Hindawi
Mathematical Problems in Engineering
Volume 2020, Article ID 8679465, 13 pages
https://doi.org/10.1155/2020/8679465

Research Article

On the Generalization of a Solution for a Class of Integro-Differential Equations with Nonseparated Integral Boundary Conditions

Yanyuan Xing[1,2] **Feng Jiao**[1] **and Fang Liu**[2]

[1] *School of Mathematics and Information Sciences, Guangzhou University, Guangzhou, Guangdong 510006, China*
[2] *Department of Mathematics, Luliang University, Luliang, Shanxi 033000, China*

Correspondence should be addressed to Yanyuan Xing; xingyanyuan@e.gzhu.edu.cn

Received 1 November 2019; Revised 7 January 2020; Accepted 6 February 2020; Published 28 March 2020

Academic Editor: Gisele Mophou

In this paper, the existence and uniqueness results of the generalization nonlinear fractional integro-differential equations with nonseparated type integral boundary conditions are investigated. A natural formula of solutions is derived and some new existence and uniqueness results are obtained under some conditions for this class of problems by using standard fixed point theorems and Leray–Schauder degree theory, which extend and supplement some known results. Some examples are discussed for the illustration of the main work.

1. Introduction

Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. Characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integral-order models. Fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, polymer rheology, aerodynamics, capacitor theory, chemistry, biology, control theory, and electrodynamics of complex medium. The initial and boundary value problems for nonlinear fractional differential equations arise from the study of models of viscoelasticity, electrochemistry, porous media, and electromagnetics. In consequence, the subject of fractional differential equations is gaining much importance and attention [1–4]. The recent development in the theory and methods for fractional differential equations indicates its popularity. For more details, we refer the reader to [5–9] and the references cited therein.

Moreover, the existence and uniqueness of solutions for fractional differential equations have been mathematically studied from different methods [10–15], yielding methods for solving fractional differential equations [16–19]. As we all know, boundary value problems of fractional differential equations have been investigated for many years. Now, there are many papers dealing with the problem for different kinds of boundary conditions such as periodic or antiperiodic boundary condition [20, 21], multipoint boundary condition [22, 23], and integral boundary condition [24–28] as well as stability and convergence analysis [29–32]. Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, and population dynamics. For a detailed description of some recent work on the integral boundary conditions, we refer the reader to some recent papers [33–35] and the references therein [36–39].

In [20], Ahmad and Nieto investigated the fractional differential equations with antiperiodic fractional boundary conditions as the following form:

$$\begin{cases} {}^{C}D_t^{\alpha}u(t) = f(t,u(t)), & t \in [0,T],\ T>0,\ 1<\alpha\le 2, \\ u(0) = -u(T), & {}^{C}D_t^{\beta}u(0) = -{}^{C}D_t^{\beta}u(T),\ 0<\beta<1, \end{cases} \tag{1}$$

where ${}^{C}D_t^{\alpha}$ and ${}^{C}D_t^{\beta}$ denote the Caputo fractional derivative of order α, β; $f\colon [0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given continuous function; and T is a fixed positive constant. The results are based on some standard fixed point principles.

Recently, in [24], the author discussed the nonlinear fractional differential equations with nonseparated type integral boundary conditions

$$\begin{cases} {}^{C}D_t^{\alpha}u(t) = f(t,u(t)), & t\in[0,T],\ T>0,\ 1<\alpha\le 2,\\ u(0)+\lambda_1 u(T) = \mu_1\int_0^T g(s,u(s))\mathrm{d}s, & \\ u'(0)+\lambda_2 u'(T) = \mu_2\int_0^T h(s,u(s))\mathrm{d}s, & \end{cases} \tag{2}$$

where ${}^{C}D_t^{\alpha}$ denotes the Caputo fractional derivative of order α, f, g, h: $[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ are given continuous function, and λ_1, λ_2, μ_1, μ_2 are suitably chosen real constants with $\lambda_1\neq-1$, $\lambda_2\neq-1$. By applying the Leray–Schauder degree theory and some standard fixed point theorems, some new existence and uniqueness results are obtained.

Motivated by the abovementioned papers and many known results, in this paper, we concentrate on the existence and uniqueness of solutions for the nonlinear fractional integro-differential equations and inclusions of order $\alpha\in(1,2]$, with nonseparated type integral boundary conditions

$$\begin{cases} {}^{C}D_t^{\alpha}u(t) = f(t,u(t),(\varphi u)(t),(\psi u)(t)), & t\in[0,T],\ 1<\alpha\le 2,\\ u(0)+\mu_1 u(T) = \sigma_1\int_0^T g(s,u(s))\mathrm{d}s, & T>0,\\ {}^{C}D_t^{\beta}u(0)+\mu_2{}^{C}D_t^{\beta}u(T) = \sigma_2\int_0^T h(s,u(s))\mathrm{d}s, & 0<\beta<1, \end{cases} \tag{3}$$

where ${}^{C}D_t^{\alpha}$ and ${}^{C}D_t^{\beta}$ denote the Caputo fractional derivative of order α, β; $f\colon [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given continuous function satisfying some assumptions that will be specified later; Γ is the Euler gamma function; and $\mu_1\neq-1$, $\mu_2\neq0$, κ, ξ: $[0,\ T]\times[0,\ T]\longrightarrow[0,\ \infty)$, φ, ψ are linear operators defined by

$$\begin{aligned} (\varphi u)(t) &= \int_0^t \kappa(t,s)u(s)\mathrm{d}s,\\ (\psi u)(t) &= \int_0^t \xi(t,s)u(s)\mathrm{d}s, \end{aligned} \tag{4}$$

g, h: $[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$, σ_1, $\sigma_2\in\mathbb{R}$. Here, $\mathscr{C}\colon = C([0,T],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,\ T]$ to $\mathbb{R}$ endowed with a topology of uniform convergence with the norm $\|u\| = \sup\{|u(t)|,\ t\in[0,T]\}$.

To the best of our knowledge, no paper has considered the generalization of nonlinear fractional integro-differential equations with nonseparated type integral boundary conditions (3). Our purpose here is to give some existence and uniqueness results for solution to (3).

Compared with the previous research problems, (3) has more general integral boundary value conditions. This paper is organized as follows: in Section 2, we present the notations and give some preliminary results via a sequence of definitions and lemmas. In Section 3, we prove new existence and uniqueness results for problem (3). These results are based on fixed point theorems and Leray–Schauder degree theory. In Section 4, two examples are demonstrated which support the theoretical analysis.

2. Preliminaries and Lemmas

In this section, we present some basic notations, definitions, and preliminary results which will be used throughout this paper. Let us recall some definitions of fractional calculus. For more details, see [1, 2].

Definition 1. The fractional integral of order α with the lower limit zero for a function $f\colon [0,\infty)\longrightarrow\mathbb{R}$ is defined as

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\mathrm{d}s,\quad t>0,\ n-1<\alpha<n, \tag{5}$$

provided the integral exists.

Definition 2. For a function $f\colon [0,\infty)\longrightarrow\mathbb{R}$ with the lower limit zero, the Caputo derivative of fractional order α is defined as

$$\begin{aligned} {}^{C}D_t^{\alpha}f(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)\mathrm{d}s,\\ &\qquad t>0,\ n-1<\alpha<n, \end{aligned} \tag{6}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denote the integer part of the real number α.

Definition 3. The Riemann–Liouville fractional derivative of order α with the lower limit zero for a function $f(t)$ is defined by

$$ {}^{R}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \quad n = [\alpha] + 1, \tag{7} $$

where $n = [\alpha] + 1$ and $[\alpha]$ denote the integer part of real number α, provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 1. *For $\alpha > 0$, the general solution of the fractional differential equation ${}^{C}D_t^{\alpha} u(t) = 0$ is given by*

$$ u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{8} $$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$ $(n = [\alpha] + 1)$.

In view of Lemma 1, it follows that

$$ I^{\alpha C}D_t^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{9} $$

for some $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$ $(n = [\alpha] + 1)$.

In the following, we derive a natural formula of solution to the integral boundary value problem for integro-differential equation (3).

Lemma 2. *Assume that $y \in C([0,T], \mathbb{R})$, $T > 0$, $1 < \alpha \le 2$. A function u(t) is a solution of the boundary value problem*

$$ \begin{cases} {}^{C}D_t^{\alpha} u(t) = y(t), & t \in [0,T],\ 1 < \alpha \le 2, \\ u(0) + \mu_1 u(T) = \sigma_1 \int_0^T g(s, u(s)) ds, & T > 0, \\ {}^{C}D_t^{\beta} u(0) + \mu_2 {}^{C}D_t^{\beta} u(T) = \sigma_2 \int_0^T h(s, u(s)) ds, & 0 < \beta < 1, \end{cases} \tag{10} $$

if and only if u is a solution of the integral equation

$$ \begin{aligned} u(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds - \frac{\mu_1}{1+\mu_1} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ & + \frac{\Gamma(2-\beta)[\mu_1 T - (1+\mu_1)t]}{(1+\mu_1)T^{1-\beta}} \int_0^T \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s) ds \\ & + \frac{\sigma_1}{1+\mu_1} \int_0^T g(s, u(s)) ds - \frac{\sigma_2 \Gamma(2-\beta) T^{\beta-1}}{\mu_2(1+\mu_1)} \\ & \cdot [\mu_1 T - (1+\mu_1)t] \int_0^T h(s, u(s)) ds. \end{aligned} \tag{11} $$

Proof. Assume that y satisfies (10). Using Lemma 1, for some constants $c_0, c_1 \in \mathbb{R}$, we have

$$ \begin{aligned} u(t) &= I^{\alpha} y(t) - c_0 - c_1 t \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds - c_0 - c_1 t. \end{aligned} \tag{12} $$

Using the facts that ${}^{C}D_t^{\beta} c = 0$ (c is a constant), ${}^{C}D_t^{\beta} t = t^{1-\beta}/\Gamma(2-\beta)$, and ${}^{C}D_t^{\alpha} I_t^{\beta} y(t) = I^{\beta-\alpha} y(t)$, we get

$$ {}^{C}D_t^{\beta} u(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s) ds - c_1 \frac{t^{1-\beta}}{\Gamma(2-\beta)}. \tag{13} $$

Applying the boundary conditions for problem (3), we find that

$$ \begin{aligned} c_0 = {} & \frac{\mu_1}{1+\mu_1} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds - \frac{\mu_1 T^{\beta} \Gamma(2-\beta)}{1+\mu_1} \int_0^T \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s) ds \\ & - \frac{\sigma_1}{1+\mu_1} \int_0^T g(s, u(s)) ds + \frac{\mu_1 \sigma_2 \Gamma(2-\beta) T^{\beta}}{\mu_2(1+\mu_1)} \int_0^T h(s, u(s)) ds, \\ c_1 = {} & \frac{\Gamma(2-\beta)}{T^{1-\beta}} \left(\frac{1}{\Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} y(s) ds - \frac{\sigma_2}{\mu_2} \right) \int_0^T h(s, u(s)) ds. \end{aligned} \tag{14} $$

Substituting the value of c_0 and c_1 in (12), we obtain the unique solution of (10) which is given by

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{\mu_1}{1+\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \frac{\Gamma(2-\beta)[\mu_1 T - (1+\mu_1)t]}{(1+\mu_1)T^{1-\beta}}\int_0^T \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s)ds + \frac{\sigma_1}{1+\mu_1}\int_0^T g(s,u(s))ds - \frac{\sigma_2\Gamma(2-\beta)T^{\beta-1}}{\mu_2(1+\mu_1)} \cdot [\mu_1 T - (1+\mu_1)t]\int_0^T h(s,u(s))ds. \quad (15)$$

Conversely, we assume that u is a solution of the integral equation (11), and in view of the relations ${}^C D_t^{\alpha} I_t^{\beta} y(t) = y(t)$, for $\alpha > 0$, we get

$${}^C D_t^{\alpha} u(t) = y(t), \quad t \in [0,T],\ 1 < \alpha \le 2. \quad (16)$$

Moreover, it can easily be verified that the boundary conditions

$$u(0) + \mu_1 u(T) = \sigma_1 \int_0^T g(s,u(s))ds, \qquad {}^C D_t^{\beta} u(0) + \mu_2 {}^C D_t^{\beta} u(T) = \sigma_2 \int_0^T h(s,u(s))ds, \quad (17)$$

are satisfied. The proof is completed.

By Lemma 2, problem (3) is reduced to the fixed point problem

$$u = \Phi(u), \quad (18)$$

where $\Phi: \mathscr{C} \longrightarrow \mathscr{C}$ is given by

$$(\Phi u)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),(\varphi u)(s),(\psi u)(s))ds - \frac{\mu_1}{1+\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),(\varphi u)(s),(\psi u)(s))ds + \frac{\Gamma(2-\beta)[\mu_1 T - (1+\mu_1)t]}{(1+\mu_1)T^{1-\beta}}\int_0^T \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} f(s,u(s),(\varphi u)(s),(\psi u)(s))ds + \frac{\sigma_1}{1+\mu_1}\int_0^T g(s,u(s))ds - \frac{\sigma_2\Gamma(2-\beta)T^{\beta-1}}{\mu_2(1+\mu_1)}[\mu_1 T - (1+\mu_1)t]\int_0^T h(s,u(s))ds. \quad (19)$$

□

3. Main Results

In this section, we will show the existence and uniqueness of solutions for problem (3). Now we state some known fixed point theorems which are needed to prove the existence of solutions for equation (3).

Theorem 1. *Let X be a Banach space. Assume that $\Phi: X \longrightarrow X$ is a completely continuous operator and the set $V = \{u \in X \mid u = \mu\Phi u,\ 0 < \mu < 1\}$ is bounded. Then, Φ has a fixed point in X.*

Theorem 2. *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $0 \in \Omega$ and let $\Phi: \overline{\Omega} \longrightarrow X$ be a completely continuous operator such that*

$$\|\Phi u\| \le \|u\|, \quad \forall u \in \partial\Omega. \quad (20)$$

Then Φ has a fixed point in $\overline{\Omega}$.

Theorem 3. *Suppose that $f: [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ is a jointly continuous function and maps bounded subsets of $[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ into relative compact subsets of $\mathbb{R}$, κ, $\xi: [0, T]\times[0, T] \longrightarrow [0, \infty)$ is continuous with*

$$k_0 = \max\{|\kappa(t,s)|: (t,s) \in [0,T]\times[0,T]\}, \quad (21)$$

$$\xi_0 = \max\{|\xi(t,s)|: (t,s) \in [0,T]\times[0,T]\}, \qquad K = \max\{\kappa_0, \xi_0\}, \quad (22)$$

and g, h: $[0,T]\times\mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions. Furthermore, there exist positive constants $C_i (i = 1, \ldots, 5)$ such that

(H_1) $|f(t,u_1,u_2,u_3) - f(t,v_1,v_2,v_3)| \le C_1|u_1 - v_1| + C_2|u_2 - v_2| + C_3|u_3 - v_3|$, $\forall t \in [0, T]$, $u_i, v_i \in \mathbb{R}, i = 1,2,3$

(H_2) $|g(t,u) - g(t,v)| \le C_4|u - v|$, $|h(t,u) - h(t,v)| \le C_5|u - v|$, $\forall u, v \in \mathbb{R}$

Then the boundary value problem (3) has a unique solution provided

$$r_1 = \left[\left(1 + \left|\frac{\mu_1}{1+\mu_1}\right|\right)\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\beta)T^{\alpha}}{\Gamma(\alpha-\beta+1)}\right] \cdot [C_1 + (C_2 + C_3)TK] + \left|\frac{\sigma_1}{1+\mu_1}\right| C_4 T + \left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^{\beta+1}C_5 < 1. \quad (23)$$

Proof. Setting $\sup_{t\in[0,T]}|f(t,0,0,0)| = M_1$, $\sup_{t\in[0,T]}|g(t,0)| = M_2$, and $\sup_{t\in[0,T]}|h(t,0)| = M_3$. For a positive number r, let $B_r = \{u \in \mathscr{C}: \|u\| \le r\}$ and $r \ge r_2/(1-r_1)$, with r_1 is given by (23), we will show that $\Phi B_r \subset B_r$, where Φ is defined by (19), and

$$r_2 = \left[\left(1+\left|\frac{\mu_1}{1+\mu_1}\right|\right)\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\beta)T^\alpha}{\Gamma(\alpha-\beta+1)}\right]M_1 + \left|\frac{\sigma_1}{1+\mu_1}\right|M_2T + \left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^{\beta+1}M_3. \tag{24}$$

First, $\forall u(t)\in B_r$, there exists $\{u_n\}\subset B_r$, and when $n\longrightarrow\infty$, $u_n\longrightarrow u$, it is easy to know that

$$\left\|\Phi u_n(t) - \Phi u(t)\right\| \longrightarrow 0. \tag{25}$$

Then Φ is continuous on B_r.

Furthermore, for $u\in B_r$, $t\in[0, T]$, we have

$$\begin{aligned}
|\Phi u(t)| \le{}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),(\varphi u)(s),(\psi u)(s))|ds \\
&+ \left|\frac{\mu_1}{1+\mu_1}\right|\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),(\varphi u)(s),(\psi u)(s))|ds \\
&+ \Gamma(2-\beta)T^\beta\int_0^T \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|f(s,u(s),(\varphi u)(s),(\psi u)(s))|ds \\
&+ \left|\frac{\sigma_1}{1+\mu_1}\right|\int_0^T |g(s,u(s))|ds + \left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^\beta\int_0^T |h((s,u(s))|ds \\
\le{}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[|f(s,u(s),(\varphi u)(s),(\psi u)(s)) - f(s,0,0,0)| + |f(s,0,0,0)|]ds \\
&+ \left|\frac{\mu_1}{1+\mu_1}\right|\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}[|f(s,u(s),(\varphi u)(s),(\psi u)(s)) - f(s,0,0,0)| + |f(s,0,0,0)|]ds \\
&+ \Gamma(2-\beta)T^\beta\int_0^T \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}[|f(s,u(s),(\varphi u)(s),(\psi u)(s)) - f(s,0,0,0)| + |f(s,0,0,0)|]ds \\
&+ \left|\frac{\sigma_1}{1+\mu_1}\right|\int_0^T [|g(s,u(s)) - g(s,0)| + |g(s,0)|]ds \\
&+ \left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^\beta\int_0^T [|h(s,u(s)) - h(s,0)| + |h(s,0)|]ds \\
\le{}& \left[\left(1+\left|\frac{\mu_1}{1+\mu_1}\right|\right)\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\beta)T^\alpha}{\Gamma(\alpha-\beta+1)}\right]\{[C_1 + (C_2+C_3)TK]r + M_1\} \\
&+ \left|\frac{\sigma_1}{1+\mu_1}\right|T(C_4r + M_2) + \left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^\beta T(C_5r + M_3) \\
\le{}& \left\{\left[\left(1+\left|\frac{\mu_1}{1+\mu_1}\right|\right)\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\beta)T^\alpha}{\Gamma(\alpha-\beta+1)}\right][C_1 + (C_2+C_3)TK] + \left|\frac{\sigma_1}{1+\mu_1}\right|C_4T + \left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^{\beta+1}C_5\right\}r \\
&+ \left[\left(1+\left|\frac{\mu_1}{1+\mu_1}\right|\right)\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\beta)T^\alpha}{\Gamma(\alpha-\beta+1)}\right]M_1 + \left|\frac{\sigma_1}{1+\mu_1}\right|M_2T \\
&+ \left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^{\beta+1}M_3, \quad \le r_1r + r_2 \le r.
\end{aligned} \tag{26}$$

Now, for $u, v \in \mathscr{C}$ and for each $t \in [0, T]$, we obtain

$$\begin{aligned}
&|(\Phi u)(t) - (\Phi v)(t)| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s), (\varphi u)(s), (\psi u)(s)) - f(s, v(s), (\varphi v)(s), (\psi v)(s))| \mathrm{d}s \\
&\quad + \left|\frac{\mu_1}{1+\mu_1}\right| \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s), (\varphi u)(s), (\psi u)(s)) - f(s, v(s), (\varphi v)(s), (\psi v)(s))| \mathrm{d}s \\
&\quad + \Gamma(2-\beta) T^{\beta} \int_0^T \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |f(s, u(s), (\varphi u)(s), (\psi u)(s)) - f(s, v(s), (\varphi v)(s), (\psi v)(s))| \mathrm{d}s \\
&\quad + \left|\frac{\sigma_1}{1+\mu_1}\right| \int_0^T |g(s, u(s)) - g(s, v(s))| \mathrm{d}s + \left|\frac{\sigma_2}{\mu_2}\right| \Gamma(2-\beta) T^{\beta} \int_0^T |h(s, u(s)) - h(s, v(s))| \mathrm{d}s \\
&\leq \left\{ \left[\left(1 + \left|\frac{\mu_1}{1+\mu_1}\right|\right) \frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\beta) T^{\alpha}}{\Gamma(\alpha-\beta+1)} \right] [C_1 + (C_2 + C_3) TK] + \left|\frac{\sigma_1}{1+\mu_1}\right| C_4 T + \left|\frac{\sigma_2}{\mu_2}\right| \Gamma(2-\beta) T^{\beta+1} C_5 \right\} \|u - v\| \\
&\leq r_1 \|u - v\|.
\end{aligned} \tag{27}$$

Observe that r_1 depends only on the parameters involved in the problem. As $r_1 < 1$, then Φ is a contraction map. Hence, the conclusion of the theorem follows by the contraction mapping principle, and Φ has a unique fixed point u. That is, the boundary value problem (3) has a unique solution. This completes the proof.

Our next existence results are based on Krasnoselskii's fixed point theorem [40].

Theorem 4. *Let M be a closed convex and nonempty subset of a Banach space X. Let A and B be the operators such that*

(i) $Ax + By \in M$, whenever $x, y \in M$

(ii) A is compact and continuous

(iii) B is a contraction mapping

Then, there exists $z \in M$ such that $z = Az + Bz$.

Theorem 5. *Assume (H_1) and (H_2) hold, $f: [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a jointly continuous function. Further, we assume that*

(H_3) $|f(t, u, \varphi u, \psi u)| \leq p(t)$, $|g(t, u)| \leq q(t)$, $|h(t, u)| \leq \nu(t)$, $\forall (t, u, \varphi u, \psi u) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $p, q, \nu \in C([0, T], \mathbb{R}^+)$;

(H_4) $[|\mu_1/(1+\mu_1)| T^{\alpha}/(\Gamma(\alpha+1)) + (\Gamma(2-\beta) T^{\alpha})/(\Gamma(\alpha-\beta+1))][C_1 + (C_2 + C_3) TK] + |\sigma_1/(1+\mu_1)| C_4 T + |\sigma_2/\mu_2| \Gamma(2-\beta) T^{\beta+1} C_5 < 1$.

Then, problem (3) has at least one solution on $[0, T]$.

Proof. Let $\sup_{t \in [0,T]} |p(t)| = \|p\|$, $\sup_{t \in [0,T]} |q(t)| = \|q\|$, and $\sup_{t \in [0,T]} |\nu(t)| = \|\nu\|$, we fix

$$\begin{aligned}
\overline{R} &= \left[\left(1 + \left|\frac{\mu_1}{1+\mu_1}\right|\right) \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)} \right] T^{\alpha} \|\varphi\| \\
&\quad + \left|\frac{\sigma_1}{1+\mu_1}\right| T \|\psi\| + \left|\frac{\sigma_2}{\mu_2}\right| \Gamma(2-\beta) T^{\beta+1} \|\nu\|,
\end{aligned} \tag{28}$$

and considering $B_{\overline{R}} = \{u \in \mathscr{C}: \|u\| \leq \overline{R}\}$, we define the operators Φ_1 and Φ_2 on $B_{\overline{R}}$ as

$$\begin{aligned}
(\Phi_1 u)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), (\varphi u)(s), (\psi u)(s)) \mathrm{d}s, \\
(\Phi_2 u)(t) &= -\frac{\mu_1}{1+\mu_1} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), (\varphi u)(s), (\psi u)(s)) \mathrm{d}s \\
&\quad + \frac{\Gamma(2-\beta)[\mu_1 T - (1+\mu_1) t]}{(1+\mu_1) T^{1-\beta}} \int_0^T \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} f(s, u(s), (\varphi u)(s), (\psi u)(s)) \mathrm{d}s \\
&\quad + \frac{\sigma_1}{1+\mu_1} \int_0^T g(s, u(s)) \mathrm{d}s - \frac{\sigma_2 \Gamma(2-\beta) T^{\beta-1}}{\mu_2 (1+\mu_1)} [\mu_1 T - (1+\mu_1) t] \int_0^T h(s, u(s)) \mathrm{d}s.
\end{aligned} \tag{29}$$

For $u, v \in B_{\overline{R}}$, we find that

$$\begin{aligned}\|\Phi_1 u + \Phi_2 v\| &\le \left[\left(1 + \left|\frac{\mu_1}{1+\mu_1}\right|\right)\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}\right] T^{\alpha}\|p\| \\ &\quad + \left|\frac{\sigma_1}{1+\mu_1}\right| T\|q\| + \left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^{\beta+1}\|v\| \\ &\le \overline{R}.\end{aligned} \tag{30}$$

Thus, $\Phi_1 u + \Phi_2 v \in B_{\overline{R}}$. It follows from the assumptions (H_1) and (H_2) that Φ_2 is a contraction mapping if

$$\left[\left|\frac{\mu_1}{1+\mu_1}\right|\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\beta)T^{\alpha}}{\Gamma(\alpha-\beta+1)}\right]\left[C_1 + (C_2+C_3)TK\right] + \left|\frac{\sigma_1}{1+\mu_1}\right| C_4 T + \left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^{\beta+1}C_5 < 1. \tag{31}$$

Moreover, the continuity of f implies that the operator Φ_1 is continuous. Also, Φ_1 is uniformly bounded on $B_{\overline{R}}$ as

$$\|\Phi_1\| \le \frac{\|p\|T^{\alpha}}{\Gamma(\alpha+1)}. \tag{32}$$

Now, we prove compactness of the operator Φ_1. In view of (H_3), we define

$$\sup_{(t,u)\in[0,T]\times B_{\overline{R}}} |f(t,u,\varphi u,\psi u)| = f_{\max}, \tag{33}$$

and consequently, we have

$$\begin{aligned}\|(\Phi_1 u)(t_2) - (\Phi_1 u)(t_1)\| &= \left\| \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),(\varphi u)(s),(\psi u)(s))\mathrm{d}s \right. \\ &\quad \left. + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),(\varphi u)(s),(\psi u)(s))\mathrm{d}s \right\| \\ &\le \frac{f_{\max}}{\Gamma(\alpha+1)}\left|2(t_2-t_1)^{\alpha} + t_1^{\alpha} - t_2^{\alpha}\right|,\end{aligned} \tag{34}$$

which is independent of u and tends to zero as $t_2 - t_1 \longrightarrow 0$. So Φ_1 is relatively compact on $B_{\overline{R}}$. Hence, by the Arzelá–Ascoli theorem, Φ_1 is compact on $B_{\overline{R}}$. Thus, all the assumptions of Theorem 4 are satisfied. Therefore, the conclusion of Theorem 4 applies that the fractional boundary value problem (3) has at least one solution on $[0, T]$. This completes the proof.

As an immediate consequence of Theorem 5, we have the following.

Corollary 1. *Assume that* $f\colon [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ *is a jointly continuous function. If there exists nonnegative functions,* $a_i(t) \in L[0,T] (i = 0, 1, 2, 3)$, $b_i(t), c_i(t) \in C[0,T] (i = 0, 1)$, $0 < \rho_j < 1 (j = 1, 2, 3)$, $0 < \theta_1, \theta_2 < 1$, *and* κ_0, ξ_0 *are given by (21) and (22) such that*

(H_5) $|f(t,u,\varphi u,\psi u)| \le a_0(t) + a_1(t)|u|^{\rho_1} + a_2(t)|\varphi u|^{\rho_2} + a_3(t)|\psi u|^{\rho_3}$

(H_6) $|g(t,u)| \le b_0(t) + b_1(t)|u|^{\theta_1}$, $|h(t,u)| \le c_0(t) + c_1(t)|u|^{\theta_2}$

For all $t \in [0,T]$, $u, \varphi u, \psi u \in \mathbb{R}$, *then the boundary value problem (3) has at least one solution.*

Proof. Let us define a ball in the Banach space $B = \{u \in \mathscr{C} \mid \|u\| \le \Lambda\}$, where Λ is fixed later. Setting

$$I_a^\alpha = \max_{t\in[0,T]}\left\{\left|I^\alpha a_i(t)\right|, \quad i=0,1,2,3\right\},$$

$$I_a^\alpha(T) = \max_{t\in[0,T]}\left\{\left|I^\alpha a_i(T)\right|, \quad i=0,1,2,3\right\},$$

$$\rho = \max_{i=1,2,3}\rho_i,$$

$$b_m = \max_{i=1,2} b_i(t),$$

$$c_m = \max_{i=1,2} c_i(t),$$

$$\begin{aligned}
\|\Phi u(t)\| \leq{}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(a_0(s)+a_1(s)|u|^{\rho_1}+a_2(s)|\varphi u|^{\rho_2}+a_3(s)|\psi u|^{\rho_3}\right)\mathrm{d}s \\
&+\left|\frac{\mu_1}{1+\mu_1}\right|\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left(a_0(s)+a_1(s)|u|^{\rho_1}+a_2(s)|\varphi u|^{\rho_2}+a_3(s)|\psi u|^{\rho_3}\right)\mathrm{d}s \\
&+\Gamma(2-\beta)T^\beta\int_0^T \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left(a_0(s)+a_1(s)|u|^{\rho_1}+a_2(s)|\varphi u|^{\rho_2}+a_3(s)|\psi u|^{\rho_3}\right)\mathrm{d}s \\
&+\left|\frac{\sigma_1}{1+\mu_1}\right|T\left(b_0(s)+b_1(s)|u|^{\theta_1}\right)+\left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^{\beta+1}\left(c_0(s)+c_1(s)|u|^{\theta_2}\right)\mathrm{d}s \\
\leq{}& \left|I^\alpha a_0(t)\right|+\left|I^\alpha a_1(t)\right|\Lambda^{\rho_1}+\left|I^\alpha a_2(t)\right|T^{\rho_2}k_0^{\rho_2}\Lambda^{\rho_2}+\left|I^\alpha a_3(t)\right|T^{\rho_3}\xi_0^{\rho_3}\Lambda^{\rho_3} \\
&+\left|\frac{\mu_1}{1+\mu_1}\right|\left(\left|I^\alpha a_0(T)\right|+\left|I^\alpha a_1(T)\right|\Lambda^{\rho_1}+\left|I^\alpha a_2(T)\right|T^{\rho_2}k_0^{\rho_2}\Lambda^{\rho_2}+\left|I^\alpha a_3(T)\right|T^{\rho_3}\xi_0^{\rho_3}\Lambda^{\rho_3}\right) \\
&+\Gamma(2-\beta)T^\beta\Big(\left|I^{\alpha-\beta}a_0(T)\right|+\left|I^{\alpha-\beta}a_1(T)\right|\Lambda^{\rho_1}+\left|I^{\alpha-\beta}a_2(T)\right|T^{\rho_2}k_0^{\rho_2}\Lambda^{\rho_2} \\
&+\left|I^{\alpha-\beta}a_3(T)\right|T^{\rho_3}\xi_0^{\rho_3}\Lambda^{\rho_3}\Big)+\left|\frac{\sigma_1}{1+\mu_1}\right|Tb_m\left(1+\Lambda^{\theta_1}\right)+\left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^{\beta+1}c_m\left(1+\Lambda^{\theta_2}\right) \\
\leq{}& I_a^\alpha\left(1+T^{\rho_2}k_0^{\rho_2}+T^{\rho_3}\xi_0^{\rho_3}\right)\Lambda^\rho+I^\alpha a_0 \\
&+\left|\frac{\mu_1}{1+\mu_1}\right|\left[I_a^\alpha(T)\left(1+T^{\rho_2}k_0^{\rho_2}+T^{\rho_3}\xi_0^{\rho_3}\right)\Lambda^\rho+I^\alpha a_0(T)\right] \\
&+\Gamma(2-\beta)T^\beta\left[I_a^{\alpha-\beta}(T)\left(1+T^{\rho_2}k_0^{\rho_2}+T^{\rho_3}\xi_0^{\rho_3}\right)\Lambda^\rho+I^{\alpha-\beta}a_0(T)\right] \\
&+\left|\frac{\sigma_1}{1+\mu_1}\right|Tb_m\left(1+\Lambda^{\theta_1}\right)+\left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^{\beta+1}c_m\left(1+\Lambda^{\theta_2}\right) \\
={}& I^\alpha a_0+I^\alpha a_0(T)+\Gamma(2-\beta)T^\beta I^{\alpha-\beta}a_0(T)+\left|\frac{\sigma_1}{1+\mu_1}\right|Tb_m\left(1+\Lambda^{\theta_1}\right) \\
&+\left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^{\beta+1}c_m\left(1+\Lambda^{\theta_2}\right)+\left(1+T^{\rho_2}k_0^{\rho_2}+T^{\rho_3}\xi_0^{\rho_3}\right)\left[\Lambda^\rho I_a^\alpha+\left|\frac{\mu_1}{1+\mu_1}\right|I_a^\alpha(T)+\Gamma(2-\beta)T^\beta I_a^{\alpha-\beta}(T)\right].
\end{aligned} \tag{35}$$

Choosing Λ sufficient large, then $\Phi: B \longrightarrow B$. On the other hand, the continuity of f implies that the operator Φ is continuous. Also, since $\Phi: B \longrightarrow B$, we have $\Phi(B)$ which is uniformly bounded on B.

Let

$$M = \max_{(t,u,\varphi u,\psi u)\in[0,T]\times B\times B\times B} |f(t,u,\varphi u,\psi u)|, \tag{36}$$

and consequently, we obtain

$$\begin{aligned}\|(\Phi u)(t_2)-(\Phi u)(t_1)\| &= \left\| \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),(\varphi u)(s),(\psi u)(s))\mathrm{d}s \right.\\ &\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),(\varphi u)(s),(\psi u)(s))\mathrm{d}s\\ &\quad + \Gamma(2-\beta)T^{\beta-1}(t_1-t_2)\int_0^T \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} f(s,u(s),(\varphi u)(s),(\psi u)(s))\mathrm{d}s\\ &\quad \left. -\frac{\sigma_2\Gamma(2-\beta)(t_1-t_2)}{\mu_2}T^{\beta}c_m(1+\|u\|)\right\|\\ &\le \frac{M}{\Gamma(\alpha+1)}\left|2(t_2-t_1)^{\alpha}+t_1^{\alpha}-t_2^{\alpha}\right| + \frac{M(T-s)^{\alpha-1}\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}(t_1-t_2)\\ &\quad + \left|\frac{\sigma_2\Gamma(2-\beta)}{\mu_2}\right| T^{\beta}c_m(1+\|u\|)(t_1-t_2).\end{aligned} \tag{37}$$

It follows that Φ is equicontinuous, so $\Phi(B)$ is relatively compact on B. Hence, $\Phi(B)$ is relatively compact on B by Arzelá–Ascoli theorem. Thus, by Schauder fixed-point theorem, problem (3) has at least one solution.

Theorem 6. *Assume that there exist positive constants p_i, $q_i (i = 1, 2, 3)$ such that $|f(t,u,\varphi u,\psi u)| \le (p_1/T^{\alpha})|u| + q_1$, $|g(t,u)| \le (p_2/T)|u| + q_2$, and $|h(t,u)| \le (p_3/T^{\beta+1})|u| + q_3$ for all $t \in [0, T]$, $u \in C[0, T]$. If*

$$\begin{aligned}&\left[\left(1+\left|\frac{\mu_1}{1+\mu_1}\right|\right)\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}\right]p_1\\ &\quad+\left|\frac{\sigma_1}{1+\mu_1}\right|p_2+\left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^{\beta+1}p_3<1,\end{aligned} \tag{38}$$

then the boundary value problem (3) has at least one solution.

Proof. In view of the fixed point problem (19), we just need to prove the existence of at least one solution $u \in \mathbb{R}$ satisfying (19). Define a suitable ball $B_S \in \mathscr{C}$ with radius $S > 0$ as

$$B_S = \left\{u \in \mathscr{C}: \max_{t\in[0,T]} |u(t)| < S\right\}, \tag{39}$$

where S will be fixed later. Then, it is sufficient to show that $\mathscr{F}: \overline{B_S} \longrightarrow \mathscr{C}$ satisfies

$$u \ne \lambda\mathscr{F}u, \quad \forall u \in \partial B_S, \forall\lambda \in [0,1]. \tag{40}$$

Then, by the Arzelá–Ascoli theorem, $h_{\lambda}(u) = u - H(\lambda,u) = u - \lambda\mathscr{F}u$ is completely continuous. If (40) is true, then the following Leray–Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_{\lambda},B_S,0) &= \deg(I-\lambda\mathscr{F},B_S,0) = \deg(h_1,B_S,0)\\ &= \deg(h_0,B_S,0) = \deg(I,B_S,0) = 1 \ne 0, \quad 0 \in B_S,\end{aligned} \tag{41}$$

where I denotes the unit operator. By the nonzero property of Leray–Schauder degree, we have $h_1(t) = u - \mathscr{F}u = 0$ for at least one $u \in B_S$. In order to prove (40), we assume that $u = \lambda\mathscr{F}u$ for some $\lambda \in [0, 1]$ and for all $t \in [0, T]$ so that

$$
\begin{aligned}
|u(t)| &= |\lambda \mathscr{F} u(t)| \\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s),(\varphi u)(s),(\psi u)(s))| ds \\
&\quad + \left|\frac{\mu_1}{1+\mu_1}\right| \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s),(\varphi u)(s),(\psi u)(s))| ds \\
&\quad + \Gamma(2-\beta) T^{\beta} \int_0^T \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |f(s,u(s),(\varphi u)(s),(\psi u)(s))| ds \\
&\quad + \left|\frac{\sigma_1}{1+\mu_1}\right| \int_0^T |g(s,u(s))| ds + \left|\frac{\sigma_2}{\mu_2}\right| \Gamma(2-\beta) T^{\beta} \int_0^T |h(s,u(s))| ds \\
&\le \left[\left(1+\left|\frac{\mu_1}{1+\mu_1}\right|\right)\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{(2-\beta))T^{\alpha}}{\Gamma(\alpha-\beta+1)}\right]\left(\frac{p_1}{T^{\alpha}}|u| + q_1\right) \\
&\quad + \left|\frac{\sigma_1}{1+\mu_1}\right| T\left(\frac{p_2}{T}|u| + q_2\right) + \left|\frac{\sigma_2}{\mu_2}\right| \Gamma(2-\beta) T^{\beta} T\left(\frac{p_3}{T^{\beta+1}}|u| + q_3\right) \\
&\le \left\{\left[\left(1+\left|\frac{\mu_1}{1+\mu_1}\right|\right)\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}\right] p_1 + \left|\frac{\sigma_1}{1+\mu_1}\right| p_2 + \left|\frac{\sigma_2}{\mu_2}\right| \Gamma(2-\beta) p_3\right\} |u| \\
&\quad + \left[\left(1+\left|\frac{\mu_1}{1+\mu_1}\right|\right)\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\beta)T^{\alpha}}{\Gamma(\alpha-\beta+1)}\right] q_1 + \left|\frac{\sigma_1}{1+\mu_1}\right| T q_2 + \left|\frac{\sigma_2}{\mu_2}\right| \Gamma(2-\beta) T^{\beta+1} q_3 \\
&= h_1 |u| + h_2,
\end{aligned} \tag{42}
$$

which by taking norm $\|u\| = \sup\{|u(t)|, t \in [0,T]\}$ and solving for $\|u\|$, we have

$$\|u\| \le \frac{h_2}{1-h_1}, \tag{43}$$

where

$$
\begin{aligned}
h_1 &= \left[\left(1+\left|\frac{\mu_1}{1+\mu_1}\right|\right)\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta+1)}\right] p_1 + \left|\frac{\sigma_1}{1+\mu_1}\right| p_2 + \left|\frac{\sigma_2}{\mu_2}\right| \Gamma(2-\beta) p_3, \\
h_2 &= \left[\left(1+\left|\frac{\mu_1}{1+\mu_1}\right|\right)\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\beta)T^{\alpha}}{\Gamma(\alpha-\beta+1)}\right] q_1 + \left|\frac{\sigma_1}{1+\mu_1} T q_2 + \left|\frac{\sigma_2}{\mu_2}\right| \Gamma(2-\beta) T^{\beta+1} q_3.
\end{aligned} \tag{44}
$$

Setting

$$S = \frac{h_2}{1-h_1} + 1, \tag{45}$$

it follows that (40) holds. This completes the proof.

4. Examples

Example 1. Consider the following nonlinear fractional integro-differential equation with nonseparated type integral boundary conditions of $\alpha = 3/2$, $\beta = 1/2$, and $T = 1$:

$$\begin{cases} {}^{C}D_t^{3/2}u(t) = \dfrac{1}{(t+10)^2}\left[\dfrac{|u(t)|}{1+|u(t)|}+\cos^2 t\right]+\dfrac{1}{120}\displaystyle\int_0^t \dfrac{e^{-t}}{(3+t)^2}u(s)ds+\dfrac{1}{140}\int_0^1 \dfrac{1}{(4+t)^2}u(s)ds, \\ u(0)+u(1)=\displaystyle\int_0^1 \dfrac{|u(s)|}{50+|u(s)|}ds, \\ {}^{C}D_t^{1/2}u(0)+{}^{C}D_t^{1/2}u(1)=\displaystyle\int_0^1\left(\dfrac{1}{t+10}\right)^2\dfrac{|u(s)|}{1+|u(s)|}ds. \end{cases} \tag{46}$$

Here, $\mu_1=\mu_2=\sigma_1=\sigma_2=1$ and

$$\begin{aligned} (\varphi u)(t) &= \int_0^t \frac{e^{-t}}{(3+t)^2}u(s)ds, \\ (\psi u)(t) &= \int_0^1 \frac{1}{(4+t)^2}u(s)ds, \\ g(s,u(s)) &= \frac{|u(s)|}{50+|u(s)|}, \\ h(s,u(s)) &= \left(\frac{1}{t+10}\right)^2\frac{|u(s)|}{1+|u(s)|}. \end{aligned} \tag{47}$$

For $u, v \in \mathbb{R}$ and $t \in [0,1]$, we have

$$\begin{aligned} &|f(t,u,\varphi u,\psi u)-f(t,v,\varphi v,\psi v)| \\ &\le \left(\frac{1}{t+10}\right)^2|u-v|+\frac{1}{120}|\varphi u-\varphi v| \\ &\quad+\frac{1}{140}|\psi u-\psi v| \\ &\le \frac{1}{100}\left[|u-v|+|\varphi u-\varphi v|+|\psi u-\psi v|\right], \\ &|g(s,u)-g(s,v)| \le \frac{1}{50}|u-v|, \\ &|h(s,u)-h(s,v)| \le \frac{1}{100}|u-v|. \end{aligned} \tag{48}$$

As $C_1=C_2=C_3=1/100$ and $C_4=1/50$, $C_5=1/100$, $K=1/9$, we obtain

$$\begin{aligned} r_1 &= \left[\left(1+\left|\frac{\mu_1}{1+\mu_1}\right|\right)\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{\Gamma(2-\beta)T^\alpha}{\Gamma(\alpha-\beta+1)}\right] \\ &\quad\cdot\left[C_1+(C_2+C_3)TK\right] \\ &\quad+\left|\frac{\sigma_1}{1+\mu_1}\right|C_4T+\left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^{\beta+1}C_5 \\ &= \left(\frac{2}{\sqrt{\pi}}+\frac{\sqrt{\pi}}{2}\right)\left(\frac{1}{100}+\frac{1}{450}\right)+\frac{1}{100}+\frac{\sqrt{\pi}}{200} \\ &\approx 0.043 < 1. \end{aligned} \tag{49}$$

Thus, all the assumptions of Theorem 3 hold. Consequently, the conclusion of Theorem 3 implies that problem (46) has a unique solution.

Example 2. Consider the following integro-differential fractional boundary value problem

$$\begin{cases} {}^{C}D_t^{3/2}u(t) = \dfrac{e^{-t}}{10}\dfrac{|u(t)|}{1+|u(t)|}+\displaystyle\int_0^t \dfrac{e^{-(s-t)}}{200}\dfrac{|u(t)|}{1+|u(t)|}ds+\int_0^1 \dfrac{e^{-t}(s+1)}{32}\dfrac{|u(t)|}{1+|u(t)|}ds, \\ u(0)+u(1)=\displaystyle\int_0^1 \dfrac{|u(s)|}{50+|u(s)|}ds, \\ {}^{C}D_t^{1/2}u(0)+{}^{C}D_t^{1/2}u(1)=\displaystyle\int_0^1\left(\dfrac{1}{t+10}\right)^2\dfrac{|u(s)|}{1+|u(s)|}ds. \end{cases} \tag{50}$$

Here, $\alpha = 3/2$, $\beta = 1/2$, $T=1$, $\mu_1=\mu_2=\sigma_1=\sigma_2=1$, $T=1$, and

$$\begin{aligned} K &= \sup_{t\in[0,1]}\{\kappa_0,\xi_0\} \\ &= \sup_{t\in[0,1]}\left\{\int_0^t \frac{e^{-(s-t)}}{200}\mathrm{d}s, \int_0^t \frac{e^{-t}(s+1)}{32}\mathrm{d}s\right\} = 0.08. \end{aligned} \tag{51}$$

Since

$$\begin{aligned} |f(t,u,\varphi u,\psi u)| &\le \frac{3e^{-t}}{10}, \\ |g(t,u)| &\le 1, \\ |h(t,u)| &\le \frac{1}{100}. \end{aligned} \tag{52}$$

Clearly, $C_1 = 1/10$, $C_2 = 1/200$, $C_3 = 1/32$, $C_4 = 1/50$, and $C_5 = 1/100$. Furthermore,

$$\begin{aligned} &\left[\left|\frac{\mu_1}{1+\mu_1}\right|\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\beta)T^\alpha}{\Gamma(\alpha-\beta+1)}\right]\left[C_1 + (C_2+C_3)TK\right] \\ &+ \left|\frac{\sigma_1}{1+\mu_1}\right|C_4T + \left|\frac{\sigma_2}{\mu_2}\right|\Gamma(2-\beta)T^{\beta+1}C_5 \\ &= \left(\frac{2}{3\sqrt{\pi}} + \frac{\sqrt{\pi}}{2}\right)\left[\frac{1}{10} + \left(\frac{1}{200}+\frac{1}{32}\right)\times 0.08\right] + \frac{1}{100} + \frac{\sqrt{\pi}}{200} \\ &\approx 0.15 < 1. \end{aligned} \tag{53}$$

Thus, by Theorem 5, the integro-differential boundary value problem (50) has at least one solution on [0, 1].

Data Availability

No data were used in the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Nos. 11871174, 11771104, and 11871171) and Scientific and Technological Innovation Programs of Higher Education Institutions in Shanxi (No. 2019L0964). The authors, therefore, acknowledge with thanks the NNSF and STIP for the financial support.

References

[1] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, USA, 1999.

[2] A. Kilbas, H. Srivastava, and J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006.

[3] D. Baleanu, K. Diethelm, E. Scalas, and J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific, Boston, MA, USA, 2012.

[4] K. Deimling, *Nonlinear Functional Analysis*, Springer, New York, NY, USA, 1988.

[5] S. Samko, A. Kilbas, and O. Marichev, *Fractional Integrals and Derivatives (Theory and Applications)*, Gordon and Breach Science Publishers, Montreux, Switzerland, 1993.

[6] R. Hilfer, *Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[7] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House, Redding, CT, USA, 2006.

[8] J. Sabatier and O. P. Agrawal, *Advance in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, The Netherlands, 2007.

[9] E. Hernández, D. O'Regan, and K. Balachandran, "On recent developments in the theory of abstract differential equations with fractional derivatives," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 10, pp. 3462–3471, 2010.

[10] X. Zhang, L. Liu, and Y. Wu, "Variational structure and multiple solutions for a fractional advection-dispersion equation," *Computers & Mathematics with Applications*, vol. 68, no. 12, pp. 1794–1805, 2014.

[11] T. Ren, H. Xiao, Z. Zhou, and X. Zhang, "The iterative scheme and the convergence analysis of unique solution for a singular fractional differential equation from the eco-economic complex systems Co-evolution process," *Complexity*, vol. 2019, Article ID 9278056, 15 pages, 2019.

[12] F. Jiao and Y. Zhou, "Existence of solutions for a class of fractional boundary value problems via critical point theory," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1181–1199, 2011.

[13] S. Al Mosa and P. Eloe, "Upper and lower solution method for boundary value problems at resonance," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 40, pp. 1–13, 2016.

[14] F. Jiao and Y. Zhou, "Existence results for fractional boundary value problem via critical point theory," *International Journal of Bifurcation and Chaos*, vol. 22, no. 4, 2012.

[15] T. Ren, S. Li, X. Zhang, and L. Liu, "Maximum and minimum solutions for a nonlocal p-Laplacian fractional differential system from eco-economical processes," *Boundary Value Problems*, vol. 2017, p. 118, 2017.

[16] Y. Xing and Y. Yan, "A higher order numerical method for time fractional partial differential equations with nonsmooth data," *Journal of Computational Physics*, vol. 357, pp. 305–323, 2018.

[17] Y. Yan, K. Pal, and N. J. Ford, "Higher order numerical methods for solving fractional differential equations," *BIT Numerical Mathematics*, vol. 54, no. 2, pp. 555–584, 2014.

[18] A. Khan, T. S. Khan, M. I. Syam, and H. Khan, "Analytical solutions of time-fractional wave equation by double Laplace transform method," *The European Physical Journal Plus*, vol. 134, no. 4, 2019.

[19] Y. Liu, J. Roberts, and Y. Yan, "Detailed error analysis for a fractional Adams method with graded meshes," *Numerical Algorithms*, vol. 78, no. 4, pp. 1195–1216, 2018.

[20] B. Ahmad and J. J. Nieto, "Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory,"

Topological Methods in Nonlinear Analysis, vol. 35, pp. 295–304, 2010.
[21] B. Ahmad and J. J. Nieto, "Anti-periodic fractional boundary value problems," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1150–1156, 2011.
[22] V. Daftardar-Gejji and S. Bhalekar, "Boundary value problems for multi-term fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 2, pp. 754–765, 2008.
[23] C. Guo, J. Guo, and Y. Gao, "Existence of positive solutions for two-point boundary value problems of nonlinear fractional q-difference equation," *Advances in Difference Equations*, vol. 2018, p. 180, 2018.
[24] B. Ahmad, J. J. Nieto, and A. Alsaedi, "Existence and uniqueness of solutions for nonlinear fractional differential equations with non-separated type integral boundary conditions," *Acta Mathematica Scientia*, vol. 31, no. 6, pp. 2122–2130, 2011.
[25] A. Cabada and Z. Hamdi, "Nonlinear fractional differential equations with integral boundary value conditions," *Applied Mathematics and Computation*, vol. 228, pp. 251–257, 2014.
[26] X. Zhang, L. Wang, and Q. Sun, "Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter," *Applied Mathematics and Computation*, vol. 226, pp. 708–718, 2014.
[27] Y. Cui and Y. Zou, "Existence of solutions for second-order integral boundary value problems," *Nonlinear Analysis: Modelling and Control*, vol. 21, no. 6, pp. 828–838, 2016.
[28] H. Feng and C. Zhai, "Existence and uniqueness of positive solutions for a class of fractional differential equation with integral boundary conditions," *Nonlinear Analysis: Modelling and Control*, vol. 22, no. 2, pp. 160–172, 2017.
[29] A. Khan, J. F. Gómez-Aguilar, T. Saeed Khan, and H. Khan, "Stability analysis and numerical solutions of fractional order HIV/AIDS model," *Chaos, Solitons & Fractals*, vol. 122, pp. 119–128, 2019.
[30] A. Khan, H. Khan, J. F. Gómez-Aguilar, and T. Abdeljawad, "Existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel," *Chaos, Solitons & Fractals*, vol. 127, pp. 422–427, 2019.
[31] J. Wu, X. Zhang, L. Liu, Y. Wu, and Y. Cui, "The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity," *Boundary Value Problems*, vol. 2018, p. 82, 2018.
[32] J. Wu, X. Zhang, L. Liu, Y. Wu, and Y. Cui, "Convergence analysis of iterative scheme and error estimation of positive solution for a fractional differential equation," *Mathematical Modelling and Analysis*, vol. 23, no. 4, pp. 611–626, 2018.
[33] L. Liu, H. Li, C. Liu, and Y. Wu, "Existence and uniqueness of positive solutions for singular fractional differential systems with coupled integral boundary conditions," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 1, pp. 243–262, 2017.
[34] H. Khan, F. Jarad, T. Abdeljawad, and A. Khan, "A singular ABC-fractional differential equation with p-Laplacian operator," *Chaos, Solitons & Fractals*, vol. 129, pp. 56–61, 2019.
[35] H. Khan, Y. Li, A. Khan, and A. Khan, "Existence of solution for a fractional-order Lotka-Volterra reaction-diffusion model with Mittag-Leffler kernel," *Mathematical Methods in the Applied Sciences*, vol. 42, no. 9, pp. 3377–3387, 2019.
[36] J. He, X. Zhang, L. Liu, Y. Wu, and Y. Cui, "A singular fractional Kelvin-Voigt model involving a nonlinear operator and their convergence properties," *Boundary Value Problems*, vol. 2019, p. 112, 2019.
[37] J. He, X. Zhang, L. Liu, Y. Wu, and Y. Cui, "Existence and asymptotic analysis of positive solutions for a singular fractional differential equation with nonlocal boundary conditions," *Boundary Value Problems*, vol. 2018, p. 189, 2018.
[38] X. Zhang, Y. Wu, and L. Caccetta, "Nonlocal fractional order differential equations with changing-sign singular perturbation," *Applied Mathematical Modelling*, vol. 39, no. 21, pp. 6543–6552, 2015.
[39] X. Zhang, L. Liu, and Y. Wu, "Multiple positive solutions of a singular fractional differential equation with negatively perturbed term," *Mathematical and Computer Modelling*, vol. 55, no. 3-4, pp. 1263–1274, 2012.
[40] D. Smart, *Fixed Point Theorems*, Cambridge University Press, London, UK, 1980.