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Research Article

Characterization of Skew CR-Warped Product Submanifolds in Complex Space Forms via Differential Equations

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Recently, we have obtained Ricci curvature inequalities for skew CR-warped product submanifolds in the framework of complex space form. By the application of Bochner's formula on these inequalities, we show that, under certain conditions, the base of these submanifolds is isometric to the Euclidean space. Furthermore, we study the impact of some differential equations on skew CR-warped product submanifolds and prove that, under some geometric conditions, the base is isometric to a special type of warped product.

1. Introduction

The studies [1, 2] provide both important intrinsic geometric as well as isometric properties of Riemannian manifolds via differential equations. It is well known that the classification of differential equations has a significant effect on the global study of Riemannian manifolds. In 1978, Tanno [2] studied various aspects of differential equations on Riemannian manifolds. In particular, the authors of [3, 4] characterized Euclidean sphere by the approach of differential equations. The analysis performed in [2, 5] proved that a nonconstant function λ on a complete Riemannian manifold (U^n, g) satisfies the differential equation

$$\nabla^2\lambda + cg = 0, \tag{1}$$

if and only if (U^n, g) is isometric to Euclidean space R^n, where c is constant.

Moreover, Garcia-Rio et al. [4] proved that, under some restrictions, the Riemannian manifold is isometric to warped product $U \times_f R$, where U is a complete Riemannian manifold, R is the Euclidean line, and f is the warping function. Moreover, warping function f satisfies the second-order differential equation

$$\frac{d^2 f}{dt^2} + \mu_1 f = 0, \tag{2}$$

if and only if there exist a nonconstant function ϕ: $U^n \longrightarrow R$ with a negative eigenvalue $\mu_1 \leq 0$, which is the solution of following differential equation:

$$\Delta\phi + \mu_1\phi = 0. \tag{3}$$

The categorization of differential equations on Riemannian manifold has become a fascinating topic of research and has been investigated by numerous researchers, for instance, [6–9].

Recently, Al-Dayel et al. [6] studied the impact of differential equation (2) on Riemannian manifold (L^n, g) by taking the concircular vector field and proved that, under certain conditions, the Riemannian manifold (L^n, g) is isometric to Euclidean manifold R^n. Similarly, by taking gradient conformal vector field, Chen et al. [10] identified that Riemannian manifold (N^n, g) is isometric to the Euclidean space R^n. However, in [11], it has been proved that the complete totally real submanifold in CP^n (complex projective space) with bounded Ricci curvature satisfying (3) is isometric to a special class of hyperbolic space.

On the contrary, Bishop and O'Neill [12] studied the geometry of manifolds having negative curvature and confirmed that Riemannian product manifolds always have nonnegative curvature. As a result, they proposed the idea of warped product manifolds, and these manifolds are defined as follows.

Consider two Riemannian manifolds (L_1, g_1) and (L_2, g_2) with corresponding Riemannian metrics g_1 and g_2, and let $\psi\colon L_1 \longrightarrow R$ be a positive differentiable function. If x and y are projection maps such that $x\colon L_1 \times L_2 \longrightarrow L_1$ and $y\colon L_1 \times L_2 \longrightarrow L_2$, which are defined as $x(m,n) = m$ and $y(m,n) = n \forall\ (m,n) \in L_1 \times L_2$, then $L = L_1 \times L_2$ is called warped product manifold if the Riemannian structure on L satisfies

$$g(\overline{E},\overline{F}) = g_1(x_*\overline{E}, x_*\overline{F}) + (\psi^\circ x)^2 g_2(y_*\overline{E}, y_*\overline{F}), \tag{4}$$

for all $\overline{E},\overline{F} \in TL$. The function ψ represents the warping function of $L_1 \times L_2$. The Riemannian product manifold is a special case of warped product manifold in which the warping function is constant. The study of Bishop and O'Neill [12] revealed that these types of manifolds have wide range of applications in physics and theory of relativity. It is well known that the warping function is the solution of some partial differential equations, for example, Einstein field equation can be solved by the approach of the warped product [13]. The warped product is also applicable in the study of space time near to black holes [14].

Latterly, Ali et al. [7] characterized warped product submanifolds in Sasakian space form by the approach of the differential equation. The purpose of this paper is to study the impact of differential equation on skew CR-warped product submanifolds in the framework of the complex space form.

2. Preliminaries

Let $\overline{L}$ be an almost Hermitian manifold with an almost complex structure J and a Hermitian metric g, i.e., $J^2 = -I$ and $g(JX, JY) = g(X, Y)$, for all vector fields X, Y on $\overline{L}$. If the almost complex structure J satisfies

$$(\overline{D}_X J)Y = 0, \tag{5}$$

for all $X, Y \in T\overline{L}$, where $\overline{D}$ is Levi-Civita connection on $\overline{L}$, then $(\overline{L}, J)$ is called a Kaehler manifold.

A Kaehler manifold $\overline{L}$ is called a *complex space form* if it has constant holomorphic sectional curvature and denoted by $\overline{L}(c)$. The curvature tensor of the complex space form $\overline{L}(c)$ is given by

$$\begin{aligned}\overline{R}(X,Y,Z,W) = \frac{c}{4}[&g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(X,JZ)g(JY,W)\\ &- g(Y,JZ)g(JX,W) + 2g(X,JY)g(JZ,W)],\end{aligned} \tag{6}$$

for any $X, Y, Z, W \in T\overline{L}$.

Let L be an n–dimensional Riemannian manifold isometrically immersed in a m–dimensional Riemannian manifold $\overline{L}$. Then, the Gauss and Weingarten formulas are $\overline{D}_X Y = D_X Y + h(X,Y)$ and $\overline{D}_X \xi = -A_\xi X + D_X^\perp \xi$, respectively, for all $X, Y \in TL$ and $\xi \in T^\perp L$, where D is the induced Levi-Civita connection on L, ξ is a vector field normal to L, h is the second fundamental form of L, $D^\perp$ is the normal connection in the normal bundle $T^\perp L$, and A_ξ is the shape operator of the second fundamental form. For any $X \in TL$ and $N \in T^\perp L$, JX and JN can be decomposed as follows:

$$\begin{aligned} JX &= PX + FX,\\ \text{and } JN &= tN + fN,\end{aligned} \tag{7}$$

where PX (respectively, tN) is the tangential and FX (respectively, fN) is the normal component of JE (respectively, JN).

It is evident that $g(\phi X, Y) = g(PX, Y)$, for any $X, Y \in T_x L$; this implies that $g(PX, Y) + g(X, PY) = 0$. Thus, P^2 is a symmetric operator on the tangent space $T_x L$, for all $x \in L$. The eigenvalues of P^2 are real and diagonalizable. Moreover, for each $x \in L$, one can observe

$$S_x^\lambda = \text{Ker}\{P^2 + \mu^2(x) I\}_x, \tag{8}$$

where I denotes the identity transformation on $T_x L$ and $\mu(x) \in [0,1]$ such that $-\mu^2(x)$ is an eigenvalue of $P^2(x)$. Furthermore, it is easy to observe that $\text{Ker}F = S_x^1$ and $\text{Ker}P = S_x^0$, where S_x^1 is the maximal holomorphic subspace of $T_x L$ and S_x^0 is the maximal totally real subspace of $T_x L$, and these distributions are denoted by S^T and $S^\perp$, respectively. If $-\mu_1^2(x), \ldots, -\mu_k^2(x)$ are the eigenvalues of P^2 at x, then $T_x L$ can be decomposed as

$$T_x L = S_x^{\mu_1} \oplus S_x^{\mu_2} \oplus \cdots \oplus S_x^{\mu_k}. \tag{9}$$

Every $S_x^{\lambda_i}$, $1 \le i \le k$, is a P–invariant subspace of $T_x L$. Moreover, if $\mu_i \ne 0$, then $S_x^{\lambda_i}$ is even dimensional; the submanifold L of a Kaehler manifold $\overline{L}$ is a generic submanifold if there exists an integer k and functions μ_i $1 \le i \le k$ defined on L with $\mu_i \in (0,1)$ such that

(i) Each $-\mu_i^2(x), 1 \le i \le k$, is a distinct eigenvalue of P^2 with

$$T_x L = S_x^T \oplus S_x^\perp \oplus S_x^{\lambda_1} \oplus \cdots \oplus S_x^{\lambda_k}, \tag{10}$$

for any $x \in M$.

(ii) The distributions of $S_x^T, S_x^\perp$ and $S_x^{\mu_i}, 1 \le i \le k$, are independent of $x \in L$.

If, in addition, each μ_i is constant on L, then L is called a skew CR-submanifolds [15]. It is significant to account that

CR-submanifolds are particular class of skew CR-submanifold with $k = 1, S^T = \{0\}, S^\perp = \{0\}$, and μ_1 is constant. If $S^\perp = \{0\}$, $S_T^\perp \neq \{0\}$, and $k = 1$, then L is semislant submanifold, whereas if $S_T = \{0\}, S^\perp \neq \{0\}$, and $k = 1$, then L is a hemi-slant submanifold.

Definition 1. A submanifold L of a Kaehler manifold $\overline{L}$ is said to be a skew CR-submanifold of order 1 if L is a skew CR-submanifold with $k = 1$ and μ_1 is constant. For any orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space T_xL, the mean curvature vector $\Omega(x)$ and its squared norm are defined as follows:

$$\Omega(x) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$
$$\|\Omega\|^2 = \frac{1}{n^2}\sum_{i,j=1}^{n} g\big(h(e_i, e_i), h(e_j, e_j)\big), \tag{11}$$

where n is the dimension of L. If $h = 0$, then the submanifold is said to be totally geodesic and minimal if $\Omega = 0$. If $h(E_1, E_2) = g(E_1, E_2)\Omega$, for all $E_1, E_2 \in TL$, then L is called totally umbilical.

The scalar curvature of $\overline{L}$ is denoted by $\overline{\tau}(L)$ and is defined as

$$\tau(L) = \sum_{1\le\alpha<\beta\le m} \kappa_{\alpha\beta}, \tag{12}$$

where $\kappa_{\alpha\beta} = \overline{\kappa}(e_\alpha \wedge e_\beta)$ and m is the dimension of the Riemannian manifold $\overline{L}$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_xL, and if e_γ belongs to the orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the normal space $T^\perp L$, then we have

$$h^\gamma_{\alpha\beta} = g\big(h(e_\alpha, e_\beta), e_\gamma\big),$$
$$\|h\|^2 = \sum_{\alpha,\beta=1}^{n} gh\big(e_\alpha, e_\beta\big), h\big(e_\alpha, e_\beta\big). \tag{13}$$

The global tensor field for orthonormal frame of vector field $\{e_1, \ldots, e_n\}$ on L is defined as

$$S(E_1, E_2) = \sum_{i=1}^{n}\{g(R(e_i, E_1)E_2, e_i)\}, \tag{14}$$

for all $E_1, E_2 \in T_xL$, where R is the Riemannian curvature tensor. The above tensor is called the Ricci tensor. If we fix a distinct vector e_u from $\{e_1, \ldots, e_n\}$ on L^n, which is governed by χ, then the Ricci curvature is defined by

$$\mathrm{Ric}(\chi) = \sum_{\substack{\alpha=1\\ \alpha\neq u}}^{n} \kappa(e_\alpha \wedge e_u). \tag{15}$$

A submanifold L of a Kaehler manifold $\overline{L}$ is said to be skew CR-warped product submanifolds if it is warped product of the type $L = L_1 \times_f L_\perp$, where L_1 is a semislant submanifold which was defined by N. Papaghiuc [16] and $L_\perp$ is a totally real submanifold. Sahin [17] proved the existence skew CR-warped product submanifolds. Recently, Ali Khan and Al-Dayel [18] studied Skew CR-warped product submanifolds of the form $L^d = L_1^{d_1+d_2} \times_f L_\perp^{d_3}$, where L_1 is semislant submanifold with d_1–dimensional holomorphic distribution S and d_2– dimensional slant distribution S_θ and $L_\perp^{d_3}$ is totally real submanifold. More precisely, they obtained Ricci curvature inequalities for these submanifolds as follows.

Theorem 1. *Let $L^d = L_1^{d_1+d_2} \times_f L_\perp^{d_3}$ be a S–minimal complete skew CR-warped product submanifold isometrically immersed in a Complex space form $\overline{L}^m(c)$. If the holomorphic and slant distributions S and S_θ are integrable with integral submanifolds $L_T^{d_1}$ and $L_\theta^{d_2}$, respectively, then, for each orthogonal unit vector field $\chi \in T_xL^d$, either tangent to $L_T^{d_1}$ or $L_\theta^{d_2}$ or $L_\perp^{d_3}$, the Ricci curvature satisfies the following expressions.*

(i) If $\chi \in TL_T^{d_1}$, then

$$R^L(\chi) + d_3\Delta ln f \le \frac{1}{4}d^2\|\Omega\|^2 + d_3\|\nabla ln f\|^2 - \frac{c}{4}\left(d - d_1d_2 - d_2d_3 - d_1d_3 - \frac{1}{2}\right). \tag{16}$$

(ii) If $\chi \in TL_\theta^{d_2}$, then

$$R^L(\chi) + d_3\Delta ln f \le \frac{1}{4}d^2\|\Omega\|^2 + d_3\|\nabla ln f\|^2 - \frac{c}{4}\left(d - d_1d_2 - d_2d_3 - d_1d_3 + 1 - \frac{3}{2}\cos^2\theta\right). \tag{17}$$

(iii) If $\chi \in TL_\perp^{d_2}$, then

$$R^L(\chi) + d_3\Delta ln f \le \frac{1}{4}d^2\|\Omega\|^2 + d_3\|\nabla ln f\|^2 - \frac{c}{4}\left(d - d_1d_2 - d_2d_3 - d_1d_3 + 1\right), \tag{18}$$

where d_1, d_2, and d_3 are the dimensions of $S_T^{d_1}$, $S_\theta^{d_2}$, and $S_\perp^{d_3}$, respectively, and Ω is the mean curvature vector.

Let f be a real-valued differential function on a Riemannian manifold L^n; then, the Bochner formula [19] is stated as

$$\frac{1}{2}\Delta|\nabla f|^2 = R^L(\nabla f, \nabla f) + |H(f)|^2 + g(\nabla\Delta f, \nabla f), \tag{19}$$

where R^L denotes Ricci tensor and $H(f)$ is the Hessian of the function f.

3. Main Results

In this section, we obtain some characterization by the application of Bochner formula.

Theorem 2. *Let $L^d = L_1^{d_1+d_2} \times_f L_\perp^{d_3}$ be a $S-$minimal $d-$dimensional complete skew CR-warped product submanifold in a complex space form $\overline{L}^m(c)$, where L_1 is a semislant submanifold with d_1- dimensional invariant distribution S and d_2-dimensional slant distribution S_θ, such that Ricci curvature $R^L(\chi) \geq K, K>0$. If $\chi \in S$ and satisfying the equality,*

$$(\lambda_1 + d_3)K = \lambda_1 \left[\frac{d_3}{d} + \frac{d^2}{4}\|\Omega\|^2 - \frac{c}{4}\left(d - d_1d_2 - d_2d_3 - d_1d_3 - \frac{1}{2}\right)\right], \quad (20)$$

then the base submanifold $L_1^{d_1+d_2}$ is isometric to Euclidean space $R^{d_1+d_2}$, where d_3 is the dimension of the anti-invariant submanifold $L_\perp^{d_3}$ and λ_1 is the eigenvalue corresponding to the eigenfunction lnf.

Proof. Since $\chi \in S$, by equation (16),

$$R^L(\chi) + d_3\Delta lnf \leq \frac{1}{4}d^2\|\Omega\|^2 + d_3\|\nabla lnf\|^2 - \frac{c}{4}\left(d - d_1d_2 - d_2d_3 - d_1d_3 - \frac{1}{2}\right). \quad (21)$$

By the assumption that $R_L(\chi) \geq K$, we have

$$K + d_3\Delta lnf \leq \frac{1}{4}d^2\|\Omega\|^2 + d_3\|\nabla lnf\|^2 - \frac{c}{4}\left(d - d_1d_2 - d_2d_3 - d_1d_3 - \frac{1}{2}\right). \quad (22)$$

Since $R^L(\chi) \geq K$, on applying the theorem of Myers [1], according to this, if Ricci curvature is greater than by a positive constant, then base manifold $L_1^{d_3}$ is compact. On integrating (22) and using Green's theorem, we obtain

$$\mathrm{Vol}(L_1)K \leq \frac{d^2}{4}\int_{L_t}\|\Omega\|^2 dV + d_3\int_{L^d}\|\nabla lnf\|^2 dV - \int_{L^d}\frac{c}{4}\left(d - d_1d_2 - d_2d_3 - d_1d_3 - \frac{1}{2}\right)dV, \quad (23)$$

or

$$\int_{L^d}\|\nabla lnf\|^2 dV \geq \frac{K}{d_3}\mathrm{Vol}(L_1) - \frac{d^2}{4d_3}\int_{L^d}\|\Omega\|^2 dV + \frac{1}{d_3}\int_{L^d}\frac{c}{4}\left(d - d_1d_2 - d_2d_3 - d_1d_3 - \frac{1}{2}\right)dV. \quad (24)$$

Let $H(lnf)$ be the Hessian of the warping function lnf; then, we have

$$|H(lnf) - nI|^2 = |H(lnf)|^2 + n^2|I|^2 - 2ng(I, H(lnf)), \quad (25)$$

where n is real number. The above formula provides

$$|H(lnf) - nI|^2 = 2n\Delta(lnf) + n^2(d_1+d_2) + |H(lnf)|^2. \quad (26)$$

Putting $n = (\lambda_1/(d_1+d_2))$ and integrating the last equation with respect to dV (volume element), we obtain

$$\int_{L^d}\left|H(lnf) - \frac{\lambda_1}{d_1+d_2}I\right|^2 dV = \int_{L^d}|Hlnf|^2 dV + \int_{L^d}\frac{\lambda_1^2}{d_1+d_2}dV. \quad (27)$$

Using (19), with the fact $\Delta lnf = \lambda_1 lnf$, we have

$$\int_{L^d}|H(lnf)|^2 dV = -\lambda_1\int_{L^d}|\nabla lnf|^2 dV - \int_{L^d}R^L(\nabla lnf, \nabla lnf). \quad (28)$$

Combining (27) and (28), we derive

$$\int_{L^d}\left|H(lnf) - \frac{\lambda_1}{d_1+d_2}I\right|^2 dV = \int_{L^d}\frac{\lambda_1^2}{d_1+d_2}dV - \lambda_1\int_{L^d}|\nabla lnf|^2 dV - \int_{L^d}R^L(\nabla f, \nabla f)dV. \quad (29)$$

By the assumption $R^L(\nabla f, \nabla f) \geq K$, the above equation changes to

$$\int_{L^d}\left|H(lnf) - \frac{\lambda_1}{d_1+d_2}I\right|^2 dV \leq \int_{L^d}\frac{\lambda_1^2}{d_1+d_2}dV - \lambda_1\int_{L^d}|\nabla lnf|^2 dV - K\mathrm{Vol}(L_1). \quad (30)$$

Using (24), the last inequality leads to

$$\int_{L^d}\left|H(lnf) - \frac{\lambda_1}{d_1+d_2}I\right|^2 dV \leq \int_{L^d}\frac{\lambda_1^2}{d_1+d_2}dV - \int_{L^d}\left(\frac{\lambda_1 K}{d_3} + K\right)dV + \frac{\lambda_1}{d_3}\int_{L^d}\frac{c}{4}\left(d - d_1d_2 - d_2d_3 - d_1d_3 - \frac{1}{2}\right)dV - \frac{\lambda_1 d^2}{4d_3}\int_{L^d}\|\Omega\|^2 dV. \quad (31)$$

If (20) holds, then the above inequality produces

$$\left|H(lnf) - \frac{\lambda_1}{d_1+d_2}I\right|^2 = 0. \quad (32)$$

Therefore, we have $H(lnf)(X,X) = (\lambda_1/(d_1+d_2))$. Hence, by the application of Tashiro's result [5], the fibre L_1 is isometric to Euclidean space $R^{d_1+d_2}$. □

If we consider the unit vector field $\chi \in TL_\theta^{d_2}$ or $\chi \in TL_\perp^{d_3}$, then we have the following results which can be proved by adopting the similar steps in Theorem 2.

Theorem 3. *Let $L^d = L_1^{d_1+d_2} \times_f L_\perp^{d_3}$ be a $S-$minimal $d-$dimensional complete skew CR-warped product submanifold in a complex space form $\overline{L}^m(c)$, where L_1 is a semislant submanifold with d_1- dimensional invariant distribution S and d_2-dimensional slant distribution S_θ, such that Ricci curvature $R^L(\chi) \geq K, K > 0$. If $\chi \in S_\theta$ and satisfying the following equality,*

$$(\lambda_1 + d_3)K = \lambda_1 \left[\frac{d_3}{d} + \frac{d^2}{4}\|\Omega\|^2 - \frac{c}{4}\left(d - d_1 d_2 - d_2 d_3 - d_1 d_3 + 1 - \frac{3}{2}\cos^2\theta\right)\right]. \tag{33}$$

Then, the base submanifold $L_1^{d_1+d_2}$ is isometric to Euclidean space $R^{d_1+d_2}$, where d_3 is the dimension of the anti-invariant submanifold $L_\perp^{d_3}$ and λ_1 is the eigenvalue corresponding to the eigenfunction lnf.

Theorem 4. *Let $L^d = L_1^{d_1+d_2} \times_f L_\perp^{d_3}$ be a $S-$minimal $d-$dimensional compete skew CR-warped product submanifold in a complex space form $\overline{L}^m(c)$, where L_1 is a semislant submanifold with d_1-dimensional invariant distribution S and d_2-dimensional slant distribution S_θ, such that Ricci curvature $R^L(\chi) \geq K, K > 0$. If $\chi \in S_\perp$ and satisfying the following equality,*

$$(\lambda_1 + d_3)K = \lambda_1 \left[\frac{d_3}{d} + \frac{d^2}{4}\|\Omega\|^2 - \frac{c}{4}(d - d_1 d_2 - d_2 d_3 - d_1 d_3 + 1)\right]. \tag{34}$$

Then, the base submanifold $L_1^{d_1+d_2}$ is isometric to Euclidean space $R^{d_1+d_2}$, where d_3 is the dimension of the anti-invariant submanifold $L_\perp^{d_3}$ and λ_1 is the eigenvalue corresponding to the eigenfunction lnf.

Now, we have next result which is based on the study of Garcia-Rio et al. [4].

Theorem 5. *Let $L^d = L_1^{d_1+d_2} \times_f L_\perp^{d_3}$ be a $S-$minimal complete skew CR-warped product submanifold in a complex space form $\overline{L}^m(c)$, where L_1 is a semislant submanifold with d_1-dimensional invariant distribution S and d_2-dimensional slant distribution S_θ, such that Ricci curvature $R^L(\chi) > K, K > 0$. If $\chi \in S$ and satisfying the following relation,*

$$d^2\|\Omega\|^2 + \frac{4(d_1 + d_2)d_3}{\lambda_1}|H(lnf)|^2 = \frac{4(d_1 + d_2)d_3}{\lambda_1}\left(\frac{c}{4}\left(d - d_1 d_2 - d_2 d_3 - d_1 d_3 - \frac{1}{2}\right) + K\right), \tag{35}$$

for $\lambda_1 < 0$. Then, L_1 is isometric to warped product of the type $R \times_\theta U$, where R is the Euclidean line and U is a complete Riemannian manifold with the warping function θ, which satisfies the differential equation $(d\theta^2/dt^2) + \lambda_1 \theta = 0$.

Proof. For the warping function lnf, defining the following equation on $L_1^{d_1+d_2}$,

$$|Klnf I + H(lnf)|^2 = K^2(lnf)^2|I|^2 + |H(lnf)|^2 + 2K(lnf)g(I, H(lnf)). \tag{36}$$

However, we know that $|I|^2 = \operatorname{tr}(II^*) = d_1 + d_2$ and $g(H(lnf), I^*) = \operatorname{tr}(I^* H(lnf)) = \operatorname{tr}(H(lnf))$; using these facts, the above equation leads to

$$|Klnf I + H(lnf)|^2 = |H(lnf)|^2 + (d_1 + d_2)K^2(lnf)^2 - 2Klnf\Delta lnf. \tag{37}$$

Let lnf be an eigenfunction corresponding to the eigenvalue λ_1 satisfying $\Delta lnf = \lambda_1 lnf$, and we have\scale 90%

$$|KlnfI + H(lnf)|^2 = |H(lnf)|^2 + \left((d_1 + d_2)K^2 - 2K\lambda_1\right)(lnf)^2. \tag{38}$$

Again, using $\Delta lnf = \lambda_1 lnf$, it is easy to see that

$$\nabla\frac{(lnf)^2}{2} = lnf\lambda_1 lnf - |\nabla lnf|^2, \tag{39}$$

which on integrating provides

$$\begin{aligned}\int_{L^d}(lnf)^2 dV &= \frac{1}{\lambda_1}\int_{L^d}|\nabla lnf|^2,\\ \int_{L^d}|H(lnf) + KlnfI|^2 \mathrm{d}V &= \int_{L^d}|H(lnf)|^2 \mathrm{d}V\\ &\quad + \left(\frac{(d_1 + d_2)K^2}{\lambda_1} - 2K\right)\int_{L^d}|\nabla lnf|^2 \mathrm{d}V.\end{aligned} \tag{40}$$

Putting $K = (\lambda_1/(d_1 + d_2))$ in (40), we have

$$\int_{L^d}\left|H(lnf) + \frac{\lambda_1}{d_1 + d_2}lnfI\right|^2 \mathrm{d}V = \int_{L^d}|H(lnf)|^2 \mathrm{d}V - \frac{\lambda_1}{d_1 + d_2}\int_{L^d}|\nabla lnf|^2 \mathrm{d}V. \tag{41}$$

Furthermore, integrating (16) and applying Green's Lemma, we find

$$\int_{L^d} R^L(\chi)\mathrm{d}V \leq \frac{d^2}{4}\int_{L^d}\|\Omega\|^2 \mathrm{d}V + d_3\int_{L^d}\|\nabla lnf\|^2 \mathrm{d}V - \int_{L^d}\frac{c}{4}\left(d - d_1 d_2 - d_2 d_3 - d_1 d_3 - \frac{1}{2}\right)\mathrm{d}V. \tag{42}$$

□

From the above two expressions, we have

$$\begin{aligned}\frac{1}{d_3}\int_{L^d} R^L(\chi)\mathrm{d}V \leq{}& \frac{d^2}{4d_3}\int_{L^d}\|\Omega\|^2 \mathrm{d}V + \frac{d_1 + d_2}{\lambda_1}\int_{L^d}|H(lnf)|^2 \mathrm{d}V - \frac{d_1 + d_2}{\lambda_1}\int_{L^d}\left|H(lnf) + \frac{\lambda_1}{d_1 + d_2}lnfI\right|^2 \mathrm{d}V\\ &- \int_{L^d}\frac{c}{4}\left(d - d_1 d_2 - d_2 d_3 - d_1 d_3 - \frac{1}{2}\right)\mathrm{d}V.\end{aligned} \tag{43}$$

On using the assumption that $R^L(\chi) \geq K$, for $K > 0$,

$$\begin{aligned}\int_{L^d}\left|H(lnf) + \frac{\lambda_1}{d_1 + d_2}lnfI\right|^2 \mathrm{d}V \leq{}& \frac{d^2\lambda_1}{4(d_1 + d_2)d_3}\int_{L^d}\|\Omega\|^2 \mathrm{d}V + \int_{L^d}|H(lnf)|^2 \mathrm{d}V - \frac{\lambda_1}{(d_1 + d_2)d_3}\int_{L^d}K\mathrm{d}V\\ &- \frac{\lambda_1}{d_1 + d_2}\int_{L^d}\frac{c}{4}\left(d - d_1 d_2 - d_2 d_3 - d_1 d_3 - \frac{1}{2}\right)\mathrm{d}V.\end{aligned} \tag{44}$$

Equivalently,

$$\int_{L^d}\left|H(lnf)+\frac{\lambda_1}{d_1+d_2}lnfI\right|^2 dV \leq \int_{L^d}\left\{\frac{\lambda_1}{d_1+d_2}\left(\frac{d^2}{4d_3}\|\Omega\|^2-\frac{c}{4}\left(d-d_1d_2-d_2d_3-d_1d_3-\frac{1}{2}\right)\right)+\frac{K}{d_3}+|H(lnf)|^2\right\}dV. \tag{45}$$

By assumption (35), we obtain

$$d^2\|\Omega\|^2+\frac{4(d_1+d_2)d_3}{\lambda_1}|H(lnf)|^2=\frac{4(d_1+d_2)d_3}{\lambda_1}\left(\frac{c}{4}\left(d-d_1d_2-d_2d_3-d_1d_3-\frac{1}{2}\right)+\frac{K}{d_3}\right). \tag{46}$$

In view of (45) and (46), we find

$$\left|H(lnf)+\frac{\lambda_1}{d_1+d_2}lnfI\right|^2\leq 0,$$
$$\text{or } H(lnf)+\frac{\lambda_1}{d_1+d_2}lnfI=0. \tag{47}$$

By taking trace of the above equation, we obtain

$$\Delta lnf+\lambda_1 lnf=0. \tag{48}$$

Now, by the application of the result obtained in [4], together with the fact that $L^d=L_1^{d_1+d_2}\times_f L_\perp^{d_3}$ is nontrivial, we deduced that L_1 is isometric to a warped product of the form $R\times_\theta U$, where U is complete Riemannian manifold. In addition, the warping function θ is the solution of the differential equation $(d\theta^2/dt^2)+\lambda_1\theta=0$. Hence, the proof is completed.

Similarly, we can prove the following theorems by taking the unit vector field χ tangent to $L_\theta^{d_2}$ and $L_\perp^{d_3}$, respectively.

Theorem 6. *Let $L^d=L_1^{d_1+d_2}\times_f L_\perp^{d_3}$ be a S–minimal complete skew CR-warped product submanifold in a complex space form $\overline{L}^m(c)$, where L_1 is a semislant submanifold with d_1-dimensional invariant distribution S and d_2-dimensional slant distribution S_θ, such that Ricci curvature $R^L(\chi)>K, K>0$. If $\chi\in S_\theta$ and satisfying the following relation,*

$$d^2\|\Omega\|^2+\frac{4(d_1+d_2)d_3}{\lambda_1}|H(lnf)|^2=\frac{4(d_1+d_2)d_3}{\lambda_1}\left(\frac{c}{4}\left(d-d_1d_2-d_2d_3-d_1d_3+1-\frac{3}{2}\cos^2\theta\right)+K\right), \tag{49}$$

for $\lambda_1<0$. Then, L_1 is isometric to warped product of the type $R\times_\theta U$, where R is the Euclidean line and U is a complete Riemannian manifold with the warping function θ, which satisfies the differential equation $(d\theta^2/dt^2)+\lambda_1\theta=0$.

Theorem 7. *Let $L^d=L_1^{d_1+d_2}\times_f L_\perp^{d_3}$ be a S–minimal complete Skew CR-warped product submanifold in a complex space form $\overline{L}^m(c)$, where L_1 is a semislant submanifold with d_1-dimensional invariant distribution S and d_2-dimensional slant distribution S_θ, such that Ricci curvature $R^L(\chi)>K, K>0$. If $\chi\in S_\perp$ and satisfying the following relation,*

$$d^2\|\Omega\|^2+\frac{4(d_1+d_2)d_3}{\lambda_1}|H(lnf)|^2=\frac{4(d_1+d_2)d_3}{\lambda_1}\left(\frac{c}{4}(d-d_1d_2-d_2d_3-d_1d_3+1)+K\right), \tag{50}$$

for $\lambda_1<0$. Then, L_1 is isometric to warped product of the type $R\times_\theta U$, where R is the Euclidean line and U is a complete Riemannian manifold with the warping function θ, which satisfies the differential equation $(d\theta^2/dt^2)+\lambda_1\theta=0$.

4. Conclusions

This paper studies the geometric behavior of ordinary differential equations on the skew CR-warped product submanifolds. More precisely, we obtain characterizing theorems for skew CR-warped product submanifolds of complex space forms via differential and integral theory on Riemannian manifolds. Therefore, the present article provides a wonderful correlation of theory of differential equations with the warped product submanifolds.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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