathcal{K}}^{(i)}(\delta_i)$ is a bounded nondecreasing right continuous function at the point 0 and left continuous over the interval (0, 1]

(ii) $\overline{\mathcal{K}}^{(i)}(\delta_i)$ is a bounded nonincreasing right continuous function at the point 0 and left continuous over the interval (0, 1]

(iii) $\underline{\mathcal{K}}^{(i)}(\delta_i) \le \overline{\mathcal{K}}^{(i)}(\delta_i)$

Throughout the paper, $i = 1, 2, 3, \ldots, m$.

Definition 3 (see [40]). For arbitrary $\mathcal{K} = \langle [\underline{\mathcal{K}}^{(i)}(\delta_i), \overline{\mathcal{K}}^{(i)}(\delta_i)]\rangle$, $\mathcal{J} = \langle [\underline{\mathcal{J}}^{(i)}(\delta_i), \overline{\mathcal{J}}^{(i)}(\delta_i)]\rangle$ and $a > 0$, we define $(\mathcal{K} + \mathcal{J})$, $(\mathcal{K}.\mathcal{J})$, $(\mathcal{K}/\mathcal{J})$, and scalar multiplication by a as follows:

(i) $(\underline{\mathcal{K}^{(i)} + \mathcal{J}^{(i)}})(\delta_i) = \underline{\mathcal{K}}^{(i)}(\delta_i) + \underline{\mathcal{J}}^{(i)}(\delta_i)$, $(\overline{\mathcal{K}^{(i)} + \mathcal{J}^{(i)}})(\delta_i) = \overline{\mathcal{K}}^{(i)}(\delta_i) + \overline{\mathcal{J}}^{(i)}(\delta_i)$

(ii) $(\underline{\mathcal{K}^{(i)}\mathcal{J}^{(i)}})(\delta_i) = \min\{\underline{\mathcal{K}}^{(i)}(\delta_i)\underline{\mathcal{J}}^{(i)}(\delta_i), \underline{\mathcal{K}}^{(i)}(\delta_i)\overline{\mathcal{J}}^{(i)}(\delta_i), \overline{\mathcal{K}}^{(i)}(\delta_i)\underline{\mathcal{J}}^{(i)}(\delta_i), \overline{\mathcal{K}}^{(i)}(\delta_i)\overline{\mathcal{J}}^{(i)}(\delta_i)\}$

(iii) $(\overline{\mathcal{K}^{(i)}\mathcal{J}^{(i)}})(\delta_i) = \max\{\underline{\mathcal{K}}^{(i)}(\delta_i)\underline{\mathcal{J}}^{(i)}(\delta_i), \underline{\mathcal{K}}^{(i)}(\delta_i)\overline{\mathcal{J}}^{(i)}(\delta_i), \overline{\mathcal{K}}^{(i)}(\delta_i)\underline{\mathcal{J}}^{(i)}(\delta_i), \overline{\mathcal{K}}^{(i)}(\delta_i)\overline{\mathcal{J}}^{(i)}(\delta_i)\}$

(iv) $(\underline{\mathcal{K}^{(i)}/\mathcal{J}^{(i)}})(\delta_i) = \underline{\mathcal{K}}^{(i)}(\delta_i)/\underline{\mathcal{J}}^{(i)}(\delta_i)$, $(\overline{\mathcal{K}^{(i)}/\mathcal{J}^{(i)}})(\delta_i) = \overline{\mathcal{K}}^{(i)}(\delta_i)/\overline{\mathcal{J}}^{(i)}(\delta_i)$, $\mathcal{K}, \mathcal{J} > 0$

(v) $(\underline{a\mathcal{K}}^{(i)})(\delta_i) = a(\underline{\mathcal{K}}^{(i)})(\delta_i)$, $(\overline{a\mathcal{K}}^{(i)})(\delta_i) = a(\overline{\mathcal{K}}^{(i)})(\delta_i)$, $a \ge 0$

(vi) $(\underline{a\mathcal{K}}^{(i)})(\delta_i) = a(\overline{\mathcal{K}}^{(i)})(\delta_i)$, $(\overline{a\mathcal{K}}^{(i)})(\delta_i) = a(\underline{\mathcal{K}}^{(i)})(\delta_i)$, $a < 0$

The family of all m-PFNs is denoted by Ψ.

Definition 4 (see [40]). The $n \times n$ linear system is

$$\begin{aligned} a_{11}\mathcal{K}_1 + a_{12}\mathcal{K}_2 + \cdots + a_{1n}\mathcal{K}_n &= m_1^{(i)}, \\ a_{21}\mathcal{K}_1 + a_{22}\mathcal{K}_2 + \cdots + a_{2n}\mathcal{K}_n &= m_2^{(i)}, \\ &\vdots \\ a_{n1}\mathcal{K}_1 + a_{n2}\mathcal{K}_2 + \cdots + a_{nn}\mathcal{K}_n &= m_n^{(i)}, \end{aligned} \quad (2)$$

where the coefficient matrix $\mathcal{D} = (a_{\mathbf{pq}})$, $1 \le p, q \le n$ is a crisp $n \times n$ matrix and $m_p^{(i)}$, $1 \le p \le n, 1 \le i \le m$ are known m-PFNs and $\mathcal{K}_q$, $1 \le q \le n$ are unknowns which may or may not be m-PFNs, which are called m-PFSLEs.

Definition 5. The matrix system is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} & \cdots & \mathcal{K}_{1n} \\ \mathcal{K}_{21} & \mathcal{K}_{22} & \cdots & \mathcal{K}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{K}_{n1} & \mathcal{K}_{n2} & \cdots & \mathcal{K}_{nn} \end{pmatrix} = \begin{pmatrix} m_{11}^{(i)} & m_{12}^{(i)} & \cdots & m_{1n}^{(i)} \\ m_{21}^{(i)} & m_{22}^{(i)} & \cdots & m_{2n}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1}^{(i)} & m_{n2}^{(i)} & \cdots & m_{nn}^{(i)} \end{pmatrix}, \tag{3}$$

where the coefficient elements a_{pq}, $1 \le p, q \le n$ are crisp numbers and $m_{pq}^{(i)}$ in the right-hand matrix are m-PFNs which are called a general m-PFME. By using matrix equation, we have

$$\mathcal{D}\mathcal{K} = \mathcal{W}^{(i)}. \tag{4}$$

Definition 6. An m-PFN vector $(\mathcal{K}_1^{(i)}, \mathcal{K}_2^{(i)}, \ldots, \mathcal{K}_n^{(i)})^T$ given by $(\mathcal{K}_q^{(i)})_{\delta_i} = \mathcal{K}_q^{(i)} = \langle [\underline{\mathcal{K}}_q^{(i)}(\delta_i), \overline{\mathcal{K}}_q^{(i)}(\delta_i)] \rangle$, $1 \le q \le n, \delta_i \in [0,1]^m$, $i = 1, 2, \ldots, m$, is called a solution of the m-PFSLE (2) if

$$\underline{\sum_{1\le q\le n} a_{\mathbf{pq}}\mathcal{K}_q} = \sum_{1\le q\le n} \underline{a_{\mathbf{pq}}\mathcal{K}_q} = \underline{m}_p^{(i)}, \quad \overline{\sum_{1\le q\le n} a_{\mathbf{pq}}\mathcal{K}_q} = \sum_{1\le q\le n} \overline{a_{\mathbf{pq}}\mathcal{K}_q} = \overline{m}_p^{(i)}. \tag{5}$$

For a particular p, $a_{\mathbf{pq}} > 0$, $1 \le p \le n$, we get

$$\sum_{1\le q\le n} a_{\mathbf{pq}}\underline{\mathcal{K}}_q = \underline{m}_p^{(i)}, \quad \sum_{1\le q\le n} a_{\mathbf{pq}}\overline{\mathcal{K}}_q = \overline{m}_p^{(i)}. \tag{6}$$

From the expression above, we have the following $2n \times 2n$ crisp linear system:

$$\mathcal{N}\mathcal{K} = \mathcal{W}^{(i)}, \tag{7}$$

or

$$\begin{bmatrix} \mathcal{N}_1 \ge 0 & \mathcal{N}_2 \le 0 \\ \mathcal{N}_2 \le 0 & \mathcal{N}_1 \ge 0 \end{bmatrix} \begin{bmatrix} \underline{\mathcal{K}} \\ \overline{\mathcal{K}} \end{bmatrix} = \begin{bmatrix} \underline{\mathcal{W}}^{(i)} \\ \overline{\mathcal{W}}^{(i)} \end{bmatrix},$$

$$\begin{aligned} a_{pq} \ge 0 &\Longrightarrow \mathcal{N}_{p,q} = a_{pq}, \\ \mathcal{N}_{p+n,q+n} &= a_{pq}, \\ \mathcal{N}_{p,q+n} &= 0, \\ \mathcal{N}_{p+n,q} &= 0, \\ a_{pq} < 0 &\Longrightarrow \mathcal{N}_{p,q} = 0, \\ \mathcal{N}_{p,q+n} &= -a_{pq}, \\ \mathcal{N}_{p+n,q} &= a_{pq}, \\ \mathcal{N}_{p+n,q+n} &= 0, \quad 1 \le p, q \le n. \end{aligned} \tag{8}$$

If any $\mathcal{N}_{p,q}$ is not specified, it will be perceived as 0.

So, a system in Definition 4 extended to the crisp system (8) where $A = \mathcal{N}_1 + \mathcal{N}_2$ and (8) can be written as

$$\begin{cases} \mathcal{N}_1 \underline{\mathcal{K}} + \mathcal{N}_2 \overline{\mathcal{K}} = \underline{\mathcal{W}}^{(i)}, \\ \mathcal{N}_2 \underline{\mathcal{K}} + \mathcal{N}_1 \overline{\mathcal{K}} = \overline{\mathcal{W}}^{(i)}. \end{cases} \tag{9}$$

On the base of [18, 41, 42], we investigate the following results.

Theorem 1. *The matrix* $\mathcal{N} = \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{N}_2 & \mathcal{N}_1 \end{pmatrix}$ *is nonsingular if and only if the matrices* $\mathcal{N}_1 - \mathcal{N}_2$ *and* $\mathcal{N}_1 + \mathcal{N}_2$ *are also nonsingular.*

Definition 7. If $(\mathcal{K}_q^{(i)})_\delta = (\underline{\mathcal{K}}_1^{(i)}, \underline{\mathcal{K}}_2^{(i)}, \ldots, \underline{\mathcal{K}}_n^{(i)}, \overline{\mathcal{K}}_1^{(i)}, \overline{\mathcal{K}}_2^{(i)}, \ldots, \overline{\mathcal{K}}_n^{(i)})^T$ is a solution of system (7) and holds for each $1 \le q \le n$ the inequalities $\underline{\mathcal{K}}_q^{(i)} \le \overline{\mathcal{K}}_q^{(i)}$, then the solution is called a strong system solution (7); otherwise it would be a weak solution of system (7).

Theorem 2. *Suppose that a matrix* $\mathcal{N}$ *is nonsingular and a unique solution of* $\mathcal{N}\mathcal{K}^{(i)} = \mathcal{W}^{(i)}$ *always gives m-polar fuzzy number for arbitrary vector* $\mathcal{W}^{(i)}$ *; then the necessary and sufficient condition for the inverse of nonnegative matrix* $\mathcal{N}$ *exists.*

Theorem 3. *Let* $\mathcal{P}_{n\times n}$ *be a matrix that contains all real entries except 0 for leading minors. Then, the matrix* $\mathcal{P}_{n\times n}$ *has a unique factorization:*

$$\mathcal{P} = LU, \tag{10}$$

where L and U are unit lower-triangular and upper-triangular matrices, respectively.

To eliminate the N reduction, the L and U matrices must be found such that $\mathcal{N} = LU$, which is

$$\begin{aligned} L &= \begin{pmatrix} L_{11} & O \\ L_{21} & L_{22} \end{pmatrix}, \\ U &= \begin{pmatrix} U_{11} & U_{12} \\ O & U_{22} \end{pmatrix}, \end{aligned} \tag{11}$$

where L_{11}, L_{22} and U_{11}, U_{22} are lower- and upper-triangular matrices, L_{21} and U_{12} are any matrices, and O is null matrix, respectively.

Now, we suppose that $\mathscr{P} = \mathscr{N}_1 + \mathscr{N}_2$ has LU decomposition. We have

$$\begin{aligned}\mathscr{N} &= \begin{pmatrix} \mathscr{N}_1 & \mathscr{N}_2 \\ \mathscr{N}_2 & \mathscr{N}_1 \end{pmatrix} \\ &= \begin{pmatrix} L_{11} & O \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ O & U_{22} \end{pmatrix},\end{aligned} \tag{12}$$

and then

$$\begin{aligned}&\mathscr{N}_1 = L_{11}U_{11}, \\ &\mathscr{N}_2 = L_{11}U_{12} \Longrightarrow U_{12} = L_{11}^{-1}\mathscr{N}_2, \mathscr{N}_2 = L_{21}U_{11} \Longrightarrow L_{21} \\ &\quad = \mathscr{N}_2 U_{11}^{-1}, \mathscr{N}_1 = L_{21}U_{12} + L_{22}U_{22}.\end{aligned} \tag{13}$$

Therefore, we can write

$$\mathscr{N}_1 - \mathscr{N}_2 \mathscr{N}_1^{-1} \mathscr{N}_2 = L_{22}U_{22}. \tag{14}$$

Theorem 4. *Let $\mathscr{N}_{n\times n}$ be a positive definite symmetric matrix; then there is a unique matrix L with positive diagonal entries such that*

$$\mathscr{N} = LL^T. \tag{15}$$

Proof. Suppose that $\mathscr{N}_{n\times n}$ is a positive definite symmetric matrix. We have

$$\begin{aligned}\mathscr{N} &= \begin{pmatrix} \mathscr{N}_1 & \mathscr{N}_2 \\ \mathscr{N}_2 & \mathscr{N}_1 \end{pmatrix} \\ &= \begin{pmatrix} L_{11} & O \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} L_{11}^T & L_{21}^T \\ O & L_{22}^T \end{pmatrix},\end{aligned} \tag{16}$$

and then

$$\begin{aligned}&\mathscr{N}_1 = L_{11}L_{11}^T, \\ &\mathscr{N}_2 = L_{11}L_{21}^T \Longrightarrow L_{21}^T = L_{11}^{-1}\mathscr{N}_2, \mathscr{N}_2 \\ &\quad = L_{21}L_{11}^T \Longrightarrow L_{21} = \mathscr{N}_2\left(L_{11}^T\right)^{-1}, \\ &\mathscr{N}_1 = L_{21}U_{12} + L_{22}U_{22}.\end{aligned} \tag{17}$$

Therefore, we can write

$$\mathscr{N}_1 - \mathscr{N}_2 \mathscr{N}_1^{-1} \mathscr{N}_2 = L_{22}U_{22}^T. \tag{18}$$

By following Theorem 4 in LU decomposition method, $\mathscr{N}_1$ and $\mathscr{N}_1 - \mathscr{N}_2\mathscr{N}_1^{-1}\mathscr{N}_2$ should be a positive definite symmetric matrix.

We solve some numerical examples to illustrate our scheme.

Example 1. Consider 2×23-polar fuzzy system

$$\begin{aligned}3\mathscr{K}_1 - 2\mathscr{K}_2 &= \langle [35 + 5\delta, 45 - 5\delta], [21 + 5\delta, 31 - 5\delta], [17 + 4\delta, 25 - 4\delta] \rangle, \\ \mathscr{K}_1 + 4\mathscr{K}_2 &= \langle [9 + 6\delta, 21 - 6\delta], [13 + 2\delta, 17 - 2\delta], [16 + 3\delta, 22 - 3\delta] \rangle.\end{aligned} \tag{19}$$

The extended 4×4 matrix is

$$\begin{aligned}\mathscr{N} &= \begin{pmatrix} 3 & 0 & 0 & -2 \\ 1 & 4 & 0 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{pmatrix}, \\ \mathscr{N}_1 &= \begin{pmatrix} 3 & 0 \\ 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1.0000 & 0 \\ 0.3333 & 1.0000 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, \\ \mathscr{N}_1 - \mathscr{N}_2\mathscr{N}_1^{-1}\mathscr{N}_2 &= \begin{pmatrix} 3.0000 & 0.3333 \\ 1.0000 & 4.0000 \end{pmatrix} \\ &= \begin{pmatrix} 1.0000 & 0 \\ 0.3333 & 1.0000 \end{pmatrix} \begin{pmatrix} 3.0000 & 0.3333 \\ 0 & 3.8889 \end{pmatrix},\end{aligned} \tag{20}$$

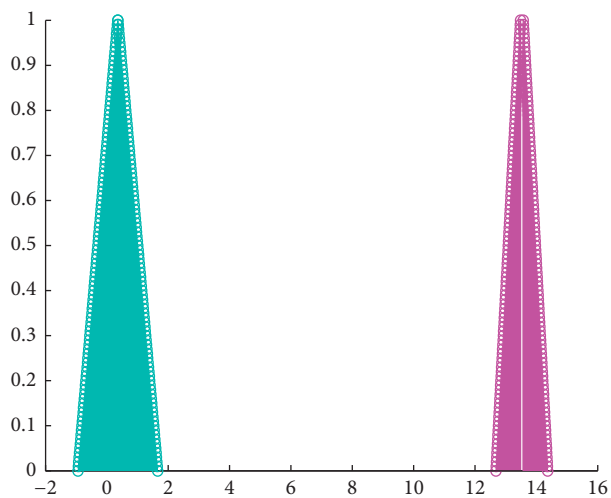

Figure 1: Exact solution.

and hence

$$\mathcal{N} = \begin{pmatrix} 1.0000 & 0 & 0 & 0 \\ 0.3333 & 1.0000 & 0 & 0 \\ 0 & -0.5000 & 1.0000 & 0 \\ 0 & 0 & 0.3333 & 1.0000 \end{pmatrix} \begin{pmatrix} 3.0000 & 0 & 0 & -2.0000 \\ 0 & 4.0000 & 0 & 0.6666 \\ 0 & 0 & 3.0000 & 0.3333 \\ 0 & 0 & 0 & 3.8889 \end{pmatrix}. \tag{21}$$

The exact solution is

$$\begin{aligned} (\mathcal{K}_1)_\delta &= \left\langle \left[\frac{447}{35} + \frac{4\delta}{5}, \frac{503}{35} - \frac{4\delta}{5}\right], \left[\frac{279}{35} + \frac{8\delta}{5}, \frac{391}{35} - \frac{8\delta}{5}\right], \left[\frac{54}{7} + \delta, \frac{68}{7} - \delta\right] \right\rangle, \\ (\mathcal{K}_2)_\delta &= \left\langle \left[\frac{-33}{35} + \frac{13\delta}{10}, \frac{58}{35} - \frac{13\delta}{10}\right], \left[\frac{44}{35} + \frac{\delta}{10}, \frac{51}{35} - \frac{\delta}{10}\right], \left[\frac{29}{14} + \frac{\delta}{2}, \frac{43}{14} - \frac{\delta}{2}\right] \right\rangle. \end{aligned} \tag{22}$$

The exact and derived solutions with LU decomposition $\langle [z_1^{(1)}, z_2^{(1)}] \rangle$ of 3-PFN are plotted in Figures 1 and 2.

The Hausdorff norm of errors is 6.3750e−008.

The exact and derived solutions with LU decomposition $\langle [z_1^{(2)}, z_2^{(2)}] \rangle$ of 3-PFN are plotted in Figures 3 and 4.

The Hausdorff norm of errors is 6.3750e−008.

The exact and derived solutions with LU decomposition $\langle [z_1^{(3)}, z_2^{(3)}] \rangle$ of 3-PFN are plotted in Figures 5 and 6.

The Hausdorff norm of errors is 6.3750e−008.

Example 2. Consider the 3×33-polar fuzzy system:

$$\begin{aligned} 6\mathcal{K}_1 + 2\mathcal{K}_2 - 4\mathcal{K}_3 &= \langle [44 + 16\delta, 76 - 16\delta], [24 + 11\delta, 46 - 11\delta], [34 + 16\delta, 66 - 16\delta] \rangle, \\ 2\mathcal{K}_1 + 16\mathcal{K}_2 + 12\mathcal{K}_3 &= \langle [86 + 24\delta, 134 - 24\delta], [66 + 19\delta, 104 - 19\delta], [76 + 24\delta, 124 - 24\delta] \rangle, \\ -4\mathcal{K}_1 + 12\mathcal{K}_2 + 24\mathcal{K}_3 &= \langle [62 + 38\delta, 138 - 38\delta], [42 + 33\delta, 108 - 33\delta], [52 + 37\delta, 126 - 37\delta] \rangle. \end{aligned} \tag{23}$$

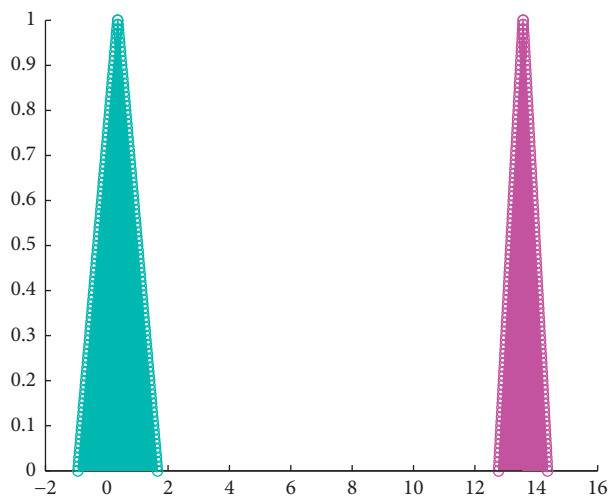

Figure 2: The solution with LU decomposition method.

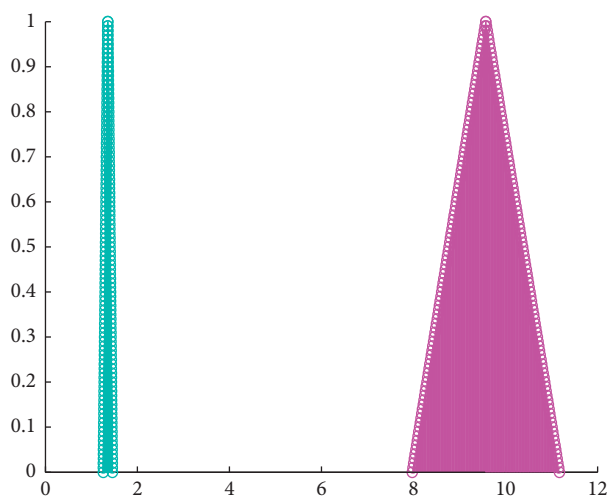

Figure 3: Exact solution.

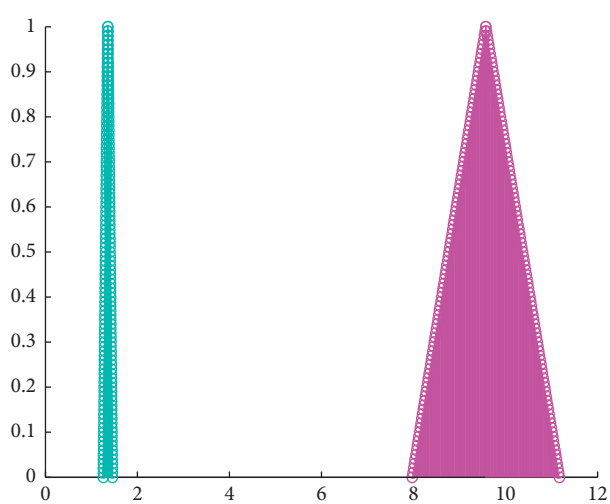

Figure 4: The solution with LU decomposition method.

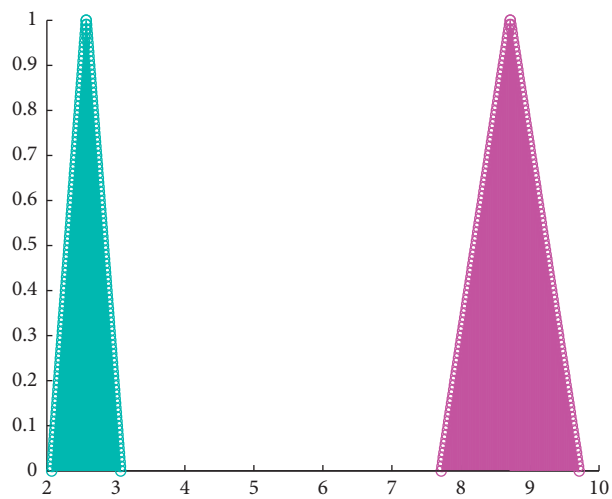

Figure 5: Exact solution.

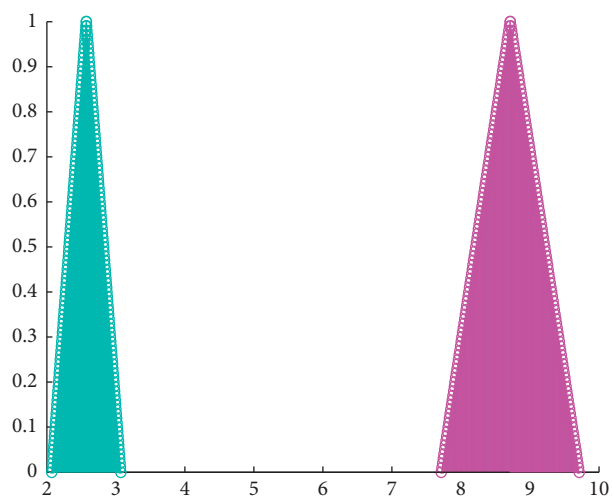

Figure 6: The solution with LU decomposition method.

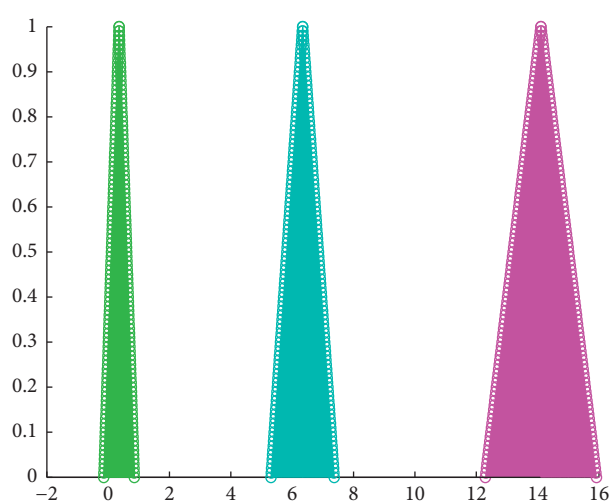

Figure 7: Exact solution.

The extended 6×6 symmetric positive definite matrix is

$$\begin{aligned}
\mathcal{N} &= \begin{pmatrix} 6 & 2 & 0 & 0 & 0 & -4 \\ 2 & 16 & 12 & 0 & 0 & 0 \\ 0 & 12 & 24 & -4 & 0 & 0 \\ 0 & 0 & -4 & 6 & 2 & 0 \\ 0 & 0 & 0 & 2 & 16 & 12 \\ -4 & 0 & 0 & 0 & 12 & 24 \end{pmatrix}, \\
\mathcal{N}_1 &= \begin{pmatrix} 6 & 2 & 0 \\ 2 & 16 & 12 \\ 0 & 12 & 24 \end{pmatrix} \\
&= \begin{pmatrix} 2.4495 & 0 & 0 \\ 0.8165 & 3.9158 & 0 \\ 0 & 3.0645 & 3.8221 \end{pmatrix} \begin{pmatrix} 2.4495 & 0.8165 & 0 \\ 0 & 3.9158 & 3.0645 \\ 0 & 0 & 3.8221 \end{pmatrix}, \\
\mathcal{N}_1 - \mathcal{N}_2 \mathcal{N}_1^{-1} \mathcal{N}_2 &= \begin{pmatrix} 4.9048 & 2.0000 & -0.2857 \\ 2.0000 & 16.0000 & 12.0000 \\ -0.2857 & 12.0000 & 21.1429 \end{pmatrix} \\
&= \begin{pmatrix} 2.2147 & 0 & 0 \\ 0.9031 & 3.8967 & 0 \\ -0.1290 & 3.1094 & 3.3849 \end{pmatrix} \begin{pmatrix} 2.2147 & 0.9031 & -0.1290 \\ 0 & 3.8967 & 3.1094 \\ 0 & 0 & 3.3849 \end{pmatrix},
\end{aligned} \tag{24}$$

and hence $\mathcal{N} = LL^T$, where

$$L = \begin{pmatrix} 2.4495 & 0 & 0 & 0 & 0 & 0 \\ 0.8165 & 3.9158 & 0 & 0 & 0 & 0 \\ 0 & 3.0645 & 3.8221 & 0 & 0 & 0 \\ 0 & 0 & -1.0465 & 2.2147 & 0 & 0 \\ 0 & 0 & 0 & 0.9031 & 3.8967 & 0 \\ -1.6330 & 0.3405 & -0.2730 & -0.1290 & 3.1094 & 3.3849 \end{pmatrix}. \tag{25}$$

Now, the exact solution is

$$\begin{aligned}
(\mathcal{K}_1)_\delta &= \left\langle \left[\frac{1377}{112} + \frac{29\delta}{16}, \frac{1783}{112} - \frac{29\delta}{16}\right], \left[\frac{1479}{224} + \frac{33\delta}{32}, \frac{1941}{224} - \frac{33\delta}{32}\right], \left[\frac{4145}{224} + \frac{59\delta}{32}, \frac{2971}{224} - \frac{59\delta}{32}\right] \right\rangle, \\
(\mathcal{K}_2)_\delta &= \left\langle \left[-\frac{1}{7} + \frac{\delta}{2}, \frac{6}{7} - \frac{\delta}{2}\right], \left[\frac{81}{56} + \frac{\delta}{4}, \frac{109}{56} - \frac{\delta}{4}\right], \left[\frac{121}{280} + \frac{11\delta}{20}, \frac{429}{280} - \frac{11\delta}{20}\right] \right\rangle, \\
(\mathcal{K}_3)_\delta &= \left\langle \left[\frac{1189}{224} + \frac{33\delta}{32}, \frac{1651}{224} - \frac{33\delta}{32}\right], \left[\frac{1107}{448} + \frac{69\delta}{64}, \frac{2073}{448} - \frac{69\delta}{64}\right], \left[\frac{9321}{2240} + \frac{307\delta}{320}, \frac{13619}{2240} - \frac{307\delta}{320}\right] \right\rangle.
\end{aligned} \tag{26}$$

The exact and derived solutions with LL^T decomposition $\langle [z_1^{(1)}, z_2^{(1)}, z_3^{(1)}] \rangle$ of 3-PFN are plotted in Figures 7 and 8.

The Hausdorff norm of errors is $3.9705e{-}004$. The exact and derived solutions with LL^T decomposition $\langle [z_1^{(2)}, z_2^{(2)}, z_3^{(2)}] \rangle$ of 3-PFN are plotted in Figures 9 and 10.

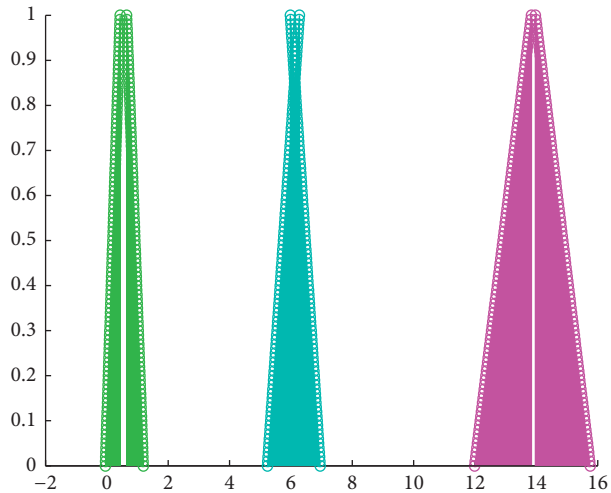

Figure 8: The solution with LU decomposition method.

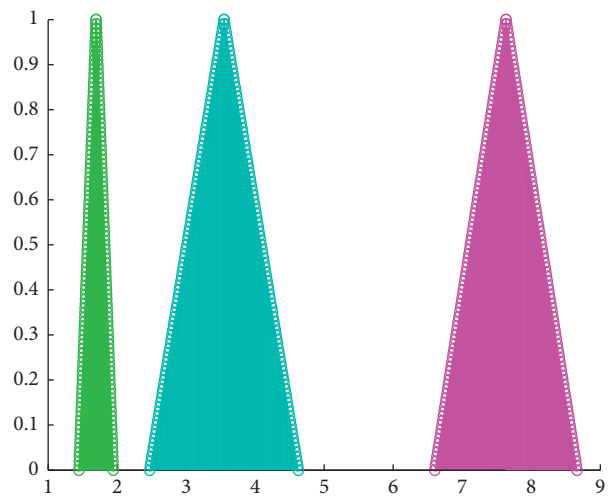

Figure 9: Exact solution.

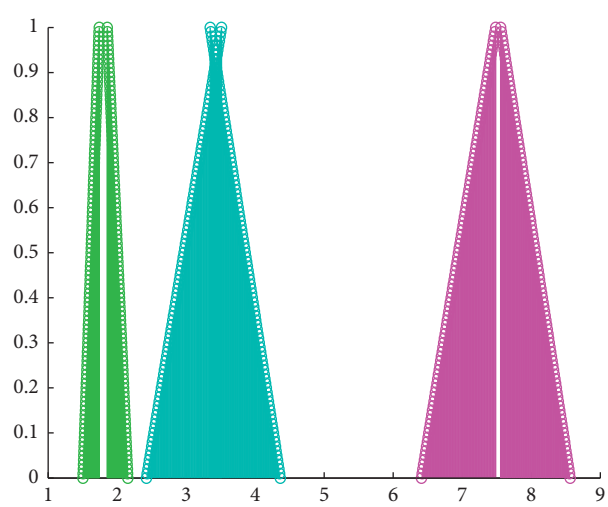

Figure 10: The solution with LU decomposition method.

The Hausdorff norm of errors is 3.9705e–004.

The exact and derived solutions with LL^T decomposition $\langle [z_1^{(3)}, z_2^{(3)}, z_3^{(3)}] \rangle$ of 3-PFN are plotted in Figures 11 and 12.

The Hausdorff norm of errors is 3.9705e–004.

Lemma 1 (see [43]). *The solution of m-PFSLEs exists if and only if the rank of $\mathcal{N}$ is equal to that of matrix $(\mathcal{N}, m^{(i)})$; that is,*

$$\operatorname{Rank}(\mathcal{N}) = \operatorname{Rank}\left(\mathcal{N}, m^{(i)}\right). \tag{27}$$

When $\operatorname{Rank}(\mathcal{N}) = \operatorname{Rank}(\mathcal{N}, m^{(i)}) = 2n$, the system has a unique solution. There are endless solutions to the system if $\operatorname{Rank}(\mathcal{N}, m^{(i)}) = \operatorname{Rank}(\mathcal{N}) < 2n$ and there is no solution if $\operatorname{Rank}(\mathcal{N}, m^{(i)}) > \operatorname{Rank}(\mathcal{N})$.

Theorem 5. *The system $\mathcal{N}\mathcal{K} = \mathcal{W}^{(i)}(\delta)$ has solution if and only if*

$$\operatorname{Rank}(\mathcal{N}) = \operatorname{Rank}\left(\mathcal{N}, m^{(i)}(\delta)\right), \quad 0 \le \delta \le 1. \tag{28}$$

Proof. Let $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n)$, $m_\delta^{(i)} = (\mathcal{K}_1(\delta), \mathcal{K}_2(\delta), \ldots, \mathcal{K}_q(\delta))$, where

$$\begin{aligned} \mathcal{K}_p &= \left(\underline{\mathcal{K}}_{1p}(\delta), \underline{\mathcal{K}}_{2p}(\delta), \ldots, \underline{\mathcal{K}}_{np}(\delta), -\overline{\mathcal{K}}_{1p}(\delta), -\overline{\mathcal{K}}_{2p}(\delta), \ldots, -\overline{\mathcal{K}}_{np}(\delta)\right)^T, \\ \left(m_p^{(i)}\right)(\delta) &= \left(\underline{m}_{1p}^{(i)}(\delta), \underline{m}_{2p}^{(i)}(\delta), \ldots, \underline{m}_{np}^{(i)}(\delta), -\overline{m}_{1p}^{(i)}(\delta), -\overline{m}_{2p}^{(i)}(\delta), \ldots, -\overline{m}_{np}^{(i)}(\delta)\right)^T, \quad p = 1, 2, \ldots, q,\ \delta \in [0, 1]. \end{aligned} \tag{29}$$

The matrices below are the same:

$$\mathcal{N}\mathcal{K} = \mathcal{W}^{(i)}(\delta) \cong \mathcal{N}\mathcal{K}_p = m_p^{(i)}(\delta), \quad p = 1, 2, 3, \ldots, q. \tag{30}$$

If $\operatorname{Rank}(\mathcal{N}, \mathcal{W}^{(i)}(\delta)) = \operatorname{Rank}(\mathcal{N})$, then we have $\operatorname{Rank}(\mathcal{N}, m_p^{(i)}(\delta)) = \operatorname{Rank}(\mathcal{N})$, $\forall p$, since $\operatorname{Rank}(\mathcal{N}, \mathcal{W}^{(i)}(\delta)) \ge \operatorname{Rank}(\mathcal{N}, m_p^{(i)}(\delta)) \ge \operatorname{Rank}(\mathcal{N})$. By Lemma 1, all linear equations $\mathcal{N}\mathcal{K}_p = m_p^{(i)}(\delta)$, $p = 1, 2, 3, \ldots, q$, have solutions. So it makes sense to have the necessary condition.

Conversely, suppose that $\mathcal{N}\mathcal{K} = \mathcal{W}^{(i)}(\delta)$ is solvable; in other words, every linear equation $\mathcal{N}\mathcal{K}_p = m_p^{(i)}(\delta)$, $p = 1, 2, 3, \ldots, q$, has solution. Let

$$\begin{aligned} \mathcal{K}_p &= \left(\underline{\mathcal{K}}_{1p}(\delta), \underline{\mathcal{K}}_{2p}(\delta), \ldots, \underline{\mathcal{K}}_{np}(\delta), -\overline{\mathcal{K}}_{1p}(\delta), -\overline{\mathcal{K}}_{2p}(\delta), \ldots, -\overline{\mathcal{K}}_{np}(\delta)\right)^T, \quad p = 1, 2, \ldots, q, \delta \in [0, 1], \\ \mathcal{N} &= (s_1, s_2, s_3, \ldots, s_{2n}), \end{aligned} \tag{31}$$

where $s_p = (s_{1p}, s_{2p}, s_{3p}, \ldots, s_{2n,p})^T$, $p = 1, 2, 3, \ldots, 2n$.

By using equation $\mathcal{N}\mathcal{K}_p = m_p^{(i)}(\delta)$, $p = 1, 2, 3, \ldots, q$, we have

$$\underline{\mathcal{K}}_{1p}(\delta)s_1, \underline{\mathcal{K}}_{2p}(\delta)s_2, \ldots, \underline{\mathcal{K}}_{np}(\delta)s_n, -\overline{\mathcal{K}}_{1p}(\delta)s_{n+1}, -\overline{\mathcal{K}}_{2p}(\delta)s_{n+1}, \ldots, -\overline{\mathcal{K}}_{np}(\delta)s_{2n} = m_p^{(i)}(\delta). \tag{32}$$

From the above equation, it follows that $m_p^{(i)}(\delta)$ can be expressed as linear combination of $s_1, s_2, s_3, \ldots, s_{2n}$; that is, from this equation, the following is that a linear combination of $m_p^{(i)}(\delta)$ can be expressed as $s_1, s_2, s_3, \ldots, s_{2n}$:

$$\begin{aligned} &\operatorname{Rank}(\mathcal{N}) = \operatorname{Rank}\left(\mathcal{N}, m_p^{(i)}(\delta)\right), \quad p = 1, 2, \ldots, q, \\ &\operatorname{Rank}(\mathcal{N}) = \operatorname{Rank}\left(\mathcal{N}, \mathcal{W}^{(i)}(\delta)\right). \end{aligned} \tag{33}$$

From Theorem 5, we can deduce the following result about the solvability of (7).

Theorem 6. *The equation has an equivalent solution to (7):*

$$\mathcal{N}\mathcal{K}_p = m_p^{(i)}(\delta), \quad \forall p = 1, 2, 3, \ldots, q. \tag{34}$$

Theorem 7. *Equation (7) has solution in which the necessary and sufficient conditions for the rows of $\mathcal{W}^{(i)}(\delta)$ have the same linear relation as the rows of the $\mathcal{N}$ matrix.*

Theorem 8. *If (7) has no solution, then the corresponding m-PFME also has no solution.*

Corollary 1. *Consider the condition*

$$\operatorname{Rank}\left(\mathcal{N}, \mathcal{W}^{(i)}(\delta)\right) = \operatorname{Rank}(\mathcal{N}). \tag{35}$$

If $\operatorname{Rank}(\mathcal{N}) = 2n$, then (7) has unique solution; otherwise an infinite number of solutions exist.

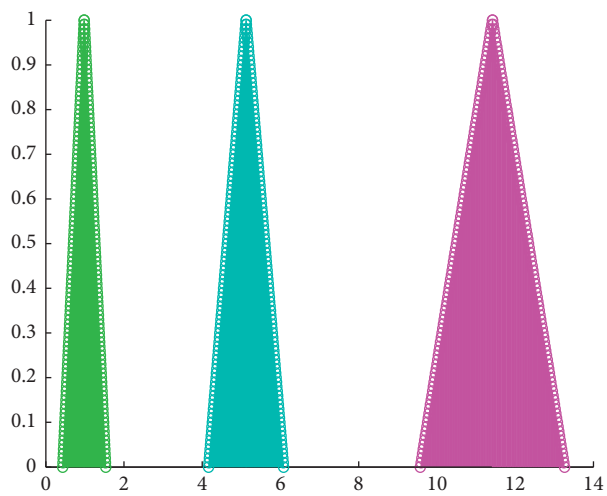

Figure 11: Exact solution.

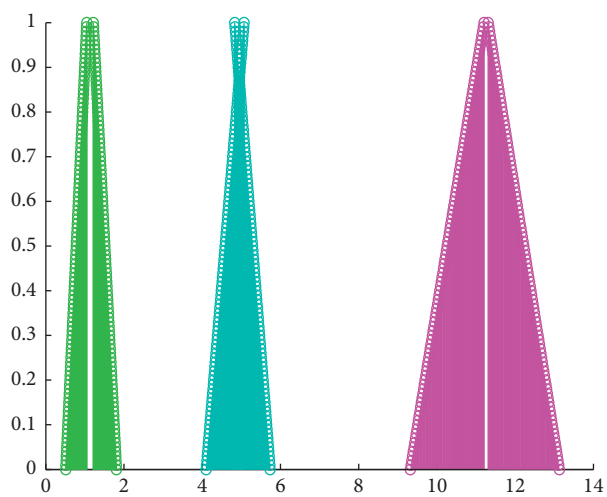

Figure 12: The solution with LU decomposition method.

Corollary 2. *If there is only one solution in the crisp system (7), then it is equivalent to m-PFSLE:*

$$\mathcal{N}\mathcal{K}_p = m_p^{(i)}(\delta), \quad \forall p = 1, 2, 3, \ldots, q, \tag{36}$$

which has only one solution.

3. Inconsistent m-Polar Fuzzy Matrix Equation

Definition 8. If the crisp matrix equation (7) has no solution, then the associated m-PFME is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & m2 & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} & \cdots & \mathcal{K}_{1p} \\ \mathcal{K}_{21} & \mathcal{K}_{22} & \cdots & \mathcal{K}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{K}_{n1} & \mathcal{K}_{n2} & \cdots & \mathcal{K}_{np} \end{pmatrix} = \begin{pmatrix} m_{11}^{(i)} & m_{12}^{(i)} & \cdots & m_{1p}^{(i)} \\ m_{21}^{(i)} & m_{22}^{(i)} & \cdots & m_{2p}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ m_{m1}^{(i)} & m_{m2}^{(i)} & \cdots & m_{mp}^{(i)} \end{pmatrix}, \tag{37}$$

where the coefficient matrix $\mathcal{D} = (a_{pq}^{(i)})$, $1 \le p \le m, 1 \le q \le n$, is crisp matrix, the right-hand matrix $\mathcal{W}^{(i)} = (m_{pq}^{(i)})$ is m-PFN called an inconsistent m-PFME.

We consider the following examples.

Example 3. 3-polar fuzzy matrix system

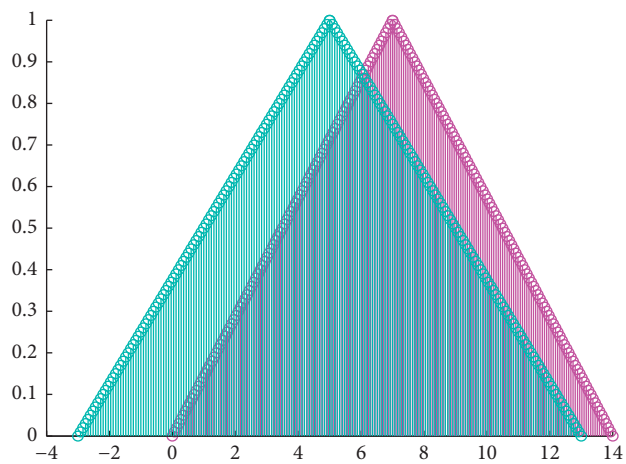

Figure 13: 3-polar fuzzy LSS (a).

$$\begin{pmatrix} 7 & 7 & 7 \\ 7 & 7 & -7 \\ 7 & -7 & -7 \end{pmatrix}\begin{pmatrix} \mathscr{K}_{11} & \mathscr{K}_{12} & \mathscr{K}_{13} \\ \mathscr{K}_{21} & \mathscr{K}_{22} & \mathscr{K}_{23} \\ \mathscr{K}_{31} & \mathscr{K}_{32} & \mathscr{K}_{33} \end{pmatrix}$$
$$= \begin{pmatrix} (5+3\delta, 11-3\delta, 16+2\delta, 20-2\delta, 6+4\delta, 14-4\delta) & (15+4\delta, 23-4\delta, 11+3\delta, 17-3\delta, 22+6\delta, 34-6\delta) & (17+5\delta, 27-5\delta, 13+3\delta, 19-3\delta, 7+\delta, 9-\delta) \\ (4+\delta, 6-\delta, 13+4\delta, 21-4\delta, 32+6\delta, 44-6\delta) & (25+2\delta, 29-2\delta, 35+7\delta, 49-7\delta, 14+5\delta, 24-5\delta) & (8+2\delta, 12-2\delta, 23+3\delta, 29-3\delta, 19+5\delta, 29-5\delta) \\ (15+6\delta, 27-6\delta, 26+3\delta, 32-3\delta, 8+\delta, 10-\delta) & (25+3\delta, 31-3\delta, 19+4\delta, 27-4\delta, 7+3\delta, 13-3\delta) & (18+2\delta, 22-2\delta, 28+5\delta, 38-5\delta, 9+4\delta, 17-4\delta) \end{pmatrix}, \tag{38}$$

is nonsingular, while the extended 6×6 matrix

$$\mathscr{N} = \begin{pmatrix} 7 & 7 & 7 & 0 & 0 & 0 \\ 7 & 7 & 0 & 0 & 0 & 7 \\ 7 & 0 & 0 & 0 & 7 & 7 \\ 0 & 0 & 0 & 7 & 7 & 7 \\ 0 & 0 & 7 & 7 & 7 & 0 \\ 0 & 7 & 7 & 7 & 0 & 0 \end{pmatrix}, \tag{39}$$

is singular. This example shows that even though we represent a nonsingular system, then an extended m-polar fuzzy matrix system can have infinite or no solutions.

Example 4. Consider the 3-polar fuzzy matrix system

$$\begin{pmatrix} 7 & 7 & -7 \\ 7 & -7 & 7 \\ -7 & 7 & 7 \end{pmatrix}\begin{pmatrix} \mathscr{K}_{11} & \mathscr{K}_{12} & \mathscr{K}_{13} \\ \mathscr{K}_{21} & \mathscr{K}_{22} & \mathscr{K}_{23} \\ \mathscr{K}_{31} & \mathscr{K}_{32} & \mathscr{K}_{33} \end{pmatrix}$$
$$= \begin{pmatrix} (13+2\delta, 17-2\delta, 23+3\delta, 29-3\delta, 5+\delta, 7-\delta) & (8+4\delta, 16-4\delta, 21+5\delta, 31-5\delta, 32+2\delta, 36-2\delta) & (14+5\delta, 24-5\delta, 23+4\delta, 31-4\delta, 17+\delta, 19-\delta) \\ (14+3\delta, 20-3\delta, 33+4\delta, 41-4\delta, 12+6\delta, 22-6\delta) & (5+2\delta, 9-2\delta, 15+7\delta, 29-7\delta, 24+5\delta, 34-5\delta) & (18+2\delta, 22-2\delta, 13+3\delta, 19-3\delta, 29+5\delta, 39-5\delta) \\ (15+6\delta, 27-6\delta, 22+3\delta, 28-3\delta, 38+\delta, 40-\delta) & (15+4\delta, 23-4\delta, 29+4\delta, 37-4\delta, 17+3\delta, 23-3\delta) & (28+2\delta, 32-2\delta, 8+5\delta, 18-5\delta, 19+4\delta, 27-4\delta) \end{pmatrix}. \tag{40}$$

The extended 6×6 matrix is

$$\mathcal{N} = \begin{pmatrix} 7 & 7 & 0 & 0 & 0 & 7 \\ 7 & 0 & 7 & 0 & 7 & 0 \\ 0 & 7 & 7 & 7 & 0 & 0 \\ 0 & 0 & 7 & 7 & 7 & 0 \\ 0 & 7 & 0 & 7 & 0 & 7 \\ 7 & 0 & 0 & 0 & 7 & 7 \end{pmatrix}, \tag{41}$$

and the augmented matrices are

$$\left(\mathcal{N}, \mathcal{W}^{(1)}(\delta)\right) = \begin{pmatrix} 7 & 7 & 0 & 0 & 0 & 7 & 13+2\delta & 8+4\delta & 14+5\delta \\ 7 & 0 & 7 & 0 & 7 & 0 & 14+3\delta & 5+2\delta & 18+2\delta \\ 0 & 7 & 7 & 7 & 0 & 0 & 15+6\delta & 15+4\delta & 28+2\delta \\ 0 & 0 & 7 & 7 & 7 & 0 & 2\delta-17 & 4\delta-16 & 5\delta-24 \\ 0 & 7 & 0 & 7 & 0 & 7 & 3\delta-20 & 2\delta-9 & 2\delta-22 \\ 7 & 0 & 0 & 0 & 7 & 7 & 6\delta-27 & 4\delta-23 & 2\delta-32 \end{pmatrix},$$

$$\left(\mathcal{N}, \mathcal{W}^{(2)}(\delta)\right) = \begin{pmatrix} 7 & 7 & 0 & 0 & 0 & 7 & 23+3\delta & 21+5\delta & 23+4\delta \\ 7 & 0 & 7 & 0 & 7 & 0 & 33+4\delta & 15+7\delta & 13+3\delta \\ 0 & 7 & 7 & 7 & 0 & 0 & 22+3\delta & 29+4\delta & 8+5\delta \\ 0 & 0 & 7 & 7 & 7 & 0 & 3\delta-29 & 5\delta-31 & 4\delta-31 \\ 0 & 7 & 0 & 7 & 0 & 7 & 4\delta-41 & 7\delta-29 & 3\delta-19 \\ 7 & 0 & 0 & 0 & 7 & 7 & 3\delta-28 & 4\delta-37 & 5\delta-18 \end{pmatrix},$$

$$\left(\mathcal{N}, \mathcal{W}^{(3)}(\delta)\right) = \begin{pmatrix} 7 & 7 & 0 & 0 & 0 & 7 & 5+\delta & 32+2\delta & 17+\delta \\ 7 & 0 & 7 & 0 & 7 & 0 & 12+6\delta & 24+5\delta & 29+5\delta \\ 0 & 7 & 7 & 7 & 0 & 0 & 38+\delta & 17+3\delta & 19+4\delta \\ 0 & 0 & 7 & 7 & 7 & 0 & \delta-7 & 2\delta-36 & \delta-19 \\ 0 & 7 & 0 & 7 & 0 & 7 & 6\delta-22 & 5\delta-34 & 5\delta-39 \\ 7 & 0 & 0 & 0 & 7 & 7 & \delta-40 & 3\delta-23 & 4\delta-27 \end{pmatrix}. \tag{42}$$

Since, $\mathrm{Rank}(\mathcal{N}) \neq \mathrm{Rank}(\mathcal{N}, \mathcal{W}^{(i)}(\delta))$, $i = 1, 2, 3$, the original system is therefore inconsistent. Examples 3 and 4 show that m-PFME exists without a solution for some time. The approximate solution to this m-PFME type is essential. If system (7) is not consistent, then the approximate solution we want can be found by reducing some norm of $(\mathcal{W}^{(i)}(\delta) - \mathcal{N}\mathcal{K}(\delta))$. We often use the least square solution of (7) for an approximation solution that is described by minimizing Frobenius norms $(\mathcal{W}^{(i)}(\delta) - \mathcal{N}\mathcal{K}(\delta))$,

$$\left\|\mathcal{W}^{(i)}(\delta) - \mathcal{N}\mathcal{K}(\delta)\right\|_F = \min, \quad \delta \in [0, 1]. \tag{43}$$

This means to minimize the sum of the module squares $(\mathcal{W}^{(i)}(\delta) - \mathcal{N}\mathcal{K}(\delta))$

$$\left\|\mathcal{W}^{(i)}(\delta) - \mathcal{N}\mathcal{K}(\delta)\right\|_F^2 = \sum_{q=1}^{q}\sum_{p=1}^{u}\left(\left|\underline{m}_{p,q}^{(i)}(\delta) - \sum_{t=1}^{n}\left[s_{pt}\underline{\mathcal{K}}_{t,q}(\delta) - s_{p,n+t}\overline{\mathcal{K}}_{t,q}(\delta)\right]\right|^2 + \left|\overline{m}_{p,q}^{(i)}(\delta) + \sum_{t=1}^{n}\left[s_{p+n,t}\underline{\mathcal{K}}_{t,q}(\delta) - s_{p+n,n+t}\overline{\mathcal{K}}_{t,q}(\delta)\right]\right|^2, \quad \delta \in [0,1]\right). \tag{44}$$

Now, we define the m-polar fuzzy LSS to the inconsistent m-PFME by Definition 8.

3.1. m-Polar Fuzzy Least Square Solution. We analyze from this investigation that the m-PFME is inconsistent if $\mathrm{Rank}(\mathcal{N}) \neq \mathrm{Rank}(\mathcal{N}, m^{(i)}(\delta))$ of its extended crisp system (7). When m-PFME is inconsistent, then the least square solution may be considered. However, the m-PFLSS may not have m-PFN matrix. We are limiting our conversation to quadruple m-PFNs, that is, $\underline{m}_{pq}^{(i)}(\delta), \overline{m}_{pq}^{(i)}(\delta)$, $1 \le p \le m, 1 \le q \le n$, and therefore, $\underline{\mathcal{K}}_{pq}(\delta), \overline{\mathcal{K}}_{pq}(\delta)$ are all linear functions of r. We can then describe the m-polar fuzzy solution to the m-polar fuzzy matrix by calculating $\mathcal{K}$ which are solved by system (7).

Definition 9. Let $\mathcal{R}_\delta = \left\{\underline{\mathcal{K}}_{pq}(\delta), -\overline{\mathcal{K}}_{pq}(\delta)\right), 1 \le p \le m, 1 \le q \le n\}$ represent the LSS of system (7). The m-PFN matrix $U_\delta = \left\{\underline{t}_{pq}(\delta), \overline{t}_{pq}(\delta), 1 \le p \le m, 1 \le q \le n\right\}$ defined by

$$\begin{aligned} \underline{t}_{pq}(\delta) &= \min\left\{\underline{t}_{pq}(\delta), \overline{t}_{pq}(\delta), \underline{t}_{pq}(1), \overline{t}_{pq}(1)\right\}, \\ \overline{t}_{pq}(\delta) &= \max\left\{\underline{t}_{pq}(\delta), \overline{t}_{pq}(\delta), \underline{t}_{pq}(1), \overline{t}_{pq}(1)\right\}, \end{aligned} \tag{45}$$

is called the m-FLSS of $\mathcal{N}\mathcal{K} = \mathcal{W}^{(i)}(\delta)$. If $\underline{\mathcal{K}}_{pq}(\delta), \overline{\mathcal{K}}_{pq}(\delta)$, $1 \le p \le m, 1 \le q \le n$, are all m-PFNs, then $\underline{t}_{pq}(\delta) = \underline{\mathcal{K}}_{pq}(\delta), \overline{t}_{pq}(\delta) = \overline{\mathcal{K}}_{pq}(\delta)$, $1 \le p \le m, 1 \le q \le n$, and

U_δ is called a strong m-polar fuzzy LSS. Otherwise, U_δ is called a weak m-polar fuzzy LSS.

4. Least Square Solution of Fuzzy Matrix Equation in m-Polar Fuzzy Environment

We analyze the following Lemma.

Lemma 2. *Let $\mathcal{N} \in \mathbf{R}^{2n\times 2n}$. A vector $\mathcal{K}(\delta)$ is a m-polar fuzzy LSS of the extended crisp function linear equation $\mathcal{N}\mathcal{K} = \mathcal{W}^{(i)}(\delta)$, that is,*

$$\begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1,2n} \\ s_{21} & s_{22} & \cdots & s_{2,2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{2n,1} & s_{2n,2} & \cdots & s_{2n,2n} \end{pmatrix} \begin{pmatrix} \underline{\mathcal{K}}_1(\delta) \\ \vdots \\ \underline{\mathcal{K}}_n(\delta) \\ -\overline{\mathcal{K}}_1(\delta) \\ \vdots \\ -\overline{\mathcal{K}}_n(\delta) \end{pmatrix} = \begin{pmatrix} \underline{m}_1^{(i)}(\delta) \\ \vdots \\ \underline{m}_m^{(i)}(\delta) \\ -\overline{m}_1^{(i)}(\delta) \\ \vdots \\ -\overline{m}_m^{(i)}(\delta) \end{pmatrix}, \tag{46}$$

which is transformed from inconsistent m-PFME (4), if and only if

$$\mathcal{N}\mathcal{K} = \mathcal{K}\mathcal{N}^{\{1,3\}} m^{(i)}(\delta). \tag{47}$$

The LSSs of the abovementioned matrix equation may be expressed in this case by

$$\mathcal{K}(\delta) = \mathcal{N}^{\{1,3\}} m^{(i)}(\delta) + \left(\mathcal{B}_{2n} - \mathcal{N}^{\{1,3\}}\mathcal{N}\right) z^{(i)}(\delta), \tag{48}$$

where $\mathcal{N}^{\{1,3\}}$ is the least squares generalized inverse of matrix $\mathcal{N}$, $\mathcal{B}_{2n}$ is unit matrix of order $2n$, and $z^{(i)}(\delta)$ are arbitrary vectors with parameter δ. According to Lemma 2 and the hypothesis of generalized inverse theory, we have the following theorems about the LSS for (7).

Theorem 9. *Let $\mathcal{N} \in \mathbf{R}^{2n\times 2n}$. The matrix $\mathcal{K}(\delta)$ is the LSS of the matrix system (7), if and only if*

$$\mathcal{N}\mathcal{K} = \mathcal{K}\mathcal{N}^{\{1,3\}} m^{(i)}(\delta). \tag{49}$$

The general LSS of system (7) of the crisp matrix equation can be defined by the following

$$\mathcal{K}(\delta) = \mathcal{N}^{\{1,3\}} m^{(i)}(\delta) + \left(\mathcal{B}_{2n} - \mathcal{N}^{\{1,3\}} N\right) z^{(i)}(\delta), \tag{50}$$

where $\mathcal{N}^{\{1,3\}}$ is the least squares generalized inverse of the matrix $\mathcal{N}$ and $z^{(i)}(\delta)$ are $2m \times n$ any matrices with the parameters δ.

Proof. First, we consider the crisp matrix equation (7) in block forms of the matrix

$$\mathcal{N}\mathcal{K}_q(\delta) = \mathcal{W}_q^{(i)}(\delta), \quad q = 1, 2, 3, \ldots, n, \tag{51}$$

where $\mathcal{K}(\delta) = [\mathcal{K}_1(\delta), \ldots, \mathcal{K}_n(\delta)]$, $\mathcal{W}^{(i)}(\delta) = [m_1^{(i)}(\delta), \ldots, m_n^{(i)}(\delta)]$. Let $\mathcal{N}_q^*(\delta)$, $q = 1, 2, 3, \ldots, n$, be the LSS of (51). By following the matrix theory [44], the matrix equations $\mathcal{N}\mathcal{K}(\delta) = \mathcal{W}^{(i)}(\delta)$ are inconsistent if and only if at least one of the linear equations $\mathcal{N}\mathcal{K}_q(\delta) = \mathcal{W}_q^{(i)}(\delta)$, $q = 1, 2, 3, \ldots, n$, is inconsistent. By following Lemma 2, we have

$$\begin{aligned} \mathcal{N}\mathcal{K}_q^*(\delta) &= \mathcal{K}\mathcal{N}^{\{1,3\}}\mathcal{W}_q^{(i)}(\delta), \\ \mathcal{K}_q^*(\delta) &= \mathcal{N}^{\{1,3\}}\mathcal{W}_q^{(i)}(\delta) + \left(\mathcal{B}_{2n} - \mathcal{N}^{\{1,3\}} N\right) Z^{(i)}(\delta), \end{aligned} \tag{52}$$

where $Z^{(i)}(\delta)$ are $2m \times n$ any matrix with the parameter δ. Since $\mathcal{K}_q^*(\delta)$ is the LSSs of the linear equation $\mathcal{N}\mathcal{K}_q(\delta) = \mathcal{W}_q^{(i)}(\delta)$, $q = 1, 2, 3, \ldots, n$, we have

$$\begin{aligned} &\left\|\mathcal{W}_q^{(i)}(\delta) - \mathcal{N}\mathcal{K}_q^*(\delta)\right\|_F^2 \\ &\quad = \min\left(\left\|\mathcal{W}_q^{(i)}(\delta) - \mathcal{N}\mathcal{K}_q(\delta)\right\|_F^2\right), \quad q = 1, 2, 3, \ldots, n, \end{aligned} \tag{53}$$

where

$$\begin{aligned} \left\|\mathcal{W}_q^{(i)}(\delta) - \mathcal{N}\mathcal{K}_q^*(\delta)\right\|_F^2 = \sum_{p=1}^{u} \Bigg(&\left|\underline{m}_{p,q}^{(i)}(\delta) - \sum_{t=1}^{n}\left[s_{pt}\underline{\mathcal{K}}_{t,q}(\delta) - s_{p,n+t}\overline{\mathcal{K}}_{t,q}(\delta)\right]\right|^2 \\ &+ \left|\overline{m}_{p,q}^{(i)}(\delta) + \sum_{t=1}^{n}\left[s_{p+m,t}\underline{\mathcal{K}}_{t,q}(\delta) - s_{p+m,n+t}\overline{\mathcal{K}}_{t,q}(\delta)\right]\right|^2\Bigg), \quad \delta \in [0,1]. \end{aligned} \tag{54}$$

In fact,

$$\left\|\mathcal{W}^{(i)}(\delta) - \mathcal{N}\mathcal{K}\right\|_F^2 = \sum_{q=1}^{n}\left\|\mathcal{W}_q^{(i)}(\delta) - \mathcal{N}\mathcal{K}_q(\delta)\right\|_F^2. \tag{55}$$

The expression

$$\left\|\mathcal{W}^{(i)}(\delta) - \mathcal{N}\mathcal{K}^*(\delta)\right\|_F^2 = \min\left\|\mathcal{W}^{(i)}(\delta) - \mathcal{N}\mathcal{K}(\delta)\right\|_F^2, \quad \delta \in [0,1], \tag{56}$$

holds which corresponds to the following conditions:

$$\left\|\mathcal{N}\mathcal{K}_q^*(\delta) - \mathcal{W}_q^{(i)}(\delta)\right\|_F^2 = \min\left(\left\|\mathcal{N}_q\mathcal{K}_q(\delta) = \mathcal{W}^{(i)}(\delta)\right\|_F^2\right), \tag{57}$$

where

$$\mathscr{K}^{*}(\delta)=\left[\mathscr{K}_{1}^{*}(\delta),\mathscr{K}_{2}^{*}(\delta),\mathscr{K}_{3}^{*}(\delta),\ldots,\mathscr{K}_{n}^{*}(\delta)\right],$$
$$\left\|\mathscr{W}^{(i)}(\delta)-\mathscr{N}\mathscr{K}\right\|_{F}^{2}=\sum_{q=1}^{n}\sum_{p=1}^{u}\left|\underline{m}_{p,q}^{(i)}(\delta)-\sum_{t=1}^{n}\left[s_{pt}\underline{\mathscr{K}}_{t,q}(\delta)-s_{p,n+t}\overline{\mathscr{K}}_{t,q}(\delta)\right]\right|^{2}+\left|\overline{m}_{p,q}^{(i)}(\delta)+\sum_{t=1}^{n}\left[s_{p+m,t}\underline{\mathscr{K}}_{t,q}(\delta)-s_{p+m,n+t}\overline{\mathscr{K}}_{t,q}(\delta)\right]\right|^{2},\quad \delta\in[0,1]. \tag{58}$$

Thus, the matrix

$$\mathscr{K}^{*}(\delta)=\left[\mathscr{K}_{1}^{*}(\delta),\mathscr{K}_{2}^{*}(\delta),\mathscr{K}_{3}^{*}(\delta),\ldots,\mathscr{K}_{n}^{*}(\delta)\right], \tag{59}$$

is the LSS of (7). The following results are significant based on the operation of block forms of the matrix

$$\left(\mathscr{N}\mathscr{K}^{*}(\delta)\right)=\left(\mathscr{K}\mathscr{N}^{\{1,3\}}\mathscr{W}^{(i)}(\delta)\right), \tag{60}$$

where

$$\mathscr{K}_{q}^{*}(\delta)=\mathscr{N}^{\{1,3\}}m_{q}^{(i)}(\delta)+\left(\mathscr{B}_{2n}-\mathscr{N}^{\{1,3\}}N\right)Z(\delta). \tag{61}$$

Remark 1. It is observed that the LSS is unique only when the full rank is $\mathscr{N}$; i.e., the matrix equation LSS (7) is

$$\mathscr{K}(\delta)=\begin{cases}\left(\mathscr{N}^{\top}\mathscr{N}\right)^{-1}\mathscr{N}^{\top}\mathscr{W}^{(i)}(\delta), & \operatorname{Rank}(\mathscr{N})=2n,\\ \mathscr{N}^{\top}\left(\mathscr{N}\mathscr{N}^{\top}\right)^{-1}\mathscr{W}^{(i)}(\delta), & \operatorname{Rank}(\mathscr{N})=2n.\end{cases} \tag{62}$$

Otherwise, an infinite set of such solutions can occur in (7).

Theorem 10. *Among the general LSSs to system (7),*

$$\mathscr{K}(\delta)=\mathscr{N}^{\dagger}\mathscr{W}^{(i)}(\delta), \tag{63}$$

is one of the minimum norms, where $\mathscr{N}^{\dagger}$ is the Moore–Penrose inverse of the matrices $\mathscr{N}$. We know that it is unique $\mathscr{N}^{\dagger}$. System (63) is, therefore, unique. Since the LSS is shown as an m-polar fuzzy matrix, the general inverse of the $\mathscr{N}$ matrix is now considered in an exceptional structure. And

$$\mathscr{N}=\begin{pmatrix}\mathscr{N}_{1} & \mathscr{N}_{2}\\ \mathscr{N}_{2} & \mathscr{N}_{1}\end{pmatrix}. \tag{64}$$

We must follow the following statement.

Lemma 3. *Let $\mathscr{N}$ be in the form of (64). Then, the matrix*

$$\mathscr{N}^{\{1,3\}}=\frac{1}{2}\begin{pmatrix}\left(\mathscr{N}_{1}+\mathscr{N}_{2}\right)^{\{1,3\}}+\left(\mathscr{N}_{1}-\mathscr{N}_{2}\right)^{\{1,3\}} & \left(\mathscr{N}_{1}+\mathscr{N}_{2}\right)^{\{1,3\}}-\left(\mathscr{N}_{1}-\mathscr{N}_{2}\right)^{\{1,3\}}\\ \left(\mathscr{N}_{1}+\mathscr{N}_{2}\right)^{\{1,3\}}-\left(\mathscr{N}_{1}-\mathscr{N}_{2}\right)^{\{1,3\}} & \left(\mathscr{N}_{1}+\mathscr{N}_{2}\right)^{\{1,3\}}+\left(\mathscr{N}_{1}-\mathscr{N}_{2}\right)^{\{1,3\}}\end{pmatrix}, \tag{65}$$

is $\{1,3\}$-inverse of the matrix $\mathscr{N}$, where $(\mathscr{N}_{1}+\mathscr{N}_{2})^{\{1,3\}}$, $(\mathscr{N}_{1}-\mathscr{N}_{2})^{\{1,3\}}$ are $\{1,3\}$-inverse of matrices $(\mathscr{N}_{1}+\mathscr{N}_{2})$ and $(\mathscr{N}_{1}-\mathscr{N}_{2})$, respectively. In particular, the Moore–Penrose inverse of the matrix $\mathscr{N}$ is

$$\mathscr{N}^{\dagger}=\frac{1}{2}\begin{pmatrix}\left(\mathscr{N}_{1}+\mathscr{N}_{2}\right)^{\dagger}+\left(\mathscr{N}_{1}-\mathscr{N}_{2}\right)^{\dagger} & \left(\mathscr{N}_{1}+\mathscr{N}_{2}\right)^{\dagger}-\left(\mathscr{N}_{1}-\mathscr{N}_{2}\right)^{\dagger}\\ \left(\mathscr{N}_{1}+\mathscr{N}_{2}\right)^{\dagger}-\left(\mathscr{N}_{1}-\mathscr{N}_{2}\right)^{\dagger} & \left(\mathscr{N}_{1}+\mathscr{N}_{2}\right)^{\dagger}+\left(\mathscr{N}_{1}-\mathscr{N}_{2}\right)^{\dagger}\end{pmatrix}. \tag{66}$$

The following theorem provided the necessary or sufficient condition for the LSS matrix to the m-PFN matrix, given with the arbitrary input m-PFN $\mathscr{W}^{(i)}(\delta)$, and the next theorem provides the sufficient condition for one LSS to be the m-PFN matrix.

Theorem 11. *For the inconsistent m-PFME (7) and any least squares inverse $\mathscr{N}^{\{1,3\}}$ of the coefficient matrix $\mathscr{N}$, the expression $\mathscr{K}(\delta)=\mathscr{N}^{\{1,3\}}\mathscr{W}^{(i)}(\delta)$ is the solution of system and, therefore, it admits a strong or weak m-polar fuzzy LSS. Specifically, if $\mathscr{N}^{\{1,3\}}$ is nonnegative with the structure (65), the expression $\mathscr{K}(\delta)=\mathscr{N}^{\{1,3\}}\mathscr{W}^{(i)}(\delta)$ admits the strong m-polar fuzzy solution for arbitrary m-polar fuzzy matrices $\mathscr{W}^{(i)}(\delta)$.*

Proof. From Theorem 9 and the theory of generalized inverses, the expression is the LSS to the inconsistent m-PFME (7). We used Theorem 9 and one LSS of Definition 5 accompanied by a solution of (7) (from the previous analysis in Theorem 9). Therefore, by Definition 9, it admits a strong or weak m-polar fuzzy LSS. It is sufficient to prove this theorem where the m-PFN definition is set to $\mathscr{K}(\delta)$. To prove this

theorem, it is enough to show that the definition of bipolar fuzzy number holds for $\mathscr{K}(\delta)$. Let

$$\mathscr{K}(\delta) = \left[\overset{(i)}{\underline{\mathscr{K}}}(\delta), \overline{\mathscr{K}}^{(i)}(\delta)\right]^T,$$
$$\mathscr{W}^{(i)}(\delta) = \left[\overset{(i)}{\underline{\mathscr{W}}}(\delta), -\overline{\mathscr{W}}^{(i)}(\delta)\right]^T. \tag{67}$$

We denote

$$\mathscr{N}^{\{1,3\}} = \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}, \tag{68}$$

and we can obtain LSS of system (7), i.e.,

$$\begin{aligned} \begin{pmatrix} \overset{(i)}{\underline{\mathscr{K}}}(\delta) \\ \overline{\mathscr{K}}^{(i)}(\delta) \end{pmatrix} &= \mathscr{N}^{\{1,3\}}\mathscr{W}^{(i)}(\delta) \\ &= \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}\begin{pmatrix} \overset{(i)}{\underline{\mathscr{W}}}(\delta) \\ -\overline{\mathscr{W}}^{(i)}(\delta) \end{pmatrix} \\ &= \begin{pmatrix} X\overset{(i)}{\underline{\mathscr{W}}}(\delta) - Y\overline{\mathscr{W}}^{(i)}(\delta) \\ X\overline{\mathscr{W}}^{(i)}(\delta) - Y\overset{(i)}{\underline{\mathscr{W}}}(\delta) \end{pmatrix}. \end{aligned} \tag{69}$$

Since,
$X+Y, \overline{\mathscr{W}}^{(i)}(\delta) - \underline{\mathscr{W}}^{(i)}(\delta)$, $\delta \in [0,1]$, are nonnegative,

$$\overline{\mathscr{K}}^{(i)}(\delta) - \overset{(i)}{\underline{\mathscr{K}}}(\delta) = (X+Y)\left(\overline{\mathscr{W}}^{(i)}(\delta) - \overset{(i)}{\underline{\mathscr{W}}}(\delta)\right) \geq 0. \tag{70}$$

Since $\underline{\mathscr{W}}^{(i)}(\delta)$ is nondecreasing and $\overline{\mathscr{W}}^{(i)}(\delta)$ is nonincreasing, also the bounded left continuity of $\underline{\mathscr{K}}^{(i)}(\delta), \overline{\mathscr{K}}^{(i)}(\delta)$ is quite simple, and they are in the form of the linear combinations $\underline{\mathscr{W}}^{(i)}(\delta), \overline{\mathscr{W}}^{(i)}(\delta)$.

Remark 2. From Theorem 11, if $\mathscr{N}$ has a least squares generalized inverse $\mathscr{N}^{\{1,3\}}$ such as (65) with $\mathscr{N}^{\{1,3\}} \geq 0$, the system $\mathscr{K}(\delta) = \mathscr{N}^{\{1,3\}}\mathscr{W}^{(i)}(\delta)$ has strong m-polar fuzzy LSS. Specifically, if $\mathscr{N}^{\dagger}$ (Moore–Penrose inverse) such as (66) are nonnegative, the system $\mathscr{K}(\delta) = \mathscr{N}^{\dagger}\mathscr{W}^{(i)}(\delta)$ has also strong m-polar fuzzy LSS. By Theorem 10, it is the minimum norm m-polar fuzzy LSS. Now we are providing a few result for such $\mathscr{N}^{\{1,3\}}$ and $\mathscr{N}^{\dagger}$ are nonnegative. Usually, $(\cdot)^{\top}$ denotes the transpose of a matrix $(\cdot)$.

Theorem 12 (see [45]). *The matrix $\mathscr{N}$ of rank l except zero columns or rows, which admit the condition $\mathscr{N}^{\{1,3\}} \geq 0$, is necessary and sufficient where there exist certain permutation matrices D, E such that*

$$DNE = [L, *], \tag{71}$$

where the direct sum of l positive is L, and matrices are ranked one.

Theorem 13 (see [46]). *Let $\mathscr{N}^{\dagger}$ be the nonnegative matrix inverse of $\mathscr{N}$, if and only if*

$$\mathscr{N}^{\dagger} = \begin{pmatrix} SX^{\top} & SY^{\top} \\ SY^{\top} & SX^{\top} \end{pmatrix}, \tag{72}$$

for some positive diagonal matrix S. In this case,

$$\begin{aligned} (X+Y)^{\top} &= S(X+Y)^{\top}, \\ (X-Y)^{\top} &= S(X-Y)^{\top}. \end{aligned} \tag{73}$$

Example 5. Consider the following 3-polar fuzzy systems:

$$\begin{aligned} &\begin{pmatrix} 2\mathscr{K}_{11} - 2\mathscr{K}_{21} & 2\mathscr{K}_{12} - 2\mathscr{K}_{22} \\ -2\mathscr{K}_{11} + 2\mathscr{K}_{21} & -2\mathscr{K}_{12} + 2\mathscr{K}_{22} \end{pmatrix} \\ &= \begin{pmatrix} 10+5\delta, 20-5\delta, 7+3\delta, 13-3\delta, 8+4\delta, 16-4\delta & 15+3\delta, 21-3\delta, 17+6\delta, 29-6\delta, 9+5\delta, 19-5\delta \\ 6+2\delta, 10-2\delta, 3+\delta, 5-\delta, 12+4\delta, 20-4\delta & 8+5\delta, 18-5\delta, 11+2\delta, 15-2\delta, 6+3\delta, 12-3\delta \end{pmatrix}. \end{aligned} \tag{74}$$

The extended 4×4 matrix $\mathscr{N}$ is

$$\mathscr{N} = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}. \tag{75}$$

The augmented matrices for $m = 1, 2, 3$ are

$$\left(\mathcal{N}, \mathcal{W}^{(1)}(\delta)\right) = \begin{pmatrix} 2 & 0 & 0 & 2 & 10+5\delta & 15+3\delta \\ 0 & 2 & 2 & 0 & 6+2\delta & 8+5\delta \\ 0 & 2 & 2 & 0 & 5\delta-20 & 3\delta-21 \\ 2 & 0 & 0 & 2 & 2\delta-10 & 5\delta-18 \end{pmatrix},$$
$$\left(\mathcal{N}, \mathcal{W}^{(2)}(\delta)\right) = \begin{pmatrix} 2 & 0 & 0 & 2 & 7+3\delta & 17+6\delta \\ 0 & 2 & 2 & 0 & 3+\delta & 11+2\delta \\ 0 & 2 & 2 & 0 & 3\delta-13 & 6\delta-29 \\ 2 & 0 & 0 & 2 & \delta-5 & 2\delta-15 \end{pmatrix}, \quad (76)$$
$$\left(\mathcal{N}, \mathcal{W}^{(3)}(\delta)\right) = \begin{pmatrix} 2 & 0 & 0 & 2 & 8+4\delta & 9+5\delta \\ 0 & 2 & 2 & 0 & 12+4\delta & 6+3\delta \\ 0 & 2 & 2 & 0 & 4\delta-16 & 5\delta-19 \\ 2 & 0 & 0 & 2 & 4\delta-20 & 3\delta-12 \end{pmatrix}.$$

Since $\operatorname{Rank}(\mathcal{N}) \neq \operatorname{Rank}(\mathcal{N}, \mathcal{W}^{(i)}(\delta))$, the original system is inconsistent. One $\{1,3\}$-inverse of $\mathcal{N}$ is

$$\mathcal{N}^{\{1,3\}} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (77)$$

which is nonnegative, and the corresponding solution is given by

$$\begin{aligned} \mathcal{K}^{(1)} &= \begin{pmatrix} \mathcal{K}_{11}^{(1)} & \mathcal{K}_{12}^{(1)} \\ \mathcal{K}_{21}^{(1)} & \mathcal{K}_{22}^{(1)} \end{pmatrix} \\ &= \begin{pmatrix} (7\delta, 14-7\delta) & (-3+8\delta, 13-8\delta) \\ (0,0) & (0,0) \end{pmatrix}, \\ \mathcal{K}^{(2)} &= \begin{pmatrix} \mathcal{K}_{11}^{(2)} & \mathcal{K}_{12}^{(2)} \\ \mathcal{K}_{21}^{(2)} & \mathcal{K}_{22}^{(2)} \end{pmatrix} \\ &= \begin{pmatrix} (2+4\delta, 10-4\delta) & (2+8\delta, 18-8\delta) \\ (0,0) & (0,0) \end{pmatrix}, \\ \mathcal{K}^{(3)} &= \begin{pmatrix} \mathcal{K}_{11}^{(3)} & \mathcal{K}_{12}^{(3)} \\ \mathcal{K}_{21}^{(3)} & \mathcal{K}_{22}^{(3)} \end{pmatrix} \\ &= \begin{pmatrix} (-12+8\delta, 4-8\delta) & (-3+8\delta, 13-8\delta) \\ (0,0) & (0,0) \end{pmatrix}, \end{aligned} \quad (78)$$

and it is a strong 3-polar fuzzy LSS plotted in Figures 13–15.

We have $(a) = \mathcal{K}^{(1)}$, $(b) = \mathcal{K}^{(2)}$, $(c) = \mathcal{K}^{(3)}$. The Moore–Penrose inverse of $\mathcal{N}$ is

$$\mathcal{N}^{\dagger} = \begin{pmatrix} 0.5000 & 0 & 0 & 0.5000 \\ 0 & 0.5000 & 0.5000 & 0 \\ 0 & 0.5000 & 0.5000 & 0 \\ 0.5000 & 0 & 0 & 0.5000 \end{pmatrix}, \quad (79)$$

which is nonnegative. Therefore the original system has a strong 3-polar fuzzy solution:

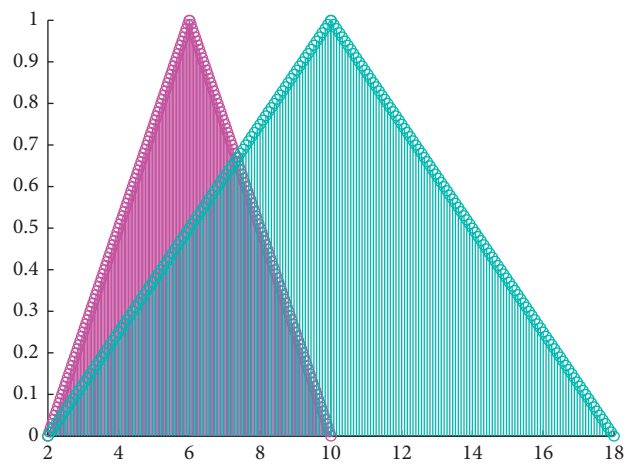

Figure 14: 3-polar fuzzy LSS (b).

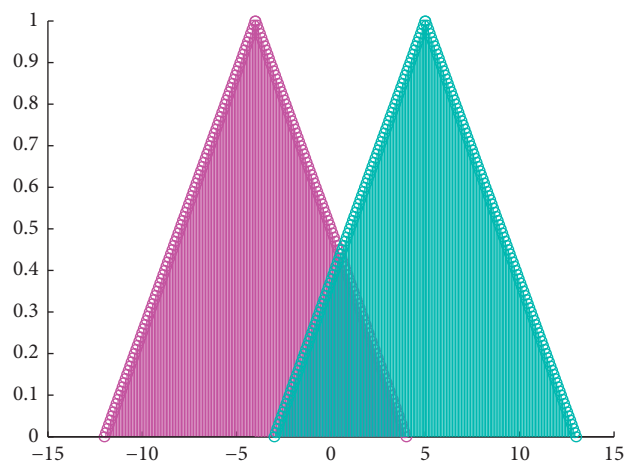

Figure 15: 3-polar fuzzy LSS (c).

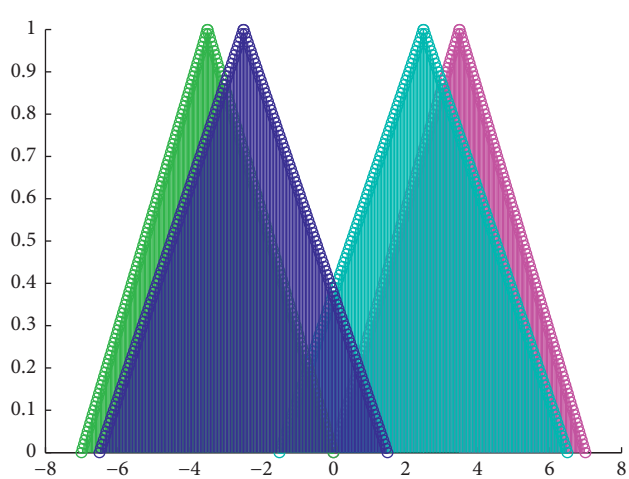

Figure 16: Moore–Penrose bipolar fuzzy LSS (d).

$$\mathcal{K}^{(1)} = \begin{pmatrix} \underline{\mathcal{K}}_{11}^{(1)}(\delta) & \underline{\mathcal{K}}_{12}^{(1)}(\delta) \\ \underline{\mathcal{K}}_{21}^{(1)}(\delta) & \underline{\mathcal{K}}_{22}^{(1)}(\delta) \\ -\overline{\mathcal{K}}_{11}^{(1)}(\delta) & -\overline{\mathcal{K}}_{12}^{(1)}(\delta) \\ -\overline{\mathcal{K}}_{21}^{(1)}(\delta) & -\overline{\mathcal{K}}_{22}^{(1)}(\delta) \end{pmatrix}$$
$$= \mathcal{N}^{\dagger}\mathcal{W}^{(1)}(\delta)$$
$$= \begin{pmatrix} 0.5000 & 0 & 0 & 0.5000 \\ 0 & 0.5000 & 0.5000 & 0 \\ 0 & 0.5000 & 0.5000 & 0 \\ 0.5000 & 0 & 0 & 0.5000 \end{pmatrix} \begin{pmatrix} 10+5\delta & 15+3\delta \\ 6+2\delta & 8+5\delta \\ 5\delta-20 & 3\delta-21 \\ 2\delta-10 & 5\delta-18 \end{pmatrix},$$
$$\mathcal{K}^{(1)} = \begin{pmatrix} \mathcal{K}_{11}^{(1)} & \mathcal{K}_{12}^{(1)} \\ \mathcal{K}_{21}^{(1)} & \mathcal{K}_{22}^{(1)} \end{pmatrix}$$
$$= \begin{pmatrix} \left(\frac{7\delta}{2}, 7-\frac{7\delta}{2}\right) & \left(-\frac{3}{2}+4\delta, \frac{13}{2}-4\delta\right) \\ \left(-7+\frac{7\delta}{2}, -\frac{7\delta}{2}\right) & \left(-\frac{13}{2}+4\delta, \frac{3}{2}-4\delta\right) \end{pmatrix},$$
$$\mathcal{K}^{(2)} = \begin{pmatrix} \underline{\mathcal{K}}_{11}^{(2)}(s) & \underline{\mathcal{K}}_{12}^{(2)}(s) \\ \underline{\mathcal{K}}_{21}^{(2)}(s) & \underline{\mathcal{K}}_{22}^{(2)}(s) \\ -\overline{\mathcal{K}}_{11}^{(2)}(s) & -\overline{\mathcal{K}}_{12}^{(2)}(s) \\ -\overline{\mathcal{K}}_{21}^{(2)}(s) & -\overline{\mathcal{K}}_{22}^{(2)}(s) \end{pmatrix}$$
$$= \mathcal{N}^{\dagger}\mathcal{W}^{(2)}(\delta)$$
$$= \begin{pmatrix} 0.5000 & 0 & 0 & 0.5000 \\ 0 & 0.5000 & 0.5000 & 0 \\ 0 & 0.5000 & 0.5000 & 0 \\ 0.5000 & 0 & 0 & 0.5000 \end{pmatrix} \begin{pmatrix} 7+3\delta & 17+6\delta \\ 3+\delta & 11+2\delta \\ 3\delta-13 & 6\delta-29 \\ \delta-5 & 2\delta-15 \end{pmatrix},$$
$$\mathcal{K}^{(2)} = \begin{pmatrix} \mathcal{K}_{11}^{(2)} & \mathcal{K}_{12}^{(2)} \\ \mathcal{K}_{21}^{(2)} & \mathcal{K}_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} (1+2\delta, 5-2\delta) & (1+4\delta, 9-4\delta) \\ (-5+2\delta, -1-2\delta) & (-9+4\delta, -1-4\delta) \end{pmatrix},$$
$$\mathcal{K}^{(3)} = \begin{pmatrix} \underline{\mathcal{K}}_{11}^{(3)}(\delta) & \underline{\mathcal{K}}_{12}^{(3)}(\delta) \\ \underline{\mathcal{K}}_{21}^{(3)}(\delta) & \underline{\mathcal{K}}_{22}^{(3)}(\delta) \\ -\overline{\mathcal{K}}_{11}^{(3)}(\delta) & -\overline{\mathcal{K}}_{12}^{(3)}(\delta) \\ -\overline{\mathcal{K}}_{21}^{(3)}(\delta) & -\overline{\mathcal{K}}_{22}^{(3)}(\delta) \end{pmatrix}$$
$$= \mathcal{N}^{\dagger}\mathcal{W}^{(3)}(\delta)$$
$$= \begin{pmatrix} 0.5000 & 0 & 0 & 0.5000 \\ 0 & 0.5000 & 0.5000 & 0 \\ 0 & 0.5000 & 0.5000 & 0 \\ 0.5000 & 0 & 0 & 0.5000 \end{pmatrix} \begin{pmatrix} 8+4\delta & 9+5\delta \\ 12+4\delta & 6+3\delta \\ 4\delta-16 & 5\delta-19 \\ 4\delta-20 & 3\delta-12 \end{pmatrix},$$
$$\mathcal{K}^{(3)} = \begin{pmatrix} \mathcal{K}_{11}^{(3)} & \mathcal{K}_{12}^{(3)} \\ \mathcal{K}_{21}^{(3)} & \mathcal{K}_{22}^{(3)} \end{pmatrix}$$
$$= \begin{pmatrix} (-6+4\delta, 2-4\delta) & \left(-\frac{3}{2}+4\delta, \frac{13}{2}-4\delta\right) \\ (-2+4\delta, 6-4\delta) & \left(-\frac{13}{2}+4\delta, \frac{3}{2}-4\delta\right) \end{pmatrix}. \tag{80}$$

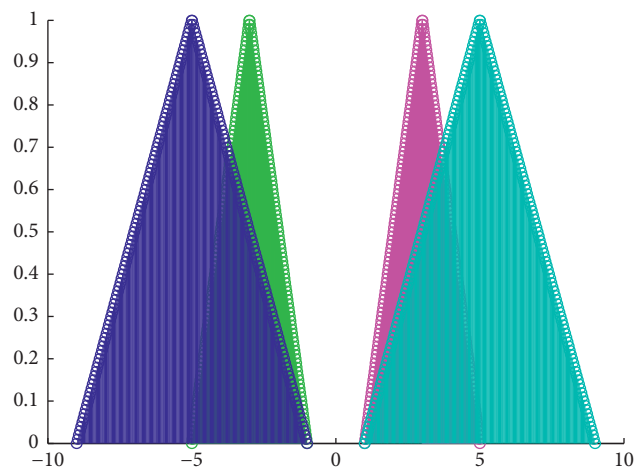

Figure 17: Moore–Penrose 3-polar fuzzy LSS (e).

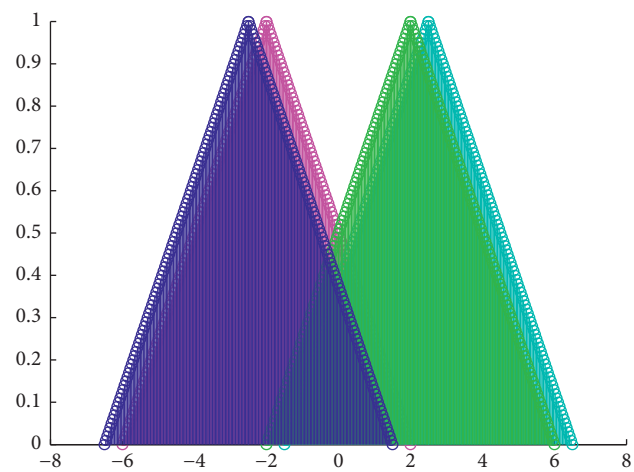

Figure 18: Moore–Penrose 3-polar fuzzy LSS (f).

which leads to the minimum norm 3-polar fuzzy LSS plotted in Figures 16–18. We have $(d) = \mathcal{K}^{(1)}$, $(e) = \mathcal{K}^{(2)}$, $(f) = \mathcal{K}^{(3)}$.

Remark 3. Notice that Figures 1–3 and 4–6 are plotted to show the differences of the solutions using $\{1,3\}$-inverse and Moore–Penrose inverse of the matrix $\mathcal{N}$, respectively. Moreover, we obtain the strong m-polar fuzzy least square solution for $m = 1, 2, 3$ by using $\{1,3\}$-inverse and Moore–Penrose inverse of the matrix $\mathcal{N}$.

5. Conclusion

We have solved m-polar fuzzy system of linear equations by using LU decomposition method. We have analyzed that if the matrices $\mathcal{N}_1$ and $\mathcal{N}_1 - \mathcal{N}_2\mathcal{N}_1^{-1}\mathcal{N}_2$ have LU or LL^T decomposition, then $\mathcal{N}$ is also decomposition, and if $\mathcal{N}$ is positive definite symmetric matrix, then it has LL^T decomposition. The solvability of the LU decomposition method has been discussed in detail and the concept of

inconsistent m-PFME was presented. Moreover, we have discussed a class of inconsistent m-PFMEs $\mathscr{D}\mathscr{K} = \mathscr{W}^{(i)}$ in which $\mathscr{D}$ is an $m \times n$ crisp matrix, and the right-hand side vector $\mathscr{W}^{(i)}$ is $m \times n$ arbitrary m-PFN matrix. We also found the m-polar fuzzy least square solution of m-polar fuzzy inconsistent matrix by using the theory of generalized inverse matrix on $\mathscr{N}$. Finally, the strong m-polar fuzzy LSS has been obtained and we illustrated this concept with an example. In the future, this work can help to determine the flow rate of the traffic on the road by using m-polar fuzzy linear system of equations. Moreover, this work can be used in circuit analysis to balance the flow of current in circuit of the system [47–51].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of the research article.

Acknowledgments

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