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Research Article

Numerical Solution of Multidimensional Stochastic Itô-Volterra Integral Equation Based on the Least Squares Method and Block Pulse Function

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In this paper, a method based on the least squares method and block pulse function is proposed to solve the multidimensional stochastic Itô-Volterra integral equation. The Itô-Volterra integral equation is transformed into a linear algebraic equation. Furthermore, the error analysis is given by the isometry property and Doob's inequality. Numerical examples verify the effectiveness and precision of this method.

1. Introduction

Stochastic Volterra integral equations have been generally used in many fields such as mechanics, medicine, materials, finance, and physical science, especially in the process of engineering practice, because many material coefficients are uncertain, it is necessary to introduce stochastic integral equations to establish models. Meanwhile, the stochastic integral equation is used to solve a variety of engineering problems, for example, material science modeling [1], automated systems science [2], dynamics [3], and biology [4]. However, most of the stochastic Volterra integral equations cannot be solved explicitly. Therefore, it is essential to provide numerical solutions to these equations. Many scholars have developed various methods to study the stochastic Volterra integral equation. Different Volterra integral equations are handled by Bernstein polynomials, block pulse functions, least squares method, Haar wavelets, Walsh functions, Chebyshev and Legendre polynomials, and other functions. We only mentioned the referenced such as [5–21] and other relevant literatures. On the other hand, some authors obtained the numerical solution of stochastic Volterra integral equation by Euler–Maruyama approximation or iterative algorithm, for example [22–27].

Recently, there have been many articles on the numerical solution of the stochastic Itô-Volterra integral equation. For example, in [13], Maleknejad et al. considered linear stochastic Itô-Volterra integral equations (SIVIEs) by block pulse functions (BPFs). In [15], Wu et al. used Haar wavelets (HWs) to solve nonlinear SIVIEs. Mirzaee et al. studied nonlinear SIVIEs of fractional order based on the hybrid of block pulse and parabolic functions [16]. In [28], Jiang et al. used BPFs to obtain the numerical solution of two-dimensional nonlinear SIVIEs. Ahmadinia et al. presented a new method that applies the least squares method to study SIVIEs [29]. Moreover, Maleknejad et al. simplified m-dimensional SIVIEs into linear lower trigonometric equations by using BPFs and stochastic integral operation matrix, and then the equations were solved [30].

Motivated by the abovementioned literatures, we consider the following multidimensional linear SIVIEs which are studied relatively little and propose a method that combines BPFs and least squares method.

$$X(t) = f(t) + \int_0^t \widetilde{S}(u,t)X(u)\mathrm{d}u + \sum_{k=1}^{q}\int_0^t \widehat{S}_k(u,t)X(u)\mathrm{d}B_k(u), \quad t \in [0,T), \tag{1}$$

where $X(t)$, $\widetilde{S}(u,t)$, and $\widehat{S}_k(u,t)$, $k = 1,2,\ldots,q$, for $0 < u < t$, are stochastic processes on the same probability space of $(\Omega, \mathscr{F}, P)$, $f(t)$ is an initial function, and $X(t)$ is unknown. $B_k(u)$ are Brownian motions and $\int_0^t \widehat{S}_k(u,t)X(u)\mathrm{d}B_k(u)$ are Itô integrals.

In contrast to the articles [13, 29], this paper's difference is to study linear SIVIEs driven by multiple independent Brownian motions. In addition, the numerical solution based on the least squares method and BPFs is more accurate than other methods in the reference [30]. Finally, this method can be applied to the models of engineering and material science, which makes the practical data more meaningful.

In Section 2, BPFs are introduced. In Section 3, stochastic integral operation matrix is given. In Section 4, the least squares method is recommended. In Section 5, the error analysis is obtained. In Section 6, the accuracy of the method is verified by two numerical examples. In Section 7, the conclusion is given.

2. Preliminaries

Definition 1 (see [31]) (BPFs).

BPFs have been considered and used to solve various equations by numerous scholars, such as [8, 12, 13, 28, 30]. BPFs are denoted as

$$\psi_i(t) = \begin{cases} 1, & (i-1)h \le t < ih, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $t \in [0,T)$, $i = 1,\ldots,m$, and $h = T/m$.

The basic properties of BPFs are described as follows:

(1) Disjointness:

$$\psi_i(t)\psi_j(t) = \delta_{ij}\psi_i(t), \quad i,j = 1,2,\ldots,m, \tag{3}$$

where δ_{ij} is the Kronecker delta.

(2) Orthogonality:

$$\int_0^T \psi_i(t)\psi_j(t)\mathrm{d}t = h\delta_{ij}. \tag{4}$$

(3) Completeness: Parseval's identity is true for every $f \in L^2([0,T))$ and $f(t) = \sum_{i=1}^{\infty} f_i\psi_i(t)$,

$$\int_0^T f^2(t)\mathrm{d}t = \sum_{i=1}^{\infty} f_i^2 \left\|\psi_i(t)\right\|, \tag{5}$$

where $f_i = (1/h)\int_0^T f(t)\psi_i(t)\mathrm{d}t$.

BPFs are expressed as a vector

$$\Psi(t) = \left(\psi_1(t), \psi_2(t), \ldots, \psi_m(t)\right)^T. \tag{6}$$

It can be concluded from the above description that

$$\Psi_m(t)\Psi_m^T(t) = \begin{pmatrix} \psi_1(t) & 0 & \cdots & 0 \\ 0 & \psi_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \psi_m(t) \end{pmatrix}_{m\times m},$$
$$\Psi_m^T(t)\Psi_m(t) = 1,$$
$$\Psi_m(t)\Psi_m^T(t)F_m = \mathbf{D}_{F_m}\Psi_m(t). \tag{7}$$

where $F_m = (f_1, f_2, \ldots, f_m)^T$ and $\mathbf{D}_{F_m} = \mathrm{diag}(F_m)$.

Any function $X(t) \in L^2([0,T))$ can be approximately expressed as

$$X(t) \simeq X_m(t) = \sum_{i=1}^{m} c_i\psi_i(t) = \Psi_m^T(t)C_m, \quad C_m = (c_1, c_2, \ldots, c_m)^T, \tag{8}$$

where $X_m(t)$ is a linear combination of BPFs.

Let $S(u,t) \in L^2([0,T_1) \times [0,T_2))$; it can be extended as

$$S(u,t) = \Psi_{m_1}^T(u)\mathbf{S}\Psi_{m_2}(t) = \Psi_{m_2}^T(t)\mathbf{S}^T\Psi_{m_1}(u), \tag{9}$$

where $\mathbf{S} = (s_{ij})_{m_1\times m_2}$,

$$s_{ij} = \frac{1}{h_1h_2}\int_0^{T_1}\int_0^{T_2} S(u,t)\psi_i(u)\phi_j(t)\mathrm{d}u\mathrm{d}t, \tag{10}$$

and $h_1 = (T_1/m_1)$, $h_2 = (T_2/m_2)$. In order to facilitate, we put $m_1 = m_2 = m$ in the following sections.

3. Stochastic Integration Operational Matrix

The section gives the relevant lemmas.

Lemma 1 (see [8, 13]). *Supposing $\Psi_m(t)$ is given in (6), we obtain*

$$\int_0^t \Psi_m(u)\mathrm{d}u \simeq \mathbf{Q}\Psi_m(t), \tag{11}$$

where

$$\mathbf{Q} = \frac{h}{2}\begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{m\times m}. \tag{12}$$

Therefore, each function $X(t) \in L^2([0,T))$ can be approximately expressed as

$$\int_0^t X(u)\mathrm{d}u \simeq C_m^T\mathbf{Q}\Psi_m(t). \tag{13}$$

Lemma 2 (see [8, 13]). *Supposing $\Psi_m(t)$ is given in (6), we obtain*

$$\int_0^t \Psi_m(u)\mathrm{d}B(u) \simeq \mathbf{Q}_B \Psi_m(t), \tag{14}$$

where $\mathbf{Q}_B$ can be expressed as

$$\mathbf{Q}_B = \begin{pmatrix} \beta_1 & \alpha_1 & \alpha_1 & \cdots & \alpha_1 \\ 0 & \beta_2 & \alpha_2 & \cdots & \alpha_2 \\ 0 & 0 & \beta_3 & \cdots & \alpha_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_m \end{pmatrix}_{m\times m}, \tag{15}$$

where $\alpha_i = B(ih) - B((i-1)h)$, $i = 1, 2, \ldots, m-1$; $\beta_j = B(ih/2) - B((i-1)h/2)$, $j = 1, 2, \ldots, m$.

Therefore, each function $X(t) \in L^2([0,T))$ can be approximately expressed as

$$\int_0^t X(u)\mathrm{d}B(u) \simeq C_m^T \mathbf{Q}_B \Psi_m(t). \tag{16}$$

4. Method Description

In this section, the multidimensional linear SIVIEs are transformed into linear algebraic equations by the least square method and BPFs.

The linear operator L is taken into account,

$$L(X(t)) := X(t) - \int_0^t \tilde{S}(u,t)X(u)\mathrm{d}u - \sum_{k=1}^{q} \int_0^t \hat{S}_k(u,t)X(u)\mathrm{d}B_k(u). \tag{17}$$

If $X(t)$ is the exact solution to (1), then the residual norm disappears:

$$\|L(X) - f\|_{L^2} = 0. \tag{18}$$

For $\varepsilon > 0$, $X_\varepsilon(t)$ is the approximate solution, so that the residual norm of $X_\varepsilon(t)$ is less than ε,

$$\left\|X_\varepsilon(t) - f\right\|_{L^2} < \varepsilon. \tag{19}$$

An approximate solution can be obtained by (19), and X_m is a linear combination by BPFs,

$$X_m = \sum_{i=1}^{m} c_i \psi_i(t), \tag{20}$$

where c_i is the unknown coefficient.

The minimization problem is studied below,

$$\min_{c_1,\ldots,c_m} \left\|L(X_m) - f\right\|_{L^2}. \tag{21}$$

Minimizing (21) to get a set of values of $\hat{c}_1, \ldots, \hat{c}_m$, then $\sum_{i=1}^m \hat{c}_i \psi_i(t)$ is an approximate solution to (1), and we'll prove it.

$$\begin{aligned} &\mathbb{E}\left[\min_{c_1,\ldots,c_m} \left\|L(X_m) - f\right\|_{L^2}\right] \\ &\quad = \mathbb{E}\left[\left\|L\left(\sum_{i=1}^m \hat{c}_i \psi_i\right) - f\right\|_{L^2}\right] \longrightarrow 0, \end{aligned} \tag{22}$$

where $m \longrightarrow \infty$ (or $h \longrightarrow 0$).

To obtain the minimum value of the approximate solution in (21), we have to take the partial derivative in c_i, and let

$$\frac{\partial}{\partial c_i} \int_0^T \left(\sum_{j=1}^m c_j L\left(\psi_j(t)\right) - f(t)\right)^2 \mathrm{d}t = 0, \tag{23}$$

where $i = 1, 2, \ldots, m$.

Then,

$$\begin{aligned} &\int_0^T \sum_{j=1}^m c_j L\left(\psi_j(t)\right) L\left(\psi_i(t)\right)\mathrm{d}t \\ &\quad = \int_0^T f(t) L\left(\psi_i(t)\right)\mathrm{d}t, \end{aligned} \tag{24}$$

that is,

$$\sum_{j=1}^m c_j \left\langle L\left(\psi_i(t)\right), L\left(\psi_j(t)\right)\right\rangle = \left\langle L\left(\psi_i(t)\right), f(t)\right\rangle, \tag{25}$$

where $\langle L(\psi_i(t)), L(\psi_j(t))\rangle := \int_0^T L(\psi_j(t))L(\psi_i(t))\mathrm{d}t$.

The following is expressed as a matrix:

$$c = \mathbf{G}^{-1} a, \tag{26}$$

where $c = (c_1, c_2, \ldots, c_m)^T \in \mathbb{R}^m$, $a = (a_1, a_2, \ldots, a_m)^T \in \mathbb{R}^m$, and $\mathbf{G} = (g_{ij}) \in \mathbb{R}^{m\times m}$ such that

$$\begin{aligned} a_i &= \left\langle L\left(\psi_i(t)\right), f(t)\right\rangle, \\ g_{ij} &= \left\langle L\left(\psi_i(t)\right), L\left(\psi_j(t)\right)\right\rangle, \quad i, j = 1, 2, \ldots, m. \end{aligned} \tag{27}$$

For solving equation (1), $X(t)$, $f(t)$, $\tilde{S}(u,t)$, and $\hat{S}(u,t)$ can be approximated by BPFs in the following forms:

$$X(t) \simeq X_m(t) = C_m^T \Psi_m(t) = \Psi_m^T(t) C_m, \tag{28}$$

$$f(t) \simeq f_0(t) = F_m^T \Psi_m(t) = \Psi_m^T(t) F_m, \tag{29}$$

$$\tilde{S}(u,t) = \Psi_m^T(u)\mathbf{S}_1\Psi_m(t) = \Psi_m^T(t)\mathbf{S}_1^T\Psi_m(u), \tag{30}$$

$$\hat{S}(u,t) = \Psi_m^T(u)\mathbf{S}_2\Psi_m(t) = \Psi_m^T(t)\mathbf{S}_2^T\Psi_m(u), \tag{31}$$

where C_m and F_m are BPF coefficient vectors; $\mathbf{S}_1$ and $\mathbf{S}_2$ are BPF coefficient matrices by the equation of (10).

Next, for obtaining a_i and g_{ij}, by (17) and (28)–(31), we have

$$\begin{aligned} L(\psi_i(t)) &= \psi_i(t) - \int_0^t \widetilde{S}(u,t)\psi_i(u)\mathrm{d}u \\ &\quad - \sum_{k=1}^{q}\int_0^t \widehat{S}_k(u,t)\psi_i(u)\mathrm{d}B_k(u) \\ &= I_i\Psi_m(t) - \int_0^t I_i\Psi_m(u)\Psi_m^T(u)\mathbf{S}_1\Psi_m(t)\mathrm{d}u \\ &\quad - \sum_{k=1}^{q}\int_0^t I_i\Psi_m(u)\Psi_m^T(u)\mathbf{S}_2\Psi_m(t)\mathrm{d}B_k(u) \\ &= I_i\Psi_m(t) - I_i\int_0^t \Psi_m(u)\Psi_m^T(u)\mathrm{d}u\mathbf{S}_1\Psi_m(t) \\ &\quad - I_i\sum_{k=1}^{q}\int_0^t \Psi_m(u)\Psi_m^T(u)\mathrm{d}B_k(u)\mathbf{S}_2\Psi_m(t), \end{aligned} \tag{32}$$

where I_i is the ith row of an $m\times m$ identity matrix.

Let S_1^i and S_2^i be the ith row of the matrix $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively, R^i and R_B^i be the ith row of the integral operation matrix $\mathbf{Q}$ and the stochastic integral matrix $\mathbf{Q}_B$, respectively, and $\mathbf{D}_{S_1^i}$ and $\mathbf{D}_{S_2^i}$ be a diagonal matrix with S_1^i and S_2^i as the diagonal elements, respectively. By (11) and (14), we get

$$\begin{aligned} &\left(\int_0^t \Psi_m(u)\Psi_m(u)^T\mathrm{d}u\right)\mathbf{S}_1\Psi_m(t), \\ &= \begin{pmatrix} R^1\Psi_m(t)S_1^1\Psi_m(t) \\ R^2\Psi_m(t)S_1^2\Psi_m(t) \\ \vdots \\ R^m\Psi_m(t)S_1^m\Psi_m(t) \end{pmatrix} \\ &= \begin{pmatrix} R^1\mathbf{D}_{S_1^1} \\ R^2\mathbf{D}_{S_1^2} \\ \vdots \\ R^m\mathbf{D}_{S_1^m} \end{pmatrix}\Psi_m(t) = \widetilde{\mathbf{S}}\Psi_m(t), \end{aligned} \tag{33}$$

where

$$\widetilde{\mathbf{S}} = \frac{h}{2}\begin{pmatrix} \widetilde{S}_{11} & 2\widetilde{S}_{12} & 2\widetilde{S}_{13} & \cdots & 2\widetilde{S}_{1m} \\ 0 & \widetilde{S}_{22} & 2\widetilde{S}_{23} & \cdots & 2\widetilde{S}_{2m} \\ 0 & 0 & \widetilde{S}_{33} & \cdots & 2\widetilde{S}_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \widetilde{S}_{mm} \end{pmatrix}_{m\times m},$$

$$\begin{aligned} &\left(\int_0^t \Psi_m(u)\Psi_m(u)^T\mathrm{d}B(u)\right)\mathbf{S}_2\Psi_m(t) \\ &= \begin{pmatrix} R_B^1\Psi_m(t)S_2^1\Psi_m(t) \\ R_B^2\Psi_m(t)S_2^2\Psi_m(t) \\ \vdots \\ R_B^m\Psi_m(t)S_2^m\Psi_m(t) \end{pmatrix} \\ &= \begin{pmatrix} R_B^1\mathbf{D}_{S_2^1} \\ R_B^2\mathbf{D}_{S_2^2} \\ \vdots \\ R_B^m\mathbf{D}_{S_2^m} \end{pmatrix}\Psi_m(t) = \widehat{\mathbf{S}}\Psi_m(t), \end{aligned} \tag{34}$$

where

$$\widehat{\mathbf{S}}_k = \begin{pmatrix} \widehat{S}_{11}^k\beta_1 & \widehat{S}_{12}^k\alpha_1 & \widehat{S}_{13}^k\alpha_1 & \cdots & \widehat{S}_{1m}^k\alpha_1 \\ 0 & \widehat{S}_{22}^k\beta_2 & \widehat{S}_{23}^k\alpha_2 & \cdots & \widehat{S}_{2m}^k\alpha_2 \\ 0 & 0 & \widehat{S}_{33}^k\beta_3 & \cdots & \widehat{S}_{3m}^k\alpha_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \widehat{S}_{mm}^k\beta_m \end{pmatrix}_{m\times m}. \tag{35}$$

According to Lemmas 1 and 2,

$$L(\psi_i(t)) = I_i\Psi_m(t) - I_i\widetilde{\mathbf{S}}\Psi_m(t) - I_i\sum_{k=1}^{q}\widehat{\mathbf{S}}_k\Psi_m(t) = \left(I_i - I_i\widetilde{\mathbf{S}} - I_i\sum_{k=1}^{q}\widehat{\mathbf{S}}_k\right)\Psi_m(t). \quad (36)$$

Then,

$$\begin{aligned}
a_i &= \langle L(\psi_i(t)), f(t)\rangle = \int_0^T L(\psi_i(t))f(t)\mathrm{d}t \\
&= \int_0^T\left(I_i - I_i\widetilde{\mathbf{S}} - I_i\sum_{k=1}^{q}\widehat{\mathbf{S}}_k\right)\Psi_m(t)F_m^T\Psi_m(t)\mathrm{d}t \\
&= \int_0^T\left(I_i - I_i\widetilde{\mathbf{S}} - I_i\sum_{k=1}^{q}\widehat{\mathbf{S}}_k\right)F_m\Psi_m^T(t)\Psi_m(t)\mathrm{d}t \\
&= \left(I_i - I_i\widetilde{\mathbf{S}} - I_i\sum_{k=1}^{q}\widehat{\mathbf{S}}_k\right)F_m\int_0^T\Psi_m^T(t)\Psi_m(t)\mathrm{d}t \\
&= \left(I_i - I_i\widetilde{\mathbf{S}} - I_i\sum_{k=1}^{q}\widehat{\mathbf{S}}_k\right)F_mT, \\
g_{ij} &= \langle L(\psi_i(t)), L(\psi_j(t))\rangle \\
&= \int_0^T L(\psi_i(t))L(\psi_j(t))\mathrm{d}t \\
&= \int_0^T\left(I_i - I_i\widetilde{\mathbf{S}} - I_i\sum_{k=1}^{q}\widehat{\mathbf{S}}_k\right)\Psi_m(t)\left(I_j - I_j\widetilde{\mathbf{S}} - I_j\sum_{k=1}^{q}\widehat{\mathbf{S}}_k\right)\Psi_m(t)\mathrm{d}t \\
&= \int_0^T\left(I_i - I_i\widetilde{\mathbf{S}} - I_i\sum_{k=1}^{q}\widehat{\mathbf{S}}_k\right)\left(I_j - I_j\widetilde{\mathbf{S}} - I_j\sum_{k=1}^{q}\widehat{\mathbf{S}}_k\right)^T\Psi_m^T(t)\Psi_m(t)\mathrm{d}t \\
&= \left(I_i - I_i\widetilde{\mathbf{S}} - I_i\sum_{k=1}^{q}\widehat{\mathbf{S}}_k\right) \\
&\quad\cdot\left(I_j - I_j\widetilde{\mathbf{S}} - I_j\sum_{k=1}^{q}\widehat{\mathbf{S}}_k\right)^T\int_0^T\Psi_m^T(t)\Psi_m(t)\mathrm{d}t \\
&= \left(I_i - I_i\widetilde{\mathbf{S}} - I_i\sum_{k=1}^{q}\widehat{\mathbf{S}}_k\right)\left(I_j - I_j\widetilde{\mathbf{S}} - I_j\sum_{k=1}^{q}\widehat{\mathbf{S}}_k\right)^T T.
\end{aligned} \quad (37)$$

5. Error Analysis

We will give the error analysis in this section.

Lemma 3 (see [29]) (continuous module). *f with respect to ϱ in $[0,T]$ is the definition of continuous modulus $\omega(f,\varrho)$*

$$\omega(f,\varrho) = \sup\{|f(x) - f(y)| x, y \in [0,T], |x-y| \le \varrho\}. \quad (38)$$

As shown in the reference [32], if and only if $\lim_{\rho\longrightarrow 0}\omega(f,\varrho) = 0$, on $[0,T]$, $f(t)$ is uniformly continuous.

Lemma 4 (see [29]). *Suppose $f \in C[0,T]$, $Q_h = \sum_{i=1}^{m} e_i\psi_i(t)$, where $e_i = f((t_{i-1}+t_i)/2)$, $t_i = ih$, $h = T/m$, $i = 0,1,\ldots,m$. Then,*

$$\|f - Q_h\|_\infty \le \omega(f,h). \quad (39)$$

Theorem 1. *Suppose $X(t)$ be the exact solution of (1) and $\widehat{X}_m(t) = \sum_{i=1}^{m} e_i\psi_i(t)$, where $e_i = X((t_{i-1}+t_i)/2)$, $\widetilde{S}(u,t)$, and $\widehat{S}_k(u,t)$ are deterministic functions, $\|\widetilde{S}\|_\infty \le M$, $\|\widehat{S}_k\|_\infty \le M$, where M is a positive constant. Then, we get the following conclusions, when $h \longrightarrow 0$:*

TABLE 1: When $m = 2^4$, the table shows error means E_m, error standard deviations E_s, and confidence intervals for different time t.

t	E_m	E_s	95% confidence interval for error mean Lower	Upper
1/16	$1.36075252 \times 10^{-4}$	$6.80376264 \times 10^{-5}$	2.7215050×10^{-6}	2.6942900×10^{-4}
3/16	$1.13092940 \times 10^{-4}$	$5.65464702 \times 10^{-5}$	2.2618588×10^{-6}	2.2392402×10^{-4}
5/16	$5.33574672 \times 10^{-5}$	$2.66787336 \times 10^{-5}$	1.0671493×10^{-6}	1.0564778×10^{-4}
7/16	$2.67128325 \times 10^{-5}$	$1.33564162 \times 10^{-5}$	5.3425665×10^{-7}	5.2891408×10^{-5}
9/16	$2.53074880 \times 10^{-6}$	$1.26537440 \times 10^{-6}$	5.0614976×10^{-8}	5.0108826×10^{-6}

TABLE 2: When $m = 2^5, n = 100$, this table shows error means E_m, error standard deviations E_s, and confidence intervals for different time t.

t	E_m	E_s	95% confidence interval for error mean Lower	Upper
1/32	$1.47378183 \times 10^{-4}$	$7.36890915 \times 10^{-5}$	2.9475636×10^{-6}	2.9180880×10^{-4}
7/32	$9.11708782 \times 10^{-5}$	$4.55854391 \times 10^{-5}$	1.8234175×10^{-6}	1.8051833×10^{-4}
13/32	$5.04388689 \times 10^{-5}$	$2.52194344 \times 10^{-5}$	1.0087773×10^{-6}	9.9868960×10^{-5}
19/32	$9.47544810 \times 10^{-6}$	$4.73772405 \times 10^{-6}$	1.8950896×10^{-7}	1.8761387×10^{-5}
25/32	$4.47443350 \times 10^{-5}$	$2.23721675 \times 10^{-5}$	8.9488670×10^{-7}	8.8593783×10^{-5}

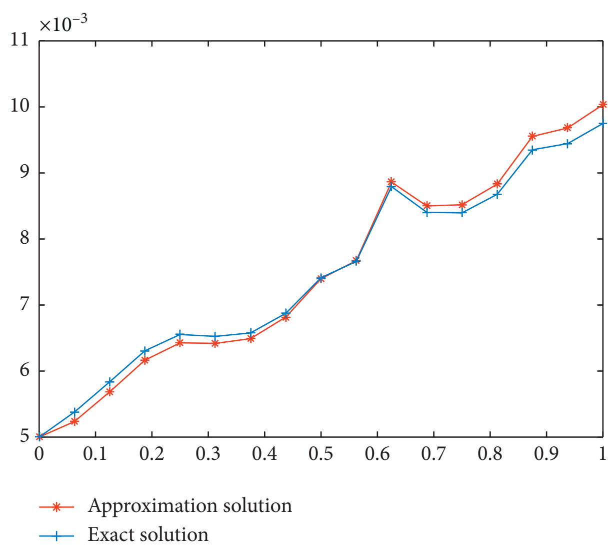

FIGURE 1: $m = 2^4$, the approximate solution and exact solution for Example 1.

$$
\begin{aligned}
&(i)\ \left\|\widehat{X}_m(t) - X(t)\right\|_{\infty,E} = \mathbb{E}\left[\sup_{t\in[0,T)}\left|\widehat{X}_m(t) - X(t)\right|\right] \longrightarrow 0,\\
&(ii)\ \mathbb{E}\left[\min_{c_1,\dots,c_m}\left\|X_m(t) - f(t) - \int_0^t \widetilde{S}(u,t)X_m(u)\mathrm{d}u - \sum_{k=1}^{q}\int_0^t \widehat{S}_k(u,t)X_m(u)\mathrm{d}B_k(u)\right\|_{L^2}^2\right] \longrightarrow 0.
\end{aligned} \tag{40}
$$

Proof
(i) By Lemma 4, we have

$$
\begin{aligned}
&\left\|\widehat{X}_m(t) - X(t)\right\|_{\infty,E}\\
&= \mathbb{E}\left[\sup_{t\in[0,T)}\left|\widehat{X}_m(t) - X(t)\right|\right] \le \omega(X,h) \longrightarrow 0.
\end{aligned} \tag{41}
$$

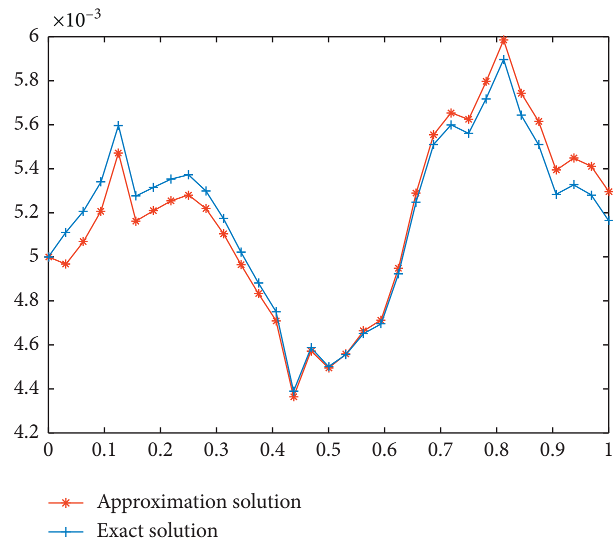

Figure 2: $m = 2^5$, the approximate solution and exact solution for Example 1.

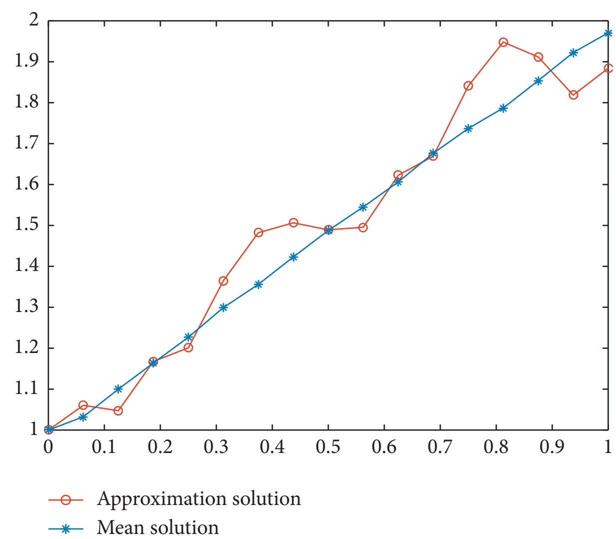

Figure 3: $m = 2^4$, the approximate solution and mean solution for Example 2.

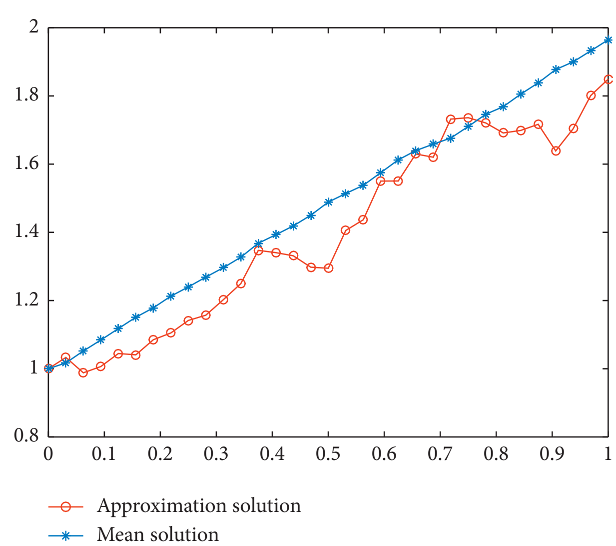

Figure 4: $m = 2^5$, the approximate solution and mean solution for Example 2.

(ii)

$$\begin{aligned}
&\mathbb{E}\left[\min_{c_1,\ldots,c_m}\left\|X_m(t)-f(t)-\int_0^t\widetilde{S}(u,t)X_m(u)\mathrm{d}u-\sum_{k=1}^{q}\int_0^t\widehat{S}_k(u,t)X_m(u)\mathrm{d}B_k(u)\right\|_{L^2}^2\right]\\
&\le\mathbb{E}\left[\left\|\widehat{X}_m(t)-f(t)-\int_0^t\widetilde{S}(u,t)\widehat{X}_m(u)\mathrm{d}u-\sum_{k=1}^{q}\int_0^t\widehat{S}(u,t)\widehat{X}_m(u)\mathrm{d}B_k(u)\right\|_{L^2}^2\right]\\
&\le T^2\left\|\widehat{X}_m(t)-f(t)-\int_0^t\widetilde{S}(u,t)\widehat{X}_m(u)\mathrm{d}u-\sum_{k=1}^{q}\int_0^t\widehat{S}(u,t)\widehat{X}_m(u)\mathrm{d}B_k(u)\right\|_{\infty,E}^2\\
&=T^2\left\|\widehat{X}_m(t)-X(t)+\int_0^t\widetilde{S}(u,t)\left(X(u)-\widehat{X}_m(u)\right)\mathrm{d}u+\sum_{k=1}^{q}\int_0^t\widehat{S}(u,t)\left(X(u)-\widehat{X}_m(u)\right)\mathrm{d}B_k(u)\right\|_{\infty,E}^2\\
&\le 3T^2\left(\left\|\widehat{X}_m(t)-X(t)\right\|_{\infty,E}^2+\left\|\int_0^t\widetilde{S}(u,t)\left(X(u)-\widehat{X}_m(u)\right)\mathrm{d}u\right\|_{\infty,E}^2+\left\|\sum_{k=1}^{q}\int_0^t\widehat{S}(u,t)\left(X(u)-\widehat{X}_m(u)\right)\mathrm{d}B_k(u)\right\|_{\infty,E}^2\right)\\
&\le 3T^2\left\|\widehat{X}_m(t)-X(t)\right\|_{\infty,E}^2+3T^2M^2\left\|X(t)-\widehat{X}_m(t)\right\|_{\infty,E}^2\\
&\quad+3T^2\left\|\sum_{k=1}^{q}\int_0^t\widehat{S}(u,t)\left(X(u)-\widehat{X}_m(u)\right)\mathrm{d}B_k(u)\right\|_{\infty,E}^2.
\end{aligned}\tag{42}$$

According to isometry property and Doob's inequality,

$$\begin{aligned}
&\left\|\sum_{k=1}^{q}\int_0^t\widehat{S}(u,t)\left(X-\widehat{X}_m\right)(u)\mathrm{d}B_k(u)\right\|_{\infty,E}^2\\
&=\mathbb{E}\left[\sup_{0\le\tau\le t}\left|\sum_{k=1}^{q}\int_0^\tau\widehat{S}(u,t)\left(X(u)-\widehat{X}_m(u)\right)\mathrm{d}B_k(u)\right|^2\right]\\
&\le 4\mathbb{E}\left[\left|\sum_{k=1}^{q}\int_0^t\widehat{S}(u,t)\left(X(u)-\widehat{X}_m(u)\right)\mathrm{d}B_k(u)\right|^2\right]\\
&\le 4\mathbb{E}q\left[\sum_{k=1}^{q}\left|\int_0^t\widehat{S}(u,t)\left(X(u)-\widehat{X}_m(u)\right)\mathrm{d}B_k(u)\right|^2\right]\\
&=4\mathbb{E}q^2\left[\int_0^t|\widehat{S}(u,t)|^2\left|X(u)-\widehat{X}_m(u)\right|^2\mathrm{d}u\right]\\
&\le 4TM^2q^2\left\|X(t)-\widehat{X}_m(t)\right\|_{\infty,E}^2.
\end{aligned}\tag{43}$$

By (41)–(43), we get

$$\begin{aligned}
&\mathbb{E}\left[\min_{c_1,\ldots,c_m}\left\|X_m(t)-f(t)-\int_0^t\widetilde{S}(u,t)X_m(u)\mathrm{d}u-\sum_{k=1}^{q}\int_0^t\widehat{S}_k(u,t)X_m(u)\mathrm{d}B_k(u)\right\|_{L^2}^2\right]\\
&\le\left(3T^2+3T^4M^2+12T^3M^2q^2\right)\left\|\widehat{X}_m(t)-X(t)\right\|_{\infty,E}^2\\
&\le\left(3T^2+3T^4M^2+12T^3M^2q^2\right)\omega(X,h)\longrightarrow 0,
\end{aligned}\tag{44}$$

and the proof of the theorem is completed. □

6. Numerical Examples

The following two examples show the availability and accuracy of the method mentioned in Section 4. The algorithms were implemented by MATLAB 2016a.

Example 1. The linear SIVIE is studied as follows:

$$X(t) = X_0 + \int_0^t rX(u)\mathrm{d}u + \sum_{k=1}^{q} \int_0^t a_k X(u)\mathrm{d}B_k(u), \quad u, t \in [0, 1), \tag{45}$$

where $X(t) = X_0 e^{(r-(1/2)\sum_{k=1}^{q} a_k^2)t + \sum_{k=1}^{q} a_k B_k(t)}$, $0 \le t < 1$, and $B_k(u)$ are q-dimensional Brownian motions. When $X_0 = 1/200, r = 1/20, a_1 = 1/50, a_2 = 2/50, a_3 = 4/50$, and $a_4 = 9/50$, the numerical results are shown in Tables 1 and 2. The trajectories are shown in Figures 1 and 2, where $n = 100$ is the number of iterations.

We can see from Figures 1 and 2 and Tables 1 and 2 that the approximate solution and the exact solution are very close. This method is more accurate than the method in [30], and this method is feasible and effective for solving the q-dimensional SIVIEs.

Example 2. The linear SIVIE is studied as follows:

$$X(t) = X_0 + \int_0^t e^{-(t-u)} X(u)\mathrm{d}u + \sum_{k=1}^{q} \int_0^t a_k e^{-(t-u)} X(u)\mathrm{d}B_k(u), \quad u, t \in [0, 1), \tag{46}$$

where $B_k(u)$ are q-dimensional Brownian motions. When $X_0 = 1, a_1 = 1/50, a_2 = 2/50, a_3 = 4/50$, and $a_4 = 9/50$, the trajectories are shown in Figures 3 and 4.

7. Conclusion

In this paper, a numerical method based on the least squares method and BPFs is proposed to solve multidimensional linear SIVIEs different from that of Maleknejad et al., who only used BPFs to solve equations [30]. The least squares method is a mathematical optimization technique that seeks the best function match of data by minimizing errors and quickly obtaining unknown coefficients. Combining it with BPFs can make the error of the numerical solution of multidimensional linear SIVIEs smaller. In Section 6, the numerical simulations are carried out and compared with the literature of [30]; this paper concludes that this method is better than applying only the block pulse functions directly.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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