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Research Article

Certain Bounds of Regularity of Elimination Ideals on Operations of Graphs

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Elimination ideals are regarded as a special type of Borel type ideals, obtained from degree sequence of a graph, introduced by Anwar and Khalid. In this paper, we compute graphical degree stabilities of $K_n \vee C_m$ and $K_n * C_m$ by using the DVE method. We further compute sharp upper bound for Castelnuovo–Mumford regularity of elimination ideals associated to these families of graphs.

1. Introduction

Let $S = k[x_1, \ldots, x_n]$, $n \geq 2$ be a polynomial ring in n variables over an infinite field k. Bayer and Stillman (see [1]) noted that a monomial ideal $I \subset S$ is of Borel type, if it satisfies the following condition:

$$\left(I\colon x_j^{\infty}\right) = \left(I\colon \left(x_1, \ldots, x_j\right)^{\infty}\right), \quad \text{for all } 1 \leq j \leq n. \tag{1}$$

The Castelnuovo–Mumford regularity (or simply regularity) gives an estimate of the complexity of computing Grö bner bases. The regularity of an ideal I is defined as $\operatorname{reg}(I) = \left\{\max j\colon \beta_{i,i+j}(I) \neq 0\right\}$, where β_{ij}'s are the graded Betti numbers of ideal I. The regularity of monomial ideals of Borel type was extensively studied (see for instance [2–4]).

The study of degree sequence started in 1981 by Bollobas [5]. Tyshkevich et al. in [6] established a correspondence between degree sequence of a graph and some structural properties of this graph. Havel–Hakimi criterion (see [7]) states that a sequence $(d_1, d_2, \ldots, d_n)$ of non-negative integers such that $d_1 \geq d_2 \geq \cdots \geq d_n$ is graphic if and only if the sequence $(d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is graphic. Given a non-negative sequence $(d_1, d_2, \ldots, d_n)$, this sequence will be called realizable if it is a degree sequence of any graph.

Throughout this paper, we assume H to be a simple, finite, and connected graph and label its vertices according to the degree sequence in descending order. In figures, we identify the vertex v_i of H with the positive integer i. Further, we identify a vertex v_i of a graph H with variable x_i of the polynomial ring $S = k[x_1, \ldots, x_n]$. Recently, Anwar and Khalid in [2] introduced a new class of monomial ideals, namely, elimination ideals $I_D(H)$, associated to a graph H. They showed that elimination ideals are the monomial ideals of Borel type. They gave the description of graphical degree stability of graph H denoted by $\mathrm{Stab}_d(H)$, a combinatorial measure associated to H. They gave a systematic procedure to compute the graphical degree stability, namely, dominating vertex elimination method (DVE method). In [2, 8], the authors computed the upper bounds of the Castelnuovo–Mumford regularity of elimination ideals associated to some families of graphs [9–11].

Motivated by the studies in [2, 8], we further extend this study for join and corona product of complete and cyclic graphs. We compute the graphical degree stabilities of join and corona product of above graphs by using the dominating vertex elimination method (see Lemmas 1 and 2). We give a description on computing the sharp upper bounds for the regularity of elimination ideals associated to these families of graphs respectively (see Theorems 1 and 2).

2. Preliminaries

Definition 1. Let H be a simple, finite, and connected graph with vertex set $V(H)$. The degree of a vertex $v \in V(H)$, denoted by $d(v)$, is the number of edges at v.

A vertex v_i of a graph H is called dominating vertex, if $\deg(v_i) \geq \deg(v_j)$ for all $v_j \in V(H)$ with $i \neq j$. The set of all dominating vertices of a graph H is called dominating set and is denoted by $D(H)$. A graph H is called scattered graph if it has at least one vertex with degree zero; otherwise, it is called non-scattered graph.

Here, we introduce different operations on graphs.

Definition 2. Let G and H be two graphs with mutually disjoint vertex sets $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$. A graph denoted by $G \vee H$ is called the join of G and H if $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{u_i v_j | u_i \in V(G), v_j \in V(H)\}$.

Definition 3. The corona product of two graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ disjoint copies of H and then joining the j^{th} vertex of G to every vertex in the j^{th} copy of H. We denote this graph as $G * H$. Corona product of two graphs G and H with n and m vertices, respectively, has $n + nm$ vertices.

Definition 4. Let H_i be a graph and pick a vertex $x \in D(H_i)$ such that $H_{i+1} = H_i - \{x\}$ is an induced, non-scattered subgraph of H_i. This method of obtaining an induced, non-scattered subgraph H_{i+1} from H_i by eliminating a vertex from the dominating set $D(H_i)$ is called the dominating vertex elimination method (DVE method) (see [2] for more details). Let H be a simple finite and connected graph; then, the length of the maximum chain of induced, non-scattered, subgraphs of H obtained by successively applying DVE method is called graphical degree stability and is denoted as $\text{Stab}_d(H)$. In other words, if $H = {H_0}^{H_1^{\cdots^{H_r}}}$ is the maximum chain of induced, non-scattered, subgraphs of H obtained by using DVE method, then $\text{Stab}_d(H)$. Note that the vertex set of H_j consists of $[n-j]$ vertices where $1 \leq j \leq r$.

Definition 5. Let H be a simple, finite, and connected graph with vertex set $V(H) = \{x_1, x_2, \ldots, x_n\}$ having degree sequence of vertices $(d_1, d_2, \ldots, d_n)$ such that $d_1 \geq d_2 \geq \cdots \geq d_n$ and $d_i \geq 0$, $\forall\, 1 \leq i \leq n$; then, the ideal $Q(H) = \langle x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n} \rangle$ is called the sequential ideal associated to the graph H. Let $H = {H_0}^{H_1^{\cdots^{H_r}}}$ be the maximum chain of induced, non-scattered, subgraphs of H obtained by using the DVE method with $\text{Stab}_d(H) = r$ and $Q(H_i) = \langle x_1^{d_{i_1}}, x_2^{d_{i_2}}, \ldots, x_{n-i}^{d_{i_{n-i}}} \rangle$ be the sequential ideal corresponding to H_i, $0 \leq i \leq r$; then, the elimination ideal of H is defined as

$$I_D(H) = Q_{H_0} \cap Q_{H_1} \cap \cdots \cap Q_{H_r}. \tag{2}$$

Let $S = K[x_1, x_2, \ldots, x_n]$, $n \geq 2$, be a polynomial ring over infinite field K and $I \subseteq S$ be a monomial ideal of S. Let $G(I)$ be the minimal set of monomial generators of I and the highest degree of monomial in $G(I)$ is denoted as $\deg(I)$. Also, let $I_{\geq t}$ be an ideal generated by monomial of I of degree $\geq t$. Given a monomial $u \in S$, set $m(u) = \max\{i: x_i | u\}$ and $m(I) = \max_{u \in G(I)} m(u)$. A monomial ideal I is stable if for any $u \in G(I)$ we have $x_i . u / x_{m(u)} \in I$ for all $1 \leq i < m(u)$.

Eisenbud et al. proved the following result in [12].

Proposition 1. *Let I be a monomial ideal such that $I_{\geq e}$ is stable, where $e \geq \deg(I)$ is an integer. Then, $\text{reg}(I) \leq e$.*

Remark 1. Let I and J are two monomial ideals with $r \geq \deg(I)$ and $s \geq \deg(J)$, where r and s are two integers such that $I_{\geq r}$ and $J_{\geq s}$ are stable ideals; then, $(I \cap J)_{\geq \max\{r,s\}}$ is stable ideal.

3. Regularity of $K_n \vee C_m$

In this section, we present our results regarding the Castelnuovo–Mumford regularity of elimination ideals associated to join operation of complete graph and cyclic graph.

In [2], the authors gave the following formula to compute the graphical degree stability for cyclic graphs.

Proposition 2. *If C_m, $m \geq 3$, is the cyclic graph, then*

$$\text{Stab}_d(C_m) = \begin{cases} \dfrac{m}{3}, & \text{if } m \equiv 0 \pmod 3, \\[2ex] \dfrac{m-1}{3}, & \text{if } m \equiv 1 \pmod 3, \\[2ex] \dfrac{m-2}{3}, & \text{if } m \equiv 2 \pmod 3. \end{cases} \tag{3}$$

Lemma 1. *Let K_n, $n \geq 2$, be a complete graph and C_m, $m \geq 4$, be a cyclic graph; then,*

$$\text{Stab}_d(K_n \vee C_m) = n + \text{Stab}_d(C_m). \tag{4}$$

Proof. We shall prove it by induction on n. Let $n = 2$; then, $H_0 = K_2 \vee C_m$ with degree sequence

$$\left(m+1, m+1, \underbrace{4, \ldots, 4}_{m\text{-tuple}} \right), \tag{5}$$

and $D(H_0) = \{x_1, x_2\}$. Without loss of generality, pick $x_1 \in D(H_0)$, and using the DVE method, we get H_1 with the degree sequence (always rearrange the sequence after applying DVE method) $(m, \underbrace{3, \ldots, 3}_{m\text{-tuple}})$ and $D(H_1) = \{x_1\}$. Again applying the DVE method, we get $H_2 = C_m$. So,

$\mathrm{Stab}_d(K_2 \vee C_m) = 2 + \mathrm{Stab}_d(C_m)$. Suppose that result is true for $n = p$; then, the degree sequence $K_p \vee C_m$ will be

$$\Big(\underbrace{p+m-1,\dots,p+m-1}_{p\text{-tuple}},\underbrace{p+2,\dots,p+2}_{m\text{-tuple}}\Big),$$
$$\mathrm{Stab}_d\big(K_p \vee C_m\big) = p + \mathrm{Stab}_d(C_m). \tag{6}$$

Consider $n = p+1$; then, $H_0 = K_{p+1} \vee C_m$ with degree sequence

$$\Big(\underbrace{p+m,\dots,p+m}_{p+1\text{-tuple}},\underbrace{p+3,\dots,p+3}_{m\text{-tuple}}\Big), \tag{7}$$

and $|V(H_0)| = p+m+1$, with $D(H_0) = \{x_1,\dots,x_{p+1}\}$ as $m \geq 4$ which are precisely the vertices that initially belonged to K_{p+1}. Without loss of generality, pick $x_1 \in D(H_0)$, and applying the DVE method, we get $H_1 = K_p \vee C_m$.

$$\begin{aligned}&\Rightarrow \mathrm{Stab}_d\big(K_{p+1}\vee C_m\big) = 1 + \mathrm{Stab}_d\big(K_p \vee C_m\big) = 1 + p + \mathrm{Stab}_d(C_m) = (1+p) + \mathrm{Stab}_d(C_m),\\ &\Rightarrow \mathrm{Stab}_d\big(K_{p+1}\vee C_m\big) = (p+1) + \mathrm{Stab}_d(C_m).\end{aligned} \tag{8}$$

□

Corollary 1. *Let K_n, $n \geq 2$, be a complete graph and C_m, $m \geq 4$, be a cyclic graph; then, the sequential ideal of $K_n \vee C_m$ is given as*

$$Q_i = \begin{cases} \langle x_1^{n+m-i-1},\dots,x_{n-i}^{n+m-i-1},x_{n-i+1}^{n-i+2},\dots,x_{n+m-i}^{n-i+2}\rangle, & 0 \leq i \leq n,\\ \langle x_1^2,\dots,x_{m-3(i-n)}^2,x_{m-3(i-n)+1},\dots,x_{n+m-i}\rangle, & n+1 \leq i \leq n + Stab_d(C_m).\end{cases} \tag{9}$$

Proof. This proof is immediate from Lemma 1. □

Proposition 3. *Let $K_n \vee C_m$, $n \geq 2$ and $m \geq 4$, be the join of complete and cyclic graphs, respectively, and*

$$Q_i = \langle x_1^{n+m-i-1},\dots,x_{n-i}^{n+m-i-1},x_{n-i+1}^{n-i+2},\dots,x_{n+m-i}^{n-i+2}\rangle \tag{10}$$

be the sequential ideal associated to H_i; then, $Q_{i_{\geq\gamma(i)}}$ is stable ideal, where $\gamma(i) = (n-i)^2 + m(2n-2i+1) + 2(i-n) + 1$ and $0 \leq i \leq n$.

Proof. Let $H_0 = K_n \vee C_m$, $n \geq 2$ and $m \geq 4$, be the join of complete and cyclic graphs, respectively; then, by Corollary 1, sequential ideal associated to H_i, $0 \leq i \leq n$, is given as $Q_i = \langle x_1^{a_1},\dots,x_{n+m-i}^{a_{n+m-i}}\rangle$ where

$$a_k = \begin{cases} n+m-i-1, & 1 \leq k \leq n-i,\\ n-i+2, & n-i+1 \leq k \leq n+m-i.\end{cases} \tag{11}$$

Let $\gamma(i) = (n-i)^2 + m(2n-2i+1) + 2(i-n) + 1$, for all $0 \leq i \leq n$. We shall show that $Q_{i_{\geq\gamma(i)}}$ is a stable ideal. Take $u \in Q_{i_{\geq\gamma(i)}}$; then, $u = v x_k^{a_k}$ for some $1 \leq k \leq n+m-i$ where $v \in \langle x_1,\dots,x_{n+m-i}\rangle^{\gamma(i)-a_k}$.

If $m(u) > k$, then $x_l u / x_{m(u)} = x_l v / x_{m(u)} x_k^{a_k} \in Q_{i_{\geq\gamma(i)}}$ for all $l < m(u)$. So, $Q_{i_{\geq\gamma(i)}}$ is stable.

If $m(u) = k$, then clearly $u \in \langle x_1,\dots,x_{n+m-i}\rangle^{\gamma(i)}$.

$$\Rightarrow Q_{i_{\geq\gamma(i)}} \subseteq \langle x_1,\dots,x_{n+m-i}\rangle^{\gamma(i)}. \tag{12}$$

We show that $\langle x_1,\dots,x_{n+m-i}\rangle^{\gamma(i)} \subseteq Q_{i_{\geq\gamma(i)}}$. Let $w \in \langle x_1,\dots,x_{n+m-i}\rangle^{\gamma(i)}$; then, $w = x_1^{\beta_1}x_2^{\beta_2},\dots,x_{n+m-i}^{\beta_{n+m-i}}$ with $\beta_s \geq 0$ for all $1 \leq s \leq n+m-i$ and $\sum_{s=1}^{n+m-i}\beta_s \geq \gamma(i)$. Therefore, there exists at least one $s \in \{1,\dots,n+m-i\}$ such that $\beta_s \geq a_s$ and $w = (x_1^{\beta_1},\dots,x_s^{\beta_s-a_s},\dots,x_{n+m-i}^{\beta_{n+m-i}})x_s^{a_s} \in Q_{i_{\geq\gamma(i)}}$, and the result follows. □

Proposition 4. *Let $K_n \vee C_m$, $n \geq 2$ and $m \geq 4$, be the join of complete and cyclic graphs, respectively, and*

$$Q_i = \langle x_1^2,\dots,x_{m-3(i-n)}^2,x_{m-3(i-n)+1},\dots,x_{n+m-i}\rangle \tag{13}$$

be the sequential ideal associated to $G_i, n+1 \leq i \leq n + Stab_d(C_m)$; then, $Q_{i_{\geq\gamma(i)}}$ is stable ideal, where $\gamma(i) = m + 3n - 3i + 1$.

Proof. Let $H_0 = K_n \vee C_m$, $n \geq 2$ and $m \geq 4$, be the join of complete and cyclic graphs, respectively; then, by Corollary 1, sequential ideal associated to $H_i, n+1 \leq i \leq n + \mathrm{Stab}_d(C_m)$, is given as $Q_i = \langle x_1^{a_1},\dots,x_{n+m-i}^{a_{n+m-i}}\rangle$ where

$$a_k = \begin{cases} 2, & 1 \leq k \leq m-3(i-n),\\ 1, & m-3(i-n)+1 \leq k \leq n+m-i.\end{cases} \tag{14}$$

Let $\gamma(i) = m + 3n - 3i + 1$, for all $n+1 \leq i \leq n + \mathrm{Stab}_d(C_m)$; then, by similar arguments as above, $Q_{i_{\geq\gamma(i)}}$ is a stable ideal. □

Theorem 1. *Let K_n, $n \geq 2$, be a complete graph and C_m, $m \geq 4$, be a cyclic graph. Then, $reg(I_D(K_n \vee C_m)) \leq n^2 + m(2n+1) - 2n + 1$.*

Proof. If $\text{Stab}_d(C_m) = p$, then by using Proposition 2, $\text{Stab}_d(K_n \vee C_m) = n + p$, so the corresponding elimination ideal is given as $I_D(K_n \vee C_m) = \cap_{i=0}^{n+p} Q_i$, and by using Propositions 3 and 4 and Remark 1, $I_D(K_n \vee C_m)$ is stable for γ, where

$$\gamma = \max\{\gamma(i), |0 \le i \le n+p\} = n^2 + m(2n+1) - 2n + 1. \tag{15}$$

By using Proposition 1,

$$\text{reg}(I_D(K_n \vee C_m)) \le n^2 + m(2n+1) - 2n + 1. \tag{16}$$

□

Example 1. Consider $K_2 \vee C_4$, where $n = 2$ and $m = 4$ (see Figure 1).

$D(H_0) = \{x_1, x_2\}$, $Q_0 = \langle x_1^5, x_2^5, x_3^4, x_4^4, x_5^4, x_6^4 \rangle$, and $\text{reg}(Q_0) = 21$.

$D(H_1) = \{x_1\}$, $Q_1 = \langle x_1^4, x_2^3, x_3^3, x_4^3, x_5^3 \rangle$, and $\text{reg}(Q_1) = 12$.

$D(H_2) = \{x_1, x_2, x_3, x_4\}$, $Q_2 = \langle x_1^2, x_2^2, x_3^2, x_4^2 \rangle$, and $\text{reg}(Q_2) = 5$.

$D(H_3) = \{x_1\}$, $Q_3 = \langle x_1^2, x_2, x_3 \rangle$, and $\text{reg}(Q_3) = 2$.

Remark 2. In general, $Q_{i_{\ge \gamma(i)-1}}$ is not stable, where Q_i are sequential ideals of $K_n \vee C_m$, where $0 \le i \le n + \text{Stab}_d(C_m)$. As in Example 1, $Q_{1_{\ge 11}}$ is not a stable ideal for $i = 1$ and $\gamma(1) = 12$.

4. Regularity of $K_n * C_m$

In this section, we present our results concerning the Castelnuovo–Mumford regularity of elimination ideals associated to corona product operation of complete graph and cyclic graphs.

Lemma 2. *Let K_n, $n \ge 2$, be a complete graph and C_m, $m \ge 4$, be a cyclic graph; then,*

$$\text{Stab}_d(K_n * C_m) = n + n\text{Stab}_d(C_m). \tag{17}$$

Proof. We shall prove it by induction on n. Let $n = 2$; then, $H_0 = K_2 * C_m$ with degree sequence

$$\left(m+1, m+1, \underbrace{3, \ldots, 3}_{m-tuple}, \underbrace{3, \ldots, 3}_{m-tuple} \right), \tag{18}$$

and $D(H_0) = \{x_1, x_2\}$. Without loss of generality, pick $x_1 \in D(H_0)$, and using the DVE method, we get H_1 with the degree sequence

$$\left(m, \underbrace{3, \ldots, 3}_{m-tuple}, \underbrace{2, \ldots, 2}_{m-tuple} \right). \tag{19}$$

Now $D(H_1) = \{x_1\}$, and again applying the DVE method, we have only two copies of C_m in H_2 with degree sequence

$$\left(\underbrace{2, \ldots, 2}_{m-tuple}, \underbrace{2, \ldots, 2}_{m-tuple} \right), \\ \Rightarrow Stab_d(K_2 * C_m) = 2 + 2Stab_d(C_m). \tag{20}$$

Suppose that result is true for $n = p$; then, the degree sequence of $K_p * C_m$ will be

$$\left(\underbrace{p+m-1, \ldots, p+m-1}_{p-tuple}, \underbrace{\underbrace{3, \ldots, 3}_{m-tuple}, \ldots, \underbrace{3, \ldots, 3}_{m-tuple}}_{pm-tuple} \right), \\ \Rightarrow Stab_d(K_p * C_m) = p + pStab_d(C_m). \tag{21}$$

Consider $n = p + 1$; then, $H_0 = K_{p+1} * C_m$ with degree sequence

$$\left(\underbrace{p+m, \ldots, p+m}_{p+1-tuple}, \underbrace{\underbrace{3, \ldots, 3}_{m-tuple}, \ldots, \underbrace{3, \ldots, 3}_{m-tuple}}_{m(p+1)-tuple} \right), \tag{22}$$

and $|V(H_0)| = pm + p + m + 1$ with $D(H_0) = \{x_1, \ldots, x_{p+1}\}$ which are precisely the vertices that initially belonged to K_{p+1} because $m \ge 4$. By applying the DVE method, we get $K_p * C_m$ and one copy of C_m, and we call this graph H_1.

$$\Rightarrow \text{Stab}_d(K_{p+1} * C_m) = 1 + \text{Stab}_d(K_p * C_m) + \text{Stab}_d(C_m) = 1 + p + p\text{Stab}_d(C_m) + \text{Stab}_d(C_m), \\ \Rightarrow \text{Stab}_d(K_{p+1} * C_m) = (p+1) + (p+1)\text{Stab}_d(C_m). \tag{23}$$

□

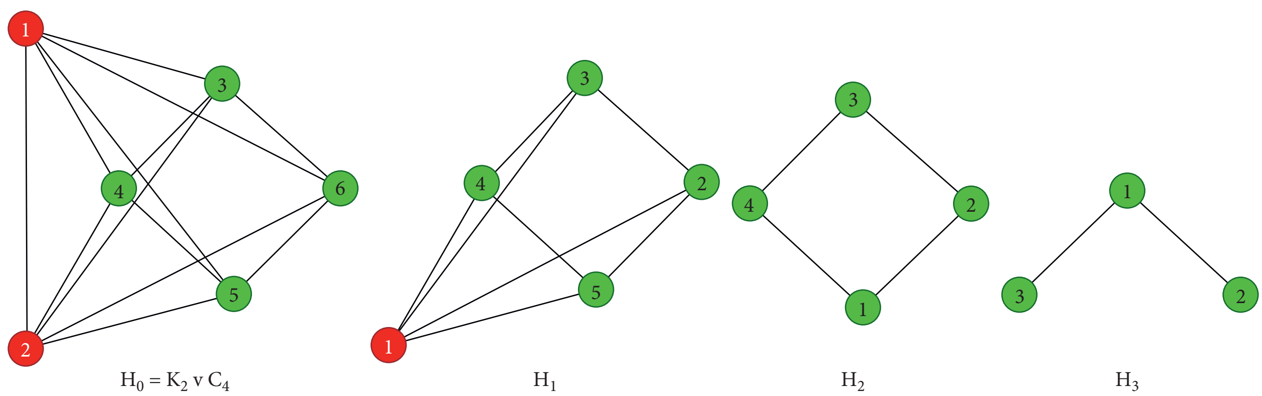

FIGURE 1: H_0, H_1, H_2, H_3.

Corollary 2. *Let K_n, $n \geq 2$, be a complete graph and C_m, $m \geq 4$, be a cyclic graph; then, the sequential ideal of $K_n * C_m$ is given as*

$$Q_i = \begin{cases} \langle x_1^{n+m-1}, \ldots, x_n^{n+m-1}, x_{n+1}^3, \ldots, x_r^3 \rangle, & i = 0, \\ \langle x_1^{n+m-i-1}, \ldots, x_{n-i}^{n+m-i-1}, x_{n-i+1}^3, \ldots, x_{r-i(m+1)}^3, x_{r-i(m+1)+1}^2, \ldots, x_{r-i}^2 \rangle, & 1 \leq i \leq n-1, \\ \langle x_1^2, \ldots, x_{nm}^2 \rangle, & i = n, \\ \langle x_1^2, \ldots, x_{nm-3(i-n)}^2, x_{nm-3(i-n)+1}, \ldots, x_{r-i} \rangle, & n+1 \leq i \leq n + n\mathrm{Stab}_d(C_m), \end{cases} \tag{24}$$

where $r = n + nm$

Proof. The proof is immediate from Lemma 2. □

Proposition 5. *Let $K_n * C_m$, $n \geq 2$ and $m \geq 4$, be the corona product of complete and cyclic graphs, respectively, and*

$$Q_0 = \langle x_1^{n+m-1}, \ldots, x_n^{n+m-1}, x_{n+1}^3, \ldots, x_r^3 \rangle, \tag{25}$$

be the sequential ideal associated to H_0; then, $Q_{0_{\geq \gamma(0)}}$ is stable ideal, where $r = n + nm$ and $\gamma(0) = n^2 + n(3m-2) + 1$.

Proof. Let $H_0 = K_n * C_m$, $n \geq 2$ and $m \geq 4$, be the corona product of complete and cyclic graphs, respectively; then, by Corollary 2, the sequential ideal associated to H_0 is given as $Q_0 = \langle x_1^{a_1}, \ldots, x_r^{a_r} \rangle$ where

$$a_j = \begin{cases} n+m-1, & 1 \leq j \leq n, \\ 3, & n+1 \leq j \leq r. \end{cases} \tag{26}$$

Let $\gamma(0) = n^2 + n(3m-2) + 1$. We show that $Q_{0_{\geq \gamma(0)}}$ is a stable ideal. Take $u \in Q_{0_{\geq \gamma(0)}}$; then, $u = v x_k^{a_k}$ for some $1 \leq k \leq r$ where $v \in (x_1, \ldots, x_r)^{\gamma(0) - a_k}$.

If $m(u) > k$, then $x_l u / x_{m(u)} = (x_l v / x_{m(u)}) x_k^{a_k} \in Q_{0_{\geq \gamma(0)}}$ for all $l < m(u)$. So, $Q_{0_{\geq \gamma(0)}}$ is stable.

If $m(u) = k$, then clearly $u \in \langle x_1, \ldots, x_r \rangle^{\gamma(0)}$.

$$\Rightarrow Q_{0_{\geq \gamma(0)}} \subseteq \langle x_1, \ldots, x_r \rangle^{\gamma(0)}. \tag{27}$$

We show that $\langle x_1, \ldots, x_r \rangle^{\gamma(0)} \subseteq Q_{0_{\geq \gamma(0)}}$. Let $w \in \langle x_1, \ldots, x_r \rangle^{\gamma(0)}$; then, $w = x_1^{\beta_1} x_2^{\beta_2} \cdots x_r^{\beta_r}$ with $\beta_s \geq 0$ for all $1 \leq s \leq r$ and $\sum_{s=1}^r \beta_s \geq \gamma(0)$. Therefore, there exists at least one $s \in \{1, \ldots, r\}$ such that $\beta_s \geq a_s$ and $w = \langle x_1^{\beta_1}, \ldots, x_s^{\beta_s - a_s}, \ldots, x_r^{\beta_r} \rangle x_s^{a_s} \in Q_{0_{\geq \gamma(0)}}$, and the result follows. □

Proposition 6. *Let $K_n * C_m$, $n \geq 2$ and $m \geq 4$, be the corona product of complete and cyclic graphs, respectively, and*

$$Q_i = \langle x_1^{n+m-i-1}, \ldots, x_{n-i}^{n+m-i-1}, x_{n-i+1}^3, \ldots, x_{r-i(m+1)}^3, x_{r-i(m+1)+1}^2, \ldots, x_{r-i}^2 \rangle, \tag{28}$$

be the sequential ideal associated to H_i; then, $Q_{i_{\geq \gamma(i)}}$ is stable ideal, where $r = n + nm$, $\gamma(i) = (n-i)^2 - 2(n + mi - i) + 3nm + 1$ and $1 \leq i \leq n-1$.

Proof. Let $H_0 = K_n * C_m$, $n \geq 2$ and $m \geq 4$, be the corona product of complete and cyclic graphs, respectively; then, by Corollary 2, sequential ideal associated to H_i, $1 \leq i \leq n-1$, is given as $Q_i = \langle x_1^{a_1}, \ldots, x_{r-i}^{a_{r-i}} \rangle$ where

$$a_j = \begin{cases} n+m-i-1, & 1 \le j \le n-i, \\ 3, & n-i+1 \le j \le r-i(m+1), \\ 2, & r-i(m+1)+1 \le j \le r-i. \end{cases} \quad (29)$$

Let $\gamma(i) = (n-i)^2 - 2(n+mi-i) + 3nm + 1$, for all $1 \le i \le n-1$. We shall show that $Q_{i_{\ge\gamma(i)}}$ is a stable ideal. Take $u \in Q_{i_{\ge\gamma(i)}}$; then, $u = vx_k^{a_k}$ for some $1 \le k \le r-i$ where $v \in (x_1, \ldots, x_{r-i})^{\gamma(i)-a_k}$.

If $m(u) > k$, then $x_l u / x_{m(u)} = x_l v / x_{m(u)} x_k^{a_k} \in Q_{i_{\ge\gamma(i)}}$ for all $l < m(u)$. So, $Q_{i_{\ge\gamma(i)}}$ is stable.

If $m(u) = k$, then clearly $u \in \langle x_1, \ldots, x_{r-i} \rangle^{\gamma(i)}$.

$$\Rightarrow Q_{i_{\ge\gamma(i)}} \subseteq \langle x_1, \ldots, x_{r-i} \rangle^{\gamma(i)}. \quad (30)$$

We show that $\langle x_1, \ldots, x_{r-i} \rangle^{\gamma(i)} \subseteq Q_{i_{\ge\gamma(i)}}$. Let $w \in \langle x_1, \ldots, x_{r-i} \rangle^{\gamma(i)}$; then, $w = x_1^{\beta_1} x_2^{\beta_2}, \ldots, x_{r-i}^{\beta_{r-i}}$ with $\beta_s \ge 0$ for all $1 \le s \le r-i$ and $\sum_{s=1}^{r-i} \beta_s \ge \gamma(i)$. Therefore, there exists at least one $s \in \{1, \ldots, r-i\}$ such that $\beta_s \ge a_s$ and $w = \langle x_1^{\beta_1}, \ldots, x_s^{\beta_s - a_s}, \ldots, x_{r-i}^{\beta_{r-i}} \rangle x_s^{a_s} \in Q_{i_{\ge\gamma(i)}}$, and the result follows. □

Proposition 7. *Let $K_n * C_m$, $n \ge 2$ and $m \ge 4$, be the corona product of complete and cyclic graphs, respectively, and $Q_n = \langle x_1^2, \ldots, x_{nm}^2 \rangle$ be the sequential ideal associated to H_n; then, $Q_{n_{\ge\gamma(n)}}$ is stable ideal, where $r = n + nm$ and $\gamma(n) = nm + 1$.*

Proof. Let $K_n * C_m$, $n \ge 2$ and $m \ge 4$, be the corona product of complete and cyclic graphs, respectively; then, by Corollary 2, sequential ideal associated to H_n is given as $Q_n = \langle x_1^2, \ldots, x_{nm}^2 \rangle$. Let $\gamma(n) = nm + 1$; then, by similar arguments as above, $Q_{n_{\ge\gamma(n)}}$ is a stable ideal. □

Proposition 8. *Let $K_n * C_m$, $n \ge 2$ and $m \ge 4$, be the corona product of complete and cyclic graphs, respectively, and*

$$Q_i = \langle x_1^2, \ldots, x_{nm-3(i-n)}^2, x_{nm-3(i-n)+1}, \ldots, x_{r-i} \rangle, \quad (31)$$

be the sequential ideal associated to H_i; then, $Q_{i_{\ge\gamma(i)}}$ is stable ideal, where $r = n + nm$, $\gamma(i) = nm + 3(n-i) + 1$, and $n + 1 \le i \le n + n\mathrm{Stab}_d(C_m)$.

Proof. Let $K_n * C_m$, $n \ge 2$ and $m \ge 4$, be the corona product of complete and cyclic graphs, respectively; then, by Corollary 2, sequential ideal associated to H_i, $n+1 \le i \le n + n\mathrm{Stab}_d(C_m)$, is given as $Q_i = \langle x_1^{a_1}, \ldots, x_{r-i}^{a_{r-i}} \rangle$, where

$$a_j = \begin{cases} 2, & 1 \le j \le nm - 3(i-n), \\ 1, & nm - 3(i-n) + 1 \le j \le r-i. \end{cases} \quad (32)$$

Let $\gamma(i) = nm + 3(n-i) + 1$, for all $n + 1 \le i \le n + n\mathrm{Stab}_d(C_m)$. We shall show that $Q_{i_{\ge\gamma(i)}}$ is a stable ideal. Take $u \in Q_{i_{\ge\gamma(i)}}$; then, $u = vx_k^{a_k}$ for some $1 \le k \le r-i$ where $v \in \langle x_1, \ldots, x_{r-i} \rangle^{\gamma(i)-a_k}$.

If $m(u) > k$, then $x_l u / x_{m(u)} = (x_l v / x_{m(u)}) x_k^{a_k} \in Q_{i_{\ge\gamma(i)}}$ for all $l < m(u)$. So, $Q_{i_{\ge\gamma(i)}}$ is stable.

If $m(u) = k$, then clearly $u \in \langle x_1, \ldots, x_{r-i} \rangle^{\gamma(i)}$.

$$\Rightarrow Q_{i_{\ge\gamma(i)}} \subseteq \langle x_1, \ldots, x_{r-i} \rangle^{\gamma(i)}. \quad (33)$$

We show that $\langle x_1, \ldots, x_{r-i} \rangle^{\gamma(i)} \subseteq Q_{i_{\ge\gamma(i)}}$. Let $w \in \langle x_1, \ldots, x_{r-i} \rangle^{\gamma(i)}$; then, $w = x_1^{\beta_1} x_2^{\beta_2}, \ldots, x_{r-i}^{\beta_{r-i}}$ with $\beta_s \ge 0$ for all $1 \le s \le r-i$ and $\sum_{s=1}^{r-j} \beta_s \ge \gamma(i)$. Therefore, there exists at least one $s \in \{1, \ldots, r-i\}$ such that $\beta_s \ge a_s$ and $w = \langle x_1^{\beta_1}, \ldots, x_s^{\beta_s - a_s}, \ldots, x_{r-i}^{\beta_{r-i}} \rangle x_s^{a_s} \in Q_{i_{\ge\gamma(i)}}$, and the result follows. □

Theorem 2. *Let K_n, $n \ge 2$, be a complete graph and C_m, $m \ge 4$, be a cyclic graph; then, we have*

$$\mathrm{reg}(I_D(K_n * C_m)) \le n^2 + n(3m-2) + 1. \quad (34)$$

Proof. If $\mathrm{Stab}_d(C_m) = p$, then by using Proposition 2, $\mathrm{Stab}_d(K_n * C_m) = n + np$, so the corresponding elimination ideal is given as $I_D(K_n * C_m) = \cap_{i=0}^{n+np} Q_i$, and by using Propositions 5–8 and Remark 1, $I_D(K_n * C_m)$ is stable for γ, where

$$\gamma = \max\{\gamma(i) | 0 \le i \le n + np\} = n^2 + n(3m-2) + 1. \quad (35)$$

By using Proposition 1,

$$\mathrm{reg}(I_D(K_n * C_m)) \le n^2 + n(3m-2) + 1. \quad (36)$$

□

Example 2. Consider $K_2 * C_4$, where $n = 2$ and $m = 4$ (see Figure 2).

$D(H_0) = \{x_1, x_2\}$, $Q_0 = \langle x_1^5, x_2^5, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3, x_9^3, x_{10}^3 \rangle$, and $\mathrm{reg}(Q_0) = 25$.

$D(H_1) = \{x_1\}$, $Q_1 = \langle x_1^4, x_2^3, x_3^3, x_4^3, x_5^3, x_6^2, x_7^2, x_8^2, x_9^2 \rangle$, and $\mathrm{reg}(Q_1) = 16$.

$D(H_2) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, $Q_2 = \langle x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_7^2, x_8^2 \rangle$, and $\mathrm{reg}(Q_2) = 9$.

$D(H_3) = \{x_1, x_2, x_3, x_4, x_5\}$, $Q_3 = \langle x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6, x_7 \rangle$, and $\mathrm{reg}(Q_3) = 6$.

$D(H_4) = \{x_1, x_2\}$, $Q_4 = \langle x_1^2, x_2^2, x_3, x_4, x_5, x_6 \rangle$, and $\mathrm{reg}(Q_4) = 3$.

Remark 3. In general, one cannot get $Q_{i_{\ge\gamma(i)-1}}$ stable, where Q_i are the sequential ideals of $K_n * C_m$, $0 \le i \le n + n\mathrm{Stab}_d(C_m)$. As in Example 2, for $i = 2\,\gamma(2) = 9$ and $Q_{2_{\ge 8}}$ is not a stable ideal.

Remark 4. Elimination ideals are Borel type ideals, and upper bound for Borel type ideals was discussed in [3, 13]; it

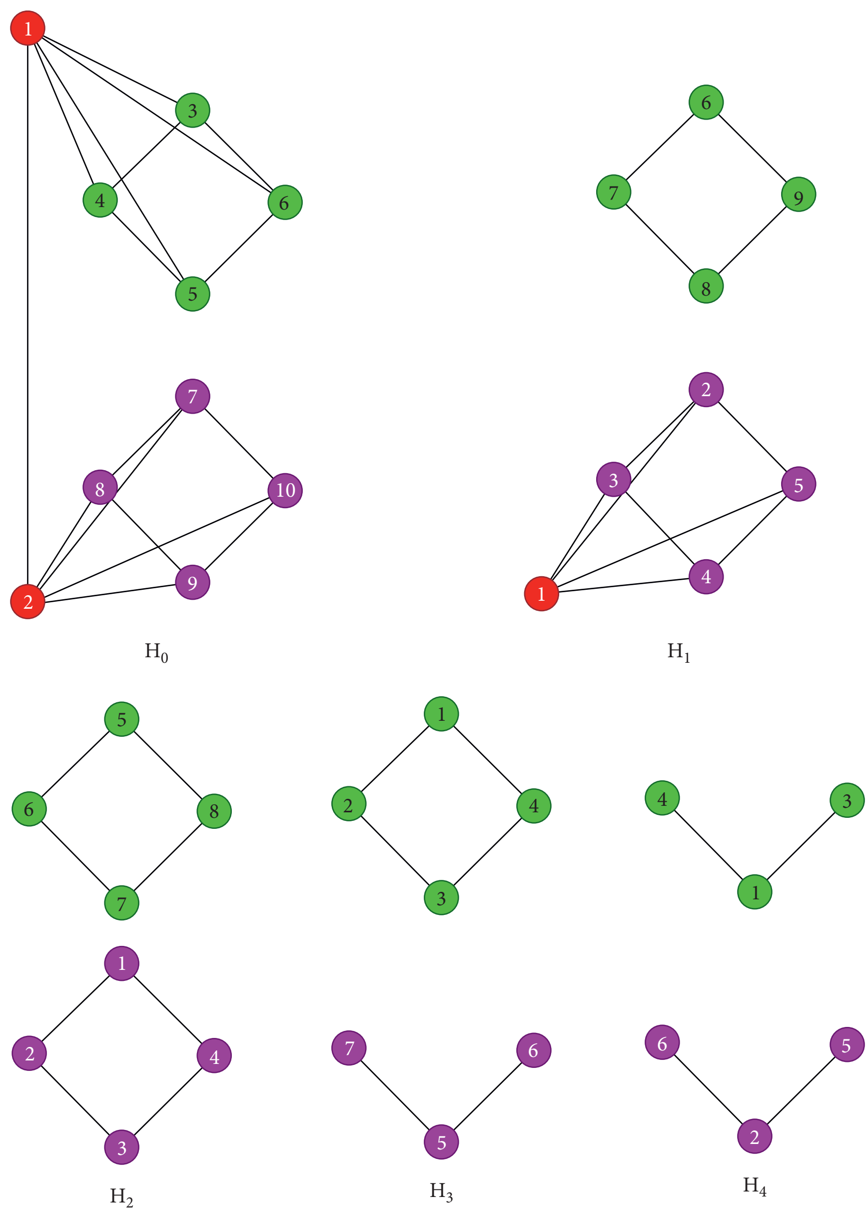

Figure 2: H_0, H_1, H_2, H_3, H_4.

is worthy to note that our given bounds are sharper than the ones given in [3, 13].

Data Availability

The data used to support the findings of this study are included within this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Zongming Lv verified and analyzed the results and arranged the funding for this study. Muhammad Junaid Ali Junjua proved the results. Muhammad Tajammal Tahir wrote the first version of the paper. Khurram Shabbir proposed the problem and supervised this work.

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