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Research Article

On Opial-Type Inequalities for Superquadratic Functions and Applications in Fractional Calculus

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We have studied the Opial-type inequalities for superquadratic functions proved for arbitrary kernels. These are estimated by applying mean value theorems. Furthermore, by analyzing specific functions, the fractional integral and fractional derivative inequalities are obtained.

1. Introduction and Preliminary Results

Opial's inequality is very important and useful in the study of differential and difference equations. It was introduced by Opial in 1960. It is stated in the following theorem.

Theorem 1 (see [1]). *Let $\psi \in C^1[0,c]$ be such that $\psi(0) = 0$ and $\psi(t) > 0$ for $t \in (0,c)$. Then,*

$$\int_0^c \left|\psi(t)\psi'(t)\right| \mathrm{d}t \leq \frac{c}{4}\int_0^h \left(\psi'(t)^2\right)\mathrm{d}t. \tag{1}$$

Here, $(c/4)$ is the best possible constant.

Many authors have been working continuously on Opial's inequality and succeeded to establish very interesting results. For its numerous generalizations and extensions, we refer the readers to [2–14] and references therein. Opial-type inequalities are useful in the study of difference and differential equations, for example, in the uniqueness of initial value problems, in the existence and uniqueness of boundary value problems, and setting upper bounds of their solutions. For a historical survey of the inequalities given after the publication of inequality (1), one can study the book by Agarwal and Pang; see [2]. Mitrinović and Pečarić, in 1988, proved Opial-type inequalities for convex functions with respect to the power function by defining two classes of functions involving general kernels; see [15, 16]. Motivated by these inequalities, recently, we have studied such inequalities for convex functions by considering a generalized class of functions with arbitrary kernel [17]. Also, for special kernels, fractional Opial-type inequalities are proved for different fractional integral and derivative operators [18, 19].

Our aim in this paper is to study Opial-type inequalities for superquadratic functions which are also connected with inequalities that hold for convex functions. The established results estimate the Opial-type inequalities with the help of some suitably defined functions. The definitions and results needful for the presentation of results of this paper are given as follows.

Definition 1 (see [4]). A function $\psi\colon I \longrightarrow \mathbb{R}$ is said to be convex if for all $x, y \in I$ and all $t \in [0,1]$,

$$\psi(tx + (1-t)y) \leq t\psi(x) + (1-t)\psi(y), \tag{2}$$

holds, where I is an interval in $\mathbb{R}$.

Definition 2 (see [20]). A function $\psi\colon [0,\infty) \longrightarrow \mathbb{R}$ is called a superquadratic function provided for all $x \geq 0$, there exists a real constant C_x such that

$$\psi(y) \geq \psi(x) + C_x(y-x) + \psi(|y-x|), \quad \forall y \geq 0. \tag{3}$$

The following lemma explores convexity from the superquadratic function.

Lemma 1 (see [20]). *Let ψ be a superquadratic function with C_x as in Definition 2. Then, we have*

(i) $\psi(0) \leq 0$

(ii) $C_x = \psi'(x)$ *where* ψ *is differentiable for* $x > 0$ *and* $\psi(0) = \psi'(0) = 0$

(iii) ψ *is convex and* $\psi(0) = \psi'(0) = 0$ *if* $\psi \geq 0$

Let $U_1(u,k)$ denote the class of functions w: $[a,b] \longrightarrow \mathbb{R}$ having representation

$$w(x) = \int_a^x k(x,t)u(t)\mathrm{d}t, \tag{4}$$

where u is a continuous function and k is an arbitrary nonnegative kernel such that $k(x,t) = 0$ for $t < x$ and $u(x) > 0$ implies $w(x) > 0$ for every $x \in [a,b]$. Let $U_2(u,k)$ denote the class of functions w: $[a,b] \longrightarrow \mathbb{R}$ having representation

$$w(x) = \int_x^b k(x,t)u(t)\mathrm{d}t, \tag{5}$$

where u is a continuous function and k is an arbitrary nonnegative kernel such that $k(x,t) = 0$ for $t < x$ and $u(x) > 0$ implies $w(x) > 0$ for every $x \in [a,b]$.

Theorem 2 (see [15, 16]). *Let ψ: $[0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that, for $p_2 > 1$, the function $\psi(x^{(1/p_2)})$ is convex and $\psi(0) = 0$. Let $w \in U_1(u,k)$ where $(\int_a^x (k(x,t)^{p_1} dt)^{(1/p_1)} \leq K_1$ and $(1/p_1) + (1/p_2) = 1$. Then,*

$$\int_a^b |w(x)|^{1-p_2}\psi'(|w(x)|)|u(x)|^{p_2}\mathrm{d}x \leq \frac{p_2}{K_1^{p_2}}\psi\left(K_1\left(\int_a^b |u(x)|^{p_2}\mathrm{d}x\right)^{(1/p_2)}\right). \tag{6}$$

The reverse of the above inequality holds when $\psi(x^{(1/p_2)})$ is concave.

A similar result for the class $U_2(u,k)$ is given as follows.

Theorem 3 (see [15, 16]). *Let ψ: $[0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that, for $p_2 > 1$, the function $\psi(x^{(1/p_2)})$ is convex and $\psi(0) = 0$. Let $w \in U_2(u,k)$ where $(\int_x^b (k(x,t)^{p_1} dt)^{(1/p_1)} \leq K_2$ and $(1/p_1) + (1/p_2) = 1$. Then,*

$$\int_a^b |w(x)|^{1-p_2}\psi'(|w(x)|)|u(x)|^{p_2}\mathrm{d}x \leq \frac{p_2}{K_2^{p_2}}\psi\left(K_2\left(\int_a^b |u(x)|^{p_2}\mathrm{d}x\right)^{(1/p_2)}\right). \tag{7}$$

The reverse of the above inequality holds when $\psi(x^{(1/p_2)})$ is concave.

Recently, the following Opial-type inequalities for superquadratic functions are established.

Theorem 4 (see [21]). *Let ψ: $[0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that $(\psi'(x)/x)$ is increasing and $\psi(0) = 0$. Let $w \in U_1(u,k)$, $|k(x,t)| \leq K$, and $M < \int_a^x |u(t)|dt$. Then, the following inequality holds for the superquadratic function ψ:*

$$\int_a^b \frac{\psi'(|w(x)|)|u(x)|}{|w(x)|}\mathrm{d}x \leq \frac{1}{K^2 M}\psi\left(K\int_a^b (|u(t)|\mathrm{d}t)\right). \tag{8}$$

Theorem 5 (see [21]). *Under the conditions of ieorem 4, in addition, if ψ is nonnegative, then we have*

$$\begin{aligned}\int_a^b \frac{\psi'(|w(x)|)|u(x)|}{|w(x)|}\mathrm{d}x &\leq \frac{1}{K^2 M}\psi\left(K\int_a^b (|u(t)|\mathrm{d}t)\right)\\ &\leq \frac{1}{(b-a)MK^2}\int_a^b \psi(K(b-a)|u(t)|)\mathrm{d}t.\end{aligned} \tag{9}$$

Theorem 6 (see [21]). *Let ψ: $[0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that $(\psi'(x)/x)$ is increasing and $\psi(0) = 0$. Let $w \in U_1(u,k)$, $(\int_a^x (k(x,t))^{p_1} dt)^{(1/p_1)} \leq K$, $M < (\int_a^x |u(t)|^{p_2} dt)^{(2/p_2)-1}$, and $(1/p_1) + (1/p_2) = 1$. Then, the following inequality holds for superquadratic function ψ:*

$$\int_a^b \frac{\psi'(|w(x)|)|u(x)|^{p_2}}{|w(x)|}\mathrm{d}x \le \frac{p_2}{K^2M}\psi\left(K\left(\int_a^b |u(t)|^{p_2}\mathrm{d}t\right)^{(1/p_2)}\right). \tag{10}$$

Theorem 7 (see [21]). *Under the conditions of Theorem 6, in addition, if ψ is nonnegative, then we have*

$$\begin{aligned}\int_a^b \frac{\psi'(|w(x)|)|u(x)|^{p_2}}{|w(x)|}\mathrm{d}x &\le \frac{p_2}{K^2M}\psi\left(K\left(\int_a^b |u(t)|^{p_2}\mathrm{d}t\right)^{(1/p_2)}\right)\\ &\le \frac{p_2}{K^2M(b-a)}\int_a^b \Psi\left((b-a)^{(1/p_2)}K|u(t)|\right)\mathrm{d}t.\end{aligned} \tag{11}$$

The following lemmas provide classes of convex as well as superquadratic functions which will be useful to establish results of this paper.

Lemma 2 (see [22]). *Let $\psi \in C^2(I)$, where $I \subseteq (0,\infty)$ and $m' \le \psi''(\eta) \le M'$ for all $\eta \in I$. Then, the functions ψ_1 and ψ_2 defined by*

$$\begin{aligned}\psi_1(x) &= \frac{M'x^2}{2} - \psi(x),\\ \psi_2(x) &= \psi(x) - \frac{m'x^2}{2},\end{aligned} \tag{12}$$

are convex.

Lemma 3 (see [23]). *Let $\psi \in C^2(I)$, $I \subseteq (0,\infty)$ such that $m' \le (\eta\psi''(\eta) - \psi'(\eta)/\eta^2) \le M'$, for all $\eta \in I$. Consider the functions ψ_1 and ψ_2 defined by*

$$\begin{aligned}\psi_1(x) &= \frac{M'x^3}{2} - \psi(x),\\ \psi_2(x) &= \psi(x) - \frac{m'x^3}{2}.\end{aligned} \tag{13}$$

Then, the functions $(\psi_1'(x)/x)$ and $(\psi_1'(x)/x)$ are increasing. Also, if $\psi_j(0) = 0, j = 1, 2$, then the functions ψ_j, $j = 1, 2$, are superquadratic.

Lemma 4 (see [22]). *Let $I \subseteq (0,\infty)$, $\psi \in C^2(I)$, $h(x) = x^{p_2}$, $p_2 > 1$, and $m' \le ((\eta\psi''(\eta) - (p_2-1)\psi'(\eta))/p_2\eta^{2p_2-1}) \le M'$ for all $\eta \in I$. Then, the functions ψ_1 and ψ_2 defined by*

$$\begin{aligned}\psi_1(x) &= \frac{M'x^{2p_2}}{2} - \psi(x),\\ \psi_2(x) &= \psi(x) - \frac{m'x^{2p_2}}{2},\end{aligned} \tag{14}$$

are convex with respect to $h(x) = x^{p_2}$; that is, $\psi_j(x^{(1/p_2)})$, $j = 1, 2$, are convex.

Next, we define the Riemann–Liouville fractional integral and Caputo fractional derivative as follows.

Definition 3 (see [24]). Let $-\infty < a < b < \infty$ and $h \in L_1[a,b]$. Then, the left-sided and right-sided Riemann–Liouville fractional integrals of order $\gamma > 0$ are defined by

$$\begin{aligned}I_{a+}^{\gamma}h(x) &= \frac{1}{\Gamma(\gamma)}\int_a^x (x-t)^{\gamma-1}h(t)\mathrm{d}t, \quad x > a,\\ I_{b-}^{\gamma}h(x) &= \frac{1}{\Gamma(\gamma)}\int_x^b (t-x)^{\gamma-1}h(t)\mathrm{d}t, \quad x < b,\end{aligned} \tag{15}$$

where $\Gamma(.)$ is the gamma function.

Definition 4 (see [24]). Let $\gamma > 0$ and $\gamma \notin \{1,2,3,\ldots\}$, $n = [\gamma] + 1$, and $h \in AC^n[a,b]$. Then, the left-sided and right-sided Caputo fractional derivatives of order γ are defined by

$$\begin{aligned}{}^{C}D_{a+}^{\gamma}h(x) &= \frac{1}{\Gamma(n-\gamma)}\int_a^x \frac{h^n(t)}{(x-t)^{\gamma-n+1}}\mathrm{d}t, \quad x > a,\\ {}^{C}D_{b-}^{\gamma}h(x) &= \frac{(-1)^n}{\Gamma(n-\gamma)}\int_x^b \frac{h^n(t)}{(t-x)^{\gamma-n+1}}\mathrm{d}t, \quad x < b.\end{aligned} \tag{16}$$

In [25], Andrić et al. presented the composition identities for the left-sided and right-sided Caputo fractional derivatives; these are stated in the following two lemmas.

Lemma 5. *Let $\mu > \gamma \ge 0$, $l = [\mu] + 1$, and $m = [\gamma] + 1$ for $\gamma, \mu \notin N_0$; $m = [\gamma]$ and $l = [\mu]$ for $\gamma, \mu \in N_0$. Let $h \in AC^m[a,b]$ be such that $h^j(a) = 0$ for $j = m, m+1, \ldots, l-1$. Let ${}^{C}D_{a+}^{\mu}h, {}^{C}D_{a+}^{\gamma}h \in L_1[a,b]$. Then, we have*

$$ {}^{C}D_{a+}^{\gamma}h(x) = \frac{1}{\Gamma(\mu-\gamma)} \int_a^x (x-t)^{\mu-\gamma-1} {}^{C}D_{a+}^{\gamma}h(t)\mathrm{d}t, \quad x \in [a,b]. \tag{17} $$

Lemma 6. *Let $\mu > \gamma \geq 0$, $l = [\mu] + 1$, and $m = [\gamma] + 1$ for $\gamma, \mu \notin N_0$; $m = [\gamma]$ and $l = [\mu]$ for $\gamma, \mu \in N_0$. Let $h \in AC^m[a,b]$ be such that $h^j(b) = 0$ for $j = m, m+1, \ldots, l-1$. Let ${}^{C}D_{b-}^{\mu}h, {}^{C}D_{b-}^{\gamma}h \in L_1[a,b]$. Then, we have*

$$ {}^{C}D_{b-}^{\gamma}h(x) = \frac{1}{\Gamma(\mu-\gamma)} \int_x^b (t-x)^{\mu-\gamma-1} {}^{C}D_{b-}^{\gamma}h(t)\mathrm{d}t, \quad x \in [a,b]. \tag{18} $$

In the upcoming section, we prove the mean value theorems for the estimation of the nonnegative differences of the inequalities given in Theorems 4–7. In Section 3, we prove the fractional versions of the results of Section 2 for Riemann–Liouville integrals and Caputo fractional derivatives.

2. Main Results

First, we define the linear functional $\Phi_j^{\psi}(w,u;K)$, $j = 1,2$, from nonnegative differences of Opial-type inequalities for superquadratic functions given in Theorems 4 and 5 as follows:

$$ \begin{aligned} \Phi_1^{\psi}(w,u;K) &= \psi\left(K\int_a^b (|u(t)|\mathrm{d}t)\right) - K^2 M \int_a^b \frac{\psi'(|w(x)|)|u(x)|}{|w(x)|}\mathrm{d}x, \\ \Phi_2^{\psi}(w,u;K) &= \int_a^b \psi(K(b-a)|u(t)|)\mathrm{d}t - (b-a)MK^2 \int_a^b \frac{\psi'(|w(x)|)|u(x)|}{|w(x)|}\mathrm{d}x. \end{aligned} \tag{19} $$

Remark 1. Under the assumptions of Theorems 4 and 5, we have $\Phi_j^{\psi}(w,u;K) \geq 0$ for $j = 1,2$.

Theorem 8. *With the same assumptions as given in Theorem 4, furthermore, let $I \subseteq (0,\infty)$ be a compact interval and $\psi \in C^2(I)$. Also, let $m' \leq (\psi'(x)/x) \leq M'$ where $\inf_{x\in I}(\psi''(x)) = m'$ and $\sup_{x\in I}(\psi''(x)) = M'$. Then, there exists $\eta_1 \in I$ such that the following result holds:*

$$ \Phi_1^{\psi}(w,u;K) = \frac{\eta_1\psi''(\eta_1) - \psi'(\eta_1)}{3\eta_1^2}\Phi_1^{x^3}(w,u;K). \tag{20} $$

Proof. By replacing ψ with ψ_1 (defined in Lemma 3) in Theorem 4, one can have the following inequality:

$$ \begin{aligned} &\int_a^b \frac{\left(M'|w(x)|^2 - \psi'(|w(x)|)\right)|u(x)|}{|w(x)|}\mathrm{d}x \\ &\leq \frac{1}{MK^2}\left(\frac{M'}{3}\left(K\int_a^b |u(t)|\mathrm{d}t\right)^3 - \psi\left(K\left(\int_a^b |u(t)|\mathrm{d}t\right)\right)\right). \end{aligned} \tag{21} $$

From inequality (21), after simplification, one can obtain

$$ \Phi_1^{\psi}(w,u;K) \leq \frac{M'}{3}\Phi_1^{x^3}(w,u;K). \tag{22} $$

Similarly, if we take ψ_2 from Lemma 3 instead of ψ in Theorem 4, then the following inequality holds:

$$ \Phi_1^{\psi}(w,u;K) \geq \frac{m'}{3}\Phi_1^{x^3}(w,u;K). \tag{23} $$

The above two inequalities lead to the following inequality:

$$ m' \leq \frac{3\left(\Phi_1^{\psi}(w,u;K)\right)}{\Phi_1^{x^3}(w,u;K)} \leq M'. \tag{24} $$

Therefore, there exists $\eta_1 \in I$ such that the following equation is obtained:

$$ \frac{\eta_1\psi''(\eta_1) - \psi'(\eta_1)}{\eta_1^2} = \frac{3\Phi_1^{\psi}(w,u;K)}{\Phi_1^{x^3}(w,u;K)}, \tag{25} $$

which gives equation (20). □

Theorem 9. *With the same assumptions on ψ_1 and ψ_2 as given for ψ in Theorem 4, furthermore, if $I \subseteq (0,\infty)$ is a compact interval and $\psi_1, \psi_2 \in C^2(I)$ where $\Phi_1^{x^3}(w,u;K) \neq 0$, then there exists $\eta_1 \in I$ such that the following result holds:*

$$ \frac{\Phi_1^{\psi_1}(w,u;K)}{\Phi_1^{\psi_2}(w,u;K)} = \frac{\eta_1\psi_1''(\eta_1) - \psi_1'(\eta_1)}{\eta_1\psi_2''(\eta_1) - \psi_2'(\eta_1)}, \tag{26} $$

where denominators should not be zero.

Proof. Let us define the function f by $f = \lambda_1\psi_1 - \lambda_2\psi_2$, where λ_1 and λ_2 are given by

$$ \begin{aligned} \lambda_1 &= \phi_1^{\psi_2}(w,u;K), \\ \lambda_2 &= \phi_1^{\psi_1}(w,u;K). \end{aligned} \tag{27} $$

Then, $f \in C^2(I)$; by applying Theorem 8 for f, it follows that there exists η_1 such that we have

$$ \begin{aligned} 0 = &(\lambda_1(\eta_1\psi_1''(\eta_1) - \psi_1'(\eta_1)) - \lambda_2(\eta_1\psi_2''(\eta_1) \\ &- \psi_2'(\eta_1)))\Phi_1^{x^3}(w,u;K). \end{aligned} \tag{28} $$

From this, one can get the required equation. □

Theorem 10. *With the same assumptions as given in Theorem 5, furthermore, if $I\subseteq(0,\infty)$ is a compact interval and $\psi\in C^2(I)$, then there exists $\eta_2\in I$ such that the following result holds:*

$$\Phi_2^{\psi}(w,u;K)=\frac{\psi''(\eta_2)}{2}\Phi_2^{x^2}(w,u;K). \tag{29}$$

Proof. By replacing ψ with ψ_1 from Lemma 2 in Theorem 5, we get the following inequality:

$$\int_a^b \frac{(M'|w(x)|-\psi'|w(x)|)|u(x)|}{|w(x)|}\mathrm{d}x\le\frac{1}{(b-a)MK^2}\times\left(\left(\frac{M'}{2}\left(\int_a^b(K(b-a)|u(t)|\mathrm{d}t)\right)^2\right)-\int_a^b\psi(K(b-a)|u(t)|)\mathrm{d}t\right). \tag{30}$$

Similarly, adopting the method for functional $\Phi_2^{\psi}(w,u;K)$ as we did for $\Phi_1^{\psi}(w,u;K)$ in the proof of Theorem 8, one can see there exists $\eta_2\in I$ such that equation (29) holds. □

Theorem 11. *With the same assumptions on ψ_1 and ψ_2 as given for ψ in Theorem 5, furthermore if $I\subseteq(0,\infty)$ is a compact interval and $\psi_1,\psi_2\in C^2(I)$ where $\Phi_2^{x^2}(w,u;K)\neq0$, then there exists $\eta_2\in I$ such that we have*

$$\frac{\Phi_2^{\psi_1}(w,u;K)}{\Phi_2^{\psi_2}(w,u;K)}=\frac{\psi_1''(\eta_2)}{\psi_2''(\eta_2)}, \tag{31}$$

where denominators should not be zero.

Proof. Let us define the function g by $g=\lambda_1\psi_1-\lambda_2\psi_2$, where λ_1 and λ_2 are given by

$$\begin{aligned}\lambda_1&=\Phi_2^{\psi_1}(w,u;K),\\ \lambda_2&=\Phi_2^{\psi_2}(w,u;K).\end{aligned} \tag{32}$$

Then, $g\in C^2(I)$; by applying Theorem 10 for g, it follows that there exists η_2 such that we have

$$0=(\lambda_1\psi_1''(\eta_2)-\lambda_2\psi_2''(\eta_2))\Phi_2^{x^2}(w,u;K). \tag{33}$$

From this, one can get the required equation.

Next, we define functional $\Pi_j^{\psi}(w,u;K)$, $j=1,2$, from nonnegative differences of Opial-type inequalities for superquadratic functions given in Theorems 6 and 7 as follows:

$$\begin{aligned}\Pi_1^{\psi}(w,u;K)&=\psi\left(K\left(\int_a^b|u(t)|^{p_2}\mathrm{d}t\right)^{(1/p_2)}\right)-\frac{K^2M}{p_2}\int_a^b\frac{\psi'(|w(x)|)|u(x)|^{p_2}}{|w(x)|}\mathrm{d}x,\\ \Pi_2^{\psi}(w,u;K)&=\int_a^b\psi\left(K(b-a)^{(1/p_2)}|u(t)|\right)\mathrm{d}t-\frac{K^2M(b-a)}{p_2}\int_a^b\frac{\psi'(|w(x)|)|u(x)|^{p_2}}{|w(x)|}\mathrm{d}x.\end{aligned} \tag{34}$$

□

Remark 2. Under the assumptions of Theorems 6 and 7, we have $\Pi_j^{\psi}(w,u;K)\ge0$ for $j=1,2$.

Theorem 12. *With the same assumptions of Theorem 6 on ψ, furthermore, if $I\subseteq(0,\infty)$ is a compact interval and $\psi\in C^2(I)$, then there exists $\eta_3\in I$ such that we have*

$$\Pi_1^{\psi}(w,u;K)=\frac{\eta_3\psi''(\eta_3)-\psi'(\eta_3)}{3\eta_3^2}\Pi_1^{x^3}(w,u;K). \tag{35}$$

Proof. By replacing ψ with ψ_1 from Lemma 3 in Theorem 6, we get the following inequality:

$$\int_a^b \frac{\left(M'|w(x)|^2 - \psi'\left(|w(x)|\right)\right)|u(x)|^{p_2}}{|w(x)|}\mathrm{d}x \le \frac{p_2}{MK^2}\left(\frac{M'}{3}\left(K\left(\int_a^b |u(t)|^{p_2}\mathrm{d}t\right)^{(1/p_2)}\right)^3 - \psi\left(K\left(\int_a^b |u(t)|^{p_2}\mathrm{d}t\right)^{(1/p_2)}\right)\right). \tag{36}$$

From inequality (36), after simplification, one can obtain

$$\Pi_1^{\psi}(w,u;K) \le \frac{M'}{3}\Pi_1^{x^3}(w,u;K). \tag{37}$$

Similarly, if we apply ψ_2 from Lemma 3 instead of ψ in Theorem 6, then the following inequality holds:

$$\Pi_1^{\psi}(w,u;K) \ge \frac{m'}{3}\Pi_1^{x^3}(w,u;K). \tag{38}$$

The above two inequalities lead to the following inequality:

$$m' \le \frac{3\Pi_1^{\psi}(w,u;K)}{\Pi_1^{x^3}(w,u;K)} \le M'. \tag{39}$$

Therefore, there exists $\eta_3 \in I$ such that the following equation is obtained:

$$\frac{\eta_3\psi''(\eta_3) - \psi'(\eta_3)}{\eta_3^2} = \frac{3\Pi_1^{\psi}(w,u;K)}{\Pi_1^{x^3}(w,u;K)}, \tag{40}$$

which gives the required equation. □

Theorem 13. *With the same assumptions on ψ_1 and ψ_2 as given for ψ in Theorem 6, furthermore, if $I \subseteq (0,\infty)$ is a compact interval and $\psi_1,\psi_2 \in C^2(I)$ where $\Pi_1^{x^3}(w,u) \neq 0$, then there exists $\eta_3 \in I$ such that we have*

$$\frac{\Pi_1^{\psi_1}(w,u;K)}{\Pi_1^{\psi_2}(w,u;K)} = \frac{\eta_3\psi_1''(\eta_3) - \psi_1(\eta_3)}{\eta_3\psi_2''(\eta_3) - \psi_2(\eta_3)}, \tag{41}$$

where denominators should not be zero.

Proof. Let us define the function h by $h = \omega_1\psi_1 - \omega_2\psi_2$, where ω_1 and ω_2 are given by

$$\omega_1 = \Pi_2^{\psi_1}(w,u;K), \quad \omega_2 = \Pi_2^{\psi_2}(w,u;K). \tag{42}$$

Then, $h \in C^2(I)$; by applying Theorem 12 for h, it follows that there exists η_3 such that we have

$$0 = \omega_1\frac{\eta_3\psi_1''(\eta_3) - \psi_1(\eta_3)}{3\eta_3^2} - \omega_2\frac{\eta_3\psi_2''(\eta_3) - \psi_2(\eta_3)}{3\eta_3^2}\Pi_2^{x^3}(w,u;K). \tag{43}$$

From this, one can get the required equation. □

Theorem 14. *With the same assumptions of Theorem 7 on ψ, furthermore, if $I \subseteq (0,\infty)$ is a compact interval and $\psi \in C^2(I)$, then there exists $\eta_4 \in I$ such that the following result holds:*

$$\Pi_2^{\psi}(w,u;K) = \frac{\eta_4\psi''(\eta_4) - (p_2-1)\psi(\eta_4)}{2p_2\eta_4^{2p_2-1}}\Pi_2^{x^{2p_2}}(w,u;K). \tag{44}$$

Proof. By replacing ψ with ψ_1 from Lemma 3 in Theorem 4, we get the following inequality:

$$\int_a^b \frac{\left(M' p_2|w(x)|^{2p_2-1} - \psi'|w(x)|\right)|u(x)|^{p_2}}{|w(x)|}\mathrm{d}x \le \frac{p_2}{(b-a)MK^2} \times\left(\frac{M'}{2}\left(\int_a^b\left(K(b-a)^{(1/p_2)}|u(t)|\mathrm{d}t\right)\right)^{2p_2} - \int_a^b \psi\left(K(b-a)^{(1/p_2)}|u(t)|\right)\mathrm{d}t\right). \tag{45}$$

Rest of the proof can be followed from the proof of Theorem 12. □

Theorem 15. *With the same assumptions on ψ_1 and ψ_2 as given for ψ in Theorem 7, furthermore, if $I \subseteq (0,\infty)$ is a compact interval and $\psi_1,\psi_2 \in C^2(I)$ where $\Pi_1^{x^{2p_2}}(w,u;K) \neq 0$, then there exists $\eta_4 \in I$ such that we have*

$$\frac{\Pi_1^{\psi_1}(w,u;K)}{\Pi_2^{\psi_2}(w,u;K)} = \frac{\eta_4\psi_1''(\eta_4) - (p_2-1)\psi_1'(\eta_4)}{\eta_4\psi_2''(\eta_4) - (p_2-1)\psi_2'(\eta_4)}, \tag{46}$$

where denominators should not be zero.

Proof. The proof can be followed from the proof of Theorem 13. □

3. Fractional Versions of Mean Value Theorems

In this section, we give applications of results proved in Section 2 by choosing particular kernels. We get fractional versions of mean value theorems by applying definitions of fractional integral/derivative operators. First, we give results for Riemann–Liouville fractional integrals.

Theorem 16. *With same assumptions of Theorem 4 on ψ, furthermore, let $I\subseteq(0,\infty)$ be a compact interval and $\psi\in C^2(I)$. Also, let $u\in L_1[a,b]$ be a Riemann–Liouville fractional integral of order γ, where $\gamma\geq 1$; then, there exists $\eta_1\in I$ such that the following result holds:*

$$\Phi_1^{\psi}\left(I_{a+}^{\gamma}u,u;\frac{(b-a)^{\gamma-1}}{\Gamma(\gamma)}\right)=\frac{\eta_1\psi''(\eta_1)-\psi'(\eta_1)}{3\eta_1^2}\Phi_1^{x^3}\left(I_{a+}^{\gamma}u,u;\frac{(b-a)^{\gamma-1}}{\Gamma(\gamma)}\right). \tag{47}$$

Proof. Let us define the kernel $k(x,t)$ as follows:

$$k(x,t)=\begin{cases}\dfrac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}, & t\in[a,x],\\ 0, & t\in(x,b].\end{cases} \tag{48}$$

Furthermore, we take w as follows:

$$w(x)=I_{a+}^{\gamma}u(x)=\frac{1}{\Gamma(\gamma)}\int_a^x(x-t)^{\gamma-1}u(t)\mathrm{d}t. \tag{49}$$

It is clear that, for $\gamma\geq 1$, $0\leq k(x,t)\leq((x-a)^{\gamma-1}/\Gamma(\gamma))\leq((b-a)^{\gamma-1}/\Gamma(\gamma))$, $t\in[a,x]$, $x\in[a,b]$. By choosing $K=((b-a)^{\gamma-1}/\Gamma(\gamma))$ and applying Theorem 8 for the kernel given in (48), one can get the required result. □

Theorem 17. *With the same assumptions of Theorem 4 on ψ_1 and ψ_2, furthermore, let $I\subseteq(0,\infty)$ be a compact interval and $\psi_1,\psi_2\in C^2(I)$. Also, let $u\in L_1[a,b]$ be a Riemann–Liouville fractional integral of order γ where $\gamma\geq 1$; then, there exists η_1 such that we have*

$$\frac{\phi_1^{\psi_1}\left(I_{a+}^{\gamma}u,u;\left((b-a)^{\gamma-1}/\Gamma(\gamma)\right)\right)}{\phi_1^{\psi_2}\left(I_{a+}^{\gamma}u,u;\left((b-a)^{\gamma-1}/\Gamma(\gamma)\right)\right)}=\frac{\eta_1\psi_1''(\eta_1)-\psi_1'(\eta_1)}{\eta_1\psi_2''(\eta_1)-\psi_2'(\eta_1)}, \tag{50}$$

where denominators should not be zero.

Proof. It is easy to prove by applying Theorem 9 for the kernel defined in (48) and using the function w given by (49). □

Theorem 18. *With the same assumptions of Theorem 5 on ψ, furthermore, let $I\subseteq(0,\infty)$ be a compact interval and $\psi\in C^2(I)$. Also, let $u\in L_1[a,b]$ be a Riemann–Liouville fractional integral of order γ where $\gamma\geq 1$; then, there exists $\eta_2\in I$ such that the following result holds:*

$$\Phi_2^{\psi}\left(I_{a+}^{\gamma}u,u;\frac{(b-a)^{\gamma-1}}{\Gamma(\gamma)}\right)=\frac{\psi''(\eta_2)}{2}\Phi_2^{x^2}\left(I_{a+}^{\gamma}u,u;\frac{(b-a)^{\gamma-1}}{\Gamma(\gamma)}\right). \tag{51}$$

Proof. The proof is similar to the proof of Theorem 16. □

Theorem 19. *With the same assumptions of Theorem 5 on ψ_1 and ψ_2, furthermore, let $I\subseteq(0,\infty)$ be a compact interval and $\psi_1,\psi_2\in C^2(I)$. Also, let $u\in L_1[a,b]$ be a Riemann–Liouville fractional integral of order γ where $\gamma\geq 1$; then, there exists η_2 such that we have*

$$\frac{\Phi_2^{\psi_1}\left(I_{a+}^{\gamma}u,u;\left((b-a)^{\gamma-1}/\Gamma(\gamma)\right)\right)}{\Phi_2^{\psi_2}\left(I_{a+}^{\gamma}u,u;\left((b-a)^{\gamma-1}/\Gamma(\gamma)\right)\right)}=\frac{\psi_1''(\eta_2)}{\psi_2''(\eta_2)}, \tag{52}$$

where denominators should not be zero.

Proof. It is easy to prove by applying Theorem 11 for the kernel defined in (48) and using the function w given by (49). □

Theorem 20. *With the same assumptions of Theorem 6 on ψ, furthermore, let $I\subseteq(0,\infty)$ be a compact interval and $\psi\in C^2(I)$. Let $u\in L_1[a,b]$ be a Riemann–Liouville fractional integral of order γ where $\gamma\geq(1/p_2)$; then, there exists $\eta_3\in I$ such that the following result holds:*

$$\begin{aligned}\Pi_1^{\psi}\left(I_{a+}^{\gamma}u,u;\frac{(b-a)^{\gamma-(1/p_2)}}{\Gamma(\gamma)(p_1(\gamma-1)+1)^{(1/p_1)}}\right)&=\frac{\eta_3\psi''(\eta_1)-\psi'(\eta_3)}{3\eta_3^2}\\&\times\Pi_1^{x^3}\left(I_{a+}^{\gamma}u,u;\frac{(b-a)^{\gamma-(1/p_2)}}{\Gamma(\gamma)(p_1(\gamma-1)+1)^{(1/p_1)}}\right).\end{aligned} \tag{53}$$

Proof. Let us consider the kernel $k(x,t)$ as defined in (48) and w given in (49). If we set $Q(x)=(\int_a^x(k(x,t))^{p_1}\mathrm{d}t)^{(1/p_1)}$, then $Q(x)=((x-a)^{\gamma-(1/p_2)}/\Gamma(\gamma)(p_1(\gamma-1)+1)^{(1/p_1)})$. Furthermore, Q is increasing for $\gamma>(1/p_2)$, on $[a,b]$;

therefore, we have $Q(x) \leq ((b-a)^{\gamma-(1/p_2)}/\Gamma(\gamma)(p_1(\gamma-1)+1)^{(1/p_1)}) = K$. By applying Theorem 12, we get the required result. ☐

Theorem 21. *With the same assumptions of Theorem 6 on ψ_1 and ψ_2, furthermore, let $I \subseteq (0,\infty)$ be a compact interval and $\psi_1, \psi_2 \in C^2(I)$. Furthermore, let $u \in L_1[a,b]$ be a Riemann–Liouville fractional integral of order γ. If $\gamma > (1/p_2)$ and $\Pi_2^{x^3}(I_{a+}^{\gamma}u, u) \neq 0$, then there exists $\eta_3 \in I$ such that the following equality holds:*

$$\frac{\Pi_1^{\psi_1}\left(I_{a+}^{\gamma}u, u; \left((b-a)^{\gamma-(1/p_2)}/\Gamma(\gamma)(p_1(\gamma-1)+1)^{(1/p_1)}\right)\right)}{\Pi_1^{\psi_2}\left(I_{a+}^{\gamma}u, u; \left((b-a)^{\gamma-(1/p_2)}/\Gamma(\gamma)(p_1(\gamma-1)+1)^{(1/p_1)}\right)\right)} = \frac{\eta_3\psi_1''(\eta_3) - \psi_1(\eta_3)}{\eta_3\psi_2''(\eta_3) - \psi_2(\eta_3)}, \tag{54}$$

where denominators should not be zero.

Proof. It is easy to prove by applying Theorem 13 for the kernel defined in (48) and using the function w given by (49). ☐

Theorem 22. *With the same assumptions of Theorem 7 on ψ, furthermore, let $I \subseteq (0,\infty)$ be a compact interval and $\psi \in C^2(I)$. Let $u \in L_1[a,b]$ be a Riemann–Liouville fractional integral of order γ. If $\gamma > (1/p_2)$, then there exists $\eta_4 \in I$ such that the following result holds:*

$$\Pi_2^{\psi}\left(I_{a+}^{\gamma}u, u; \frac{(b-a)^{\gamma-(1/p_2)}}{\Gamma(\gamma)(p_1(\gamma-1)+1)^{(1/p_1)}}\right) = \frac{\eta_4\psi''(\eta_4) - (p_2-1)\psi'(\eta_4)}{2p_2\eta_4^{2p_2-1}} \times \Pi_2^{x^{2p_2}}\left(I_{a+}^{\gamma}u, u; \frac{(b-a)^{\gamma-(1/p_2)}}{\Gamma(\gamma)(p_1(\gamma-1)+1)^{(1/p_1)}}\right). \tag{55}$$

Proof. The proof is similar to the proof of Theorem 20. ☐

Theorem 23. *With the same assumptions of Theorem 7 on ψ_1 and ψ_2, furthermore, let $I \subseteq (0,\infty)$ be a compact interval and $\psi_1, \psi_2 \in C^2(I)$. Let $u \in L_1[a,b]$ be a Riemann–Liouville fractional integral of order γ. If $\gamma > (1/p_2)$ and $\Pi_2^{x^{2p_2}}(I_{a+}^{\gamma}u, u) \neq 0$, then there exists $\eta_4 \in I$ such that the following equality holds:*

$$\frac{\Pi_1^{\psi_1}\left(I_{a+}^{\gamma}u, u; \left((b-a)^{\gamma-(1/p_2)}/\Gamma(\gamma)(p_1(\gamma-1)+1)^{(1/p_1)}\right)\right)}{\Pi_1^{\psi_2}\left(I_{a+}^{\gamma}u, u; \left((b-a)^{\gamma-(1/p_2)}/\Gamma(\gamma)(p_1(\gamma-1)+1)^{(1/p_1)}\right)\right)} = \frac{\eta_4\psi_1''(\eta_4) - (p_2-1)\psi_1(\eta_4)}{\eta_4\psi_2''(\eta_4) - (p_2-1)\psi_2(\eta_4)}, \tag{56}$$

where denominators should not be zero.

Proof. It is easy to prove by applying Theorem 15 for the kernel defined in (48) and using the function w given by (49).

Next, we give the results for Caputo fractional derivatives using their composition identities. ☐

Theorem 24. *With the same assumptions of Theorem 4 on ψ, furthermore, let $I \subseteq (0,\infty)$ be a compact interval and $\Psi \in C^2(I)$. Also, let $l = [\mu]+1$ and $m = [\gamma]+1$, for $\gamma, \mu \notin N_0$ and $u \in AC^m[a,b]$ such that $u^j(a) = 0$ for $j = m, m+1, \ldots, l-1$. Let ${}^CD_{a+}^{\mu}u, {}^CD_{a+}^{\gamma}u \in L_1[a,b]$. Then, for $\gamma \leq \mu - 1$, the following result holds:*

$$\Phi_1^{\psi}\left({}^CD_{a+}^{\gamma}u, {}^CD_{a+}^{\mu}u; \frac{(b-a)^{\mu-\gamma-1}}{\Gamma(\mu-\gamma)}\right) = \frac{\eta_1\psi_1''(\eta_1) - \psi_1'(\eta_1)}{2} \times \Phi_1^{x^3}\left({}^CD_{a+}^{\gamma}u, {}^CD_{a+}^{\mu}u; \frac{(b-a)^{\mu-\gamma-1}}{\Gamma(\mu-\gamma)}\right). \tag{57}$$

Proof. Let us consider the kernel $k(x,t)$ as follows:

$$k(x,t) = \begin{cases} \dfrac{(x-t)^{\mu-\gamma-1}}{\Gamma(\mu-\gamma)}, & t \in [a,x], \\ 0, & t \in (x,b]. \end{cases} \tag{58}$$

Furthermore, we take w as follows:

$$w(x) = {}^{C}D_{a+}^{\gamma}u(x) = \frac{1}{\Gamma(\mu-\gamma)}\int_a^x (x-t)^{\mu-\gamma-1}\,{}^{C}D_{a+}^{\mu}u(t)\mathrm{d}t. \tag{59}$$

It is clear that, for $\gamma \le \mu - 1$, $0 \le k(x,t) \le ((x-a)^{\mu-\gamma-1}/\Gamma(\mu-\gamma)) \le ((b-a)^{\mu-\gamma-1}/\Gamma(\mu-\gamma))$, $t \in [a,x]$, $x \in [a,b]$. By choosing $K = ((b-a)^{\mu-\gamma-1}/\Gamma(\mu-\gamma))$ and applying Theorem 8 for the kernel given in (59), one can get the required result. □

Theorem 25. *With the same assumptions of Theorem 4 on ψ_1 and ψ_2, furthermore, let $I \subseteq (0,\infty)$ be a compact interval and $\psi_1, \psi_2 \in C^2(I)$. Let $l = [\mu]+1$ and $m = [\gamma]+1$, for $\gamma, \mu \notin N_0$. Also, let ${}^{C}D_{a+}^{\mu}u, {}^{C}D_{a+}^{\mu}u \in L_1[a,b]$, $\gamma \le \mu - 1$, and $\phi_1^{x^3}(D_{a+}^{\gamma}u, D_{a+}^{\mu}u) \ne 0$. Then, there exists $\eta_1 \in I$ such that we have*

$$\frac{\Phi_1^{\psi_1}\left({}^{C}D_{a+}^{\gamma}u, {}^{C}D_{a+}^{\mu}u; \left((b-a)^{\mu-\gamma-1}/\Gamma(\mu-\gamma)\right)\right)}{\Phi_1^{\psi_2}\left({}^{C}D_{a+}^{\gamma}u, {}^{C}D_{a+}^{\mu}u; \left((b-a)^{\mu-\gamma-1}/\Gamma(\mu-\gamma)\right)\right)} = \frac{\eta_1\psi_1''(\eta_1) - \psi_1'(\eta_1)}{\eta_1\psi_2''(\eta_2) - \psi_2'(\eta_1)}, \tag{60}$$

where denominators should not be zero.

Proof. It is easy to prove by applying Theorem 9 for the kernel defined in (58) and using function w given by (59). □

Theorem 26. *With the same assumptions of Theorem 5 on ψ, furthermore, let $I \subseteq (0,\infty)$ be a compact interval and $\psi \in C^2(I)$. Also, let $l = [\mu]+1$ and $m = [\gamma]+1$, for $\gamma, \mu \notin N_0$, and $u \in AC^m[a,b]$ such that $u^j(a) = 0$ for $j = m, m+1, \ldots, l-1$. Let ${}^{C}D_{a+}^{\mu}u, {}^{C}D_{a+}^{\gamma}u \in L_1[a,b]$. Then, the following result holds for $\gamma \le \mu - 1$:*

$$\Phi_2^{\psi}\left({}^{C}D_{a+}^{\gamma}u, {}^{C}D_{a+}^{\mu}u; \frac{(b-a)^{\mu-\gamma-1}}{\Gamma(\mu-\gamma)}\right) = \frac{\psi''(\eta_2)}{2}\Phi_2^{x^2}\left({}^{C}D_{a+}^{\gamma}u, {}^{C}D_{a+}^{\mu}u\right). \tag{61}$$

Proof. The proof is similar to the proof of Theorem 24. □

Theorem 27. *With the same assumptions of Theorem 5 on ψ_1 and ψ_2, furthermore, let $I \subseteq (0,\infty)$ be a compact interval and $\psi_1, \psi_2 \in C^2(I)$. Let $l = [\mu]+1$ and $m = [\gamma]+1$, for $\gamma, \mu \notin N_0$. Also, let ${}^{C}D_{a+}^{\mu}u, {}^{C}D_{a+}^{\mu}u \in L_1[a,b]$, $\gamma \le \mu - 1$, and $\phi_2^{x^2}(D_{a+}^{\gamma}u, D_{a+}^{\mu}u) \ne 0$. Then, there exists $\eta_2 \in I$ such that we have*

$$\frac{\Phi_2^{\psi_1}\left({}^{C}D_{a+}^{\gamma}u, {}^{C}D_{a+}^{\mu}u; \left((b-a)^{\mu-\gamma-1}/\Gamma(\mu-\gamma)\right)\right)}{\Phi_2^{\psi_2}\left({}^{C}D_{a+}^{\gamma}u, {}^{C}D_{a+}^{\mu}u; \left((b-a)^{\mu-\gamma-1}/\Gamma(\mu-\gamma)\right)\right)} = \frac{\psi_1''(\eta_2)}{\psi_2''(\eta_2)}, \tag{62}$$

where denominators should not be zero.

Proof. It is easy to prove by applying Theorem 11 for the kernel defined in (58) and using function w given by (59). □

Theorem 28. *With the same assumptions of Theorem 6 on ψ, furthermore, let $I \subseteq (0,\infty)$ be a compact interval and $\psi \in C^2(I)$. Also, let $l = [\mu]+1$ and $m = [\gamma]+1$, for $\gamma, \mu \notin N_0$, and $u \in AC^m[a,b]$ such that $u^j(a) = 0$ for $j = m, m+1, \ldots, l-1$. Let ${}^{C}D_{a+}^{\mu}u, {}^{C}D_{a+}^{\gamma}u \in L_1[a,b]$. Then, for $\gamma < \mu - (1/p_2)$, the following result holds:*

$$\Pi_1^{\psi}\left({}^{C}D_{a+}^{\gamma}u, {}^{C}D_{a+}^{\mu}u; \frac{(b-a)^{\mu-\gamma-(1/p_2)}}{\Gamma(\mu-\gamma)\left(p_1(\gamma-1)+1\right)^{(1/p_1)}}\right) = \frac{\eta_3\psi''(\eta_3) - \psi'(\eta_3)}{3\eta_3^2} \times \Pi_1^{x^3}\left({}^{C}D_{a+}^{\gamma}u, {}^{C}D_{a+}^{\mu}u; \frac{(b-a)^{\mu-\gamma-(1/p_2)}}{\Gamma(\mu-\gamma)\left(p_1(\gamma-1)+1\right)^{(1/p_1)}}\right). \tag{63}$$

Proof. Let us consider the kernel $k(x,t)$ as defined in (58) and w given in (59). If we set $Q(x) = (\int_a^x (k(x,t))^{p_1}\mathrm{d}t)^{(1/p_1)}$, then $Q(x) = ((x-a)^{\mu-\gamma-(1/p_2)}/\Gamma(\mu-\gamma)(p_1(\mu-\gamma-1)+1)^{(1/p_1)})$. Furthermore, Q is increasing for $\gamma < \mu - (1/p_2)$, on $[a,b]$; therefore, we have $Q(x) \le ((b-a)^{\mu-\gamma-(1/p_2)}/\Gamma(\mu-\gamma)(p_1(\mu-\gamma-1)+1)^{(1/p_1)}) = K$. By applying Theorem 12, we get the required result. □

Theorem 29. *With the same assumptions of Theorem 6 on ψ_1 and ψ_2, furthermore, let $I \subseteq (0,\infty)$ be a compact interval and $\psi_1, \psi_2 \in C^2(I)$. Let $l = [\mu]+1$ and $m = [\gamma]+1$, for*

$\gamma,\mu \notin N_0$. Also, let ${}^C D^{\mu}_{a+}u, {}^C D^{\mu}_{a+}u \in L_1[a,b]$, $\gamma \le \mu - 1$, and $\Pi_2^{x^3}(D^{\gamma}_{a+}u, D^{\mu}_{a+}u) \neq 0$. Then, there exists $\eta_3 \in I$ such that, for $\gamma < \mu - (1/p_2)$, we have

$$\frac{\Pi_1^{\psi_1}\left({}^C D^{\gamma}_{a+}u, {}^C D^{\mu}_{a+}u; \left((b-a)^{\mu-\gamma-(1/p_2)}/\Gamma(\mu-\gamma)\left(p_1(\gamma-1)+1\right)^{(1/p_1)}\right)\right)}{\Pi_1^{\psi_2}\left({}^C D^{\gamma}_{a+}u, {}^C D^{\mu}_{a+}u; \left((b-a)^{\mu-\gamma-(1/p_2)}/\Gamma(\mu-\gamma)\left(p_1(\gamma-1)+1\right)^{(1/p_1)}\right)\right)} = \frac{\eta_3\psi_1''(\eta_3)-\psi_1'(\eta_3)}{\eta_3\psi_2''(\eta_3)-\psi_2'(\eta_3)}, \tag{64}$$

where denominators should not be zero.

Proof. It is easy to prove by applying Theorem 13 for the kernel defined in (58) and using function w given by (59). □

Theorem 30. *With the same assumptions of Theorem 7 on ψ, furthermore, let $I \subseteq (0,\infty)$ be a compact interval and $\psi \in C^2(I)$. Also, let $l = [\mu]+1$ and $m = [\gamma]+1$, for $\gamma,\mu \notin N_0$, and $u \in AC^m[a,b]$ such that $u^j(a) = 0$ for $j = m, m+1, \ldots, l-1$. Let ${}^C D^{\mu}_{a+}u, {}^C D^{\gamma}_{a+}u \in L_1[a,b]$. Then, for $\gamma < \mu - (1/p_2)$, the following result holds:*

$$\Pi_2^{\psi}\left({}^C D^{\gamma}_{a+}u, {}^C D^{\mu}_{a+}u; \frac{(b-a)^{\mu-\gamma-(1/p_2)}}{\Gamma(\mu-\gamma)\left(p_1(\gamma-1)+1\right)^{(1/p_1)}}\right) = \frac{\eta_4\psi''(\eta_4)-(p_2-1)\psi_1(\eta_4)}{2p_2\eta_4^{2p_2-1}} \times \Pi_2^{x^{2p_2}}\left({}^C D^{\gamma}_{a+}u, {}^C D^{\mu}_{a+}u; \frac{(b-a)^{\mu-\gamma-(1/p_2)}}{\Gamma(\mu-\gamma)\left(p_1(\gamma-1)+1\right)^{(1/p_1)}}\right). \tag{65}$$

Proof. The proof is similar to the proof of Theorem 28. □

Theorem 31. *With the same assumptions of Theorem 7 on ψ_1 and ψ_2, furthermore, let $I \subseteq (0,\infty)$ be a compact interval and $\psi_1,\psi_2 \in C^2(I)$. Let $l = [\mu]+1$ and $m = [\gamma]+1$, for $\gamma,\mu \notin N_0$. Also, let ${}^C D^{\mu}_{a+}u \in L_q[a,b]$ and ${}^C D^{\mu}_{a+} \in L_1[a,b]$, $\gamma \le \mu - 1$, and $\Pi_j^{x^{2p_2}}(D^{\gamma}_{a+}u, D^{\mu}_{a+}u) \neq 0$. Then, there exists $\eta_4 \in I$ such that, for $\gamma < \mu - (1/p_2)$, we have*

$$\frac{\Pi_2^{\psi_1}\left({}^C D^{\gamma}_{a+}u, {}^C D^{\mu}_{a+}u; \left((b-a)^{\mu-\gamma-(1/p_2)}/\Gamma(\mu-\gamma)\left(p_1(\gamma-1)+1\right)^{(1/p_1)}\right)\right)}{\Pi_2^{\psi_1}\left({}^C D^{\gamma}_{a+}u, {}^C D^{\mu}_{a+}u; \left((b-a)^{\mu-\gamma-(1/p_2)}/\Gamma(\mu-\gamma)\left(p_1(\gamma-1)+1\right)^{(1/p_1)}\right)\right)} = \frac{\eta_4\psi_1''(\eta_4)-(p_2-1)\psi_1'(\eta_4)}{\eta_4\psi_2''(\eta_4)-(p_2-1)\psi_2'(\eta_4)}, \tag{66}$$

where denominators should not be zero.

Proof. It is easy to prove by applying Theorem 15 for the kernel defined in (58) and using function w given by (59). □

Data Availability

No additional data were required for the findings of results of this paper.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All the authors contributed equally to this article.

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