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Research Article

Ruin Problems of Multidimensional Risk Models under Constant Interest Rates and Dependent Risks with Heavy Tails

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Consider a discrete-time multidimensional risk model with constant interest rates where capital transfers between lines are partially allowed over each period. By assuming a large initial capital and regularly varying distributions for the losses, we derive asymptotic estimates for the ruin probability under some dependence structure and study the optimal allocation of the initial reserve. Some numerical simulations are provided to illuminate our main results.

1. Introduction

Consider an insurer with multiple business lines against catastrophic risks (e.g., earthquakes, floods, hurricane, or terrorist attacks). The eventual occurrence of these events usually has a substantial effect on some lines of business simultaneously. Thus, the statistical dependence among claims in these business lines should be considered. Moreover, the insurer may allow restricted capital transfers to reduce the risk of ruin in insurance practice. Particularly, the surplus of those profitable lines is partially allowed to be transferred to cover the total deficit of the other lines in deficit.

The reserve of the insurer can be described as the following multidimensional discrete-time risk model:

$$\mathbf{U}_0 = \begin{pmatrix} ub_1 \\ ub_2 \\ \vdots \\ ub_d \end{pmatrix},$$
$$\mathbf{U}_l = \begin{pmatrix} U_{1,l} \\ U_{2,l} \\ \vdots \\ U_{d,l} \end{pmatrix} = \begin{pmatrix} \left(U_{1,l-1} + e_{1,l-1}\right)\left(1+\rho_l\right) - Z_{1,l} \\ \left(U_{2,l-1} + e_{2,l-1}\right)\left(1+\rho_l\right) - Z_{2,l} \\ \vdots \\ \left(U_{d,l-1} + e_{d,l-1}\right)\left(1+\rho_l\right) - Z_{d,l} \end{pmatrix}, \quad l = 1, 2, \ldots, \tag{1}$$

where $u \geq 0$ is the global initial reserve, $b_i \in [0, 1]$ is the proportion of capital allocated to the i-th business line so that $\sum_{i=1}^{d} b_i = 1$, and the constant interest rate over the l-th period is denoted by $\rho_l\,(\rho_l > -1)$ and $\rho_0 = 0$. For an insurer with $d\,(d \geq 2)$ lines of business, suppose now that the claim size with finite mean is denoted by $Z_{i,l}$ and the constant premium income is $e_{i,l-1}\,(e_{i,l-1} = (1+\lambda)EZ_{i,l}, \lambda > 0)$ from the i-th business line over the l-th period, that is, the claims are paid at the end of each period while the premiums are paid at the beginning of each period.

Throughout this paper, for the conciseness in expression, we shall mainly formulate our problems in a vector notation. Unless otherwise stated, random vectors and real vectors will be written as bold letters and will be assumed to be d-dimensional. For example, $\mathbf{X} = (X_1, \ldots, X_d)^T$ and the L_1 norm $\|\mathbf{X}\| = |X_1| + \cdots + |X_d|$. Particularly, we also use a bold Arabic number to stand for the vector with all components being that number, e.g., $\mathbf{1} = (1, \ldots, 1)^T$, $\mathbf{0} = (0, \ldots, 0)^T$. For two vectors $\mathbf{a}$ and $\mathbf{b}$ of the same dimension, we understand relations such as $\mathbf{a} \geq \mathbf{b}$ and $\mathbf{a} \pm \mathbf{b}$ as componentwise. Additionally, the scalar multiplication is defined as usual, i.e., $y\mathbf{a} = (ya_1, \ldots, ya_d)^T$ with y being a real number. Furthermore, for a set K and any real number u, $uK = \{u\mathbf{x}\colon \mathbf{x} \in K\}$.

Let

$$y_l = \prod_{k=0}^{l} \left(1+\rho_k\right)^{-1}, \quad l = 0, 1, \ldots, \tag{2}$$

which stand for the discount factors. With the above conventions, multidimensional risk model (1) can be rewritten as

$$\mathbf{U}_l = y_l^{-1}\left(u\mathbf{b} - \sum_{k=1}^{l}\left(\mathbf{Z}_k y_k - \mathbf{e}_{k-1} y_{k-1}\right)\right), \quad l = 1,2,\ldots. \tag{3}$$

Ruin theory is a hot topic in risk theory and has been widely studied, for example, valuing death benefits of a Lévy model was studied by Yu et al. [1] and Zhang et al. [2]; the compound Poisson risk model with threshold dividend strategy was considered by Peng et al. [3] and Yu et al. [4]; and the compound Poisson risk model with discounted penalty function was considered by Ruan et al. [5]. In the classical risk theory, ruin probabilities with respect to the unidimensional risk process have been intensively investigated in the past decades. However, the unidimensional risk models cannot provide the whole picture for assessing the solvency ability of an insurer with multiple business lines, and the study on ruin probabilities of the multidimensional risk model has become a hot topic recently. Compared to fruitful results on the study of the unidimensional risk model, the investigation for the multidimensional risk model is quite limited. During the few works, Collamore [6, 7] addressed the multidimensional ruin problems with the general ruin set and studied the sampling techniques in a light-tailed case. Picard et al. [8] introduced a multidimensional discrete-time model and developed a sample recursive method for numerical evaluation of the corresponding ruin probability. Hult et al. [9] firstly studied the ruin probability for multidimensional heavy-tailed processes. Subsequently, under the assumption that the claim size vectors have multivariate regularly varying distribution, Hult and Lindskog [10] derived the asymptotic decay of the ruin probability as the initial capital tends to infinity and analysed the impact of rules for capital transfer on ruin probability. Biard et al. [11] extended some asymptotic results on finite-time probabilities with heavy-tailed claim sizes for the renewal risk model, in which the assumption of independence and stationarity was relaxed. Recently, Huang et al. [12] considered a discrete-time multidimensional risk model with the assumption of regularly varying distribution for net losses and established asymptotic estimates for finite-time ruin probabilities in terms of the upper tail dependence function. Under the framework of heavy-tailed and non-identically distributed claim sizes with some dependence structure, Li et al. [13] focused on the finite-time ruin probability of the continuous-time multidimensional model in which capital transfers were partially allowed, and they studied the optimal allocation of the initial reserve. For more recent studies on bidimensional risk models with heavy-tailed distribution about the ruin problem, see Yang and Yuen [14]; Yang et al. [15]; Chen and Yang [16]; Chen et al. [17]; Cheng and Yu [18], and Yang et al. [19], among many others.

Motivated by Li et al. [13], we consider the following ruin probability that the total surplus of a fraction of each profitable line fails to cover the total deficit of the others over n periods and the fraction can be different by lines:

$$\psi_{\boldsymbol{\omega}}(u,n) = \mathrm{P}\left(\mathbf{U}_l \in \Gamma_{\boldsymbol{\omega}} \text{ for some } l = 1,\ldots,n \,\middle|\, \mathbf{U}_0 = u\mathbf{b}\right), \tag{4}$$

where the ruin set is defined as follows:

$$\Gamma_{\boldsymbol{\omega}} = \left\{\mathbf{x}\colon \sum_{k=1}^{d}\omega_k\left(x_k\right)_+ < \sum_{k=1}^{d}\left(-x_k\right)_+\right\}, \tag{5}$$

for some $\boldsymbol{\omega} = (\omega_1,\omega_2,\ldots,\omega_d)^T \in [0,1]^d$, $\mathbf{b} = (b_1,b_2,\ldots,b_d)^T$, and $(x)_+ = \max\{x,0\}$. The ruin set $\Gamma_{\boldsymbol{\omega}}$ denotes all possible events that the sum of some portions of the corresponding positive lines fails to cover the sum of the negative position of other lines. Particularly, for $\boldsymbol{\omega} = 0$, no transfer is allowed, and the corresponding ruin set $\Gamma_{\mathbf{0}} = \left\{\mathbf{x}\colon \sum_{k=1}^{d}(-x_k)_+ > 0\right\}$ denotes that the surplus of some business lines becomes negative. For $\boldsymbol{\omega} = 1$, the corresponding ruin set $\Gamma_{\mathbf{1}} = \left\{\mathbf{x}\colon \sum_{k=1}^{d}x_k < 0\right\}$ denotes the total surplus of all lines becomes negative.

The present paper focuses on establishing asymptotic estimates for ruin probability in a discrete-time model with constant interest rates. Under the assumption that the claim size vector follows a multivariate regularly varying distribution, capital transfers between business lines are partially allowed with different fractions $\{\omega_i \in [0,1], i = 1,\ldots,d\}$ for each line. Furthermore, the obtained asymptotic formula for ruin probability can be used to study the optimal allocation of the initial reserve in order to minimize the ruin probability.

The paper proceeds as follows. After introducing the concepts of multivariate regular variation, Section 2 states the main results for the multidimensional discrete-time risk model and then shows some specific examples for which the underlying dependence assumptions are satisfied. Section 3 proposes applications of the study to obtain the initial reserve allocation. Some numerical simulations on the ruin probability are presented to illuminate the main results in Section 4.

2. Multidimensional Finite-Time Ruin Probability

2.1. Preliminaries. We first recall the concept of regular variation. A positive measurable function $L(\cdot)$ on $[0,\infty)$ is said to be regularly varying at ∞ with regularity index $\rho \in \mathbb{R}$, written as $L \in RV_\rho(\infty)$, if it holds

$$\lim_{t\longrightarrow\infty}\frac{L(tx)}{L(t)} = x^{-\rho}, \quad x > 0. \tag{6}$$

We often suppress the argument ∞ in $RV_\rho(\infty)$ if there is no confusion in the context. Particularly, if $\rho = 0$, then L is said to be slowly varying.

Definition 1. A random vector $\mathbf{X}$ is said to be multivariate regularly varying (MRV) if there exists a nonzero Radon measure μ called intensity measure on $\overline{\mathbb{R}}^d\backslash\{\mathbf{0}\}$ such that

$$\lim_{t\longrightarrow\infty}\frac{P(\mathbf{X}\in tA)}{P(\|\mathbf{X}\|>t)}=\mu(A), \quad (7)$$

for every Borel set $A\subset\overline{\mathbb{R}}^d/\{\mathbf{0}\}$ with $\mu(\partial A)=0$.

By Theorem 1.14 of Lindskog [20], for such a Radon measure μ, there exists $\alpha>0$ such that $\mu(tA)=t^{-\alpha}\mu(A)$ holds for any $t>0$, and each Borel set $A\subseteq\overline{\mathbb{R}}^d\backslash\{\mathbf{0}\}$ with $\mu(\partial A)=0$. Hence, $\mathbf{X}$ is said to be regularly varying with index α and limiting measure μ meanwhile denoted by $\mathbf{X}\in MRV_{\alpha,\mu}$. The detailed discussions on multivariate regular variation including equivalent statements of MRV and its various applications can be found in Resnick's study [21].

2.2. Assumptions. In the present paper, we focus on risk vectors with MRV tails, comparable marginal tails, and asymptotic independence. Before giving the following assumptions, we have the setting that the claim sizes $\{\mathbf{Z}_l:=(Z_{1,l},\ldots,Z_{d,l})^T, l\geq 1\}$ are independent and identically distributed (i.i.d.) as a generic random vector $\mathbf{Z}=(Z_1,\ldots,Z_d)^T$. For a distribution function $F(x)$, let $\overline{F}(x)=1-F(x)$.

Assumption 1. The random vector $\mathbf{Z}\in MRV_{\alpha,\mu}$ for some $\alpha>1$.

Assumption 2. The random vector $\mathbf{Z}$ has the univariate marginal distributions $F_1,\ldots,F_d$ satisfying $\overline{F}_1(x)\in RV_{-\alpha}$ for some $\alpha>1$ and

$$\lim_{x\longrightarrow\infty}\frac{\overline{F}_i(x)}{\overline{F}_1(x)}=c_i, \quad (8)$$

for some $0\leq c_i<\infty, i=2,\ldots,d$ and $c_1=1$.

Remark 1. According to Section 6.5.6 of Resnick [21], if $\mathbf{Z}\in MRV_{\alpha,\mu}$ with univariate marginal distributions $F_1,\ldots,F_d$, then

$$\lim_{x\longrightarrow\infty}\frac{\overline{F}_i(x)}{\overline{F}_j(x)}=\frac{\lambda_i}{\lambda_j}, \quad (9)$$

for some $0\leq\lambda_i\leq\infty, i=1,2,\ldots,d$, which means $\mathbf{Z}$ has comparable marginal tails. Particularly, the case that $\lambda_i=0$ or ∞ for some $i=1,2,\ldots,d$ means some marginal tails are heavier than the others, so the index α of marginal tails may not be the same. But, with some transformation, we can bring it to the standard case of Theorem 6.1 of Resnick [21]. Throughout the paper, without loss of generality, suppose that $0\leq c_i<\infty, i=2,\ldots,d$ and $c_1=1$ in Assumption 2 which is a special case of the above relation (9). Moreover, $\alpha>1$ is assumed such that $E(\mathbf{Z})$ can be well defined.

Assumption 3. The components of the random vector $\mathbf{Z}$ are bivariate asymptotically independent, that is,

$$\lim_{x\longrightarrow\infty}P\left(Z_i>x\,\middle|\,Z_j>x\right)=0,\quad i\neq j. \quad (10)$$

It is well known that the Clayton family of Archimedean copulas with generator $\varphi(u)=(1/\theta)(u^{-\theta}-1)$ is asymptotically independent for each $\theta>0$. Other important families of copulas which are asymptotically independent include Gaussian copulas, and Farlie–Gumbel–Morgenstern copulas.

2.3. Main Results. We consider the d-dimensional discrete-time risk model $\mathbf{U}_l$ shown by (3) with the same constant interest rates over one period for all business lines. Due to the impact of interest rates in the above model, some results of the i.i.d. random vector sequence cannot be used directly to derive the asymptotic relation of the ruin probability. Under the assumption that the claim sizes follow a general MRV structure with capital transfer, the following Theorem 1 derives the asymptotic estimate of $\psi_{\boldsymbol{\omega}}(u,n)$.

Theorem 1. *Suppose that the multidimensional risk model $\mathbf{U}_l$ shown by (3) satisfies Assumption 1. Then, for any fixed integer $n\in\mathbb{N}$,*

$$\lim_{u\longrightarrow\infty}\frac{\psi_{\boldsymbol{\omega}}(u,n)}{P(\|\mathbf{Z}\|>u)}=\sum_{m=1}^{n}y_m^{\alpha}\mu(\mathbf{b}-\Gamma_{\boldsymbol{\omega}}). \quad (11)$$

Proof. Note that $u\Gamma_{\boldsymbol{\omega}}=\Gamma_{\boldsymbol{\omega}}$ for every real number $u>0$, and we obtain

$$\begin{aligned}\psi_{\boldsymbol{\omega}}(u,n)&=P\left(\mathbf{U}_l\in\Gamma_{\boldsymbol{\omega}}\text{ for some } l=1,\ldots,n\,\middle|\,\mathbf{U}_0=u\mathbf{b}\right)\\&\geq P\left(\mathbf{U}_n\in\Gamma_{\boldsymbol{\omega}}\,\middle|\,\mathbf{U}_0=u\mathbf{b}\right)\\&=P\left(\sum_{i=1}^{n}(\mathbf{Z}_i y_i-\mathbf{e}_{i-1}y_{i-1})\in u(\mathbf{b}-\Gamma_{\boldsymbol{\omega}})\right),\end{aligned} \quad (12)$$

where

$$\{\mathbf{b}-\Gamma_{\boldsymbol{\omega}}\}=\left\{\mathbf{x}:\ \sum_{k=1}^{d}\omega_k(b_k-x_k)_+<\sum_{k=1}^{d}(x_k-b_k)_+\right\}. \quad (13)$$

Since $\mathbf{Z}\in MRV_{\alpha,\mu}$, we have that $\mathbf{Z}-\mathbf{M}\in\mathbb{R}^d$ follows the same $MRV_{\alpha,\mu}$ structure where $\mathbf{M}$ is a constant vector. Note that $\mu(\partial\{\mathbf{b}-\Gamma_{\boldsymbol{\omega}}\})=0$. Then, we shall apply Proposition A.1 of Hult and Lindskog [22] to obtain

$$\begin{aligned}&\liminf_{u\longrightarrow\infty}\frac{\psi_{\boldsymbol{\omega}}(u,n)}{P(\sum_{i=1}^{n}\mathbf{Z}_i y_i\in u(\mathbf{b}-\Gamma_{\boldsymbol{\omega}}))}\\&\quad\geq\liminf_{u\longrightarrow\infty}\frac{P(\sum_{i=1}^{n}(\mathbf{Z}_i y_i-\mathbf{e}_{i-1}y_{i-1})\in u(\mathbf{b}-\Gamma_{\boldsymbol{\omega}}))}{P(\sum_{i=1}^{n}\mathbf{Z}_i y_i\in u(\mathbf{b}-\Gamma_{\boldsymbol{\omega}}))}=1.\end{aligned} \quad (14)$$

On the contrary, denote

$$\widetilde{U}_l=y_l^{-1}\left(u\mathbf{b}-\sum_{k=1}^{l}\mathbf{Z}_k y_k\right),\quad l=1,2,\ldots,n. \quad (15)$$

Now, we have

$$\psi_{\boldsymbol{\omega}}(u,n)\leq P\left(\widetilde{U}_l\in\Gamma_{\boldsymbol{\omega}}\text{ for some } l=1,\ldots,n\,\middle|\,\mathbf{U}_0=u\mathbf{b}\right). \quad (16)$$

Note that $\{\widetilde{U}_i \in \Gamma_{\boldsymbol{\omega}}, i = 1, 2, \ldots, n\}$ is a nondecreasing sequence of events. Then, we have

$$\begin{aligned}\psi_{\boldsymbol{\omega}}(u,n) &\le P\left(\widetilde{U}_n \in \Gamma_{\boldsymbol{\omega}} \mid \mathbf{U}_0 = u\mathbf{b}\right) \\ &= P\left(\sum_{i=1}^n \mathbf{Z}_i y_i \in u\left(\mathbf{b} - \Gamma_{\boldsymbol{\omega}}\right)\right).\end{aligned} \tag{17}$$

Therefore,

$$\limsup_{u\longrightarrow\infty} \frac{\psi_{\boldsymbol{\omega}}(u,n)}{P\left(\sum_{i=1}^n \mathbf{Z}_i y_i \in u\left(\mathbf{b} - \Gamma_{\boldsymbol{\omega}}\right)\right)} \le 1. \tag{18}$$

Combining (14) and (18) implies that

$$\lim_{u\longrightarrow\infty} \frac{\psi_{\boldsymbol{\omega}}(u,n)}{P\left(\sum_{i=1}^n \mathbf{Z}_i y_i \in u\left(\mathbf{b} - \Gamma_{\boldsymbol{\omega}}\right)\right)} = 1. \tag{19}$$

Finally, we apply Proposition A.1 of Hult and Lindskog [22] to derive that, for $u \longrightarrow \infty$,

$$\begin{aligned}\lim_{u\longrightarrow\infty} \frac{\psi_{\boldsymbol{\omega}}(u,n)}{P(\|\mathbf{Z}\| > u)} &= \lim_{u\longrightarrow\infty} \frac{P\left(\sum_{i=1}^n \mathbf{Z}_i y_i \in u\left(\mathbf{b} - \Gamma_{\boldsymbol{\omega}}\right)\right)}{P(\|\mathbf{Z}\| > u)} \\ &= \sum_{i=1}^n y_i^\alpha \mu\left(\mathbf{b} - \Gamma_{\boldsymbol{\omega}}\right).\end{aligned} \tag{20}$$

We obtain the desired result. □

Next, under the assumptions of asymptotic independence and comparable marginal tails, the following Proposition 1 gives a more explicit asymptotic estimate of the ruin probability.

Proposition 1. *Suppose that the multidimensional risk model $\mathbf{U}_l$ shown by (3) satisfies Assumptions 2 and 3. Then, for any fixed integer $n \in \mathbb{N}$,*

$$\lim_{u\longrightarrow\infty} \frac{\psi_{\boldsymbol{\omega}}(u,n)}{\overline{F}_1(u)} = \sum_{m=1}^n y_m^\alpha \sum_{i=1}^d c_i \left(b_i + \sum_{k\neq i}^d \omega_k b_k\right)^{-\alpha}. \tag{21}$$

Proof. Firstly, going along the same lines of the proofs of Lemma 3.2 of Li et al. [13] but with some modifications due to Assumption 2, by the inclusion-exclusion principle and Assumption 3, for any $\mathbf{x} > \mathbf{0}$,

$$\lim_{t\longrightarrow\infty} \frac{P\left(\mathbf{X}/t \in [\mathbf{0},\mathbf{x}]^c\right)}{P\left(\mathbf{X}/t \in [\mathbf{0},\mathbf{1}]^c\right)} = \lim_{t\longrightarrow\infty} \frac{P\left(\cup_{i=1}^d \{X_i > t x_i\}\right)}{P\left(\cup_{i=1}^d \{X_i > t\}\right)} = \frac{\sum_{i=1}^n c_i x_i^{-\alpha}}{\sum_{i=1}^n c_i}. \tag{22}$$

Then, by Theorem 6.1 of Resnick [21], we have $\mathbf{Z} \in MRV_{\alpha,\mu}$.

Next, according to Resnick [21], Assumption 3 means that μ puts zero mass in the interior of the positive quadrant, that is, the measure μ is concentrated on each axis. Thus,

$$\begin{aligned}\mu\left(\mathbf{b} - \Gamma_{\boldsymbol{\omega}}\right) &= \mu\left(\left\{\mathbf{x}\colon \sum_{k=1}^d \omega_k \left(b_k - x_k\right)_+ < \sum_{k=1}^d \left(x_k - b_k\right)_+\right\}\right) \\ &= \mu\left(\left\{\mathbf{x}\colon \bigcup_{i=1}^d \left\{x_i > b_i + \sum_{k\neq i}^d \omega_k b_k\right\}\right\}\right).\end{aligned} \tag{23}$$

Then, we shall apply Assumption 2 in the third step of the following display to obtain

$$\begin{aligned}\mu\left(\mathbf{b} - \Gamma_{\boldsymbol{\omega}}\right) &= \sum_{i=1}^d \mu\left(\left\{\mathbf{x}\colon x_i > b_i + \sum_{k\neq i}^d \omega_k b_k\right\}\right) \\ &= \sum_{i=1}^d \lim_{u\longrightarrow\infty} \frac{P\left(Z_i > u\left(b_i + \sum_{k\neq i}^d \omega_k b_k\right)\right)}{P(\|\mathbf{Z}\| > u)} \\ &= \sum_{i=1}^d c_i \left(b_i + \sum_{k\neq i}^d \omega_k b_k\right)^{-\alpha} \lim_{u\longrightarrow\infty} \frac{\overline{F}_1(u)}{P(\|\mathbf{Z}\| > u)}.\end{aligned} \tag{24}$$

Combining Theorem 1 and (24), we obtain the desired result. □

2.4. Examples. In this section, we propose to use a copula to model the dependence among the components of the claim size vector with a continuous joint distribution. The following two examples take the form of excess of loss ratio reinsurance and proportional reinsurance policy, respectively.

Example 1. Suppose that an insurer with d lines of business purchases a proportional reinsurance policy, and the reinsurer takes a stated percentage share of each policy that the insurer issues. Particularly, under quota share arrangement, a fixed proportion of each business line is reinsured, that is, the insurer retains proportion $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_d)^T \in (0,1)^d$ of the claim sizes. Suppose the claim size vectors are i.i.d. as a generic random vector $\mathbf{Z} = (Z_1, \ldots, Z_d)^T$ which meets the following conditions:

(1) The components of $\mathbf{Z}$ follow the Pareto distribution for shape parameter $\alpha > 1$ and especially $Z_i \sim$ Pareto (γ_i, α) for some $\gamma_i > 0$, $i = 1, \ldots, d$, that is,

$$F_i(x_i) = \begin{cases} 0, & x_i < \gamma_i, \\ 1 - \left(\dfrac{\gamma_i}{x_i}\right)^\alpha, & x_i \ge \gamma_i. \end{cases} \tag{25}$$

(2) The underlying copula of $\mathbf{Z}$ is a Clayton copula with generator $\varphi(u) = (1/2)(u^{-2} - 1)$.

It is easy to verify that $\mathbf{Z}$ has the joint distribution

$$G(\mathbf{x}) = \left(1 - d + \sum_{i=1}^d F_i^{-2}(x_i)\right)^{-(1/2)}, \quad x_i > 0,\ i = 1, \ldots, d, \tag{26}$$

and the components of $\mathbf{Z}$ are tail equivalent and bivariate asymptotically independent. Let $\widetilde{Z} = (\eta_1 Z_1, \eta_2 Z_2, \ldots, \eta_d Z_d)^T$. Under the reinsurance policy, the claim sizes that the insurer should pay are i.i.d. as the random vector $\widetilde{Z}$. The distributions of the components of $\widetilde{Z}$ are still tail equivalent, and the dependence structure of $\widetilde{Z}$ is the same as $\mathbf{Z}$. Thus, according to Proposition 1, when the initial reserve u is large enough, the asymptotic estimate of ruin probability of the insurer under quota share arrangement is

$$\psi_{\boldsymbol{\omega}}(u,n) \sim \sum_{m=1}^{n} y_m^{\alpha} \sum_{i=1}^{d} \gamma_i^{\alpha} \eta_i^{\alpha} \left(b_i + \sum_{k \neq i}^{d} \omega_k b_k \right)^{-\alpha} u^{-\alpha}. \quad (27)$$

Similarly, the asymptotic estimate of ruin probability of the reinsurer under quota share arrangement is

$$\psi_{\widetilde{\omega}}(\widetilde{u},n) \sim \sum_{m=1}^{n} y_m^{\alpha} \sum_{i=1}^{d} \gamma_i^{\alpha} (1-\eta_i)^{\alpha} \left(\widetilde{b}_i + \sum_{k \neq i}^{d} \widetilde{\omega}_k \widetilde{b}_k \right)^{-\alpha} \widetilde{u}^{-\alpha}, \quad (28)$$

where $\widetilde{u}$, $\widetilde{\omega} = (\widetilde{\omega}_1, \widetilde{\omega}_2, \ldots, \widetilde{\omega}_d)^T$, and $\widetilde{b} = (\widetilde{b}_1, \widetilde{b}_2, \ldots, \widetilde{b}_d)^T$ are the initial reserve, the capital transfer ratio vector, and the allocation vector for the reinsurer, respectively.

Example 2. Suppose that the reinsurer sells an excess of loss ratio reinsurance policy to an insurer with d lines of business. The reinsurer only pays out if the total claims suffered by the insurer in a given period exceed a predetermined loss ratio $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_d)^T \in (0,1)^d$ of the premium $\mathbf{e} = (e_1, e_2, \ldots, e_d)^T$ collected by the insurer, that is, the reinsurer has agreed to bear any balance so that the insurer's gross loss ratio is maintained at $\boldsymbol{\eta}$. Suppose the claim size vectors are i.i.d. as a generic random vector $\mathbf{Z} = (Z_1, \ldots, Z_d)^T$ with $Z_i \sim \text{Pareto}(\gamma_i, \alpha)$ for some $\gamma_i > 0$, $i = 1, \ldots, d$, and $\mathbf{Z}$ has the joint distribution

$$G(\mathbf{x}) = \left(1 + \theta \prod_{i=1}^{d} \overline{F}_i(x_i) \right) \prod_{i=1}^{d} F_i(x_i), \quad \theta \in [-1, 1], \quad (29)$$

which implies that the underlying copula of $\mathbf{Z}$ is a Farlie–Gumbel–Morgenstern copula. It is clear that the components of $\mathbf{Z}$ are tail equivalent and bivariate asymptotically independent. So, the claim sizes that the reinsurer affords are i.i.d. as $\widetilde{Z} = ((Z_1 - \eta_1 e_1)_+, (Z_2 - \eta_2 e_2)_+, \ldots, (Z_d - \eta_d e_d)_+)^T$. The distributions of the components of $\widetilde{Z}$ are still tail equivalent, and the dependence structure of $\widetilde{Z}$ is the same as $\mathbf{Z}$. According to Proposition 1, when the initial reserve u is large enough, the asymptotic estimate of ruin probability of the reinsurer under excess of loss ratio reinsurance is

$$\psi_{\boldsymbol{\omega}}(u,n) \sim \sum_{m=1}^{n} y_m^{\alpha} \sum_{i=1}^{d} \gamma_i^{\alpha} \left(b_i + \sum_{k \neq i}^{d} \omega_k b_k \right)^{-\alpha} u^{-\alpha}. \quad (30)$$

From the above formula, we can observe that the excess of loss ratio $\boldsymbol{\eta}$ has no effect on reducing risk for the reinsurer, and they may turn to the other types of reinsurance such as quota share arrangement or design the capital transfer ratio $\boldsymbol{\omega}$ and the allocation $\mathbf{b}$.

3. Initial Reserve Allocation

In this section, the optimal allocation of the initial reserve is investigated. For some similar discussions on the optimization problem in insurance, see Biard [23]; Li et al. [13]; and Yu et al. [24], among many others. Suppose that the insurer owns a global initial reserve u to allocate to d lines of business. Let $b_i \in [0,1]$ be the proportion of capital allocated to the i-th business line. To minimize the asymptotic finite-time ruin probability, we have to find the optimal allocation of the initial reserve u. The problem can be formulated as the computation of the following optimization problem:

$$\begin{aligned} &\min_{\mathbf{b} \in [0,1]^d} \psi_{\boldsymbol{\omega}(u,n)} \\ &\text{s.t.} \sum_{i=1}^{d} b_i = 1. \end{aligned} \quad (31)$$

According to Proposition 1, when the initial reserve u is large enough,

$$\psi_{\boldsymbol{\omega}}(u,n) \sim \overline{F}_1(u) \left(\sum_{m=1}^{n} y_m^{\alpha} \sum_{i=1}^{d} c_i \left(b_i + \sum_{k \neq i}^{d} \omega_k b_k \right)^{-\alpha} \right). \quad (32)$$

Given $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d)^T \in [0,1]^d$, it is sufficient to find the minimizer in $[0,1]^d$ of the following function

$$f(b_1, \ldots, b_d) = \sum_{i=1}^{d} c_i \left(b_i + \sum_{k \neq i}^{d} \omega_k b_k \right)^{-\alpha}, \quad (33)$$

where $\sum_{i=1}^{d} b_i = 1$.

First, consider the case that the insurer owns only two business lines with the same interest rate over one period.

Proposition 2. *Suppose that the risk model*

$$\mathbf{U}_l = \begin{pmatrix} y_l^{-1} \left(u b_1 - \sum_{k=1}^{l} \left(Z_{1,k} y_k - e_{1,k-1} y_{k-1} \right) \right) \\ y_l^{-1} \left(u b_2 - \sum_{k=1}^{l} \left(Z_{2,k} y_k - e_{2,k-1} y_{k-1} \right) \right) \end{pmatrix}, \quad l = 1, 2, \ldots, \quad (34)$$

satisfies Assumptions 2 and 3. For $\omega_1, \omega_2 \in [0,1]$ and $\alpha > 1$,

(a) If $\omega_2 = \omega_1 = 1$, then any $b_1 \in [0,1]$ is the optimal solution for the problem of (31).

(b) If $\omega_2 \neq 1$ or $\omega_1 \neq 1$, then the problem of (31) has the optimal solution

$$b^* = \begin{cases} 0, & a > \dfrac{1}{\omega_2} \text{ or } \omega_2 = 1, \\ \dfrac{1 - a\omega_2}{1 - a\omega_2 + a - \omega_1}, & \omega_1 \leq a \leq \dfrac{1}{\omega_2}, \\ 1, & a < \omega_1 \text{ or } \omega_1 = 1, \end{cases} \quad (35)$$

with $a = (c_2(1-\omega_1)/c_1(1-\omega_2))^{(1/\alpha+1)}$.

Proof. By Proposition 1, it suffices to find the critical points in $[0,1]$ of the following function:

$$\begin{aligned} L(b_1) &= c_1 (b_1 + \omega_2 b_2)^{-\alpha} + c_2 (b_2 + \omega_1 b_1)^{-\alpha} \\ &= c_1 (\omega_2 + (1-\omega_2) b_1)^{-\alpha} + c_2 (1 + (\omega_1 - 1) b_1)^{-\alpha}. \end{aligned} \tag{36}$$

Let $L(x) = c_1 (\omega_2 + (1-\omega_2)x)^{-\alpha} + c_2 (1 + (\omega_1 - 1)x)^{-\alpha}$; then, we have

$$\begin{aligned} \frac{\partial L(x)}{\partial x} = (-\alpha)\big[& c_1 (\omega_2 + (1-\omega_2)x)^{-\alpha-1} (1-\omega_2) \\ & + c_2 (1 + (\omega_1 - 1)x)^{-\alpha-1} (\omega_1 - 1)\big]. \end{aligned} \tag{37}$$

For $\omega_2 = \omega_1 = 1$, we have $L(x) = c_1 + c_2$; thus, any $x \in [0,1]$ is the optimal solution.

We now consider the case $\omega_1 \neq 1$ or $\omega_2 \neq 1$. Note that $a = (c_2 (1-\omega_1)/c_1 (1-\omega_2))^{(1/\alpha+1)}$.

For $a > (1/\omega_2)$ or $\omega_2 = 1$, the function $L(x)$ increases in $x \in [0,1]$ and takes minimum at $x = 0$.

For $\omega_1 \leq a \leq (1/\omega_2)$, we have $(\partial L(x)/\partial x) < 0$ in $x \in [0, x^*)$ and $(\partial L(x)/\partial x) > 0$ in $x \in (x^*, 1]$, where $x^* = (1 - a\omega_2 / 1 - a\omega_2 + a - \omega_1)$. Thus, the function $L(x)$ is minimized at x^*.

For $a < \omega_1$ or $\omega_1 = 1$, $L(x)$ decreases in $x \in [0,1]$ and takes minimum at $x = 1$. □

We now move forward to the general case. The optimal solution for the problem must exist because the Hessian matrix of $f(b_1, \ldots, b_d)$ can be expressed as $\alpha(\alpha+1)\mathbf{W}[\mathbf{C}\mathbf{H}^{-\alpha-2}]\mathbf{W}^T$ which is positively definite in $[0,1]^d$, where $h_i = b_i + \sum_{k \neq i}^{d} \omega_k b_k$, $i = 1, \ldots, d$ and

$$\mathbf{W} = \begin{pmatrix} 1 & \omega_1 & \omega_1 & \ldots & \omega_1 \\ \omega_2 & 1 & \omega_2 & \ldots & \omega_2 \\ \omega_3 & \omega_3 & 1 & \ldots & \omega_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_d & \omega_d & \omega_d & \ldots & 1 \end{pmatrix}_{d \times d},$$
$$\mathbf{C} = \begin{pmatrix} c_1 & 0 & \ldots & 0 \\ 0 & c_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & c_d \end{pmatrix}_{d \times d}, \tag{38}$$
$$\mathbf{H} = \begin{pmatrix} h_1 & 0 & \ldots & 0 \\ 0 & h_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & h_d \end{pmatrix}_{d \times d}.$$

Subject to the equality constraint $\sum_{i=1}^{d} b_i = 1$, the method of Lagrange multipliers is a good strategy to find the minimum of the function $f(b_1, \ldots, b_d)$. The optimal solution in $(0,1)^d$ for the general case is given in Proposition 3 in the following. See also Proposition 4.2 of Li et al. [13] for the similar result. The algorithm was sketched to reach the optimal solution of allocation $\mathbf{b}$ in Li et al. [13].

Proposition 3. *Under Assumptions 2 and 3, if* $(\mathbf{W}^T)^{-1} (\mathbf{C}^{-1}\mathbf{W}^{-1}\mathbf{1}^T)^{-(1/\alpha+1)} > \mathbf{0}$ *and* $\mathbf{C}^{-1}\mathbf{W}^{-1}\mathbf{1}^T > \mathbf{0}$, *then the optimal solution in* $(0,1)^d$ *for problem (31) can be expressed as*

$$\mathbf{b}^* = \frac{(\mathbf{W}^T)^{-1} (\mathbf{C}^{-1}\mathbf{W}^{-1}\mathbf{1}^T)^{-(1/\alpha+1)}}{\mathbf{1}(\mathbf{W}^T)^{-1} (\mathbf{C}^{-1}\mathbf{W}^{-1}\mathbf{1}^T)^{-(1/\alpha+1)}}. \tag{39}$$

Proof. Going along the same lines of the proofs of Proposition 4.2 of Li et al. [13] but with some obvious modifications, we can prove Proposition 3 immediately. Thus, we omit the details here. □

4. Numerical Studies

In this section, we conduct numerical studies to examine the accuracy of the asymptotic relations. We choose the Clayton copula with generator $\varphi(u) = (1/2)(u^{-2} - 1)$. Assume that the claim size vectors are i.i.d. as $\mathbf{Z}$ whose components follow the Pareto distribution for shape parameter $\alpha > 1$, and especially, $Z_1 \sim \text{Pareto}(1, 1.5)$. With the settings above, $\mathbf{Z}$ has the joint distribution

$$G(\mathbf{x}) = \left(-2 + \sum_{i=1}^{3} F_i^{-2}(x_i) \right)^{-1/2}, \quad x_i > 0,\ i = 1, 2, 3. \tag{40}$$

Various parameters are set to be $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T = (0.5, 0.5, 0.5)^T$, $\mathbf{b} = (b_1, b_2, b_3)^T = (0.3, 0.4, 0.3)^T$, $c_1 = 1$, $c_2 = 0.9$, and $c_3 = 1.3$. The premium charged is assumed to be $1.2E(Z_i)$, $i = 1, 2, 3$.

We generate samples in the R environment using the copula package, and the ruin probability is obtained by crude Mote Carlo simulation. Letting insurance period $n = 10$, we compare asymptotic estimate given by (21) and $\psi_{\boldsymbol{\omega}}(u, n)$ by a simulation of 1,000,000 rounds on the left and show their ratios on the right in Figure 1. As is seen, the asymptotic estimate and the ruin probability are relatively close on the left, and the ratios do converge to 1 on the right when u is less than 4000. However, the ratio of Figure 1 seems not to converge to 1 when u goes to infinity. Our explanation for the phenomenon is the limitation of crude Mote Carlo simulation since the ruin probability becomes too small as u goes to infinity. With the sample size being increased to 10,000,000 in Figure 2, the convergence of ratio is much improved compared with Figure 1. In conclusion, the convergence is stable, and this confirms the efficiency of the asymptotic estimate in Proposition 1.

Next, we turn to consider the optimal allocation of the initial reserve given $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T = (0.5, 0.5, 0.5)^T$ and $c_1 = c_2 = c_3 = 1$. Under different initial reserves with different allocations $\mathbf{b}$, Table 1 lists ruin probabilities by a simulation of 1,000,000 rounds of the case $n = 10$ and the case $n = 30$, respectively. It is easy to verify that $b^* = ((1/3), (1/3), (1/3))^T$ is the optimal allocation by Proposition 3. As is seen in Table 1, the ruin probability achieves the smallest value at b^* with different initial reserves u in both cases.

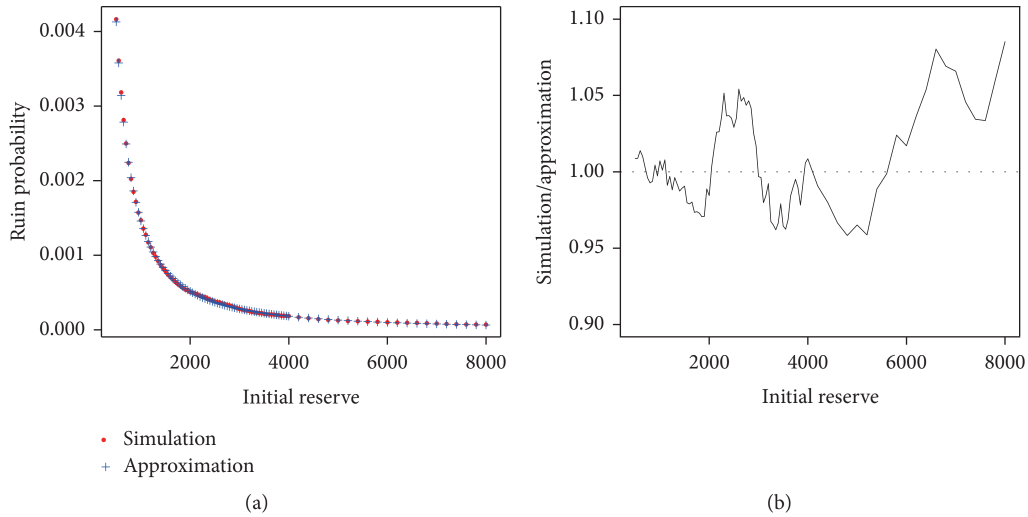

Figure 1: Comparison of estimates and approximate of ruin probability ($n = 10$, 1,000,000 rounds).

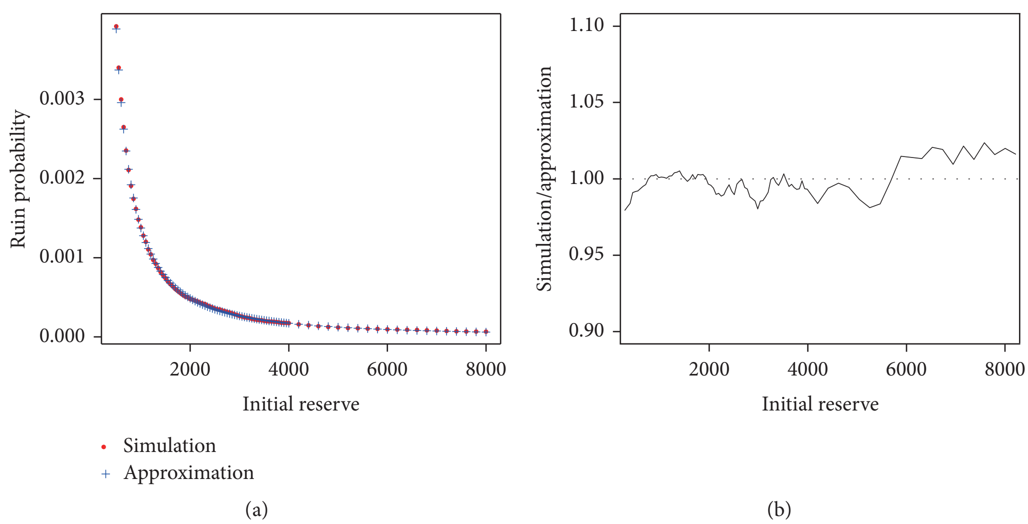

Figure 2: Comparison of estimates and approximate of ruin probability ($n = 10$, 10,000,000 rounds).

Table 1: Ruin probability under allocation of **b**.

Insurance period	Initial reserve	Allocation of **b**				
		$((1/3), (1/3), (1/3))^T$	$(0.4, 0.3, 0.3)^T$	$(0.5, 0.2, 0.3)^T$	$(0.2, 0.6, 0.2)^T$	$(0.1, 0.4, 0.5)^T$
10	1000	0.001081	0.001092	0.001104	0.001110	0.001134
	3000	0.000192	0.000193	0.000198	0.000209	0.000204
	5000	0.000076	0.000076	0.000077	0.000078	0.000087
30	1000	0.001860	0.001877	0.001919	0.001914	0.001938
	3000	0.000379	0.000384	0.000382	0.000393	0.000385
	5000	0.000160	0.000161	0.000166	0.000168	0.000169

Data Availability

All data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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