**Research Article**

**On the Generalization of a Solution for a Class of Integro-Differential Equations with Nonseparated Integral Boundary Conditions**

Yanyuan Xing, Feng Jiao, and Fang Liu

*1School of Mathematics and Information Sciences, Guangzhou University, Guangzhou, Guangdong 510006, China
2Department of Mathematics, Luliang University, Luliang, Shanxi 033000, China*

Correspondence should be addressed to Yanyuan Xing; xingyanyuan@e.gzhu.edu.cn

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In this paper, the existence and uniqueness results of the generalization nonlinear fractional integro-differential equations with nonseparated type integral boundary conditions are investigated. A natural formula of solutions is derived and some new existence and uniqueness results are obtained under some conditions for this class of problems by using standard fixed point theorems and Leray–Schauder degree theory, which extend and supplement some known results. Some examples are discussed for the illustration of the main work.

1. Introduction

Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. Characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integral-order models. Fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, polymer rheology, aerodynamics, capacitor theory, chemistry, biology, control theory, and electrodynamics of complex medium. The initial and boundary value problems for nonlinear fractional differential equations arise from the study of models of viscoelasticity, electrochemistry, porous media, and electromagnetics. In consequence, the subject of fractional differential equations is gaining much importance and attention [1–4]. The recent development in the theory and methods for fractional differential equations indicates its popularity. For more details, we refer the reader to [5–9] and the references cited therein.

Moreover, the existence and uniqueness of solutions for fractional differential equations have been mathematically studied from different methods [10–15], yielding methods for solving fractional differential equations [16–19]. As we all know, boundary value problems of fractional differential equations have been investigated for many years. Now, there are many papers dealing with the problem for different kinds of boundary conditions such as periodic or antiperiodic boundary condition [20, 21], multipoint boundary condition [22, 23], and integral boundary condition [24–28] as well as stability and convergence analysis [29–32]. Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, and population dynamics. For a detailed description of some recent work on the integral boundary conditions, we refer the reader to some recent papers [33–35] and the references therein [36–39].

In [20], Ahmad and Nieto investigated the fractional differential equations with antiperiodic fractional boundary conditions as the following form:

\[
\begin{align*}
\frac{d^{\alpha}}{dt^{\alpha}} u(t) &= f(t, u(t)), & t \in [0, T], T > 0, & 1 < \alpha \leq 2, \\
-u(0) &= -u(T), & \frac{d^{\beta}}{dt^{\beta}} u(0) &= -\frac{d^{\beta}}{dt^{\beta}} u(T), & 0 < \beta < 1,
\end{align*}
\]

(1)
where \( CD^\alpha \) and \( CD^\beta \) denote the Caputo fractional derivative of order \( \alpha, \beta \); \( f: [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) is a given continuous function; and \( T \) is a fixed positive constant. The results are based on some standard fixed point principles.

Recently, in [24], the author discussed the nonlinear fractional differential equations with nonseparated type integral boundary conditions

\[
\begin{aligned}
  &CD^\alpha_t u(t) = f(t, u(t), (\psi u)(t)), \\
  &u(0) + \lambda_1 u(T) = \mu_1 \int_0^T g(s, u(s))ds, \\
  &u(0) + \lambda_2 u(T) = \mu_2 \int_0^T h(s, u(s))ds,
\end{aligned}
\]

where \( CD^\alpha_t \) denotes the Caputo fractional derivative of order \( \alpha \), \( f, g, h: [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) are given continuous functions, and \( \lambda_1, \lambda_2, \mu_1, \mu_2 \) are suitably chosen real constants with \( \lambda_1 \neq -1, \lambda_2 \neq -1 \). By applying the Leray–Schauder degree theory and some standard fixed point theorems, some new existence and uniqueness results are obtained.

Motivated by the abovementioned papers and many known results, in this paper, we concentrate on the existence and uniqueness of solutions for the nonlinear fractional integro-differential equations and inclusions of order \( \alpha \in (1, 2] \), with nonseparated type integral boundary conditions

\[
\begin{aligned}
  &CD^\alpha_t u(t) = f(t, u(t), (\psi u)(t), (\sigma u)(t)), \\
  &u(0) + \mu_1 u(T) = \sigma_1 \int_0^T g(s, u(s))ds, \\
  &CD^\beta_t u(0) + \mu_2 CD^\beta_t u(T) = \sigma_2 \int_0^T h(s, u(s))ds, \\
\end{aligned}
\]

where \( CD^\alpha_t \) and \( CD^\beta_t \) denote the Caputo fractional derivative of order \( \alpha, \beta \); \( f: [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) is a given continuous function satisfying some assumptions that will be specified later; \( \Gamma \) is the Euler gamma function; and \( \mu_1 \neq -1, \mu_2 \neq 0 \), \( \kappa, \xi: [0, T] \times [0, T] \rightarrow [0, \infty) \), \( \varphi, \psi \) are linear operators defined by

\[
\begin{aligned}
  &\varphi u(t) = t \int_0^t \kappa(t, s)u(s)ds, \\
  &\psi u(t) = t \int_0^t \xi(t, s)u(s)ds,
\end{aligned}
\]

\( g, h: [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \), \( \sigma_1, \sigma_2 \in \mathbb{R} \). Here, \( \mathcal{C} = C([0, T], \mathbb{R}) \) denotes the Banach space of all continuous functions from \([0, T]\) to \( \mathbb{R} \) endowed with a topology of uniform convergence with the norm \( \| u \| = \sup \{|u(t)|, t \in [0, T]\} \).

To the best of our knowledge, no paper has considered the generalization of nonlinear fractional integro-differential equations with nonseparated type integral boundary conditions (3). Our purpose here is to give some existence and uniqueness results for solution to (3).

Compared with the previous research problems, (3) has more general integral boundary value conditions. This paper is organized as follows: in Section 2, we present the notations and give some preliminary results via a sequence of definitions and lemmas. In Section 3, we prove new existence and uniqueness results for problem (3). These results are based on fixed point theorems and Leray–Schauder degree theory. In Section 4, two examples are demonstrated which support the theoretical analysis.

## 2. Preliminaries and Lemmas

In this section, we present some basic notations, definitions, and preliminary results which will be used throughout this paper. Let us recall some definitions of fractional calculus. For more details, see [1, 2].

**Definition 1.** The fractional integral of order \( \alpha \) with the lower limit zero for a function \( f: [0, \infty) \rightarrow \mathbb{R} \) is defined as

\[
I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s)ds, \quad t > 0, \quad n - 1 < \alpha < n,
\]

provided the integral exists.

**Definition 2.** For a function \( f: [0, \infty) \rightarrow \mathbb{R} \) with the lower limit zero, the Caputo derivative of fractional order \( \alpha \) is defined as

\[
CD^\alpha_t f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} f^{(n)}(s)ds, \quad t > 0, \quad n - 1 < \alpha < n,
\]
where \( n = [\alpha] + 1 \) and \([\alpha]\) denote the integer part of the real number \( \alpha \).

**Definition 3.** The Riemann–Liouville fractional derivative of order \( \alpha \) with the lower limit zero for a function \( f(t) \) is defined by

\[
^{\alpha}D_t f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s) \, ds,
\]

(7)

where \( n = [\alpha] + 1 \),

where \( n = [\alpha] + 1 \) and \([\alpha]\) denote the integer part of real number \( \alpha \), provided that the right side is pointwise defined on \((0, \infty)\).

\[
\begin{aligned}
^{\alpha}D_t u(t) &= y(t), \\
u(0) + \mu_1 u(T) &= \sigma_1 \int_0^T g(s, u(s)) \, ds, \\
^{\beta}D_t u(0) + \mu_2 ^{\beta}D_t u(T) &= \sigma_2 \int_0^T h(s, u(s)) \, ds,
\end{aligned}
\]

(10)

if and only if \( u \) is a solution of the integral equation

\[
u(t) = \int_0^t \frac{(t-s)^{n-1}}{\Gamma(n)} y(s) \, ds - \frac{\mu_1}{1 + \mu_1} \int_0^T \frac{(T-s)^{n-1}}{\Gamma(n)} y(s) \, ds
\]

+ \( \frac{\Gamma(2-\beta)}{(1 + \mu_1)^{1-\beta}} \frac{\mu_1 T - (1 + \mu_1) t}{1 + \mu_1} \int_0^T \frac{(T-s)^{n-1}}{\Gamma(n)} y(s) \, ds
\]

+ \( \frac{\sigma_1}{1 + \mu_1} \int_0^T g(s, u(s)) \, ds - \frac{\sigma_1 \Gamma(2-\beta)}{\mu_2 (1 + \mu_1)} T^{1-\beta}
\]

\[
\cdot \left[ \frac{\mu_1 T - (1 + \mu_1) T}{1 + \mu_1} \right] \int_0^T h(s, u(s)) \, ds.
\]

(11)

**Lemma 1.** For \( \alpha > 0 \), the general solution of the fractional differential equation \( ^{\alpha}D_t^\alpha u(t) = 0 \) is given by

\[
u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},
\]

(8)

where \( c_i \in \mathbb{R}, i = 0, 1, \ldots, n-1 \) \((n = [\alpha] + 1)\).

In view of Lemma 1, it follows that

\[
I^\alpha u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},
\]

(9)

for some \( c_i \in \mathbb{R}, i = 0, 1, \ldots, n-1 \) \((n = [\alpha] + 1)\).

In the following, we derive a natural formula of solution to the integral boundary value problem for integro-differential equation (3).

**Lemma 2.** Assume that \( y \in C([0, T], \mathbb{R}), T > 0, 1 < \alpha \leq 2 \). A function \( u(t) \) is a solution of the boundary value problem

\[
\begin{aligned}
u(0) + \mu_1 u(T) &= \sigma_1 \int_0^T g(s, u(s)) \, ds, \\
^{\beta}D_t u(0) + \mu_2 ^{\beta}D_t u(T) &= \sigma_2 \int_0^T h(s, u(s)) \, ds,
\end{aligned}
\]

where \( \mu_1, \mu_2 \) are constants, \( \sigma_1, \sigma_2, \Gamma(0) \) are positive constants.

**Proof.** Assume that \( y \) satisfies (10). Using Lemma 1, for some constants \( c_0, c_1 \in \mathbb{R} \), we have

\[
u(t) = I^\alpha y(t) - c_0 - c_1 t
\]

(12)

Using the facts that \( ^{\alpha}D_t^\beta c = 0 \) \((c \) is a constant), \( ^{\alpha}D_t^\beta t = \Gamma(1-\beta) \frac{t^{1-\beta}}{\Gamma(1-\beta)} \), and \( ^{\alpha}D_t^\beta y(t) = \Gamma(1-\beta) y(t) \), we get

\[
^{\beta}D_t^\beta u(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{1-\beta} y(s) \, ds - c_1 t
\]

(13)

Applying the boundary conditions for problem (3), we find that

\[
\begin{aligned}
c_0 &= \frac{\mu_1}{1 + \mu_1} \int_0^T \frac{(T-s)^{n-1}}{\Gamma(n)} y(s) \, ds - \frac{\mu_1 T \Gamma(2-\beta)}{1 + \mu_1} \int_0^T \frac{(T-s)^{n-1}}{\Gamma(n)} y(s) \, ds
\]

\[- \frac{\sigma_1}{1 + \mu_1} \int_0^T g(s, u(s)) \, ds + \frac{\mu_1 \sigma_1 \Gamma(2-\beta)}{\mu_2 (1 + \mu_1)} T^{1-\beta}
\]

\[
\cdot \left[ \frac{\mu_1 T - (1 + \mu_1) T}{1 + \mu_1} \right] \int_0^T h(s, u(s)) \, ds,
\]

(14)

\[
c_1 = \frac{\Gamma(2-\beta)}{T^{1-\beta}} \left( \frac{1}{1 + \mu_1} \int_0^T (T-s)^{n-1} y(s) \, ds - \frac{\sigma_2}{\mu_2} T^{1-\beta} \right) \int_0^T h(s, u(s)) \, ds.
\]
Substituting the value of $c_0$ and $c_1$ in (12), we obtain the unique solution of (10) which is given by

$$u(t) = \int_0^t \frac{(t-s)^{P-1}}{\Gamma(\alpha)} y(s)ds - \frac{\mu_1}{1 + \mu_1} \int_0^T \frac{(T-s)^{P-1}}{\Gamma(\alpha)} y(s)ds + \frac{\Gamma(2-\beta)}{(1 + \mu_1)T^{1-\beta}} \int_0^T (T-s)^{\beta-1} y(s)ds + \frac{\sigma_1}{1 + \mu_1} \int_0^T g(s, u(s))ds - \frac{\sigma_2}{\mu_2(1 + \mu_1)} \int_0^T h(s, u(s))ds.$$

(15)

Conversely, we assume that $u$ is a solution of the integral equation (11), and in view of the relations $C D_0^\beta D_1^\alpha y(t) = y(t)$, for $\alpha > 0$, we get

$$(\Phi u)(t) = \int_0^t \frac{(t-s)^{P-1}}{\Gamma(\alpha)} f(s, u(s), (\psi u)(s), (\psi u)(s))ds - \frac{\mu_1}{1 + \mu_1} \int_0^T \frac{(T-s)^{P-1}}{\Gamma(\alpha)} f(s, u(s), (\psi u)(s), (\psi u)(s))ds + \frac{\Gamma(2-\beta)}{(1 + \mu_1)T^{1-\beta}} \int_0^T (T-s)^{\beta-1} f(s, u(s), (\psi u)(s), (\psi u)(s))ds + \frac{\sigma_1}{1 + \mu_1} \int_0^T g(s, u(s))ds - \frac{\sigma_2}{\mu_2(1 + \mu_1)} \int_0^T h(s, u(s))ds.$$

(19)

3. Main Results

In this section, we will show the existence and uniqueness of solutions for problem (3). Now we state some known fixed point theorems which are needed to prove the existence of solutions for equation (3).

**Theorem 1.** Let $X$ be a Banach space. Assume that $\Phi: X \rightarrow X$ is a completely continuous operator and the set $V = \{u \in X | u = \mu u, 0 < \mu < 1\}$ is bounded. Then, $\Phi$ has a fixed point in $X$.

**Theorem 2.** Let $X$ be a Banach space. Assume that $\Omega$ is an open bounded subset of $X$ with $0 \in \Omega$ and let $\Phi: \Omega \rightarrow X$ be a completely continuous operator such that

$$\|\Phi u\| \leq \|u\|, \quad \forall u \in \partial \Omega.$$

Then $\Phi$ has a fixed point in $\Omega$.

**Theorem 3.** Suppose that $f: [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a jointly continuous function and maps bounded subsets of $[0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ into relative compact subsets of $\mathbb{R}$, $\kappa, \xi: [0, T] \times [0, T] \rightarrow [0, \infty)$ is continuous with

$$C D_1^\alpha u(t) = y(t), \quad t \in [0, T], \quad 1 < \alpha \leq 2.$$

Moreover, it can easily be verified that the boundary conditions

$$u(0) + \mu_1 u(T) = \sigma_1 \int_0^T g(s, u(s))ds,$$

$$C D_1^\beta u(0) + \mu_2 C D_1^\alpha u(T) = \sigma_2 \int_0^T h(s, u(s))ds,$$

are satisfied. The proof is completed.

By Lemma 2, problem (3) is reduced to the fixed point problem

$$u = \Phi(u),$$

where $\Phi: \mathbb{C} \rightarrow \mathbb{C}$ is given by

$$k_0 = \max\{\|\kappa(t, s)\| : (t, s) \in [0, T] \times [0, T]\},$$

$$\xi_0 = \max\{\|\xi(t, s)\| : (t, s) \in [0, T] \times [0, T]\},$$

$$K = \max\{k_0, \xi_0\},$$

and $g, h: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. Furthermore, there exist positive constants $C_i (i = 1, \ldots, 5)$ such that

$$(H_1) \quad |f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)| \leq C_1|u_1 - v_1| + C_2|u_2 - v_2| + C_3|u_3 - v_3|, \quad \forall t \in [0, T], \quad u_1, v_1, u_2, v_2, u_3, v_3 \in \mathbb{R},$$

$$(H_2) \quad |g(t, u) - g(t, v)| \leq C_4|u - v|, \quad |h(t, u) - h(t, v)| \leq C_5|u - v|, \quad \forall u, v \in \mathbb{R}.$$

Then the boundary value problem (3) has a unique solution provided

$$r_1 = \left[1 + \frac{\mu_1}{1 + \mu_1}\right] \frac{T^\alpha}{\Gamma(\alpha + 1)} + \frac{\Gamma(2-\beta)T^\alpha}{\Gamma(\alpha + 1 - \beta)}$$

$$\cdot \left[C_1 + (C_2 + C_3)T\kappa + \frac{\sigma_1}{\mu_1}C_4 T + \frac{\sigma_2}{\mu_2}(2-\beta)T^{\beta-1}C_5 < 1.\right]$$

(23)
Proof. Setting \( \sup_{t \in [0, T]}|f(t, 0, 0, 0)| = M_1 \), \( \sup_{t \in [0, T]}|g(t, 0)| = M_2 \), and \( \sup_{t \in [0, T]}|h(t, 0)| = M_3 \). For a positive number \( r \), let \( B_r = \{ u \in \mathcal{C} : \| u \| \leq r \} \) and \( r \geq r_2/(1 - r_1) \), with \( r_1 \) is given by (23), we will show that \( \Phi B_r \subset B_{r_n} \) where \( \Phi \) is defined by (19), and

\[
\begin{align*}
  r_2 &= \left[ 1 + \frac{\mu_1}{1 + \mu_1} \right] \frac{T^\alpha}{\Gamma(\alpha + 1)} + \frac{\Gamma(2 - \beta)T^n}{\Gamma(\alpha - \beta + 1)} M_1 + \frac{\sigma_1}{1 + \mu_1} M_2 T + \frac{\sigma_2}{\mu_2} \Gamma(2 - \beta)T^{\beta + 1} M_3. 
\end{align*}
\]

First, \( \forall u(t) \in B_{r_n} \), there exists \( \{ u_n \} \subset B_{r_n} \) and when \( n \to \infty, u_n \to u \), it is easy to know that

\[
\left\| \Phi u_n(t) - \Phi u(t) \right\| \to 0. 
\]

Then \( \Phi \) is continuous on \( B_r \).

Furthermore, for \( u \in B_r, t \in [0, T] \), we have

\[
|\Phi u(t)| \leq \int_0^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,u(s),(\varphi u)(s),(\psi u)(s)) \right| ds 
+ \left| \frac{\mu_1}{1 + \mu_1} \right| \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,u(s),(\varphi u)(s),(\psi u)(s)) \right| ds 
+ \frac{\Gamma(2 - \beta)T^\beta}{\Gamma(\alpha - \beta)} \int_0^T \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha - \beta)} \left| f(s-u(s),(\varphi u)(s),(\psi u)(s)) \right| ds 
+ \frac{\sigma_1}{1 + \mu_1} \int_0^T |g(s,u(s))| ds + \frac{\sigma_2}{\mu_2} \Gamma(2 - \beta)T^\beta \int_0^T |h((s,u(s))| ds 
\leq \int_0^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,u(s),(\varphi u)(s),(\psi u)(s)) - f(s,0,0,0) \right| + \left| f(s,0,0,0) \right| ds 
+ \left| \frac{\mu_1}{1 + \mu_1} \right| \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s,u(s),(\varphi u)(s),(\psi u)(s)) - f(s,0,0,0) \right| + \left| f(s,0,0,0) \right| ds 
+ \frac{\Gamma(2 - \beta)T^\beta}{\Gamma(\alpha - \beta)} \int_0^T \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha - \beta)} \left| f(s,u(s),(\varphi u)(s),(\psi u)(s)) - f(s,0,0,0) \right| + \left| f(s,0,0,0) \right| ds 
+ \frac{\sigma_1}{1 + \mu_1} \int_0^T |g(s,u(s)) - g(s,0)| + |g(s,0)| ds 
+ \frac{\sigma_2}{\mu_2} \Gamma(2 - \beta)T^\beta \int_0^T |h(s,u(s)) - h(s,0)| + |h(s,0)| ds 
\leq \left[ 1 + \frac{\mu_1}{1 + \mu_1} \right] \frac{T^\alpha}{\Gamma(\alpha + 1)} + \frac{\Gamma(2 - \beta)T^n}{\Gamma(\alpha - \beta + 1)} \left\{ [C_1 + (C_2 + C_3)TK]r + M_1 \right\} 
+ \frac{\sigma_1}{1 + \mu_1} T(C_1 r + M_2) + \frac{\sigma_2}{\mu_2} \Gamma(2 - \beta)T^\beta (C_2 r + M_3) 
\leq \left[ 1 + \frac{\mu_1}{1 + \mu_1} \right] \frac{T^\alpha}{\Gamma(\alpha + 1)} + \frac{\Gamma(2 - \beta)T^n}{\Gamma(\alpha - \beta + 1)} \left\{ [C_1 + (C_2 + C_3)TK] + \frac{\sigma_1}{1 + \mu_1} C_1 T + \frac{\sigma_2}{\mu_2} \Gamma(2 - \beta)T^{\beta + 1} C_3 \right\} r 
+ \left[ 1 + \frac{\mu_1}{1 + \mu_1} \right] \frac{T^\alpha}{\Gamma(\alpha + 1)} + \frac{\Gamma(2 - \beta)T^n}{\Gamma(\alpha - \beta + 1)} M_1 + \frac{\sigma_1}{1 + \mu_1} M_2 T 
+ \frac{\sigma_2}{\mu_2} \Gamma(2 - \beta)T^{\beta + 1} M_3, 
\]
Now, for \( u, v \in \mathcal{C} \) and for each \( t \in [0, T] \), we obtain

\[
[(\Phi u)(t) - (\Phi v)(t)]
\]
\[
\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s), (\psi u)(s), (\psi v)(s)) - f(s,v(s), (\psi v)(s), (\psi v)(s))| ds
\]
\[
+ \frac{\mu_1}{1 + \mu_1} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s), (\psi u)(s), (\psi u)(s)) - f(s,v(s), (\psi v)(s), (\psi v)(s))| ds
\]
\[
+ \Gamma (2 - \beta)^\vartheta \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha - \beta)} |f(s,u(s), (\psi u)(s), (\psi u)(s)) - f(s,v(s), (\psi v)(s), (\psi v)(s))| ds
\]
\[
+ \frac{\sigma_1}{1 + \mu_1} \int_0^T |g(s,u(s)) - g(s,v(s))| ds + \frac{\sigma_2}{\mu_2} (2 - \beta)^\vartheta \int_0^T |h(s,u(s)) - h(s,v(s))| ds
\]
\[
\leq \bigg[ \left( 1 + \frac{\mu_1}{1 + \mu_1} \right) \frac{T^\alpha}{\Gamma(\alpha + 1)} + \frac{\Gamma (2 - \beta)^\vartheta}{\Gamma(\alpha - \beta + 1)} [C_1 + (C_2 + C_3)TK] + \frac{\sigma_1}{1 + \mu_1} C_4 T + \frac{\sigma_2}{\mu_2} (2 - \beta)^\vartheta C_5 \bigg] \|u - v\|
\]
\[
\leq r_1 \|u - v\|.
\]
For \( u, v \in B_R \), we find that
\[
\|\Phi_1 u + \Phi_2 v\| \leq \left( 1 + \frac{\mu_1}{1 + \mu_1} \right) \frac{1}{\Gamma(\alpha + 1)} + \frac{\Gamma(2 - \beta)}{\Gamma(\alpha - \beta + 1)} T^\alpha \| p \|
\]
\[+ \left| \frac{\sigma_1}{1 + \mu_1} \right| T\| q \| + \left| \frac{\sigma_2}{\mu_1} \right| (2 - \beta) T^{\beta + 1} \| y \|
\]
\[\leq \mathcal{R}.
\]
(30)

Moreover, the continuity of \( f \) implies that the operator \( \Phi_1 \) is continuous. Also, \( \Phi_1 \) is uniformly bounded on \( B_R \) as
\[
\|\Phi_1\| \leq \frac{\|p\| T^\alpha}{\Gamma(\alpha + 1)}.
\]
(32)

Thus, \( \Phi_1 u + \Phi_2 v \in B_R \). It follows from the assumptions \((H_1)\) and \((H_2)\) that \( \Phi_2 \) is a contraction mapping if
\[
\sup_{(t,u) \in [0,T] \times B_R} | f(t, u, \varphi u, \psi u) | = f_{\text{max}},
\]
and consequently, we have
\[
\left\| (\Phi_1 u)(t_2) - (\Phi_1 u)(t_1) \right\| = \left\| \int_0^{t_1} \frac{(t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1}}{\Gamma(\alpha)} f(s, u(s), (\varphi u)(s), (\psi u)(s)) ds \right.
\]
\[+ \left. \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha - 1}}{\Gamma(\alpha)} f(s, u(s), (\varphi u)(s), (\psi u)(s)) ds \right\|
\]
\[\leq \frac{f_{\text{max}}}{\Gamma(\alpha + 1)} [2(t_2 - t_1)^\alpha + t_1^{\alpha} - t_2^{\alpha}]
\]

which is independent of \( u \) and tends to zero as \( t_2 - t_1 \to 0 \). So \( \Phi_1 \) is relatively compact on \( B_R \). Hence, by the Arzelá–Ascoli theorem, \( \Phi_1 \) is compact on \( B_R \). Thus, all the assumptions of Theorem 4 are satisfied. Therefore, the conclusion of Theorem 4 applies that the fractional boundary value problem (3) has at least one solution on \([0, T]\). This completes the proof.

As an immediate consequence of Theorem 5, we have the following.

**Corollary 1.** Assume that \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a jointly continuous function. If there exists nonnegative functions, \( a_i(t) \in L[0, T][i = 0, 1, 2, 3] \), \( b_i(t) \), \( c_i(t) \in C[0, T][i = 0, 1] \), \( 0 < \rho_j < 1(j = 1, 2, 3), 0 < \theta_j, \theta < 1 \), and \( \kappa_0, \xi_0 \) are given by (21) and (22) such that
\[
(H_6) | f(t, u, \varphi u, \psi u) | \leq a_0(t) + a_1(t)|u|^{\rho_1} + a_2(t)|\varphi u|^{\rho_1} + a_3(t)|\psi u|^{\rho_1}
\]
\[
(H_5) | g(t, u) | \leq b_0(t) + b_1(t)|u|^{\theta_1}, |h(t, u)| \leq c_0(t) + c_1(t)|u|^{\theta_1}.
\]

For all \( t \in [0, T], u, \varphi u, \psi u \in \mathbb{R} \), then the boundary value problem (3) has at least one solution.

**Proof.** Let us define a ball in the Banach space \( B = \{ u \in \mathbb{C} \mid \| u \| \leq \Lambda \} \), where \( \Lambda \) is fixed later. Setting
\[ I^a_n = \max_{t \in [0, T]} \left\{ |I^a_n a_i(t)|, \quad i = 0, 1, 2, 3 \right\}, \]
\[ I^a_n(T) = \max_{t \in [0, T]} \left\{ |I^a_n a_i(T)|, \quad i = 0, 1, 2, 3 \right\}, \]
\[ \rho = \max_{i=1,2,3} \rho_i, \]
\[ b_m = \max_{i=1,2} b_i(t), \]
\[ c_m = \max_{i=1,2} c_i(t), \]
\[ \| \Phi u(t) \| \leq \int_0^T (t-s)^{\alpha-1} \left( a_0(s) + a_1(s)|u|^{\rho_1} + a_2(s)|\psi u|^{\rho_2} + a_3(s)|\psi u|^{\rho_3} \right) ds \]
\[ + \left| \frac{\mu_1}{1 + \mu_1} \right| \int_0^T (T-s)^{\alpha-1} \left( a_0(s) + a_1(s)|u|^{\rho_1} + a_2(s)|\psi u|^{\rho_2} + a_3(s)|\psi u|^{\rho_3} \right) ds \]
\[ + \Gamma(2-\beta)T^{\beta} \int_0^T (T-s)^{\alpha-1} \left( a_0(s) + a_1(s)|u|^{\rho_1} + a_2(s)|\psi u|^{\rho_2} + a_3(s)|\psi u|^{\rho_3} \right) ds \]
\[ + \left| \frac{\sigma_1}{1 + \mu_1} \right| T \left( b_0(s) + b_1(s)|u|^{\theta_1} + \left| \frac{\sigma_2}{\mu_2} \right| \Gamma(2-\beta)T^{\beta+1} c_0(s) + c_1(s)|u|^{\theta_1} \right) ds \]
\[ \leq I^a_n a_0(t) + I^a_n a_1(t)|\Lambda^{\rho_1} + I^a_n a_2(t)|T^{\beta_0} k_0^{\beta_0} \Lambda^{\rho_2} + I^a_n a_3(t)|T^{\beta_0} \xi_0^{\beta_0} \Lambda^{\rho_3} \]
\[ + \left| \frac{\mu_1}{1 + \mu_1} \right| \left( I^a_n a_0(T) + I^a_n a_1(T)|\Lambda^{\rho_1} + I^a_n a_2(T)|T^{\beta_0} k_0^{\beta_0} \Lambda^{\rho_2} + I^a_n a_3(T)|T^{\beta_0} \xi_0^{\beta_0} \Lambda^{\rho_3} \right) \]
\[ + \Gamma(2-\beta)T^{\beta} \left( I^a_n a_0(T) + I^a_n a_1(T)|\Lambda^{\rho_1} + I^a_n a_2(T)|T^{\beta_0} k_0^{\beta_0} \Lambda^{\rho_2} + I^a_n a_3(T)|T^{\beta_0} \xi_0^{\beta_0} \Lambda^{\rho_3} \right) \]
\[ + \left| \frac{\sigma_1}{1 + \mu_1} \right| \left( T b_0(1 + \Lambda^{\beta_0}) + \left| \frac{\sigma_2}{\mu_2} \right| \Gamma(2-\beta)T^{\beta+1} c_0(1 + \Lambda^{\beta_0}) \right) \]
\[ \leq I^a_n (1 + T^{\beta_0} k_0^{\beta_0} + T^{\beta_0} \xi_0^{\beta_0}) \Lambda^{\rho_1} + T^{\beta_0} a_0 \]
\[ + \left| \frac{\mu_1}{1 + \mu_1} \right| \left[ I^a_n \left( T^{\beta_0} k_0^{\beta_0} + T^{\beta_0} \xi_0^{\beta_0} \right) \Lambda^{\rho_1} + T^{\beta_0} a_0 \right] \]
\[ + \Gamma(2-\beta)T^{\beta} \left[ I^a_n \left( T^{\beta_0} k_0^{\beta_0} + T^{\beta_0} \xi_0^{\beta_0} \right) \Lambda^{\rho_1} + T^{\beta_0} a_0 \right] \]
\[ + \left| \frac{\sigma_1}{1 + \mu_1} \right| \left( T b_0(1 + \Lambda^{\beta_0}) + \left| \frac{\sigma_2}{\mu_2} \right| \Gamma(2-\beta)T^{\beta+1} c_0(1 + \Lambda^{\beta_0}) \right) \]
\[ = I^a_n a_0 + T^{\beta_0} a_0(T) + \Gamma(2-\beta)T^{\beta} I^{\alpha-\beta} a_0(T) + \left| \frac{\sigma_1}{1 + \mu_1} \right| \left( T b_0(1 + \Lambda^{\beta_0}) + \left| \frac{\sigma_2}{\mu_2} \right| \Gamma(2-\beta)T^{\beta+1} c_0(1 + \Lambda^{\beta_0}) \right) \]
\[ + \left| \frac{\sigma_2}{\mu_2} \right| \Gamma(2-\beta)T^{\beta+1} c_0(1 + \Lambda^{\beta_0}) + \left( 1 + T^{\beta_0} k_0^{\beta_0} + T^{\beta_0} \xi_0^{\beta_0} \right) \left[ \Lambda^{\rho_1} I^a_n + \left| \frac{\mu_1}{1 + \mu_1} \right| I^a_n(T) + \Gamma(2-\beta)T^{\beta} T^{\alpha-\beta}(T) \right]. \]
Choosing $\Lambda$ sufficient large, then $\Phi : B \rightarrow B$. On the other hand, the continuity of $f$ implies that the operator $\Phi$ is continuous. Also, since $\Phi : B \rightarrow B$, we have $\Phi(B)$ which is uniformly bounded on $B$.

Let

$$
\| (\Phi u)(t_2) - (\Phi u)(t_1) \| = \int_0^{t_1} \frac{(t_2 - s)^{n-1} - (t_1 - s)^{n-1}}{\Gamma(a)} f(s,u(s),(\psi u)(s),(\psi u)(s)) ds \\
+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{n-1}}{\Gamma(a)} f(s,u(s),(\psi u)(s),(\psi u)(s)) ds \\
+ \Gamma(2-\beta)\Phi_{\alpha-1}((t_1-t_2) \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(a-\beta)} f(s,u(s),(\psi u)(s),(\psi u)(s)) ds \\
- \sigma_2 \Gamma(2-\beta)(t_1-t_2) \Phi_{\alpha-1}(1+\|u\|) \\
\leq \frac{M}{\Gamma(a+1)} [2(t_2-t_1)^a + t_1^a - t_2^a] + \frac{M}{\Gamma(a-\beta+1)} (t_1-t_2) \\
+ \frac{\sigma_2}{\mu_2} \Gamma(2-\beta) \Phi_{\alpha-1}(1+\|u\|)(t_1-t_2)
$$

(37)

It follows that $\Phi$ is equicontinuous, so $\Phi(B)$ is relatively compact on $B$. Hence, $\Phi(B)$ is relatively compact on $B$ by Arzelà–Ascoli theorem. Thus, by Schauder fixed-point theorem, problem (3) has at least one solution.

**Theorem 6.** Assume that there exist positive constants $p_i, q_i (i = 1, 2, 3)$ such that $|f(t,u,\psi u,\psi u)| \leq (p_1/T^a)|u| + q_1$, $|g(t,u)| \leq (p_2/T)|u| + q_2$, and $|h(t,u)| \leq (p_3/T^{a+1})|u| + q_3$ for all $t \in [0, T], u \in C[0,T]$. If

$$
\left[ 1 + \frac{\mu_1}{1 + \mu_1} \right] \frac{\Gamma(2-\beta)}{\Gamma(a + 1)} P_1 + \left[ \frac{\alpha_1}{1 + \mu_1} \right] P_2 + \left[ \frac{\alpha_2}{\mu_2} \right] (2-\beta) \Phi_{\alpha-1} P_3 < 1,
$$

(38)

then the boundary value problem (3) has at least one solution.

**Proof.** In view of the fixed point problem (19), we just need to prove the existence of at least one solution $u \in \mathbb{R}$ satisfying (19). Define a suitable ball $B_S \in C$ with radius $S > 0$ as

$$
M = \max_{(t,u,\psi u,\psi u) \in [0,T] \times B_S \times B_S} |f(t,u,\psi u)|,
$$

and consequently, we obtain

$$
B_S = \left\{ u \in C : \max_{t \in [0,T]} |u(t)| < S \right\},
$$

(39)

where $S$ will be fixed later. Then, it is sufficient to show that $\mathcal{F} : B_S \rightarrow C$ satisfies

$$
u \neq \lambda \mathcal{F}u, \quad \forall u \in \partial B_S, \forall \lambda \in [0, 1].
$$

(40)

Then, by the Arzelà–Ascoli theorem, $h_1(u) = u - H(\lambda, u) = u - \lambda \mathcal{F}u$ is completely continuous. If (40) is true, then the following Leray–Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$
\deg(h_1, B_S, 0) = \deg(I - \lambda \mathcal{F}, B_S, 0) = \deg(h_1, B_S, 0)
$$

$$
= \deg(h_1, B_S, 0) = \deg(I, B_S, 0) = 1 \neq 0, \quad 0 \in B_S,
$$

(41)

where $I$ denotes the unit operator. By the nonzero property of Leray–Schauder degree, we have $h_1(t) = u - \mathcal{F}u = 0$ for at least one $u \in B_S$. In order to prove (40), we assume that $u = \lambda \mathcal{F}u$ for some $\lambda \in [0, 1]$ and for all $t \in [0, T]$ so that
\[ |u(t)| = |\lambda \mathcal{F} u(t)| \]

\[ \leq \int_0^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), (qu)(s), (yu)(s)) ds \]

\[ + \left| \frac{\mu_1}{1 + \mu_1} \right| \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), (qu)(s), (yu)(s)) ds \]

\[ + \Gamma (2-\beta) T^\beta \int_0^T \frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} f(s, u(s), (qu)(s), (yu)(s)) ds \]

\[ + \left| \frac{\sigma_1}{1 + \mu_1} \right| \int_0^T |g(s, u(s))| ds \]

\[ + \frac{\sigma_2}{\mu_2} \Gamma (2-\beta) T^\beta \int_0^T |h(s, u(s))| ds \]

\[ \leq \left[ 1 + \left| \frac{\mu_1}{1 + \mu_1} \right| \int_0^T T^{\alpha} \frac{(2-\beta) T^{\alpha}}{\Gamma(\alpha+1)} \left( p_1 \right) \frac{T^{\beta} |u|}{T^{\beta+1}} + q_1 \right] \]

\[ + \left[ 1 + \left| \frac{\mu_1}{1 + \mu_1} \right| \int_0^T T^{\alpha} \frac{(2-\beta) T^{\alpha}}{\Gamma(\alpha+1)} \right] \left( p_2 \right) T^{\beta} |u| + q_2 \]

\[ \leq \left[ \left( 1 + \left| \frac{\mu_1}{1 + \mu_1} \right| \int_0^T T^{\alpha} \frac{(2-\beta) T^{\alpha}}{\Gamma(\alpha+1)} \right) p_1 + \left| \frac{\sigma_1}{1 + \mu_1} \right| p_2 + \frac{\sigma_2}{\mu_2} \Gamma (2-\beta) p_3 \right] |u| \]

\[ + \left( 1 + \left| \frac{\mu_1}{1 + \mu_1} \right| \int_0^T T^{\alpha} \frac{(2-\beta) T^{\alpha}}{\Gamma(\alpha+1)} \right) q_1 + \left| \frac{\sigma_1}{1 + \mu_1} \right| T q_2 + \frac{\sigma_2}{\mu_2} \Gamma (2-\beta) T^{\beta+1} q_3 \]

\[ = h_1 |u| + h_2, \]

which by taking norm \( \|u\| = \sup \{|u(t)|, t \in [0, T]\} \) and solving for \( \|u\| \), we have

\[ \|u\| \leq \frac{h_2}{1 - h_1}, \]  \hspace{1cm} (43)

\[ h_1 = \left[ \left( 1 + \left| \frac{\mu_1}{1 + \mu_1} \right| \int_0^T T^{\alpha} \frac{(2-\beta) T^{\alpha}}{\Gamma(\alpha+1)} \right) p_1 + \left| \frac{\sigma_1}{1 + \mu_1} \right| p_2 + \frac{\sigma_2}{\mu_2} \Gamma (2-\beta) p_3, \] \hspace{1cm} (44)

\[ h_2 = \left[ \left( 1 + \left| \frac{\mu_1}{1 + \mu_1} \right| \int_0^T T^{\alpha} \frac{(2-\beta) T^{\alpha}}{\Gamma(\alpha+1)} \right) q_1 + \left| \frac{\sigma_1}{1 + \mu_1} \right| T q_2 + \frac{\sigma_2}{\mu_2} \Gamma (2-\beta) T^{\beta+1} q_3, \]

Setting

\[ S = \frac{h_2}{1 - h_1} + 1, \] \hspace{1cm} (45)

it follows that (40) holds. This completes the proof.

\section{4. Examples}

\textit{Example 1.} Consider the following nonlinear fractional integro-differential equation with nonseparated type integral boundary conditions of \( \alpha = 3/2, \beta = 1/2, \) and \( T = 1: \)
\[
\begin{align*}
\mathcal{C}D_t^{3/2} u(t) &= \frac{1}{(t + 10)^2} \left[ \frac{|u(t)|}{1 + |u(t)|} + \cos^2 t \right] + \frac{1}{120} \int_0^t e^{-t} u(s) ds + \frac{1}{140} \int_0^t \frac{1}{(4 + t)^2} u(s) ds, \\
\mathcal{C}D_t^{1/2} u(0) + \mathcal{C}D_t^{1/2} u(1) &= \int_0^1 \left( \frac{1}{t + 10} \right)^2 \frac{|u(s)|}{1 + |u(s)|} ds.
\end{align*}
\]

Here, \( \mu_1 = \mu_2 = \sigma_1 = \sigma_2 = 1 \) and
\[
(\varphi u)(t) = \int_0^t e^{-t} u(s) ds,
\]
\[
(\psi u)(t) = \int_0^1 \frac{1}{(t + 10)}^2 u(s) ds,
\]
\[
g(s, u(s)) = \frac{|u(s)|}{50 + |u(s)|},
\]
\[
h(s, u(s)) = \left( \frac{1}{t + 10} \right)^2 \frac{|u(s)|}{1 + |u(s)|}.
\]

For \( u, v \in \mathbb{R} \) and \( t \in [0, 1] \), we have
\[
|f(t, u, \varphi u, \psi u) - f(t, v, \varphi v, \psi v)| \leq \left( \frac{1}{t + 10} \right)^2 |u - v| + \frac{1}{120} |\varphi u - \varphi v| + \frac{1}{140} |\psi u - \psi v|,
\]
\[
|g(s, u) - g(s, v)| \leq \frac{1}{50} |u - v|,
\]
\[
|h(s, u) - h(s, v)| \leq \frac{1}{100} |u - v|.
\]

As \( C_1 = C_2 = C_3 = 1/100 \) and \( C_4 = 1/50, C_5 = 1/100, K = 1/9 \), we obtain
\[
r_1 = \left[ \left( 1 + \frac{\mu_1}{1 + \mu_1} \right) \frac{T^w}{\Gamma(\alpha + 1)} + \frac{\Gamma(2 - \beta)T^{\alpha+1}C_5}{\Gamma(\alpha - \beta + 1)} \right] + \left[ C_1 + (C_2 + C_3)TK \right]
\]
\[
= \left( \frac{2}{\sqrt{\pi} + \sqrt{\pi}} \right) \left( \frac{1}{100} + \frac{1}{450} \right) + \frac{1}{100} + \frac{\sqrt{\pi}}{200}
\]
\[
= 0.043 < 1.
\]

Thus, all the assumptions of Theorem 3 hold. Consequently, the conclusion of Theorem 3 implies that problem (46) has a unique solution.

**Example 2.** Consider the following integro-differential fractional boundary value problem
\[
\begin{align*}
\mathcal{C}D_t^{3/2} u(t) &= \frac{e^{-t}}{10} \left[ \frac{|u(t)|}{1 + |u(t)|} + \frac{|u(t)|}{200} \right] + \int_0^t \frac{e^{-(t-s)}}{32} \frac{|u(t)|}{1 + |u(t)|} ds + \int_0^1 \frac{e^{-t} (s + 1)}{32} \frac{|u(t)|}{1 + |u(t)|} ds, \\
\mathcal{C}D_t^{1/2} u(0) + \mathcal{C}D_t^{1/2} u(1) &= \int_0^1 \left( \frac{1}{t + 10} \right)^2 \frac{|u(s)|}{1 + |u(s)|} ds.
\end{align*}
\]
Here, $\alpha = 3/2$, $\beta = 1/2$, $T = 1$, $\mu_1 = \mu_2 = \sigma_1 = \sigma_2 = 1$, $T = 1$, and

$$K = \sup_{t \in [0,1)} \left\{ \int_0^t e^{-(s-t)} 200 \, ds \right\} \int_0^t e^{-(s+1)/32} \, ds \right\} = 0.08.$$  
(51)

Since

$$|f(t,u,\varphi u,\psi u)| \leq \frac{3e^{-t}}{10},$$  

$$|g(t,u)| \leq 1,$$  

$$|h(t,u)| \leq \frac{1}{100}.$$  
(52)

Clearly, $C_1 = 1/10$, $C_2 = 1/200$, $C_3 = 1/32$, $C_4 = 1/50$, and $C_5 = 1/100$. Furthermore,

$$\left[ \frac{\mu_1}{1 \left(1+\mu_1 \right)} + \Gamma \left(\alpha+1 \right) + \Gamma \left(\alpha+\beta+1 \right) \right] [C_1 + (C_2 + C_3)TK]$$

$$+ \left[ \frac{\sigma_1}{1 \left(1+\mu_1 \right)} + \frac{\sigma_2}{\mu_2} \Gamma \left(\alpha+\beta+1 \right) \right] C_4T$$

$$+ \frac{\mu_1}{1 \left(1+\mu_1 \right)} C_5T$$

$$= \left( \frac{2}{3\sqrt{\pi}} + \frac{\sqrt{\pi}}{2} \right) \left( \frac{1}{10} \left( \frac{1}{200} + \frac{1}{32} \right) \times 0.08 \right) + \frac{1}{100} + \frac{\sqrt{\pi}}{200}$$

$$\approx 0.15 < 1.$$  
(53)

Thus, by Theorem 5, the integro-differential boundary value problem (50) has at least one solution on $[0,1]$.

**Data Availability**

No data were used in the manuscript.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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