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Research Article

Near-Coincidence Point Results in Norm Interval Spaces via Simulation Functions

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Recently, Wu in 2018 established interesting results in the framework of interval spaces. He initiated the idea of near-fixed points and proved some related basic results in metric interval, norm interval, and hyperspaces. In 2015, Khojasteh et al. gave the concept of simulation functions and studied some fixed-point results in metric spaces. Motivated by this work, we give some near-coincidence point results in norm interval spaces using the concept given by Khojasteh et al. Examples are also provided for the validation of the results.

1. Introduction

Many researchers are still showing high interest in the field of metric fixed-point theory. They are working in different directions and generalizing the remarkable results in this area [1–4]. The first one who took interest in this area was Poincare. Later, Brouwer established a (topological) fixed-point theorem. The metric fixed-point theory attracts researchers due to its applications in both applied and pure mathematics. There are many applications of metric fixed-point theory in the existence of solutions for nonlinear systems. In 1922, Banach [5] established a remarkable result, known as the Banach contraction principle (BCP).

This BCP was modified and generalized in different forms and structures. Among them, there are dislocated quasi metric spaces [6], cone metric spaces [7], generalized metric spaces [8], controlled metric spaces [9], orthogonal partial b-metric spaces [10], etc.

Khojasteh et al. [11] modified the contractive condition by introducing the concept of a simulation function $S\colon [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$. Later, Roldan Lopez de Hierro et al. [12, 13] extended the stated concept and investigated some coincidence point results in metric spaces. With the help of a simulation-type function, Argoubi et al. [14] gave interesting results in partial ordered metric spaces. Alharbi et al. [15] made a generalization by combining the concept of simulation and admissible functions in the related literature. Alsubaie et al. [16] proved some common fixed-point results for two mappings in the setting of metric spaces by using the concept of a simulation function. Alqahtani et al. [17] proved fixed-point results by introducing the concept of a bilateral contraction which is a combination of Ćirić- and Caristi-type contractions. In [18], the authors studied the existence and uniqueness of a common fixed point in the setting of b-metric spaces, by using the concept of extended Z-contractions associated with an ψ–simulation function. In [19], the authors established results on the existence of best proximity points of certain mappings using simulation functions in complete metric spaces. Later, Karapinar [20] presented some fixed-point results by defining a new contractive condition via admissible mappings imbedded in a simulation function.

Recently, Wu [21] initiated the concept of interval spaces. These spaces contain all closed bounded intervals over the set $\mathbb{R}$. Over the interval spaces, he defined a metric, as well as a norm using the equivalence relation $\underset{=}{\Omega}$. These spaces are called metric intervals and norm interval spaces, respectively. He studied near-fixed-point results in metric intervals, as well as in norm interval spaces. After this, he gave the concept of a hyperspace, which is a space containing all possible subsets of a vector space. He defined null sets, as well as the equivalence relation $\underset{=}{\Omega}$ in a hyperspace and defined a norm over this type of spaces. He also presented near-fixed-point results in hyperspaces. For more details, see [22–24].

Inspired by the work done in [11, 21, 22], we established some near-coincidence point results in metric interval and hyperspaces [25] via a simulation function. We also presented some near-coincidence point results in norm interval spaces via a simulation function. For validation of results and definitions, some examples are provided.

2. Preliminaries

In this section, some basic definitions and results are stated related to the existing literature.

2.1. Interval Spaces. Let I be the set containing all close bounded intervals of the form $[\sigma, v]$, where $\sigma, v \in \mathbb{R}$ and $\sigma \leq v$. Also, $\sigma \in \mathbb{R}$ is considered as an element $[\sigma, \sigma] \in I$ [21].

The binary operation of addition and scaler multiplication is stated as follows:

$$[\sigma, v] \oplus [\sigma', v'] = [\sigma + \sigma', v + v'],$$
$$k[\sigma, v] = \begin{cases} [k\sigma, kv], & k \geq 0, \\ [kv, k\sigma], & k < 0. \end{cases} \tag{1}$$

Due to the inverse property, the above space does not fulfill the condition of a conventional vector space. For $[\sigma, v] \in I$, the subtraction

$$[\sigma, v] \ominus [\sigma, v] = [\sigma, v] \oplus [-v, -\sigma] = [\sigma - v, v - \sigma] \tag{2}$$

does not give the zero element $[0, 0]$. So the inverse of $[\sigma, v]$ does not exist. For the above deficiency, the null set was defined by Wu [21] as follows.

2.2. Null Set. The null set contains all the elements of the type $[-\sigma, \sigma]$, and so it is defined as follows:

$$\Omega = \{[\sigma, v] \ominus [\sigma, v];\ [\sigma, v] \text{ is an element of } I\}, \tag{3}$$

or

$$\Omega = \{[-\alpha, \alpha];\ \alpha \geq 0\}. \tag{4}$$

2.3. Binary Relation $\underset{=}{\Omega}$. We write $[\sigma, v] \underset{=}{\Omega} [\sigma', v']$ iff there exist $\omega_1, \omega_2 \in \Omega$ such that

$$[\sigma, v] \oplus \omega_1 = [\sigma', v'] \oplus \omega_2. \tag{5}$$

Clearly, we can have $[\sigma, v] = [\sigma', v'] \Rightarrow [\sigma, v] \underset{=}{\Omega} [\sigma', v']$ by taking $\omega_1 = \omega_2 = [0, 0]$. However, the converse is not true in general.

Proposition 1 (see [21]). *$\underset{=}{\Omega}$ is an equivalence relation.*

According to the equivalence relation $\underset{=}{\Omega}$, the equivalence class of almost identical intervals is defined as $\langle [\sigma, v] \rangle = \left\{ [\mathfrak{p}, \mathfrak{q}] \in I\colon\ [\sigma, v] \underset{=}{\Omega} [\mathfrak{p}, \mathfrak{q}] \right\}$ for any $[\sigma, v] \in I$.

2.4. Norm Interval Space. The pair $(I, \| \cdot \|)$ fulfilling the following axioms is called a norm interval space [21]:

(i) $\| [\sigma, v] \| = 0$ implies $[\sigma, v] \in \Omega$

(ii) $\| \alpha [\sigma, v] \| = |\alpha| \| [\sigma, v] \|$

(iii) $\| [\sigma, v] \oplus [\sigma', v'] \| \leq \| [\sigma, v] \| + \| [\sigma', v'] \|$ for all $[\sigma, v], [\sigma', v'] \in I$, where I contains all close bounded intervals over $\mathbb{R}$ with the null set Ω and $\| \cdot \|$ is a real-valued mapping on I

We say that the null condition is satisfied by $\| \cdot \|$ if the condition (iii) is replaced by

$$\| [\sigma, v] \| = 0 \quad \text{if and only if } [\sigma, v] \in \Omega. \tag{6}$$

$\| \cdot \|$ is said to satisfy the null equalities if for all $\omega_1, \omega_2 \in \Omega$ and $[\sigma, v], [\sigma', v'] \in I$, the following equalities hold:

(1) $\| ([\sigma, v] \oplus \omega_1) \ominus ([\sigma', v'] \oplus \omega_2) \| = \| [\sigma, v] \ominus [\sigma', v'] \|$

(2) $\| ([\sigma, v] \oplus \omega_1) \ominus ([\sigma', v']) \| = \| [\sigma, v] \ominus [\sigma', v'] \|$

(3) $\| ([\sigma, v]) \ominus ([\sigma', v'] \oplus \omega_2) \| = \| [\sigma, v] \ominus [\sigma', v'] \|$

Definition 1. If $(I, \| \cdot \|)$ is a norm interval space, then

(i) The mapping $\| \cdot \|$ is said to satisfy the null super-inequality if

$$\| [\sigma, v] \oplus \omega \| \geq \| [\sigma, v] \|, \quad \text{for any } [\sigma, v] \in I \text{ and } \omega \in \Omega. \tag{7}$$

(ii) The mapping $\| \cdot \|$ is said to satisfy the null sub-inequality if

$$\| [\sigma, v] \oplus \omega \| \leq \| [\sigma, v] \|, \quad \text{for any } [\sigma, v] \in I \text{ and } \omega \in \Omega. \tag{8}$$

(iii) The mapping $\| \cdot \|$ is said to satisfy the null equality if

$$\| [\sigma, v] \oplus \omega \| = \| [\sigma, v] \|, \quad \text{for any } [\sigma, v] \in I \text{ and } \omega \in \Omega. \tag{9}$$

Example 1. Let $\| \cdot \|$ be a nonnegative real-valued function defined on I by

$$\| [\sigma, v] \| = |\sigma + v|. \tag{10}$$

Then, $(I, \| \cdot \|)$ forms a norm interval space such that $\| \cdot \|$ satisfies the null equality.

Proposition 2 (see [21]). *Let $(I, \|\cdot\|)$ be a norm interval space such that $\|\cdot\|$ satisfies the null super-inequality. Then, for any $[\sigma, v]$, $[\sigma', v']$, $[\sigma_1, v_1]$, $[\sigma_2, v_2], \ldots, [\sigma_m, v_m]$, we have*

$$\left\|[\sigma, v]\ominus[\sigma', v']\right\| \leq \left\|[\sigma, v]\ominus[\sigma_1, v_1]\right\| + \left\|[\sigma_1, v_1]\ominus[\sigma_2, v_2]\right\| + \cdots + \left\|[\sigma_m, v_m]\ominus[v, v']\right\|. \tag{11}$$

Proposition 3. *Let $(I, \|\cdot\|)$ be a norm interval space; then, the following hold:*

(i) If $\|\cdot\|$ satisfies the null equality, then for all $[\sigma, v]$, $[\sigma', v'] \in I$,

$$[\sigma, v] \underset{=}{\Omega}[\sigma', v'] \text{ implies } \|[\sigma, v]\| = \left\|[\sigma', v']\right\|. \tag{12}$$

(ii) For any $[\sigma, v]$, $[\sigma', v'] \in I$,

$$\left\|[\sigma, v]\ominus[\sigma', v']\right\| = 0, \quad \text{implies } [\sigma, v] \underset{=}{\Omega}[\sigma', v']. \tag{13}$$

(iii) If $\|\cdot\|$ satisfies the null super-inequality and null condition, then for any $[\sigma, v]$, $[\sigma', v'] \in I$,

$$[\sigma, v] \underset{=}{\Omega}[\sigma', v'], \quad \text{implies} \left\|[\sigma, v]\ominus[\sigma', v']\right\| = 0. \tag{14}$$

For proof of the above propositions, see [21].

Definition 2. Let $(I, \|\cdot\|)$ be a norm interval space. A sequence $\{[\sigma_n, v_n]\}_{n=1}^{+\infty}$ is said to converge to a limit $[\sigma, v]$ if and only if

$$\lim_{n\longrightarrow\infty}\left\|[\sigma_n, v_n]\ominus[\sigma, v]\right\| = 0. \tag{15}$$

Proposition 4. *Consider a norm interval space $(I, \|\cdot\|)$ with the null set Ω.*

(i) If the null super-inequality holds for $\|\cdot\|$, then the convergence of the sequence $\{[\sigma_n, v_n]\}_{n=1}^{+\infty}$ to $[\sigma, v]$ and $[\sigma', v']$ simultaneously implies $\langle[\sigma, v]\rangle = \langle[\sigma', v']\rangle$

(ii) If the null equality holds for $\|\cdot\|$ and the sequence $\{[\sigma_n, v_n]\}_{n=1}^{+\infty}$ converges to $[\sigma, v]$, then for any $[\sigma', v'] \in \langle[\sigma, v]\rangle$, the given sequence will also converge to $[\sigma', v']$

Definition 3. Consider the norm interval space $(I, \|\cdot\|)$ with the null set Ω, where $\|\cdot\|$ satisfies the null equality. If $[\sigma, v] \in I$ is the limit of the sequence $\{[\sigma_n, v_n]\}_{n=1}^{+\infty}$, then $\langle[\sigma, v]\rangle$ is called the class limit. We can also write

$$\lim_{n\longrightarrow\infty}[\sigma_n, v_n] = \langle[\sigma, v]\rangle. \tag{16}$$

Proposition 5 (see [21]). *In the norm interval space $(I, \|\cdot\|)$ if the null super-inequality holds for $\|\cdot\|$, the class limit is unique.*

Definition 4. A sequence $\{[\sigma_n, v_n]\}_{n=1}^{+\infty}$ in a norm interval space $(I, \|\cdot\|)$ is called a Cauchy sequence if and only if for any $\varepsilon \geq 0$, there exists $K \in \mathbb{N}$ such that

$$\left\|[\sigma_n, v_n]\ominus[\sigma_m, v_m]\right\| < \varepsilon, \tag{17}$$

for $m, n > K$ with $m \neq n$. If every Cauchy sequence is convergent in I, then I is complete.

Definition 5. A complete norm interval space $(I, \|\cdot\|)$ is called a Banach interval space.

Example 2. Let $\|\cdot\|$ be a nonnegative real-valued function defined on I by

$$\|[\sigma, v]\| = |\sigma + v|. \tag{18}$$

Then, $(I, \|\cdot\|)$ forms a Banach interval space such that $\|\cdot\|$ satisfies the null equality.

Definition 6. Let F be a self-mapping on I. Then, the point $[\sigma_o, v_o] \in I$ is called a near-fixed point of F if and only if $F[\sigma_o, v_o] \underset{=}{\Omega}[\sigma_o, v_o]$.

Definition 7 (see [11, 12]). A function S: $[0, \infty)\times[0, \infty) \longrightarrow \mathbb{R}$ is called a simulation function if the following conditions hold:

S_1. $S(0, 0) = 0$

S_2. $S(\alpha, \beta) < \beta - \alpha$ for all $\alpha, \beta > 0$

S_3. If $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $(0, \infty)$ such that $\lim_{n\longrightarrow\infty}\alpha_n = \lim_{n\longrightarrow\infty}\beta_n > 0$ and $\alpha_n < \beta_n$ for all $n \in \mathbb{N}$, then

$$\lim_{n\longrightarrow\infty} \sup S(\alpha_n, \beta_n) < 0. \tag{19}$$

By (S_2), we must have

$$S(\alpha, \alpha) < 0. \tag{20}$$

The following are some interesting examples of simulation functions:

(i) $S(\alpha, \beta) = \chi(\beta) - \Upsilon(\alpha)$ for all $\alpha, \beta \in [0, \infty)$ where χ and γ are continuous on $[0, \infty)$ such that $\gamma(\alpha) = \chi(\alpha)$ if and only if $\alpha = 0$ and $\gamma(\alpha) < \alpha \leq \chi(\alpha)$ for all $\alpha > 0$. If we take $\gamma(\beta) = \lambda\beta$ and $\chi(\alpha) = \alpha$, then $S(\alpha, \beta) = \lambda\beta - \alpha$.

(ii) $S(\alpha, \beta) = \beta - \chi(\beta) - \alpha$ for all $\alpha, \beta \in [0, \infty)$ where χ is continuous on $[0, \infty)$ such that $\chi(\alpha) = 0$ if and only if $\alpha = 0$ (see Example 2.2 in [11]).

(iii) $S(\alpha, \beta) = \beta\chi(\beta) - \alpha$ for all $\alpha, \beta \in [0, \infty)$ where χ is a mapping such that $\lim_{\alpha\longrightarrow r^+}\chi(t) < 1$ for all $r > 0$ [12].

(iv) $S(\alpha, \beta) = \eta(\beta) - \alpha$ for all $\alpha, \beta \in [0, \infty)$, where η is an upper semicontinuous function so that $\eta(\alpha) < \alpha$ for all $\alpha > 0$ and $\eta(0) = 0$ [12].

3. Results and Discussion

Proposition 6. *In an interval space I with the null set* Ω, $[\sigma, v]\,\underset{=}{\Omega}\,[\sigma', v']$ *iff* $\sigma' - \sigma = v - v'$.

Proof. Let us suppose that $[\sigma, v]\,\Omega\,[\sigma', v']$; then, by definition, there exist $[-k, k]$ and $[-h, \bar{h}]$ in Ω such that

$$[\sigma, v] \oplus [-k, k] = [\sigma', v'] \oplus [-h, h], \tag{21}$$

that is,

$$[\sigma - k, v + k] = [\sigma' - h, v' + h]. \tag{22}$$

This implies that

$$\begin{aligned} \sigma - k &= \sigma' - h, \\ v + k &= v' + h, \\ \sigma - \sigma' &= k - h, \\ v - v' &= -k + h, \\ \sigma - \sigma' &= k - h, \\ v - v' &= -(k - h). \end{aligned} \tag{23}$$

Putting the values of $k - h$ from the 1st equality in the second equality, we have

$$v - v' = -(\sigma - \sigma'), \quad \text{implies}\, \sigma' - \sigma = v - v'. \tag{24}$$

Conversely, now, let us suppose that $\sigma' - \sigma = v - v'$; then, we have to show that

$$[\sigma, v]\,\underset{=}{\Omega}[\sigma', v']. \tag{25}$$

Hence,

$$\begin{aligned} \sigma' - \sigma &= v - v' = k, \\ \sigma' - \sigma &= k, \\ v - v' &= k, \\ \sigma &= \sigma' - k, \\ v &= v' + k, \end{aligned} \tag{26}$$

which implies

$$\begin{aligned} [\sigma, v] &= [\sigma' - k, v' + k], \\ [\sigma, v] &= [\sigma', v'] \oplus [-k, k], \end{aligned} \tag{27}$$

and so from the last equality, we have

$$[\sigma, v]\,\underset{=}{\Omega}[\sigma', v']. \tag{28}$$

□

Example 3. Taking the intervals $[3, 7]$ and $[4, 6]$, we have $4 - 3 = 7 - 6 = 1$ and hence by the above function, we have $[3, 7]\,\underset{=}{\Omega}\,[4, 6]$. For verification, take $\omega_1 = [0.0]$ and $\omega_2 = [-1, 1]$. Then,

$$[3, 7] \oplus [0, 0] = [4, 6] \oplus [-1, 1]. \tag{29}$$

Definition 8. For a point $[\sigma, v] \in I$, if $F[\sigma, v]\,\underset{=}{\Omega}\,g[\sigma, v]$, then the point $[\sigma, v]$ is called a near-coincidence point of F and g.

Example 4. Taking the function $F[x, y] = [x^2 - 1, 2y^2 + 1]$ and $g[x, y] = [x^2, 2y^2]$, then we can verify that $[3, 5]$ is the near-coincidence point for the functions defined above.

Definition 9. If F and g are two self-mappings over $(I, \|\cdot\|)$ such that

$$\lim_{n \longrightarrow \infty} \left\| Fg[\sigma_n, v_n] \ominus gF[\sigma_n, v_n] \right\| = 0, \tag{30}$$

then the mappings are called compatible.

Definition 10. If $Fg[\sigma, v]\,\underset{=}{\Omega}\,gF[\sigma, v]$ for all $[\sigma, v] \in (I, d)$, then F and g are called commuting mappings.

Definition 11. F is a $(Z_{\|\cdot\|}, g)$-contraction in $(I, \|\cdot\|)$ corresponding to a simulation function $S \in Z$ if

$$S\left(\left\| F[\sigma, v] \ominus F[\sigma', v'] \right\|, \left\| g[\sigma, v] \ominus g[\sigma', v'] \right\| \right) \geq 0, \tag{31}$$

for all $[\sigma, v], [\sigma', v'] \in I$ such that $g[\sigma, v]\,\underset{\neq}{\Omega}\,g[\sigma', v']$.

Example 5. Define the mappings F and g as $F[x, y] = [x^2 - 1, 2y^2 + 1]$ and $g[x, y] = [x^2, 2y^2]$; then, F satisfies the criteria of $(Z_{\|\cdot\|}, g)$-contraction in $(I, \|\cdot\|)$ according to the simulation function $S(s, t) = \lambda t - s$, where $\lambda \geq 1$.

Definition 12. For a sequence $\{[\sigma_n, v_n]\}$ in the Banach interval space $(I, \|\cdot\|)$, if

$$g([\sigma_{n+1}, v_{n+1}])\,\underset{=}{\Omega}\,F([\sigma_n, v_n]), \quad \text{for all}\, n \geq 0, \tag{32}$$

then the sequence is known as a Picard (F, g) sequence at the point $[\sigma_o, v_o]$.

Theorem 1. *Let* $F[\sigma, v] = [f_1(\sigma), f_2(v)]$ *and* $G[\sigma, v] = [g_1(\sigma), g_2(v)]$ *be two self-mappings over the interval space I, where* $f_1(\sigma) \leq f_2(v)$ *and* $g_1(\sigma) \leq g_2(v)$ *for all* $\sigma \leq v$. *If* σ *is a coincidence point for* f_1 *and* g_1 *and* v *is a coincidence point for* f_2 *and* g_2, *then* $[\sigma, v]$ *is a near-coincidence point for F and G.*

Proof. As σ and v are coincidence points for f_1, g_1 and f_2, g_2, respectively, we have

$$\begin{aligned} f_1(\sigma) &= g_1(\sigma), \\ f_2(v) &= g_2(v). \end{aligned} \tag{33}$$

This implies that

$$\begin{aligned} &[f_1(\sigma), f_2(v)] = [g_1(\sigma), g_2(v)], \\ &[f_1(\sigma), f_2(v)]\,\underset{=}{\Omega}[g_1(\sigma), g_2(v)], \\ &F[\sigma, v]\,\underset{=}{\Omega}\,G[\sigma, v]. \end{aligned} \tag{34}$$

Hence, a near-coincidence point for the mappings F and G is $[\sigma, v]$ over the interval space I. The converse of the above statement is not true in general, because $[f_1(\sigma), f_2(v)]\,\underset{=}{\Omega}\,[g_1(\sigma), g_2(v)]$ does not imply $f_1(\sigma) = g_1(\sigma)$ and $f_2(v) = g_2(v)$. □

Example 6. Taking the function $F[x, y] = [x^2, |y| + 1]$ and $G[x, y] = [|x|, y^2 + 1]$, then clearly -1 is a coincidence point for x^2 and $|x|$ and 1 is a coincidence point for $|y| + 1$ and $y^2 + 1$. So, $[-1, 1]$ is a near-coincidence point for F and G since $F[-1, 1] = [1, 2] = [1, 2] = G[1, 2]$. For justifying the converse of the above statement, we can verify that $[-(1/2), (1/2)]$ is a near-coincidence point for F and G, but $-(1/2)$ and $(1/2)$ are not coincidence points for x^2 and $|x|$ and $|y| + 1$ and $y^2 + 1$, respectively. As $F[-(1/2), (1/2)] = [(1/4), (3/2)]$ and $G[-(1/2), (1/2)] = [(1/2), (5/4)]$, to prove that $[-(1/2), (1/2)]$ is a near-coincidence point; we have to show $[(1/4), (3/2)]\,\underset{=}{\Omega}\,[(1/2), (5/4)]$. Taking $\omega_1 = [0, 0]$ and $\omega_2 = [-(1/4), (1/4)]$, we have

$$\left[\frac{1}{4}, \frac{3}{2}\right] \oplus [0, 0] = \left[\frac{1}{2}, \frac{5}{4}\right] \oplus \left[-\frac{1}{4}, \frac{1}{4}\right]. \tag{35}$$

Lemma 1. *Let $F[\sigma, v] = [f_1(\sigma), f_2(v)]$ and $G[\sigma, v] = [g_1(\sigma), g_2(v)]$ be two self-mappings over the interval space I, where $f_1(\sigma) \leq f_2(v)$ and $g_1(\sigma) \leq g_2(v)$ for all $\sigma \leq v$. If $g_1(\sigma) - f_1(\sigma) = f_2(v) - g_2(v)$ for some $[\sigma, v] \in I$, then $[\sigma, v]$ is a near-coincidence point for F and G.*

Proof. Using Proposition 6, $g_1(\sigma) - f_1(\sigma) = f_2(v) - g_2(v)$ implies that $[f_1(\sigma), f_2(v)]\,\underset{=}{\Omega}\,[g_1(\sigma), g_2(v)]$, i.e.,

$$F[\sigma, v]\,\underset{=}{\Omega}\,G[\sigma, v]. \tag{36}$$

Hence, it is proved that $[\sigma, v]$ is a near-coincidence point for F and G. □

Lemma 2. *Consider a Banach interval space I with a $(Z_{\|\cdot\|}, g)$-contraction F. If $[\sigma, \nu]$ and $[\sigma', \nu']$ both are the near-coincidence points for F and g, then*

$$F[\sigma, \nu]\,\underset{=}{\Omega}\,g[\sigma, \nu]\,\underset{=}{\Omega}\,g[\sigma', \nu']\,\underset{=}{\Omega}\,F[\sigma', \nu']. \tag{37}$$

Furthermore, the equivalence class of a near-coincidence point is unique if F or g is injective.

Proof. Let $[\sigma, \nu]$ and $[\sigma', \nu']$ be two near-coincidence points of F and g. Then, we have

$$\begin{aligned} &F[\sigma, \nu]\,\underset{=}{\Omega}\,g[\sigma, \nu], \\ &F[\sigma', \nu']\,\underset{=}{\Omega}\,g[\sigma', \nu']. \end{aligned} \tag{38}$$

In the above requirement, the two equalities are clear. We only need to show that $g[\sigma, \nu]\,\underset{=}{\Omega}\,g[\sigma', \nu']$. On the contrary, let us suppose that $g[\sigma, \nu]\,\underset{=}{\Omega}\,\bar{g}[\sigma', \nu']$; so we have

$$\left\|g[\sigma, \nu] \ominus g[\sigma', \nu']\right\| \geq 0. \tag{39}$$

As the mapping F is a $(Z_{\|\cdot\|}, g)$-contraction, by definition, we have

$$\begin{aligned} 0 &\leq S\left(\left\|F[\sigma, \nu] \ominus F[\sigma', \nu']\right\|, \left\|g[\sigma, \nu] \ominus g[\sigma', \nu']\right\|\right) \\ &= S\left(\left\|g[\sigma, \nu] \ominus g[\sigma', \nu']\right\|, \left\|g[\sigma, \nu] \ominus g[\sigma', \nu']\right\|\right). \end{aligned} \tag{40}$$

The last inequality is a contradiction to (20) in the definition of the simulation function, i.e., $S(r, r) < 0$, where $r > 0$. So our supposition is wrong and we accept that $g[\sigma, \nu]\,\underset{=}{\Omega}\,g[\sigma', \nu']$.

Hence, we prove that

$$F[\sigma, \nu]\,\underset{=}{\Omega}\,g[\sigma, \nu]\,\underset{=}{\Omega}\,g[\sigma', \nu']\,\underset{=}{\Omega}\,F[\sigma', \nu']. \tag{41}$$

Furthermore, let F be injective; then, the equivalence class of a near-coincidence point is unique. By the above work, we have

$$F[\sigma, \nu]\,\underset{=}{\Omega}\,g[\sigma, \nu]\,\underset{=}{\Omega}\,g[\sigma', \nu']\,\underset{=}{\Omega}\,F[\sigma', \nu']. \tag{42}$$

It implies that

$$F[\sigma, \nu]\,\underset{=}{\Omega}\,F[\sigma', \nu']. \tag{43}$$

As F is injective, $[\sigma, \nu]\,\underset{=}{\Omega}\,[\sigma', \nu']$. It further implies that $\langle[\sigma, \nu]\rangle = \langle[\sigma', \nu']\rangle$. □

Theorem 2. *Consider a $(z_{\|\cdot\|}, g)$-contraction F in the Banach interval space $(I, \|\cdot\|)$ where $\|\cdot\|$ satisfies the null equality and F and g are continuous and compatible mappings. Assume that the space is satisfying the $\text{CLR}_{(F,g)}$ property. Then, a near-coincidence point exists for F and g.*

Proof. As the space $(I, \|\cdot\|)$ satisfies the $\text{CLR}_{(F,g)}$ property, i.e., there exists a Picard sequence $\{[\sigma_n, \nu_n]\}$, such that

$$g[\sigma_{n+1}, \nu_{n+1}]\,\underset{=}{\Omega}\,F[\sigma_n, \nu_n], \quad \text{for all } n \geq 0. \tag{44}$$

There are two possibilities: either the sequence $\{[\sigma_n, \nu_n]\}$ contains a near-coincidence point, or it converges to the near-coincidence point. We will take the case that the sequence does not contain a near-coincidence point. Hence,

$$g[\sigma_n, \nu_n]\,\underset{=}{\Omega}\,F[\sigma_n, \nu_n]\,\underset{=}{\Omega}\,g[\sigma_{n+1}, \nu_{n+1}], \quad \text{for all } n \geq 0. \tag{45}$$

The result will be proved in the following steps.

First of all, we will show that

$$\lim_{n \longrightarrow \infty} \left\|g[\sigma_n, \nu_n] \ominus g[\sigma_{n+1}, \nu_{n+1}]\right\| = 0. \tag{46}$$

As F is a $(z_{\|\cdot\|}, g)$-contraction, by $\text{CLR}_{(F,g)}$ property and condition (ii) of a simulation function, we have

$$\begin{aligned} 0 &\leq S\left(\left\|F[\sigma_n, \nu_n] \ominus F[\sigma_{n+1}, \nu_{n+1}]\right\|, \left\|g[\sigma_n, \nu_n] \ominus g[\sigma_{n+1}, \nu_{n+1}]\right\|\right) \\ &= S\left(\left\|g[\sigma_{n+1}, \nu_{n+1}] \ominus g[\sigma_{n+2}, \nu_{n+2}]\right\|, \left\|g[\sigma_n, \nu_n] \ominus g[\sigma_{n+1}, \nu_{n+1}]\right\|\right) \\ &< \left\|g[\sigma_n, \nu_n] \ominus g[\sigma_{n+1}, \nu_{n+1}]\right\| - \left\|g[\sigma_{n+1}, \nu_{n+1}] \ominus g[\sigma_{n+2}, \nu_{n+2}]\right\|. \end{aligned} \tag{47}$$

This implies that

$$0 < \left\|g[\sigma_{n+1}, \nu_{n+1}] \ominus g[\sigma_{n+2}, \nu_{n+2}]\right\| < \left\|g[\sigma_n, \nu_n] \ominus g[\sigma_{n+1}, \nu_{n+1}]\right\|. \tag{48}$$

The sequence $\{\|g[\sigma_n,\nu_n]\ominus g[\sigma_{n+1},\nu_{n+1}]\|\}$ is nonnegative and decreasing, so it converges to a limit, say Ł, i.e.,

$$\lim_{n\longrightarrow\infty}\left\|g[\sigma_n,\nu_n]\ominus g[\sigma_{n+1},\nu_{n+1}]\right\| = Ł. \tag{49}$$

We have to show that Ł = 0. On the contrary, let us suppose that Ł > 0. Consider the sequences with the same limit $r_n = \{\|g[\sigma_{n+1},\nu_{n+1}]\ominus g[\sigma_{n+2},\nu_{n+2}]\|\}$ and $s_n = \{\|g[\sigma_n,\nu_n]\ominus g[\sigma_{n+1},\nu_{n+1}]\|\}$ such that $r_n < s_n$ for all $n\in\mathbb{N}$.

Now, by condition (iii) of the simulation function, we have

$$\begin{aligned}0 &> \limsup_{n\longrightarrow\infty}(S(r_n,s_n))\\ &= \limsup_{n\longrightarrow\infty}\Big(S\big(\|g[\sigma_{n+1},\nu_{n+1}]\ominus g[\sigma_{n+2},\nu_{n+2}]\|,\\ &\quad \|g[\sigma_n,\nu_n]\ominus g[\sigma_{n+1},\nu_{n+1}]\|\big)\Big).\end{aligned} \tag{50}$$

It is a contradiction because

$$S\big(\|g[\sigma_{n+1},\nu_{n+1}]\ominus g[\sigma_{n+2},\nu_{n+2}]\|, \|g[\sigma_n,\nu_n]\ominus g[\sigma_{n+1},\nu_{n+1}]\|\big) > 0. \tag{51}$$

Thus, Ł = 0. That is,

$$\lim_{n\longrightarrow\infty}\left\|g[\sigma_n,\nu_n]\ominus g[\sigma_{n+1},\nu_{n+1}]\right\| = 0. \tag{52}$$

Next, we will show that the sequence $\{g[\sigma_n,\nu_n]\}$ is a Cauchy sequence. Let us suppose, on the contrary, that $\{g[\sigma_n,\nu_n]\}$ is not Cauchy. So there will exist $\varepsilon_o > 0$ such that for all $N\in\mathbb{N}$, there exist positive integers m,n such that

$$\left\|g[\sigma_n,\nu_n]\ominus g[\sigma_m,\nu_m]\right\| > \varepsilon_o. \tag{53}$$

We can construct two partial subsequences $\{g[\sigma_{n_k},\nu_{n_k}]\}$ and $\{g[\sigma_{m_k},\nu\text{\_}\{\text{\_}k\}]\}$ such that $n_o \le n_k \le m_k$ and

$$\left\|g[\sigma_{n_k},\nu_{n_k}]\ominus g[\sigma_{m_k},\nu_{m_k}]\right\| > \varepsilon_o, \quad \text{for all } k\in\mathbb{N}. \tag{54}$$

Let m_k be the smallest positive integer in $\{n_k, n_k+1, n_k+2,\ldots\}$. Then,

$$\left\|g[\sigma_{m_k-1},\nu_{m_k-1}]\ominus g[\sigma_{n_k},\nu_{n_k}]\right\| \le \varepsilon_o, \quad \text{for all } k\in\mathbb{N}. \tag{55}$$

Also, $m_k > n_k$ from (54), so $m_k \ge n_k+1$ for all $k\in\mathbb{N}$. But $m_k = n_k+1$ is not possible taking into account (52) and (54) simultaneously. So, we have $m_k \ge n_k+2$ for any $k\in\mathbb{N}$. It follows that $n_{k+1} < m_k < m_{k+1}$ for all $k\in\mathbb{N}$. From (54) and (55), we have

$$\begin{aligned}\varepsilon_o &< \left\|g[\sigma_{m_k},\nu_{m_k}]\ominus g[\sigma_{n_k},\nu_{n_k}]\right\|\\ &\le \left\|g[\sigma_{m_k},\nu_{m_k}]\ominus g[\sigma_{m_{k-1}},\nu_{m_{k-1}}]\right\| + \left\|g[\sigma_{m_{k-1}},\nu_{m_{k-1}}]\ominus g[\sigma_{n_k},\nu_{n_k}]\right\|\\ &\le \left\|g[\sigma_{m_k},\nu_{m_k}]\ominus g[\sigma_{m_{k-1}},\nu_{m_{k-1}}]\right\| + \varepsilon_o, \quad \text{for all } k\in\mathbb{N}.\end{aligned} \tag{56}$$

Therefore,

$$\lim_{k\longrightarrow\infty}\left\|g[\sigma_{m_k},\nu_{m_k}]\ominus g[\sigma_{n_k},\nu_{n_k}]\right\| = \varepsilon_o. \tag{57}$$

Also,

$$\lim_{k\longrightarrow\infty}\left\|g[\sigma_{m_{k+1}},\nu_{m_{k+1}}]\ominus g[\sigma_{n_{k+1}},\nu_{n_{k+1}}]\right\| = \varepsilon_o. \tag{58}$$

As F is a (Z_d,g)-contraction associated with S,

$$\begin{aligned}0 &\le S\Big(\left\|F[\sigma_{m_k},\nu_{m_k}]\ominus F[\sigma_{n_k},\nu_{n_k}]\right\|, \left\|g[\sigma_{m_k},\nu_{m_k}]\ominus g[\sigma_{n_k},\nu_{n_k}]\right\|\Big)\\ &= S\Big(\left\|g[\sigma_{m_{k+1}},\nu_{m_{k+1}}]\ominus g[\sigma_{n_{k+1}},\nu_{n_{k+1}}]\right\|, \left\|g[\sigma_{m_k},\nu_{m_k}]\ominus g[\sigma_{n_k},\nu_{n_k}]\right\|\Big)\\ &< \left\|g[\sigma_{m_k},\nu_{m_k}]\ominus g[\sigma_{n_k},\nu_{n_k}]\right\| - \left\|g[\sigma_{m_{k+1}},\nu_{m_{k+1}}]\ominus g[\sigma_{n_{k+1}},\nu_{n_{k+1}}]\right\|.\end{aligned} \tag{59}$$

Thus,

$$0 < \left\|g[\sigma_{m_{k+1}},\nu_{m_{k+1}}]\ominus g[\sigma_{n_{k+1}},\nu_{n_{k+1}}]\right\| < \left\|g[\sigma_{m_k},\nu_{m_k}]\ominus g[\sigma_{n_k},\nu_{n_k}]\right\|. \tag{60}$$

Let

$$\begin{aligned}r_n &= \left\|g[\sigma_{m_{k+1}},\nu_{m_{k+1}}]\ominus g[\sigma_{n_{k+1}},\nu_{n_{k+1}}]\right\|,\\ s_n &= \left\|g[\sigma_{m_k},\nu_{m_k}]\ominus g[\sigma_{n_k},\nu_{n_k}]\right\|.\end{aligned} \tag{61}$$

Clearly, $r_n, s_n > 0$, $\lim_{n\longrightarrow\infty} r_n = \lim_{n\longrightarrow\infty} s_n = \varepsilon_o$, and $r_n < s_n$.

So by S_3,

$$\begin{aligned}0 &\le \limsup_{k\longrightarrow\infty} S\Big(\left\|g[\sigma_{m_{k+1}},\nu_{m_{k+1}}]\ominus g[\sigma_{n_{k+1}},\nu_{n_{k+1}}]\right\|,\\ &\quad \left\|g[\sigma_{m_k},\nu_{m_k}]\ominus g[\sigma_{n_k},\nu_{n_k}]\right\|\Big) < 0,\end{aligned} \tag{62}$$

which is a contradiction. Thus, $\{g[\sigma_n,\nu_n]\}$ is a Cauchy sequence in (I,d).

That is, $\{g[\sigma_n,\nu_n]\}$ is a Cauchy sequence. Now, as the space is complete, the sequence $\{g[\sigma_n,\nu_n]\}$ will converge to a limit $[\sigma,\nu]$. Since the mappings F and g are continuous, one writes

$$\begin{aligned}g[\sigma_n,\nu_n] &\longrightarrow [\sigma,\nu], \quad \text{implies } gg[\sigma_n,\nu_n] \longrightarrow g[\sigma,\nu],\\ g[\sigma_n,\nu_n] &\longrightarrow [\sigma,\nu], \quad \text{implies } Fg[\sigma_n,\nu_n] \longrightarrow F[\sigma,\nu].\end{aligned} \tag{63}$$

The compatibility of the mappings yields that

$$\lim_{n\longrightarrow\infty}\left\|Fg[\sigma_n,\nu_n]\ominus gF[\sigma_n,\nu_n]\right\| = 0. \tag{64}$$

Consider

$$\begin{aligned}\|F[\sigma,\nu]\ominus g[\sigma,\nu]\| &= \lim_{n\longrightarrow\infty}\left\|Fg[\sigma_n,\nu_n]\ominus gg[\sigma_{n+1},\nu_{n+1}]\right\|\\ &= \lim_{n\longrightarrow\infty}\left\|Fg[\sigma_n,\nu_n]\ominus gF[\sigma_n,\nu_n]\right\|,\\ \|F[\sigma,\nu]\ominus g[\sigma,\nu]\| &= 0.\end{aligned} \tag{65}$$

From the above function, we have $F[\sigma,\nu]\ \Omega\ g[\sigma,\nu]$; i.e., $[\sigma,\nu]$ is a near-coincidence point of F and g. □

Example 7. Consider the two continuous self-mappings F and g in the Banach interval space $(I,\|\cdot\|)$ defined by

$$F[\sigma, v] = [2\sigma - 4, 2v + 4].$$
$$g[\sigma, v] = [\sigma - 2, v + 2]. \tag{66}$$

The function F is a $(z_{\|\cdot\|}, g)$-contraction according to the simulation function $S(s,t) = \lambda t - s$, where $\lambda \geq 2$. Also, the functions F and g are compatible. The sequence $[(-1/n), (1/n)]$ is a Picard sequence, i.e.,

$$g([\sigma_{n+1}, v_{n+1}]) \underset{=}{\Omega} F([\sigma_n, v_n]), \quad \text{for all } n \geq 2,$$

$$g\left(\left[\frac{-1}{n+1}, \frac{1}{n+1}\right]\right) \underset{=}{\Omega} F\left(\left[\frac{-1}{n}, \frac{1}{n}\right]\right), \tag{67}$$

$$\left[\frac{-1}{n+1} - 2, \frac{1}{n+1} + 2\right] \underset{=}{\Omega} \left[\frac{-2}{n} - 4, \frac{2}{n} + 4\right].$$

We can easily show that $g([\sigma_{n+1}, v_{n+1}]) \underset{=}{\Omega} F([\sigma_n, v_n])$, for all $n \geq 0$, by taking $\omega_1 = [(-((2n^2 + 3n + 2)/(n(n+1)))), ((2n^2 + 3n + 2)/n(n+1))]$ and $\omega_2 = [0, 0]$. Then,

$$\left[\frac{-1}{n+1} - 2, \frac{1}{n+1} + 2\right] \oplus \left[-\frac{2n^2 + 3n + 2}{n(n+1)}, \frac{2n^2 + 3n + 2}{n(n+1)}\right] = \left[\frac{-2}{n} - 4, \frac{2}{n} + 4\right] \oplus [0, 0]. \tag{68}$$

If we replace the compatibility of mappings by commuting mappings, then the following corollary can be stated.

Corollary 1. *Consider the continuous and commuting mappings F and g in the Banach interval space $(I, \|\cdot\|)$ such that the criteria of Z-contraction is satisfied by F. Assume that $CLR(F, g)$ property holds in I; then, a near-coincidence point exists for F and g.*

Corollary 2. *Consider a Banach interval space $(I, \|\cdot\|)$ with two self-mappings F and g. Then, a near-coincidence point exists for F and g if*

$$\|F[\sigma, v] \ominus F[\sigma', v']\| \leq \lambda \|g[\sigma, v] \ominus g[\sigma', v']\|, \tag{69}$$

for all $[\sigma, v], [\sigma', v'] \in I$, where $g[\sigma, v] \underset{\neq}{\Omega} g[\sigma', v']$ and $\lambda \in [0, 1)$.

Proof. Taking the simulation function $S(\sigma, v) = \lambda v - \sigma$ for all $\sigma, v \in [0, \infty)$ and $\lambda \in [0, 1)$, according to the above condition, we have

$$\|F[\sigma, v] \ominus F[\sigma', v']\| \leq \lambda \|g[\sigma, v] \ominus g[\sigma', v']\|, \quad \text{for all } [\sigma, v], [\sigma', v'] \in I. \tag{70}$$

It implies that

$$0 \leq \lambda \|g[\sigma, v] \ominus g[\sigma', v']\| - \|F[\sigma, v] \ominus F[\sigma', v']\| \leq S(\|F[\sigma, v] \ominus F[\sigma', v']\|, \|g[\sigma, v] \ominus g[\sigma', v']\|). \tag{71}$$

The last inequality allows to say that F is a $Z_{\|\cdot\|}$ – contraction, and hence, by Theorem 2, there will be a near-coincidence point for F and g. □

Corollary 3. *Consider a Banach interval space $(I, \|\cdot\|)$ with self-mappings F and g such that*

$$\|F[\sigma, v] \ominus F[\sigma', v']\| \leq \|g[\sigma, v] \ominus g[\sigma', v']\| - \Phi(\|g[\sigma, v] \ominus g[\sigma', v']\|) \quad \forall [\sigma, v], [\sigma', v'] \in I, \tag{72}$$

where Φ is a lower semicontinuous function defined on $[0, \infty)$ so that $\Phi^{-1}(0) = 0$; then, F and g have a near-coincidence point in I.

Proof. It suffices to take the simulation function $S(\sigma, v) = v - \Phi(v) - \sigma$ for all $\sigma, v \in [0, \infty)$. Then, we can easily prove that F is a z – contraction. So by Theorem 2, there exists a near-coincidence point for F and g. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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