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Research Article

Portfolio Optimization Model with and without Options under Additional Constraints

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In this paper, first, we study mean-absolute deviation (MAD) portfolio optimization model with cardinality constraints, short selling, and risk-neutral interest rate. Then, in order to insure the investment against unfavorable outcomes, an extension of MAD model that includes options is considered. Moreover, since the data in financial models usually involve uncertainties, we apply robust optimization to the MAD model with options. Finally, a data set of S&P index is used to compare the effectiveness of options in the models in terms of returns and Sharpe ratios.

1. Introduction

The mean-variance (MV) model proposed by Markowitz [1] is a single-period model that provides the best trade-off between return and risk. It is a quadratic programming problem; so, when the number of stocks is large, estimating the covariance matrix could be difficult. Then, Konno and Yamazaki [2] proposed the mean-absolute deviation (MAD) model with the absolute deviation of the rate of portfolio return as a measure of the risk instead of the variance. They proved that the MAD model gives essentially the same results as the MV model if all the returns are normally distributed random variables. Feinstein and Thapa [3] reformulated the MAD model with constraints less than the model of Konno and Yamazaki. Later, Chang [4] provided a reformulation of the model proposed by Feinstein and Thapa. Gorard [5] presented a review of the MAD versus the standard deviation. Kasenbacher et al. [6] also compared the MV model and the MAD model. Further studies on the MAD model can be found in [7–16].

Some extensions of the MAD model include short selling, threshold, and cardinality constraints. Short selling is the sale of a stock that does not belong to the seller. Investors under it borrow the stock to repay it in the future when they believe that the price of the stock will decline. After a while, the seller buys the stock from the market and repays it to the lender. Lintner [17] studied the first model of short selling in the portfolio theory. Konno et al. used the MAD model with the long-short strategy and showed that the long-short strategy leads to a portfolio with significantly better risk-return structure compared to the portfolio with the long strategy.

Cardinality constraints put a limit on the stock number in the portfolio, and the constraints of the threshold restrict the weights of stock in the portfolio to lie between given lower bounds and upper bounds. If the portfolio selects a small number of stocks from a large investment space, it means sparse [18–20]. Kwon and Stoyan [21] used the MAD model with cardinality constraints. They solved the MV and MAD models with different trading constraints and observed that the MAD model is substantially more tractable. In 2014, Le Thi and Moeini [22] extended the MAD model with short selling, cardinality, and the threshold constraints. Their model is reformulated in terms of a DC (difference of convex functions) problem and applied DC algorithms to solve it [23–25]. Cardinality constraints are also discussed in the MV models, such as the works of Anagnostopoulos and Mamanis [26], Gao and Li [27], Cesarone et al. [28], and Salahi et al. [29, 30].

Another factor that can be used in the portfolio optimization model is option. It is a financial derivative that can be considered as an asset for investment [31] and expresses as a contract that gives the holder the right to exercise a deal, but the contractor is not obliged to accomplish this right [32]. A call or put option gives the contractor the right to buy or sell the underlying stock at a certain price over a specified time. European or American options are the most common options that differ in the period of exercising the option. In European option, a contractor can only be applied the option at the expiration time, and in American option, a contractor should decide whether or not to exercise the option in any time before or at the expiration time [33]. In 2011, Topaloglou et al. [34] studied the options in the single-period portfolio. They found that controlling the risk of the market with options had a significant effect on performance of portfolio relative to currency risk. Authors in [35] showed that option reduces the risk and leads to better portfolio performance. Other studies also have investigated portfolio optimization models with options, for example, see [36–42].

Since the future of the financial market is ambiguous, historical data play a key role in predicting the future of the market. The stock returns forecasting is significant for stock pricing, stock allocation, and risk management. Dai et al. [43] improved the accuracy of stock return forecasts by combining a new two-step economic constraint forecasting model and new technical indicators. Also Dai et al. [44] found that combining denoising of stock returns with wavelet transform with new technical indicators can significantly improve the accuracy of stock returns forecasting, where the new technical indicators can directly reflect the trend of stock returns series. However, along with all the advantages of these forecasting methods, they can lead to some errors. On the other hand, the solutions of optimization problems show significant sensitivity to perturbations in the input parameters. A small uncertainty in the input parameters can make the usual optimal solution practically meaningless. Then, there is a need to develop models that are as safe as possible to the data uncertainty. Robust optimization is one of the widely used approaches to deal with uncertainties. In this approach, uncertain parameters are considered within known sets that are called uncertainty sets. First, Soyster [45] studied robust counterpart optimization using interval uncertainty sets. Then, Ben-Tal and Nemirovski [46] suggested that the ellipsoid uncertainty set and the robust formulation become a conic quadratic optimization problem. Although the proposed model is less conservative than Soyster's approach, the problem is nonlinear. Further, Bertsimas and Sim [47] studied robust linear optimization with coefficient uncertainty using an uncertainty set with budgets where their model is less conservative and stays linear. Moon and Yao [48] studied the robust MAD model. Their model led to a linear programming problem and reduced computational burden of the earlier robust portfolio optimization models. Lutgens et al. [39] studied a robust optimization for option hedging problems under ellipsoidal uncertainty sets. Their model is formulated as a second-order cone problem. Zymler et al. [49] developed a robust optimization model for designing portfolio including European options that trades off strong and weak guarantees on the worst-case portfolio return. Further important studies in the subject can be found in [50–60].

The goal of this paper is to analyze the MAD model with and without options when short selling, risk-neutral interest rate, and cardinality constraints are allowed. Then, the robust formulation under the interval uncertainty sets is studied. The rest of this paper is structured as follows. In Section 2, we describe the MAD model with short selling, risk-neutral interest rate, and cardinality constraints in detail. In Section 3, we extend the MAD model to include options. Robust model under interval uncertainties is discussed in Section 4. Numerical results are given in Section 5. Finally, Section 6 concludes the paper.

2. MAD Model and Extensions

The MAD model is as follows [2]:

$$\begin{aligned}
\min_{x,u}\ & \lambda\left(\frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j=1}^{N}\left(r_{tj}-r_j\right)x_j\right|\right)-(1-\lambda)\left(\sum_{j=1}^{N} r_j x_j\right)\\
\text{s.t.}\ & \sum_{j=1}^{N} x_j = 1,\\
& \varepsilon_j \le x_j \le \delta_j,\ j=1,\ldots,N,\\
& x_j \ge 0,\ j=1,\ldots,N,
\end{aligned} \tag{1}$$

where T and N denote the end of investment time and the number of available stocks, respectively, r_{tj} is the return of the jth stock at time t, $(t=1,\ldots,T;\ j=1,\ldots,N)$, and $r_j = (1/T)\sum_{t=1}^{T} r_{tj}$ for the jth stock $(j=1,\ldots,N)$. Also, x_j is the weight of jth stock, ε_j and δ_j represent the lower bound and upper bound of the jth stock, respectively, and $\lambda \in [0,1]$ is the risk aversion parameter.

To include realistic constraints in the MAD model, by adding short selling, risk-neutral interest rate, and cardinality constraints, we get the following model [29]:

$$\begin{aligned}
\min_{x,z}\ & \lambda\left(\frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j=1}^{N}\left(r_{tj}-r_j\right)x_j\right|\right)\\
& \qquad -(1-\lambda)\left(\sum_{j=1}^{N}\left(r_j x_j - r_c h_j x_j\right)\right)\\
\text{s.t.}\ & \sum_{j=1}^{N} x_j = 1,\\
& \sum_{j=1}^{N} z_j = K,\\
& r_j x_j \ge 0,\ j=1,\ldots,N,\\
& \varepsilon_j z_j \le x_j \le \delta_j z_j,\ j=1,\ldots,N,\\
& z_j \in \{0,1\},\ j=1,\ldots,N,
\end{aligned} \tag{2}$$

where r_c is risk-neutral interest rate, K is the number of stocks in the portfolio, and z_j's are binary variables. If $z_j = 1$, stock j belongs to the portfolio, and if $z_j = 0$, it does not. The term $r_c\sum_{j=1}^{N} h_j x_j$ represents the short rebate, where

$$0 < h_j < 1, \quad \forall j, \tag{3}$$

is the investor's portion of the interest received on proceeding from the short sale of stock j. Then, $h_j = 0$ when $r_j \geq 0$ and $0 < h_j < 1$ when $r_j < 0$. The constraints $r_j x_j \geq 0$ $(j = 1, \ldots, N)$ show that, for any stock which is in the short selling position, the proportion of investment is negative. The objective function in model (2) is nonlinear; however, using auxiliary variable u_t, we get the following linear model:

$$\begin{aligned}
\min_{x,z,u}\ & \lambda\left(\frac{1}{T}\sum_{t=1}^{T} u_t\right) - (1-\lambda)\left(\sum_{j=1}^{N}\left(r_j x_j - r_c h_j x_j\right)\right) \\
\text{s.t.}\quad & u_t + \sum_{j=1}^{N}\left(r_{tj} - r_j\right)x_j \geq 0,\ t = 1, \ldots, T, \\
& u_t - \sum_{j=1}^{N}\left(r_{tj} - r_j\right)x_j \geq 0,\ t = 1, \ldots, T, \\
& \sum_{j=1}^{N} x_j = 1, \\
& \sum_{j=1}^{N} z_j = K, \\
& u_t \geq 0,\ t = 1, \ldots, T, \\
& r_j x_j \geq 0,\ j = 1, \ldots, N, \\
& \varepsilon_j z_j \leq x_j \leq \delta_j z_j,\ j = 1, \ldots, N, \\
& z_j \in \{0, 1\},\ j = 1, \ldots, N.
\end{aligned} \tag{4}$$

It should be noted that, ε_j is negative when short selling is allowed.

3. MAD Model with Options

In this section, we use options in the portfolio that ensure the investment against unfavorable outcomes. They reduce the risk and come, however, at some costs that decrease the return of the portfolio [35]. These costs (options prices) are formulated based on the risk-neutral interest rate as follows:

$$\begin{aligned}
O_{\text{put}} &= \max\{0, K - S_T\}\, e^{-r_c T}, \\
O_{\text{call}} &= \max\{0, S_T - K\}\, e^{-r_c T},
\end{aligned} \tag{5}$$

where S_T is the stock price vector in the expiration time and K is strike price such that

$$K = S_0\, e^{-r_c T}, \tag{6}$$

where S_0 is a vector of stock initial price. Since we use O_{call} and O_{put} for any stock, the total option price is

$$O = O_{\text{call}} + O_{\text{put}}. \tag{7}$$

Using these call and put options under strike price (K), the option payoff functions become

$$\begin{aligned}
V_{\text{put}}\left(S_T\right) &= \max\{0, K - S_T\}, \\
V_{\text{call}}\left(S_T\right) &= \max\{0, S_T - K\}.
\end{aligned} \tag{8}$$

Based on these payoff functions, options returns are as follows:

$$\begin{aligned}
R_{\text{put}} &= \frac{1}{S_0}\max\{0, K - S_T\}, \\
R_{\text{call}} &= \frac{1}{S_0}\max\{0, S_T - K\}.
\end{aligned} \tag{9}$$

Using S_T and K, the investor decides whether to exercise the call or put options for any stock. Therefore, model (4) under the options returns and options prices becomes

$$\begin{aligned}
\min_{x,z,u}\ & \lambda\left(\frac{1}{T}\sum_{t=1}^{T} u_t\right) - (1-\lambda)\left(\sum_{j=1}^{N}\left(r_j x_j - r_c h_j x_j\right)\right. \\
& \left. - \sum_{j=1}^{N} O_j\left|x_j\right| + \sum_{j=1}^{N}\left(\left(R_{\text{call}}\right)_j + \left(R_{\text{put}}\right)_j\right)x_j\right) \\
\text{s.t.}\quad & u_t + \sum_{j=1}^{N}\left(r_{tj} - r_j\right)x_j \geq 0,\ t = 1, \ldots, T, \\
& u_t - \sum_{j=1}^{N}\left(r_{tj} - r_j\right)x_j \geq 0,\ t = 1, \ldots, T, \\
& \sum_{j=1}^{N} x_j = 1, \\
& \sum_{j=1}^{N} z_j = K, \\
& u_t \geq 0,\ t = 1, \ldots, T, \\
& r_j x_j \geq 0,\ j = 1, \ldots, N, \\
& \varepsilon_j z_j \leq x_j \leq \delta_j z_j,\ j = 1, \ldots, N, \\
& z_j \in \{0, 1\},\ j = 1, \ldots, N,
\end{aligned} \tag{10}$$

which is a mixed-integer linear optimization problem. In this model, since short selling is allowed, the returns of options for stocks in these situations are considered negative.

Lemma 1. *Let Z^* and W^* denote the optimal objective values of optimization models (4) and (10), respectively. Then, $Z^* \geq W^*$.*

Proof. Let (x^*, u^*) be an optimal solution of model (4). We have

$$\begin{aligned}
&\frac{\lambda}{T}\sum_{t=1}^{T} u_t^* - (1-\lambda)\left(\sum_{j=1}^{N}\left(r_j x_j^* - r_c h_j x_j^*\right)\right) \\
&\quad \geq \frac{\lambda}{T}\sum_{t=1}^{T} u_t^* - (1-\lambda)\left(\sum_{j=1}^{N}\left(r_j x_j^* - r_c h_j x_j^*\right) - \sum_{j=1}^{N} O_j \left|x_j^*\right|\right. \\
&\qquad \left. + \sum_{j=1}^{N}\left(\left(R_{\text{call}}\right)_j + \left(R_{\text{put}}\right)_j\right) x_j^*\right),
\end{aligned} \tag{11}$$

since

$$0 \leq O_j \leq \left(R_{\text{call}}\right)_j + \left(R_{\text{put}}\right)_j. \tag{12}$$

Therefore, $Z^* \geq W^*$. □

4. Robust Model

In this section, since the future values of stock prices may involve uncertainties, we use robust optimization to deal with this situation. In this approach, input parameters are considered in bounded uncertainty sets that contain all or most values of uncertain data. Model (10) under uncertainty is

$$\begin{aligned}
\min_{x,z,u,r}\ & \lambda\left(\frac{1}{T}\sum_{t=1}^{T} u_t\right) - (1-\lambda)\left(\sum_{j=1}^{N}\left(\tilde{r}_j x_j - r_c h_j x_j\right) - \sum_{j=1}^{N} O_j \left|x_j\right| + \sum_{j=1}^{N}\left(\left(R_{\text{call}}\right)_j + \left(R_{\text{put}}\right)_j\right) x_j\right) \\
\text{s.t.}\ \ & u_t + \sum_{j=1}^{N}\left(\tilde{r}_{tj} - \tilde{r}_j\right) x_j \geq 0,\ t = 1, \ldots, T, \\
& u_t - \sum_{j=1}^{N}\left(\tilde{r}_{tj} - \tilde{r}_j\right) x_j \geq 0,\ t = 1, \ldots, T, \\
& \sum_{j=1}^{N} x_j = 1, \\
& \sum_{j=1}^{N} z_j = K, \\
& u_t \geq 0,\ t = 1, \ldots, T, \\
& \tilde{r}_j x_j \geq 0,\ j = 1, \ldots, N, \\
& \varepsilon_j z_j \leq x_j \leq \delta_j z_j,\ j = 1, \ldots, N, \\
& z_j \in \{0, 1\},\ j = 1, \ldots, N, \\
& \tilde{r}_j \in U_1,\ j = 1, \ldots, N, \\
& \tilde{r}_{tj} \in U_2,\ t = 1, \ldots, T,\ j = 1, \ldots, N,
\end{aligned} \tag{13}$$

where

$$U_1 = \{\tilde{r}_j \mid r_j^l \le \tilde{r}_j \le r_j^u, \quad j = 1, \dots, N\},$$
$$U_2 = \{\tilde{r}_{tj} \mid r_{tj}^l \le \tilde{r}_{tj} \le r_{tj}^u, \quad t = 1, \dots, T,\ j = 1, \dots, N\}. \tag{14}$$

Here, r_{tj}^l and r_{tj}^u denote the lower bound and upper bound of returns for stock j at time t, respectively. Also, r_j^l and r_j^u denote the lower bound and upper bound of r_j, respectively.

Theorem 1. *The robust counterpart of model (13) for uncertainty sets (14) is as follows:*

$$\begin{aligned}
\min_{x,z,u,r}\ & \lambda\left(\frac{1}{T}\sum_{t=1}^{T} u_t\right) - (1-\lambda)\left(\sum_{j=1}^{N}\left(-r_j^u s_j + r_j^l c_j\right) - \sum_{j=1}^{N} r_c h_j x_j - \sum_{j=1}^{N} O_j\left|x_j\right| + \sum_{j=1}^{N}\left(\left(R_{\text{call}}\right)_j + \left(R_{\text{put}}\right)_j\right)x_j\right) \\
\text{s.t.}\ & u_t + \sum_{j=1}^{N}\left(-r_{tj}^u \gamma_{tj} + r_{tj}^l \delta_{tj} - r_j^u \psi_j + r_j^l \varphi_j\right) \ge 0,\ t = 1, \dots, T, \\
& u_t + \sum_{j=1}^{N}\left(-r_{tj}^u \alpha_{tj} + r_{tj}^l \beta_{tj} - r_j^u \eta_j + r_j^l \kappa_j\right) \ge 0,\ t = 1, \dots, T, \\
& \sum_{j=1}^{N} x_j = 1, \\
& \sum_{j=1}^{N} z_j = K, \\
& u_t \ge 0,\ t = 1, \dots, T, \\
& -r_j^u d_j + r_j^l e_j \ge 0,\ j = 1, \dots, N, \\
& \varepsilon_j z_j \le x_j \le \delta_j z_j,\ j = 1, \dots, N, \\
& s_j - c_j = -x_j,\ j = 1, \dots, N, \\
& -\gamma_{tj} + \delta_{tj} = x_j,\ t = 1, \dots, T,\ j = 1, \dots, N, \\
& -\psi_j + \varphi_j = -x_j,\ j = 1, \dots, N, \\
& -\alpha_{tj} + \beta_{tj} = -x_j,\ t = 1, \dots, T,\ j = 1, \dots, N, \\
& -\eta_j + \kappa_j = x_j,\ j = 1, \dots, N, \\
& -d_j + e_j = x_j, \\
& s_j \ge 0, c_j \ge 0, d_j \ge 0, e_j \ge 0, \psi_j \ge 0, \eta_j \ge 0,\ \kappa_j \ge 0,\ j = 1, \dots, N, \\
& \alpha_{tj} \ge 0, \beta_{tj} \ge 0,\ \gamma_{tj} \ge 0, \delta_{tj} \ge 0,\ t = 1, \dots, T,\ j = 1, \dots, N, \\
& z_j \in \{0, 1\},\ j = 1, \dots, N.
\end{aligned} \tag{15}$$

Proof. The robust counterpart of model (13) under uncertainty sets (14) is

$$\begin{aligned}
\min_{x,z,u,r}\ & \lambda\left(\frac{1}{T}\sum_{t=1}^{T}u_t\right)+(1-\lambda)\left(\max_{\widetilde{r}_j\in U_1}\left(-\sum_{j=1}^{N}\widetilde{r}_jx_j\right)+\sum_{j=1}^{N}r_ch_jx_j+\sum_{j=1}^{N}O_j\left|x_j\right|\right.\\
&\left.-\sum_{j=1}^{N}\left(\left(R_{\text{call}}\right)_j+\left(R_{\text{put}}\right)_j\right)x_j\right)\\
\text{s.t.}\quad & u_t+\min_{\widetilde{r}_j\in U_1,\widetilde{r}_{tj}\in U_2}\left(\sum_{j=1}^{N}\left(\widetilde{r}_{tj}-\widetilde{r}_j\right)x_j\right)\geq 0,\ t=1,\ldots,T,\\
& u_t+\min_{\widetilde{r}_j\in U_1,\widetilde{r}_{tj}\in U_2}\left(-\sum_{j=1}^{N}\left(\widetilde{r}_{tj}-\widetilde{r}_j\right)x_j\right)\geq 0,\ t=1,\ldots,T,\\
& \sum_{j=1}^{N}x_j=1,\\
& \sum_{j=1}^{N}z_j=K,\\
& u_t\geq 0,\ t=1,\ldots,T,\\
& \min_{\widetilde{r}_j\in U_1}\widetilde{r}_jx_j\geq 0,\ j=1,\ldots,N,\\
& \varepsilon_jz_j\leq x_j\leq\delta_jz_j,\ j=1,\ldots,N,\\
& z_j\in\{0,1\},\ j=1,\ldots,N.
\end{aligned}\tag{16}$$

In order to simplify model (16), we need equivalent forms of inner maximization and minimization problems. Consider the inner maximization problem in the objective function:

$$\begin{aligned}
\max\ & \sum_{j=1}^{N}-\widetilde{r}_jx_j\\
\text{s.t.}\quad & r_j^l\leq\widetilde{r}_j\leq r_j^u,\ j=1,\ldots,N.
\end{aligned}\tag{17}$$

Its dual is

$$\begin{aligned}
\min_{s,c}\ & \sum_{j=1}^{N}\left(r_j^us_j-r_j^lc_j\right)\\
\text{s.t.}\quad & s_j-c_j=-x_j,\ j=1,\ldots,N,\\
& s_j\geq 0,\ j=1,\ldots,N,\\
& c_j\geq 0,\ j=1,\ldots,N.
\end{aligned}\tag{18}$$

Since the primal and dual are feasible and the duality gap is equal to zero, we can replace the maximization problem with its dual in the objective function and add its constraints to the model. Now, consider the minimization problem in the first constraint:

$$\begin{aligned}
\min\ & \sum_{j=1}^{N}\left(\widetilde{r}_{tj}-\widetilde{r}_j\right)x_j\\
\text{s.t.}\quad & r_j^l\leq\widetilde{r}_j\leq r_j^u,\ j=1,\ldots,N,\\
& r_{tj}^l\leq\widetilde{r}_{tj}\leq r_{tj}^u,\ j=1,\ldots,N.
\end{aligned}\tag{19}$$

Its dual is

$$\begin{aligned}
\max_{\gamma,\delta,\psi,\varphi}\ & \sum_{j=1}^{N}\left(-r_{tj}^u\gamma_{tj}+r_{tj}^l\delta_{tj}-r_j^u\psi_j+r_j^l\varphi_j\right)\\
\text{s.t.}\quad & -\gamma_{tj}+\delta_{tj}=x_j,\ j=1,\ldots,N,\\
& -\psi_j+\varphi_j=-x_j,\ j=1,\ldots,N,\\
& \gamma_{tj}\geq 0,\ j=1,\ldots,N,\\
& \delta_{tj}\geq 0,\ j=1,\ldots,N,\\
& \psi_j\geq 0,\ j=1,\ldots,N,\\
& \varphi_j\geq 0,\ j=1,\ldots,N.
\end{aligned}\tag{20}$$

The minimization problem in the second constraint is

Table 1: Companies and their monthly returns for 2016–2018.

Company	Return	Company	Return
The Bank of New York Mellon Corporation	0.0113	Pfizer inc	0.0089
Electronic Arts Inc.	0.0146	Philip Morris International inc	−0.0071
General Dynamics Corporation	0.0074	Qualcomm Incorporated	0.0133
Mylan N.V	−0.0064	Regeneron Pharmaceuticals Inc.	−0.0074
Newmont Corporation	0.0227	Raytheon Technologies Corporation	0.0079
Truist Financial Corporation	0.0101	AT&T Inc.	0.0003
Ulta Beauty Inc.	0.0099	T-Mobile US Inc.	0.0177
Dentsply Sirona Inc.	−0.0103	United Parcel Service Inc.	0.0044
Booking Holdings inc.	0.0137	Verizon Communications Inc.	0.0084
BlackRock Inc.	0.0116	Wells Fargo and Company	0.0026
Bristol Myers Squibb company	−0.0012	Exxon Mobil Corporation	−0.0007
Berkshire Hathaway Inc.	0.0167	Apache Corporation	−0.0034
Citigroup Inc.	0.0090	Comerica Incorporated	0.0224
Colgate-Palmolive Company	−0.0015	Hanesbrands Inc.	−0.0121
Comcast Corporation	0.0121	Zions Bancorporation, National Association	0.0217
Virgin Galactic Holdings Inc.	−0.0051	People's United Financial Inc.	0.0027
Chevron Corporation	0.0108	National Oilwell Varco Inc.	0.0007
Dominion Energy Inc.	0.0032	Marathon Oil Corporation	0.0162
The Walt Disney Company	0.0056	Devon Energy Corporation	0.0008
Duke Energy Corporation	0.0046	Virgin Galactic Holdings Inc.	0.0085
Gilead Sciences Inc.	−0.0053	Invesco Ltd.	−0.0090
The Goldman Sachs Group Inc.	0.0093	Unum Group	0.0116
International Business Machines Corporation	−0.0003	SL Green Realty Corp.	−0.0064
Johnson and Johnson	0.0063	Virgin Galactic Holdings Inc.	0.0139
The Coca-Cola company	0.0018	Discovery Inc.	0.0064
Mondelez International Inc.	−0.0004	Ralph Lauren Corporation	0.0102
Altria Group, inc	−0.0053	The Gap Inc.	0.0051
Morgan Stanley	0.0169	TechnipFMC plc.	−0.0023
Oracle Corporation	0.0140		

$$\begin{aligned} &\min -\sum_{j=1}^{N}\left(\tilde{r}_{tj}-\tilde{r}_{j}\right)x_{j} \\ &\text{s.t.}\quad r_{j}^{l}\leq\tilde{r}_{j}\leq r_{j}^{u},\ j=1,\ldots,N, \\ &\qquad r_{tj}^{l}\leq\tilde{r}_{tj}\leq r_{tj}^{u},\ j=1,\ldots,N, \end{aligned} \tag{21}$$

and its dual problem also is

$$\begin{aligned} &\max_{\alpha,\beta,\eta,\kappa}\sum_{j=1}^{N}\left(-r_{tj}^{u}\alpha_{tj}+r_{tj}^{l}\beta_{tj}-r_{j}^{u}\eta_{j}+r_{j}^{l}\kappa_{j}\right) \\ &\text{s.t.}\quad -\alpha_{tj}+\beta_{tj}=-x_{j},\ j=1,\ldots,N, \\ &\qquad -\eta_{j}+\kappa_{j}=x_{j},\ j=1,\ldots,N, \\ &\qquad \alpha_{tj}\geq 0,\ j=1,\ldots,N, \\ &\qquad \beta_{tj}\geq 0,\ j=1,\ldots,N, \\ &\qquad \eta_{j}\geq 0,\ j=1,\ldots,N, \\ &\qquad \kappa_{j}\geq 0,\ j=1,\ldots,N. \end{aligned} \tag{22}$$

Finally, the minimization problem in the sixth constraint is

$$\begin{aligned} &\min \tilde{r}_{j}x_{j} \\ &\text{s.t.}\quad r_{j}^{l}\leq\tilde{r}_{j}\leq r_{j}^{u}, \end{aligned} \tag{23}$$

and its dual is

$$\begin{aligned} &\max_{d,e} -r_{j}^{u}d_{j}+r_{j}^{l}e_{j} \\ &\text{s.t.}\quad -d_{j}+e_{j}=x_{j}, \\ &\qquad d_{j}\geq 0, \\ &\qquad e_{j}\geq 0. \end{aligned} \tag{24}$$

By replacing the objective functions of dual problems in constraints and adding their constraints to the model, we get the results. □

5. Numerical Experiments

In this section, we investigate the performance of models (4), (10), and (15). We assume that there are call and put European options on all stocks, and the expiration times of all options are the end of investment times (T). We provide numerical results for the S&P 5001 index for 2016–2018 when $N=57$, $\lambda=0.5$, $r_c=0.03$, and taking $\delta_i=-\varepsilon_i=0.1$ as the lower bound and upper bound of the proportion of investment in any stock. The monthly returns of stocks are presented in Table 1.

For the sake of simplicity, in the robust model, the uncertainty sets are defined as

$$\begin{aligned} &U_1=\left\{\tilde{r}_{j}\,|\,r_{j}-\varepsilon\leq\tilde{r}_{j}\leq r_{j}+\varepsilon,\quad j=1,\ldots,N\right\}. \\ &U_2=\left\{\tilde{r}_{tj}\,|\,r_{tj}-\varepsilon\leq\tilde{r}_{tj}\leq r_{tj}+\varepsilon,\quad t=1,\ldots,T,\ j=1,\ldots,N\right\}, \end{aligned} \tag{25}$$

Table 2: Comparison of returns of models (4), (10), and (15) with the data of S&P index data, with $N = 57$ stocks for 2016–2018 when $\lambda = 0.5$.

Number of stocks	Model (4)	Model (10)	Model (15)
$K = 10$	0.0135	0.0282	0.0214
$K = 20$	0.0227	0.0500	0.0206
$K = 30$	0.0309	0.0666	0.0291
$K = 40$	0.0367	0.0773	0.0297

Table 3: Comparison of Sharpe ratios of models (4), (10), and (15) with the data of S&P index data, with $N = 57$ stocks for 2016–2018 when $\lambda = 0.5$.

Number of stocks	Model (4)	Model (10)	Model (15)
$K = 10$	−0.8921	−0.0876	−0.1892
$K = 20$	−0.4370	0.9088	−0.2115
$K = 30$	0.0495	1.4657	0.0155
$K = 40$	0.3472	1.6589	0.0160

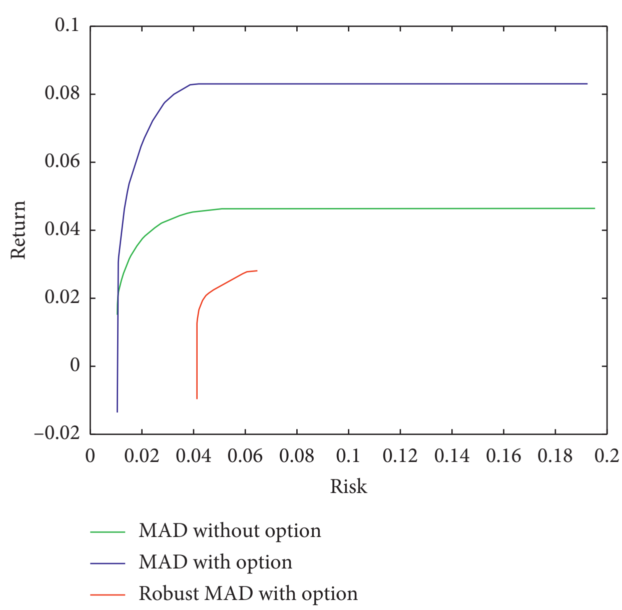

Figure 1: Comparison of efficient frontiers of the MAD model with and without options and its robust model for S&P 500 index data in 2016–2018 when $K = 40$ and $\lambda = 0.5$.

where $\varepsilon = 0.01$. To solve all models, we used CVX software using MATLAB [61].

We compare MAD model with and without options and its robust model in terms of returns and Sharpe ratios. The Sharpe ratio (SR) is calculated by

$$\mathrm{SR} = \frac{\mu - r_c}{\delta}, \tag{26}$$

where μ is the expected portfolio return, δ is the mean-absolute deviation, and r_c denotes risk-neutral interest rate. The results are summarized in Tables 2 and 3 for different K values. As we see, portfolio with options creates significant advantage in returns and Sharpe ratios compared to the portfolio without options. Also, by comparing columns 3 and 4 of these tables, we observe that the robust model (15) can be too conservative and returns, and Sharpe ratios obtained from it, are significantly less than model (10).

Further, we compare the efficient frontiers of MAD model with and without options and its robust model (models (4), (10) and (15)) in Figure 1. As we see, the efficient frontier of the MAD model with options lies above the one without options and its robust model.

6. Conclusions

In this paper, we proposed extensions of the MAD model with and without options when short selling, risk-neutral interest rate, and cardinality constraint are allowed. Also, its robust model under interval uncertainty sets is given. Numerical results for the S&P 500 index showed that using options led to better performance in terms of returns and Sharpe ratios. Moreover, numerical results of the robust model showed that uncertainty may significantly reduce portfolio's returns and Sharpe ratios. Due to the importance of forecasted data and transaction costs for portfolio optimization, studying the proposed model with these factors may be considered as a future research plan.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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