

Research Article

Robust Assessing the Lifetime Performance of Products with Inverse Gaussian Distribution in Bayesian and Classical Setup

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The inverse Gaussian (Wald) distribution belongs to the two-parameter family of continuous distributions having a range from 0 to ∞ and is considered as a potential candidate to model diffusion processes and lifetime datasets. Bayesian analysis is a modern inferential technique in which we estimate the parameters of the posterior distribution obtained by formally combining a prior distribution with an observed data distribution. In this article, we have attempted to perform the Bayesian and classical analyses of the Wald distribution and compare the results. Jeffreys' and uniform priors are used as noninformative priors, while the exponential distribution is used as an informative prior. The analysis comprises finding joint posterior distributions, the posterior means, predictive distributions, and credible intervals. To illustrate the entire estimation procedure, we have used real and simulated datasets, and the results thus obtained are discussed and compared. We have used the Bayesian specialized Open BUGS software to perform Markov Chain Monte Carlo (MCMC) simulations using a real dataset.

1. Introduction

In probability theory, the inverse Gaussian distribution (IGD), also known as the Wald distribution, belongs to the two-parameter family of continuous distributions with support 0 to ∞ [1]. The concept of Brownian motion is applicable in describing the inherent process of many phenomena, particularly in the natural and physical sciences. The time in which a Brownian motion with a positive drift reaches a fixed value is distributed as an IGD.

The probability density function, or density for short, of the IGD is given by

$$f(x; \mu, \lambda) = \left(\frac{\lambda}{2\pi x^3} \right)^{1/2} \exp \left\{ -\frac{\lambda(x - \mu)^2}{2\mu^2 x} \right\}, \quad (1)$$

$$x > 0, \mu > 0, \lambda > 0.$$

Here, μ is the location and λ is the shape parameter. The IGD approaches the normal distribution as $\lambda \rightarrow \infty$.

A brief retrospective on the IGD is now proposed. To begin, the author in [2] studied the inverse Gaussian as a model to study Brownian motion. The author in first considered its basic features of statistical properties and found certain similarities in its statistical analysis. The authors in [3] proposed it as a lifetime model to be applied in situations where the initial failure rate was high. The author in [4] considered estimating the inverse Gaussian (μ, λ) model in a Bayesian framework. He also discussed that the estimation becomes very difficult when there is no natural conjugate prior. The authors in [5] considered the parameterization when $\psi = 1/\mu$ and λ and evaluated Bayes estimates using uninformative reference and natural conjugate priors. The authors in [6] indicated that in [5] approach, the posterior mean of $1/\psi$ does not exist, so the Bayesian estimate of the mean of the distribution is not available. The author in [7] performed the estimation of the reliability function from the work done on IG (μ, λ) parameterization. The authors in [8] evaluated estimates for λ assuming Jeffreys' prior and its posterior density by using the Gibbs sampling technique when μ is known.

Because of its shapes, the related density may also be considered as a good competitor to the Gamma, Weibull, and log-normal distributions. Various sampling theory inferences with the IGD are studied by [9–12] among others. The authors in [13] treated some applications in marketing, while applications of IG in life testing are considered by [2, 3].

The authors in [14] investigated Bayesian estimation for the parameters of the IGD distribution. They emphasized the MCMC technique and gave a complete implementation of the Gibbs sampler algorithm. The author in [5] obtained some Bayesian results for the inverse Gaussian family of distributions with a noninformative reference prior as well as the natural conjugate prior. The authors in [6] derived Bayesian results for the IGD by using a proper prior under reparameterization with reference to the distribution mean and of the inverse of the squared variation coefficient, for obtaining Bayes estimates as well as of their inverses. The author in [9] presented a report on some statistical properties of the IGD distribution when the parameters are confined to $(0, \infty)$. A good review of the advantages of using Bayesian methods may be found in [15–23]. The posterior distributions often have complex multidimensional forms that require using Markov Chain Monte Carlo (MCMC) methods to get results [16, 24–27]. In recent years, the use of Markov Chain Monte Carlo (MCMC) methods has gained much popularity [28–30]. Most recently, the author in [31] has considered in detail the q-Weibull distribution for classical and Bayesian analyses, which also serves as a motivation to conduct this study. Keeping in view the extensive literature on the importance of Bayesian analysis and the importance of the Wald distribution, we have attempted to present the Bayesian analysis of the Wald distribution. Jeffreys' and uniform priors are used as noninformative priors, while the exponential distribution is used as an informative prior. The analysis comprises finding joint posterior distributions, the posterior means, predictive distributions, and credible intervals. To illustrate the entire

estimation procedure, we have used real and simulated datasets, and the results thus obtained are discussed and compared.

The literature reveals that a lot of authors have studied classical distributions, including the Wald distribution, in a classical framework. We have observed a variety of applications of the Wald distribution and the capability of the Bayes methods to incorporate the prior information of the model parameters. To the best of our knowledge and belief, the Wald distribution has not yet been studied in a Bayesian framework despite its potential applications. Therefore, to cover this gap present in the literature, we have attempted to perform the Bayesian analysis of the Wald distribution in this article.

Here is the break-up of the study. Section 2 considers the frequentist analysis of the inverse Gaussian distribution using the MLE method and computes the standard errors associated with the classical estimates. A numerical example is presented in Section 3. Section 4 carries out the Bayesian analysis of the IGD assuming the uniform, Jeffreys', and subjective informative priors. It is supposed that the parameters of the IGD follow exponential distribution(s). The convergence diagnostic is given in Section 5. The predictive inference of the inverse Gaussian distribution is presented in Section 6. Comparison between the frequentist and Bayesian approaches is performed in Section 7. The simulation study is performed in Section 8 to justify the results.

2. Maximum Likelihood (ML) Estimation

Let x_1, x_2, \dots, x_n constitute a random sample of size n from the IGD. The likelihood function is

$$p(x | \mu, \lambda) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \prod_{i=1}^n \left(\frac{1}{x_i^3}\right)^{1/2} e^{\left\{-\frac{\lambda}{2\mu^2} \sum_{i=1}^n \left(\frac{x_i - \mu}{x_i}\right)^2\right\}}. \quad (2)$$

The logarithmic form of the likelihood function is

$$l = \ln(p(x | \mu, \lambda)) = \frac{n}{2} \ln(\lambda) - \frac{n}{2} \ln(2\pi) - \frac{1}{2} \sum_{i=1}^n \ln(x_i^3) - \frac{\lambda}{2\mu^2} \sum_{i=1}^n \frac{(x_i - \mu)^2}{x_i}. \quad (3)$$

We differentiate this expression w.r.t the unknown parameters μ and λ , and equating the resulting equations to zero to maximize l , we get

$$\hat{\mu} = \frac{\sum_{i=1}^n x_i}{n} = \bar{X}, \text{ and, } \hat{\lambda} = \frac{n}{\sum_{i=1}^n x_i^{-1} - n\bar{X}^{-1}}. \quad (4)$$

They are the required ML estimates of the parameter μ and λ of the IGD.

2.1. Standard Errors of the ML Estimates. The main diagonal elements of the inverted Fisher information matrix (FIM)

designate the variances of the ML estimates. Hence, we can find the standard errors of the ML estimates by calculating the square roots of the diagonal elements as

$$s_{\hat{\mu}, \hat{\lambda}} = \frac{1}{\sqrt{I(\hat{\mu}, \hat{\lambda})}} \quad (5)$$

We find the elements of the Hessian matrix as follows. The second derivative of l with respect to λ , μ , and (μ, λ) is given as

$$\frac{\partial^2 l}{\partial \lambda^2} = \frac{-n}{2\lambda^2}, \quad \frac{\partial^2 l}{\partial \mu^2} = \frac{-n\lambda}{\mu^2} \quad \text{and} \quad \frac{\partial l}{\partial \mu \partial \lambda} = -\frac{\sum_{i=1}^n (\mu - x_i)}{\mu^3} = 0. \quad (6)$$

Fisher's information matrix (F.I.M) $I(\mu, \lambda)$ may be defined as follows:

$$I(\mu, \lambda) = -E \begin{bmatrix} \frac{\partial^2 l}{\partial \mu^2} & \frac{\partial^2 l}{\partial \mu \partial \lambda} \\ \frac{\partial^2 l}{\partial \mu \partial \lambda} & \frac{\partial^2 l}{\partial \lambda^2} \end{bmatrix}, \quad (7)$$

where

$$I(\lambda) = -E \left[\frac{\partial^2 l}{\partial \lambda^2} \right] = \frac{n}{2\lambda^2}, \quad I(\mu) = -E \left[\frac{\partial^2 l}{\partial \mu^2} \right] = \frac{n\lambda}{\mu^2} \quad \text{and} \quad I(\mu, \lambda) = -E \left[\frac{\partial^2 l}{\partial \mu \partial \lambda} \right] = 0. \quad (8)$$

Hence,

$$I(\mu, \lambda) = \begin{bmatrix} -E \left[\frac{\partial^2 l}{\partial \mu^2} \right] & -E \left[\frac{\partial^2 l}{\partial \mu \partial \lambda} \right] \\ -E \left[\frac{\partial^2 l}{\partial \mu \partial \lambda} \right] & -E \left[\frac{\partial^2 l}{\partial \lambda^2} \right] \end{bmatrix} = \begin{bmatrix} \frac{n\lambda}{\mu^2} & 0 \\ 0 & \frac{n}{2\lambda^2} \end{bmatrix}. \quad (9)$$

$$p(\mu, \lambda | x) \propto p(\mu) \cdot p(\lambda) \cdot p(x | \mu, \lambda),$$

$$\Rightarrow p(\mu, \lambda | x) \propto 1.1. \left(\frac{\lambda}{2\pi} \right)^{(n/2)} \prod_{i=1}^n \prod \left(\frac{1}{x_i^3} \right)^{(1/2)} e^{\left\{ -(\lambda/2\mu^2) \sum_{i=1}^n ((x_i - \mu)^2 / x_i) \right\}}; \quad \mu > 0, \lambda > 0. \quad (13)$$

4.2. Uninformative Bayesian Analysis Using Jeffreys' Prior.
The positive square root of the determinant of the FIM is known as Jeffreys' prior.

The variance covariance matrix $s_{\hat{\mu}, \hat{\lambda}}^2$ is

$$s_{\hat{\mu}, \hat{\lambda}}^2 = I(\mu, \lambda)^{-1} = \begin{bmatrix} \frac{n\lambda}{\mu^2} & 0 \\ 0 & \frac{n}{2\lambda^2} \end{bmatrix}^{-1} = \frac{1}{n^2/2\mu^2\lambda} \begin{bmatrix} \frac{n}{2\lambda^2} & 0 \\ 0 & \frac{n\lambda}{\mu^2} \end{bmatrix} \quad (10)$$

$$= \begin{bmatrix} \frac{\mu^2}{n\lambda^2} & 0 \\ 0 & \frac{2\lambda^2}{n} \end{bmatrix}.$$

The uncorrelated parameters are displayed in the above matrix. The results are summarized in Table 1.

3. Numerical Example

A real dataset is considered in this section that is analyzed by [14]. The dataset given in Table 2 denotes the active repair times (in hours) for an airborne communication transceiver.

Using the data given above, the ML estimates along with their variance are computed and are given in Table 3.

Here, we observe that the estimates are stable with very small standard errors.

4. Bayesian Analysis

4.1. Uninformative Bayesian Analysis Using the Uniform Prior.
The uninformative uniform prior for the both parameters μ and λ is defined as

$$p(\mu, \lambda) \propto 1, \quad \mu > 0, \lambda > 0. \quad (11)$$

The posterior distribution is given as

$$p(\mu, \lambda | x) \propto p(\mu, \lambda) \cdot p(x | \mu, \lambda). \quad (12)$$

As μ and λ are considered independent, therefore, their joint prior distribution will be the product of their individual priors and may be defined as

$$p_J(\mu, \lambda) = \sqrt{\det(\text{Fisher information matrix})} = \sqrt{|I(\mu, \lambda)|}. \quad (14)$$

TABLE 1: Maximum likelihood estimators of the parameters with variances.

Parameter	Maximum likelihood estimator	Variance of maximum likelihood estimator
μ	$\frac{\sum_{i=1}^n x_i}{n} = \bar{X}$	$\frac{\bar{X}^2}{(\sum_{i=1}^n x_i^{-1} - n\bar{X}^{-1})^2}$
λ	$\frac{n}{(\sum_{i=1}^n x_i^{-1} - n\bar{X}^{-1})}$	$2(\sum_{i=1}^n x_i^{-1} - n\bar{X}^{-1})^{-2}$

TABLE 2: Repair times (in hours) of 46 transceivers.

1.5	1.5	1.5	2.0	2.0	2.2	2.5	2.7	3.0	3.0
7.5	8.8	9.0	10.3	22.0	24.5	3.3	3.3	4.0	4.0
0.7	0.8	0.8	1.0	1.0	1.0	1.0	1.1	1.3	1.5
0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7
4.5	4.7	5.0	5.4	5.4	7.0	—	—	—	—

TABLE 3: ML estimates and variances for the repair-time data.

Parameters	Mean	Variance
λ	1.65885	0.01692
μ	3.60652	0.00260

Here, $p_J(\mu, \lambda)$ denotes for Jeffreys' prior. We carry on for Jeffreys' prior as follows:

$$p_J(\mu, \lambda | x) \propto p_J(\mu, \lambda) \cdot p(x | \mu, \lambda),$$

$$p_J(\mu, \lambda | x) \propto \sqrt{\frac{1}{\lambda\mu^2}} \cdot \left(\frac{\lambda}{2\pi}\right)^{n/2} \prod_{i=1}^n \left(\frac{1}{x_i^3}\right)^{1/2} e^{\left\{-\frac{\lambda}{2\mu^2} \sum_{i=1}^n \left(\frac{x_i - \mu}{x_i}\right)^2\right\}}, \quad (17)$$

$$p_J(\mu, \lambda | x) = K^{-1} \mu^{-1} \lambda^{(1/2)-1+(n/2)} e^{\left\{-\frac{\lambda}{2\mu^2} \sum_{i=1}^n \left(\frac{x_i - \mu}{x_i}\right)^2\right\}}; \quad \mu > 0, \lambda > 0,$$

where K is denoted as the normalizing constant. It may be defined as

$$K = \int_0^\infty \int_0^\infty \mu^{-1} \lambda^{(1/2)-1+(n/2)} e^{\left\{-\frac{\lambda}{2\mu^2} \sum_{i=1}^n \left(\frac{x_i - \mu}{x_i}\right)^2\right\}} d\mu d\lambda. \quad (18)$$

4.3. Informative Bayesian Analysis Assuming the Exponential Prior. It is known that expert opinion can be incorporated into the analysis using the informative prior of an unknown parameter about a state of nature. To achieve this, we suppose an exponential prior for both parameters of the IGD.

The determinant of the FIM of the IGD is

$$|I(\mu, \lambda)| = \begin{vmatrix} \frac{n\lambda}{\mu^2} & 0 \\ 0 & \frac{n}{2\lambda^2} \end{vmatrix} \Rightarrow |I(\mu, \lambda)| = \frac{n^2}{2\lambda\mu^2}. \quad (15)$$

Jeffreys' prior $p_J(\mu, \lambda)$ for unknown parameters (μ, λ) is given as follows:

$$p_J(\mu, \lambda) = \sqrt{\frac{n^2}{2\lambda\mu^2}} \Rightarrow p_J(\mu, \lambda) \propto \sqrt{\lambda^{-1}\mu^{-2}} = \lambda^{-1/2}\mu^{-1}. \quad (16)$$

Hence, for the unknown parameters μ and λ , the joint posterior distribution is given by

4.3.1. The Exponential Prior. Here, we assume that the prior distributions of both the parameters follow exponential distributions and are defined as

$$\begin{aligned} p(\lambda) &= \lambda e^{-h_1\lambda}, \quad \lambda > 0, h_1 > 0, \\ p(\mu) &= \mu e^{-h_2\mu}, \quad \mu > 0, h_2 > 0. \end{aligned} \quad (19)$$

Here, h_1 and h_2 are known as the hyperparameters. So, the joint posterior distribution of the unknown parameter μ and λ is

$$p(\mu, \lambda | x) \propto p(\mu, \lambda) \cdot p(x | \mu, \lambda). \quad (20)$$

As μ and λ are supposed to be independent,

$$p(\mu, \lambda) = p(\mu) \cdot p(\lambda). \quad (21)$$

So, the joint posterior distribution of the parameters μ and λ is

$$\begin{aligned} p(\mu, \lambda | x) &\propto p(\mu) \cdot p(\lambda) \cdot p(x | \mu, \lambda), \\ \Rightarrow p(\mu, \lambda | x) &\propto \lambda e^{-h_1 \lambda} \cdot \mu e^{-h_2 \mu} \cdot \left(\frac{\lambda}{2\pi}\right)^{n/2} \prod_{i=1}^n \left(\frac{1}{x_i^3}\right)^{1/2} e^{-\left(\lambda/2\mu^2\right) \sum_{i=1}^n \left((x_i - \mu)^2/x_i\right)}, \\ \Rightarrow p(\mu, \lambda | x) &\propto \mu e^{-h_2 \mu} \cdot \lambda^{n/2+2-1} \cdot e^{-\lambda \left\{h_1 + (1/2\mu^2) \sum_{i=1}^n \left((x_i - \mu)^2/x_i\right)\right\}}; \quad \mu > 0, \lambda > 0. \end{aligned} \quad (22)$$

Now, we observe that the two parameters of the IGD have independent distributions. Therefore, to make inferences, we will derive their marginal distributions regarding them. The marginal posterior distribution of μ may be defined as

$$p(\mu | x) = \int_0^\infty p(\mu, \lambda | x) d\lambda, \quad 0 \leq \mu \leq \infty. \quad (23)$$

Similarly, the marginal posterior distribution of λ may be derived as follows:

$$p(\lambda | x) = \int_0^\infty p(\mu, \lambda | x) d\mu, \quad 0 \leq \lambda \leq \infty. \quad (24)$$

Some loss function is required to find the Bayesian estimates. So, we have considered the squared error loss function (SELF), which describes the posterior means as the Bayes estimates. That is,

$$\begin{aligned} \hat{\mu} &= E(\mu | x) = \int_0^\infty \mu \cdot p(\mu | x) d\mu, \\ \hat{\lambda} &= E(\lambda | x) = \int_0^\infty \lambda \cdot p(\lambda | x) d\lambda. \end{aligned} \quad (25)$$

Such expressions generally comprise complex structures, so the numerical methods are required to solve them. Therefore, to simulate data from the posterior

distribution(s), we have used the MCMC technique by using OpenBUGS, and the resulting Bayes estimates, posterior risks, and 95% highest posterior density regions are given in Table 4.

The results show that the posterior estimates produced by the uniform, Jeffreys', and exponential priors coincide a lot and have small standard errors. The posterior marginal densities of the model parameters are presented in Figure 1.

The posterior marginal densities for the parameters of the model show a slight positive skewness for both of the parameters of the model.

5. Convergence Diagnostics

The dynamic traces and time series plots are assessed to check the convergence. They are presented in Figure 2. The simulated results confirm the convergence of the parameters of the underlying posterior distribution.

6. Predictive Inference

We evaluate the predictive distributions to study the future behaviour of data. Using the posterior distribution $p(\mu, \lambda | x)$ based on the exponential distribution defined in Section 4.3 and the Wald distribution $p(y | \mu, \lambda)$ as data model, the predictive distribution can be expressed as

$$\begin{aligned} p(y | x) &= \int_0^\infty \int_0^\infty p(y | \mu, \lambda) \cdot p(\mu, \lambda | x) d\mu d\lambda \\ &= \int_0^\infty \int_0^\infty \left(\frac{\lambda}{2\pi y^3}\right)^{(1/2)} e^{-\lambda(y-\mu)^2/2\mu^2 y} \mu e^{-h_2 \mu} \lambda^{(n/2)+1} e^{-\lambda \left\{h_1 + (1/2\mu^2) \sum_{i=1}^n \left((x_i - \mu)^2/x_i\right)\right\}} d\mu d\lambda \\ &= \left(\frac{1}{2\pi y^3}\right)^{(1/2)} \int_0^\infty \int_0^\infty \lambda^{(n+3/2)} \mu e^{-h_2 \mu} e^{-\lambda \left\{h_1 + (1/2\mu^2) \sum_{i=1}^n \left((x_i - \mu)^2/x_i\right) + (y-\mu)^2/2\mu^2 y\right\}} d\mu d\lambda \\ &= \left(\frac{1}{2\pi y^3}\right)^{(1/2)} \int_0^\infty \int_0^\infty \lambda^{(n+3/2)} \mu e^{-h_2 \mu} e^{-\lambda \left\{h_1 + (1/2\mu^2) \left[\sum_{i=1}^n \left((x_i - \mu)^2/x_i\right) + (y-\mu)^2/y\right]\right\}} d\mu d\lambda. \end{aligned} \quad (26)$$

TABLE 4: Bayes estimates of (μ, λ) under uniform, Jefferys', and exponential priors.

Measures	Uniform prior		Jefferys' prior		Exponential prior	
	μ	λ	μ	λ	μ	λ
Bayes estimates	3.60652	1.7324	3.60652	1.7324	3.60834	1.65762
Posterior risk	0.01254	0.1251	0.01254	0.1251	0.01254	0.1251
95% HDR	(2.7548, 5.5698)	(1.4822, 2.1077)	(2.7548, 5.5698)	(1.4822, 2.1077)	(2.7548, 5.5698)	(1.4822, 2.1077)

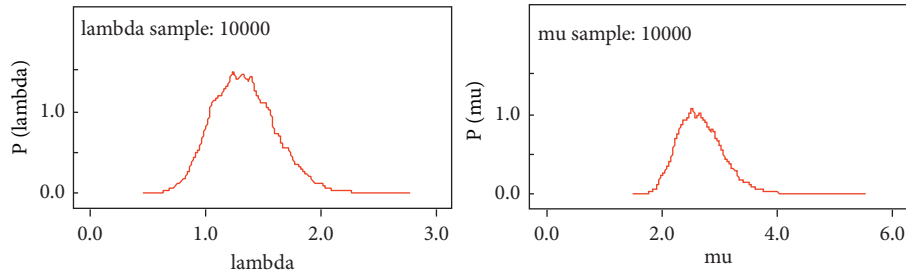


FIGURE 1: The posterior densities of the model parameters.

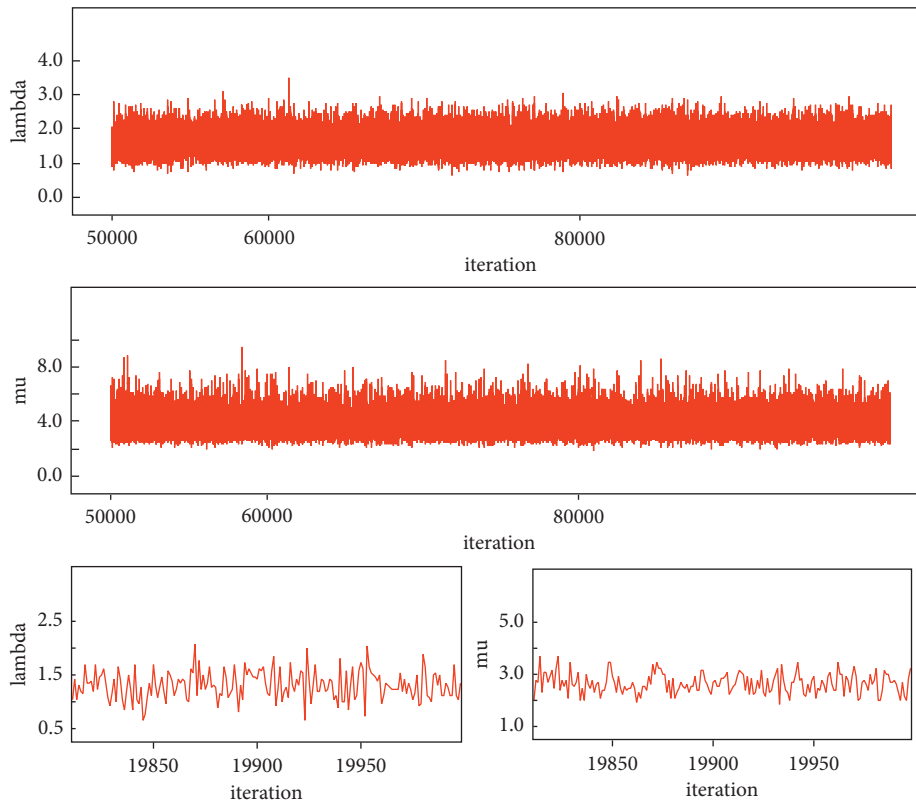


FIGURE 2: Dynamic traces and posterior densities of the parameters.

Often, the predictive distributions do not follow the baseline distributions and do not have closed forms. Here, we observed that the predictive distribution is not in closed form. So, we will require numerical methods to evaluate multiple integrals to evaluate the above defined predictive distribution. This is accomplished by using the numerical procedures, and the resulting predicted and observed datasets are summarized in Table 5 with graphical representation made in Figure 3.

It shows that the predicted and observed datasets are much identical to each other.

7. Comparison of the Frequentist and Bayesian Approaches

To make a comparison of the Bayesian estimation method with the frequentist maximum likelihood estimation method, several model selection criteria, i.e., log-likelihood

TABLE 5: Summary of the predicted and real datasets.

Types of data	Minimum	1 st quartile	Median	Mean	3 rd quartile	Maximum
Observed	0.200	0.800	1.750	3.607	4.375	24.5
Predicted	0.2137	0.7696	1.0811	2.2513	2.4510	16.1022

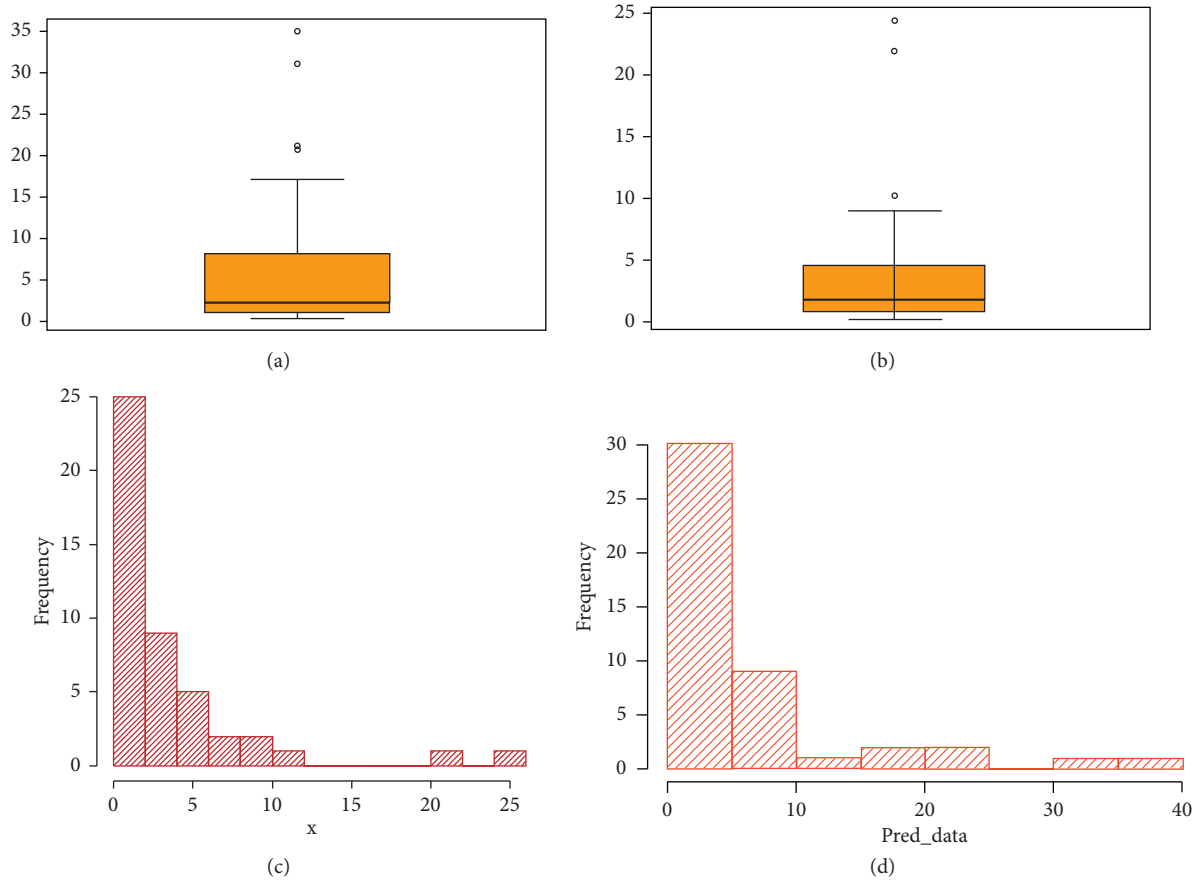


FIGURE 3: Histograms and boxplots of the predicted and real datasets. (a) Boxplot of predicted data. (b) Boxplot of real data. (c) Histogram of real data. (d) Histogram of predicted data.

TABLE 6: Summary of comparisons of estimation methods.

Estimation methods	Priors	λ	μ	LL	AIC	BIC
ML	—	1.659	3.607	-99.0477	202.0953	205.7526
UMVUE	—	1.551	3.607	-99.0988	202.1976	205.8549
Bayes Lindley	—	1.623	4.118	-99.2127	202.4255	206.0827
Bayes methods	$\lambda \sim \text{gamma}(3.3, 2), \mu \sim \text{gamma}(6.8, 1.9)$	1.859	3.354	-99.2699	202.5399	206.1971
	$\lambda \sim \text{exp}(0.6024), \mu \sim \text{exp}(0.2778)$	1.620	4.008	-99.1577	202.3155	205.9728
	Jeffrey's prior	1.658	4.210	-99.2649	202.5299	206.1872
	Uniform prior	1.668	5.346	-100.1742	204.3484	208.0057

TABLE 7: Estimates for 100000 simulations.

n	$\hat{\mu}$	$\hat{\lambda}$	$E(\hat{\mu})$	$E(\hat{\lambda})$	$SE(\hat{\mu})$	$SE(\hat{\lambda})$	LL_{μ}	UL_{μ}	LL_{λ}	UL_{λ}
5	3.6065	1.6588	3.6020	4.1478	2.3738	9.7766	0.9565	9.8878	0.74251	17.158
10	3.6065	1.6588	3.6016	2.3724	1.6716	1.5068	1.3832	7.7839	0.8687	6.2250
20	3.6065	1.6588	3.6040	1.9492	1.1889	0.7164	1.8285	6.4351	1.0078	3.7305
50	3.60652	1.6588	3.6086	1.7657	0.7510	0.3729	2.3573	5.2891	1.1830	2.6334
100	3.6065	1.6588	3.6061	1.7100	0.5302	0.2487	2.6756	4.7476	1.2912	2.2630

TABLE 8: Estimates for 500000 simulations.

n	$\hat{\mu}$	$\hat{\lambda}$	$E(\hat{\mu})$	$E(\hat{\lambda})$	$SE(\hat{\mu})$	$SE(\hat{\lambda})$	LL_{μ}	UL_{μ}	LL_{λ}	UL_{λ}
5	3.6065	1.6588	3.6033	4.1477	2.3740	9.3228	0.9558	9.8595	0.7416	17.2688
10	3.6065	1.6588	3.6057	2.3672	1.6852	1.4942	1.3842	7.8231	0.8711	6.1343
20	3.6065	1.6588	3.6044	1.9532	1.1872	0.7170	1.8348	6.4251	1.0090	3.7324
50	3.6065	1.6588	3.6070	1.7650	0.7535	0.3724	2.3564	5.2909	1.1811	2.6293
100	3.6065	1.6588	3.6075	1.7101	0.5311	0.2476	2.6776	4.7550	1.2926	2.2591

(LL), Akaike information criterion (AIC), and the Bayes information criterion (BIC) are used. That is a vital part of this study. The better the fit is, the lower the values of these criteria are. The ML estimates, noninformative Bayesian estimates, and informative Bayesian estimates with the values of the model selection criteria are evaluated and presented in Table 6.

The results reveal that the values of the AIC and BIC computed by Bayesian estimates are the smallest as compared with those produced by frequentist estimates. It is noted that the Gamma prior has minimum AIC and BIC values for Bayes estimates as the expert opinion is also involved in it.

8. Simulation Study

Simulation studies help us to understand the type and behaviour of the underlying distributions. So, we generate data values for the parameters and estimate them based on the generated data. Here, the same parametric values have been used that are obtained by using the real dataset through *R* codes, and the results are portrayed in Tables 7 and 8.

The results show that the model will produce the same results if it continues to run on a similar pattern in the future. Moreover, the estimates become more stable if the numbers of simulations are increased.

9. Conclusion and Recommendations

The Bayesian inference for the parameters of the Wald distribution has been performed in this study and also compared the results with their classical counterparts. We have evaluated the maximum likelihood estimates for the comparison purpose. We used uniform and Jeffreys' priors as noninformative priors and the exponential distribution as informative prior. We have also discussed the predictive distribution as well. Simulation studies have also been conducted to verify the results. It has been witnessed that the results produced by using the Bayesian technique produce better results by yielding smaller AIC and BIC values. We have also witnessed close coordination between the observed and predicted datasets, which indicates that the Bayes methods are the potential replacements for their classical counterparts. The Bayesian methods are best suited for evaluating the lifetime data of any type of product. Future perspectives of the research may be to conduct such studies using other distributions that can model a variety of natural phenomena. We may also extend this study to include the generalized and multivariate distributions as well.

Data Availability

The data used to support this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest with any organization or authors.

References

- [1] C. Forbes, M. Evans, N. Hastings, and B. Peacock, *Statistical Distributions*, John Wiley & Sons, Hoboken, NJ, USA, 2011.
- [2] A. Nádas, "Best tests for zero drift based on first passage times in Brownian motion," *Technometrics*, vol. 15, no. 1, pp. 125–132, 1973.
- [3] R. S. Chhikara and J. L. Folks, "The inverse Gaussian distribution as a lifetime model," *Technometrics*, vol. 19, no. 4, pp. 461–468, 1977.
- [4] J. M. Palmer, *Certain Non-Classical Inference Procedures Applied to the Inverse Gaussian Distribution*, Oklahoma State University, Stillwater, OK, USA, 1973.
- [5] A. K. Banerjee and G. K. Bhattacharyya, "Bayesian results for the inverse Gaussian distribution with an application," *Technometrics*, vol. 21, no. 2, pp. 247–251, 1979.
- [6] B. Betrò and R. Rotondi, "On Bayesian inference for the inverse Gaussian distribution," *Statistics & Probability Letters*, vol. 11, no. 3, pp. 219–224, 1991.
- [7] S. K. Sinha, "Bayesian estimation of the reliability function of the inverse Gaussian distribution," *Statistics & Probability Letters*, vol. 4, no. 4, pp. 319–323, 1986.
- [8] S. A. Ismail and H. A. Auda, "Bayesian and fiducial inference for the inverse Gaussian distribution via gibbs sampler," *Journal of Applied Statistics*, vol. 33, no. 8, pp. 787–805, 2006.
- [9] M. C. K. Tweedie, "Statistical properties of inverse Gaussian distributions. I," *The Annals of Mathematical Statistics*, vol. 28, no. 2, pp. 362–377, 1957.
- [10] R. S. Chhikara and J. L. Folks, "Estimation of the inverse Gaussian distribution function," *Journal of the American Statistical Association*, vol. 69, no. 345, pp. 250–254, 1974.
- [11] R. S. Chhikara and J. L. Floks, "Statistical distributions related to the inverse Gaussian," *Communications in Statistics*, vol. 4, no. 12, pp. 1081–1091, 1975.
- [12] R. S. Chhikara and J. L. Folks, "Optimum test procedures for the mean of first passage time distribution in Brownian motion with positive drift (inverse Gaussian distribution)," *Technometrics*, vol. 18, no. 2, pp. 189–193, 1976.
- [13] A. K. Banerjee and G. K. Bhattacharyya, "A purchase incidence model with inverse Gaussian interpurchase times," *Journal of the American Statistical Association*, vol. 71, no. 356, pp. 823–829, 1976.

- [14] B. Pandey and P. Bandyopadhyay, "Bayesian estimation of inverse gaussian distribution," 2012, <https://arxiv.org/abs/1210.4524>.
- [15] E. S. Adams, "Bayesian analysis of linear dominance hierarchies," *Animal Behaviour*, vol. 69, no. 5, pp. 1191–1201, 2005.
- [16] B. P. Carlin and T. A. Louis, *Bayes and Empirical Bayes Methods for Data Analysis*, Chapman and Hall/CRC, Boca Raton, FL, USA, 2010.
- [17] L. F. Contreras, E. T. Brown, and M. Ruest, "Bayesian data analysis to quantify the uncertainty of intact rock strength," *Journal of Rock Mechanics and Geotechnical Engineering*, vol. 10, no. 1, pp. 11–31, 2018.
- [18] C. A. dos Santos and J. A. Achcar, "A Bayesian analysis for the block and basu bivariate exponential distribution in the presence of covariates and censored data," *Journal of Applied Statistics*, vol. 38, no. 10, pp. 2213–2223, 2011.
- [19] A. Gelman, J. Carlin, A. Vehtari, H. S. Stern, D. Rubin, and D. Dunson, *Bayesian Data Analysis*, CRC Press, Boca Raton, FL, USA, 2013.
- [20] X. Jia and B. Guo, "Analysis of non-repairable cold-standby systems in Bayes theory," *Journal of Statistical Computation and Simulation*, vol. 86, no. 11, pp. 2089–2112, 2016.
- [21] X. Jia, S. Nadarajah, and B. Guo, "Bayes estimation of $P(Y < X)$ for the Weibull distribution with arbitrary parameters," *Applied Mathematical Modelling*, vol. 47, pp. 249–259, 2017.
- [22] D. P. M. Scollnik, "Bayesian inference for three bivariate beta binomial models," *The Open Statistics & Probability Journal*, vol. 8, no. 1, 2017.
- [23] D. Sivia and J. Skilling, *Data Analysis: A Bayesian Tutorial*, OUP, Oxford, UK, 2006.
- [24] D. Foreman-Mackey, D. W. Hogg, D. Lang, and J. Goodman, "emcee: the MCMC hammer," *Publications of the Astronomical Society of the Pacific*, vol. 125, no. 925, pp. 306–312, 2013.
- [25] D. Lunn, A. Thomas, N. Best, C. A. Jackson, and D. Spiegelhalter, *The BUGS Book: A Practical Introduction to Bayesian Analysis*, CRC Press, Boca Raton, FL, USA, 2012.
- [26] J. Kruschke, *Doing Bayesian Data Analysis: A Tutorial with R and BUGS*, Academic Press/Elsevier, Burlington, MA, USA, 2011.
- [27] J. Barry, "Doing Bayesian data analysis: a tutorial with R and BUGS," *Europe's Journal of Psychology*, vol. 7, no. 4, p. 778, 2011.
- [28] A. E. Gelfand and A. F. M. Smith, "Sampling-based approaches to calculating marginal densities," *Journal of the American Statistical Association*, vol. 85, no. 410, pp. 398–409, 1990.
- [29] S. Chib and E. Greenberg, "Understanding the metropolis-hastings algorithm," *The American Statistician*, vol. 49, no. 4, pp. 327–335, 1995.
- [30] J. Kruschke, *Doing Bayesian Data Analysis: A Tutorial with R, JAGS, and Stan*, Academic Press, Cambridge, MA, USA, 2014.
- [31] N. Abbas, "On examining complex systems using the q-weibull distribution in classical and Bayesian paradigms," *Journal of Statistical Theory and Applications*, vol. 19, no. 3, pp. 368–382, 2020.