Research Article

Picard Method for Existence, Uniqueness, and Gauss Hypergeomatric Stability of the Fractional-Order Differential Equations

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In this paper, we consider a class of fractional-order differential equations and investigate two aspects of these equations. First, we consider the existence of a unique solution, and then, using a new class of control functions, we investigate the Gauss hypergeometric stability. We use Chebyshev and Bielecki norms in order to prove these aspects by the Picard method. Finally, we give some examples to illustrate our results.

1. Introduction

The topic of fractional calculus and its significant applications have appeared to be an appropriate tool in study widespread fields of engineering and science. In fact, there are a lot of phenomena in which highly accurate modeling is very important to study them. Fluid mechanics, biological models, electochemistry, viscoelasticity, and electromagnetics are some examples of these fields. Also, research on fractional calculus (FC) and its applications is an important part of mathematical analysis (see [1–6] and references therein). Its importance can also be explored in other fields such as fluid dynamics traffic models, oscillation due to earthquakes, and flow in porous media due to seepage.

Researchers have always tried to create the conditions for the existence and uniqueness of the solution for fractional differential equations. Since most problems cannot be solved for exact solutions, we need powerful analytical techniques. For good results, one needs stable algorithms and methods. For such needs, the stability theory was founded [7, 8]. It has been found that the notion of fractional-order differential equations can well describe these models in science and engendering. Recently, the study of fractional differential equations has been increased among researchers.

Our article is organized as follows. In Section 2, we explain the definitions and some important results that we use in the proofs of this article. In Section 3, we consider the following fractional-order differential equation:

$$^{c}D^{\alpha}_{\mathcal{J}}\rho(\mathcal{J}) = k(\mathcal{J}, \rho(\mathcal{J}), \rho(h(\mathcal{J}))), \quad j \in J \subset \mathbb{R}, \alpha \in (0,1). \quad (1)$$

We investigate the existence of a unique solution and the Gauss hypergeometric stability of this fractional-order differential equation.

In the above equation, $^{c}D^{\alpha}_{\mathcal{J}}$ for a function ρ given on the interval $J = (0, q], q \in \mathbb{R}_{+}$ is the Caputo fractional derivative of order α and the functions $k \in C(J \times \mathbb{R}^{2}, \mathbb{R}), h \in C(\overline{J}, \overline{J})$ ($\overline{J}$ is the colure of J) with $h(\mathcal{J}) \leq \mathcal{J}$. In the first theorem, we investigate the Gauss hypergeometric stability of the fractional-order differential equation using the Chebyshev norm, and in the second theorem, we have proved the Gauss hypergeometric stability of the equation by using Bielecki norm. At the end of each theorem, we provide examples that demonstrate our results well.

2. Preliminaries

Discontinuous control strategy is a method that is used in various issues. Among its applications, we can mention the study of the dynamic behavior of the computer worm system (see [9, 10], for more information). In this paper, we consider the Gauss hypergeometric function as a control function and use this function to investigate the stability of Gauss hypergeometric. In this section, we present the definitions of the fractional integral, the Riemann–Liouville fractional derivative, and the Caputo fractional derivative of order τ, which we utilize in this paper, for more detail, we refer to [11–14]. In the continuation, by introducing the Gauss hypergeometric series, we define the Gauss hypergeometric stability of equation (1) [15–17]. Also, we consider the Picard operator and Henry–Gronwall inequality, which we use in the next section [18, 19].

Definition 1. Let $\tau > 0$ and $k \in C(J \times \mathbb{R}^2, \mathbb{R})$, and $\Gamma(\tau)$ is the Gamma function. On an interval $[0, \mathcal{J}]$, the Riemann–Liouville fractional integral of order τ is defined by

$$I_{\mathcal{J}}^{\tau} k(\mathcal{J}) = \frac{1}{\Gamma(\tau)} \int_0^{\mathcal{J}} \frac{k(\ell)}{(\mathcal{J} - \ell)^{1-\tau}} d\ell, \quad \mathcal{J} > 0, \tau > 0. \quad (2)$$

Definition 2. Assume that $n - 1 < \tau < n$, $k: [0, \infty) \longrightarrow \mathbb{R}$. The Riemann–Liouville derivative of order τ and the Caputo derivative of order τ, respectively, are defined by

$$^{L}D_{\mathcal{J}}^{\tau} k(\mathcal{J}) = \frac{1}{\Gamma(n - \tau)} \frac{d^n}{d\mathcal{J}^n} \int_0^{\mathcal{J}} \frac{k(\ell)}{(\mathcal{J} - \ell)^{\tau} + 1 - n} d\ell, \quad \mathcal{J} > 0,$$

$$^{c}D_{\mathcal{J}}^{\tau} k(\mathcal{J}) = {}^{L}D_{\mathcal{J}}^{\tau} \left(k(\mathcal{J}) - \sum_{k=0}^{n-1} \frac{\mathcal{J}^k}{k!} k^{(k)}(0) \right), \quad \mathcal{J} > 0. \quad (3)$$

Now, we consider the Gauss hypergeometric series (see [17, 20]) F by

$$F(\aleph, \beta; \wp; z) = \sum_{k=0}^{\infty} \frac{(\aleph)_k (\beta)_k}{(\wp)_k} \frac{z^k}{k!}$$

$$= \frac{\Gamma(\wp)}{\Gamma(\aleph)\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\aleph + k)\Gamma(\beta + k)}{\Gamma(\wp + k)} \frac{z^k}{k!}, \quad (4)$$

where in $\aleph, \beta, \wp \in \mathbb{R}^+$ and $z \in \mathbb{R}$.

Remark 1. We consider fractional-order differential equation (1) that is controlled by Gauss hypergeometric F. If w is a differential function, satisfying

$$\left| {}^{c}D_{\mathcal{J}}^{\alpha} w(t) - k(\mathcal{J}, w(\mathcal{J}), w(h(\mathcal{J}))) \right| \leq \varepsilon F(\aleph, \beta; \wp; \mathcal{J}^{\alpha}), \quad (5)$$

for $k \in C(J \times \mathbb{R}^2, \mathbb{R})$ and $\varepsilon > 0$, then w also satisfies the following integral inequality:

$$\left| w(\mathcal{J}) - w(0) - \frac{1}{\Gamma(\alpha)} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} k(\ell, w(\ell), w(h(\ell))) d\ell \right|$$

$$\leq \frac{\varepsilon}{\Gamma(\alpha)} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} F(\aleph, \beta; \wp, \ell^{\alpha}) d\ell$$

$$\leq \frac{\varepsilon}{\Gamma(\alpha)} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} \frac{\Gamma(c)}{\Gamma(\aleph)\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\aleph + k)\Gamma(\beta + k)}{\Gamma(\wp + k)} \frac{\ell^{\alpha k}}{k!} d\ell$$

$$= \frac{\varepsilon}{\Gamma(\alpha)} \frac{\Gamma(\wp)}{\Gamma(\aleph)\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\aleph + k)\Gamma(\beta + k)}{\Gamma(\wp + k)} \frac{1}{k!} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} \ell^{\alpha k} d\ell$$

$$= \frac{\varepsilon}{\Gamma(\alpha)} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(\aleph + k)\Gamma(\beta + k)}{\Gamma(\wp + k)} \frac{\mathcal{J}^{(k+1)\alpha}\Gamma(k\alpha + 1)\Gamma(\alpha)}{\Gamma((k+1)\alpha + 1)k!}$$

$$\leq \varepsilon \frac{\Gamma(\wp)}{\Gamma(\aleph)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\aleph + k)\Gamma(\beta + k)}{\Gamma(\wp + k)} \frac{\mathcal{J}^{n\alpha}\Gamma(n\alpha + 1)}{\Gamma((n+1)\alpha + 1)n!}$$

$$\leq \varepsilon \frac{\Gamma(\wp)}{\Gamma(\aleph)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\aleph + k)\Gamma(\beta + k)}{\Gamma(\wp + k)} \frac{1}{n!} \frac{\mathcal{J}^{n\alpha}\Gamma(n\alpha + 1)}{\Gamma(n\alpha + 1)}$$

$$= \varepsilon F(\aleph, \beta; \wp, \mathcal{J}^{\alpha}). \quad (6)$$

Remark 2. Assume that w is a function such that $w \in C(J, \mathbb{R})$. Then, w is a solution of inequality (5) if and only if we can find a function $\tilde{f}_w \in C(J, \mathbb{R})$ such that

(i) $|\tilde{f}(\mathcal{J})| \leq \varepsilon F(\aleph, \beta; \wp; \mathcal{J}^{\alpha})$, for all $\mathcal{J} \in J$

(ii) $^{c}D_{\mathcal{J}}^{\alpha} w(\mathcal{J}) = k(\mathcal{J}, w(\mathcal{J}), w(h(\mathcal{J}))) + \tilde{f}(\mathcal{J})$, for all $\mathcal{J} \in J$

Definition 3. We say that equation (1) is Gauss hypergeometric stable w.r.t $F(\aleph, \beta; \wp; \mathcal{J}^\alpha)$ if, for each $\varepsilon > 0$, there exists a constant value $c_F > 0$, such that, for a differentiable function w satisfying (5), we can find ρ such that the following holds:

$$^{c}D^\alpha_{\mathcal{J}}\rho(\mathcal{J}) = k(\mathcal{J}, \rho(\mathcal{J}), \rho(h(\mathcal{J}))), \quad \mathcal{J} \in J \subset \mathbb{R}, \alpha \in (0,1). \tag{7}$$

Then, we have

$$|w(\mathcal{J}) - \rho(\mathcal{J})| \le c_F \varepsilon F(\aleph, \beta; \wp; \mathcal{J}^\alpha), \quad \mathcal{J} \in J. \tag{8}$$

Definition 4. Consider the metric space (Y, δ) and the operator T on Y. Now, T is a Picard operator if there exists $y^* \in Y$ such that

   (i) $E_T = y^*$ wherever $E_T = \{y \in Y : T(y) = y\}$ is the fixed point set of T

   (ii) The sequence $(T^n(y_0))_{n \in \mathbb{N}}$ converges to y^*, for all $y_0 \in Y$.

Lemma 1. *If we consider an increasing Picard operator $T: Y \longrightarrow Y$, then $(Y, \delta, \le)$ is an ordered metric space. Therefore, for each $y \in Y$, $y \le T(y)$, we have $y \le y_T^*$.*

In the sequel, we express the Gronwall lemma (see Lemma 7.1.1 in [21]).

Lemma 2. *We consider the continuous functions $\mu, \nu: J \longrightarrow [0, \infty)$. For the nondecreasing function ν, there are positive constants κ and $0 < \alpha < 1$ such that*

$$\mu(\mathcal{J}) \le \nu(\mathcal{J}) + \kappa \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} \mu(\ell) d\ell, \quad \mathcal{J} \in J. \tag{9}$$

Therefore,

$$\mu(\mathcal{J}) \le \nu(\mathcal{J}) + \int_0^{\mathcal{J}} \left(\sum_{n=1}^{\infty} \frac{(\kappa \Gamma(\alpha))^n}{\Gamma(n\alpha)} (\mathcal{J} - \ell)^{n\alpha-1} \nu(\ell) \right) d\ell, \quad \mathcal{J} \in J. \tag{10}$$

Remark 3. For continuous and nondecreasing functions $\nu(j)$ and $\mu(\mathcal{J})$ on J, we have

$$\mu(\mathcal{J}) \le \nu(\mathcal{J}) F(\aleph, \beta; \wp; \kappa \Gamma(\alpha) \mathcal{J}^\alpha). \tag{11}$$

Remark 4. We propose the Picard method for the uniqueness and stability of this problem, in which case both its conditions will be weaker and it will be easier to prove it.

Remark 5 (see [22]). There are some slight mistakes and typos in [13]. In Definition 2.4 and Remark 2.6 in [13], the space $C[-h, d]$ should be $C[-h, 0] \cap C_{1-\alpha; \psi}[0, d]$. For the precise definition needed in [13], we note that the statement (some typos) and proof of Theorem 1 in [13] is fine once $X = C[-h, d]$ in [13] is replaced by $C[-h, 0] \cap C_{1-\alpha; \psi}[0, d]$ with the norm (there was accidently a typo in relation to the norm in [13]) $\max\{\|.\|C[-h, 0], \|.\|_{C1-\alpha; \psi[0,d]}\}$.

3. Main Results

As mentioned in Section 1, we use stability algorithms and methods to get good results. As we know, there are different types of stability such as exponential, Mittag-Leffler and Lyapunov type (see [9, 23], for more details). Here, we prove the Gauss hypergeometric stability of equation (1). Before proving the Gauss hypergeometric stability, we first investigate the existence of a unique solution to the fractional-order differential equation. We prove the first theorem utilizing Chebyshev norm $\|.\|_C$ ($\|\rho\|_C = \max_{\mathcal{J} \in J} |\rho(\mathcal{J})|$).

Theorem 1. *We assume that the functions $k \in C(J \times \mathbb{R}^2, \mathbb{R})$ and $h \in C(\bar{J}, \bar{J})$ exist, such that $h(\mathcal{J}) \le \mathcal{J}$. Assume the following are satisfied:*

   (i) *There exists $\mathbf{L}_k > 0$ such that $|k(\mathcal{J}, u_1, u_2) - k(\mathcal{J}, v_1, v_2)| \le \mathbf{L}_k \sum_{i=1}^2 |u_i - v_i|$, for all $\mathcal{J} \in J$, $u_i, v_i \in \mathbb{R}$, $i = 1, 2$*

   (ii) $(2\mathbf{L}_k q^\alpha / \Gamma(\alpha + 1)) < 1$

Therefore, (1) has a unique solution in $C(J, \mathbb{R})$ and is Gauss hypergeometric stable.

Proof. According to equation (1), we have

$$\rho(\mathcal{J}) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} k(\ell, \rho(\ell), \rho(h(\ell))) d\ell, \quad \mathcal{J} \in J. \tag{12}$$

It is easy to see that (1) is equivalent to (12).

We first prove the existence of solution for equation (12). We define the mapping Λ_k in $Y: = C(J, \mathbb{R})$ as

$$\Lambda_k \rho(\mathcal{J}) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} k(\ell, \rho(\ell), \rho(h(\ell))) d\ell, \quad \mathcal{J} \in J. \tag{13}$$

Now, we need to demonstrate that Λ_k determined in (13) is a contraction mapping on $Y: = C(J, \mathbb{R})$.

According (i), we have

$$
\begin{aligned}
&\left|\Lambda_k \rho(\mathcal{J}) - \Lambda_k w(\mathcal{J})\right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} \left| \begin{array}{c} k(\ell, \rho(\ell), \rho(h(\ell))) \\ -k(\ell, w(\ell), w(h(\ell))) \end{array} \right| d\ell \\
&\leq \frac{\mathbf{L}_k}{\Gamma(\alpha)} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} \left(\max_{0 \leq \ell \leq q} |\rho(\ell) - w(\ell)| + \max_{0 \leq \ell \leq q} |\rho(h(\ell)) - w(h(\ell))| \right) d\ell \\
&\leq \frac{2\mathbf{L}_k}{\Gamma(\alpha)} \|\rho - w\|_c \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} d\ell \\
&\leq \frac{2\mathbf{L}_k q^\alpha}{\Gamma(\alpha+1)} \|\rho - w\|_c,
\end{aligned}
\tag{14}
$$

for all $\mathcal{J} \in J$, which implies that

$$
\left\|\Lambda_k(\rho) - \Lambda_k(w)\right\|_C \leq \frac{2\mathbf{L}_k q^\alpha}{\Gamma(\alpha+1)} \|\rho - w\|_C, \quad \rho, w \in C(J, \mathbb{R}).
\tag{15}
$$

Thus, Λ_k is a contraction mapping w.r.t. the Chebyshev norm $\|.\|_C$ on Y. Now, to continue the proof, we utilize the Banach contraction principle.

Now, we prove Gauss hypergeometric stability. We assume that ρ is a unique solution to equation (1). Then, for $w \in C(J, \mathbb{R})$ that is a given differentiable function, satisfying (5), we have

$$
\rho(\mathcal{J}) = w(0) + \frac{1}{\Gamma(\alpha)} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} k(\ell, \rho(\ell), \rho(h(\ell))) d\ell, \quad \mathcal{J} \in J.
\tag{16}
$$

Obviously, Remark 1 implies that

$$
\left| w(\mathcal{J}) - w(0) - \frac{1}{\Gamma(\alpha)} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} k(\ell, w(\ell), w(h(\ell))) d\ell \right| \leq \varepsilon F(\aleph; \beta; \wp; \mathcal{J}^\alpha),
\tag{17}
$$

for $\mathcal{J} \in J$.

Also according to (i), we have

$$
\begin{aligned}
|w(\mathcal{J}) - \rho(\mathcal{J})| &\leq \left| w(\mathcal{J}) - w(0) - \frac{1}{\Gamma(\alpha)} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} k(\ell, w(\ell), w(h(\ell))) d\ell \right| \\
&\quad + \left| \frac{1}{\Gamma(\alpha)} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} k(\ell, w(\ell), w(h(\ell))) d\ell \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} k(\ell, \rho(\ell), \rho(h(\ell))) d\ell \right| \\
&\leq \varepsilon F(\aleph, \beta; \wp; \mathcal{J}^\alpha) + \frac{\mathbf{L}_k}{\Gamma(\alpha)} \left(\int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} [|w(\ell) - \rho(\ell)| d\ell + |w(h(\ell)) - \rho(h(\ell))|] d\ell \right),
\end{aligned}
\tag{18}
$$

for all $\mathcal{J} \in J$. Now, we consider the operator $T: C(J, \mathbb{R}_+) \longrightarrow C(J, \mathbb{R}_+)$, for each $\sigma \in C(J, \mathbb{R}_+)$, as follows:

$$
T\sigma(\mathcal{J}) = \varepsilon F(\aleph, \beta; \wp; \mathcal{J}^\alpha) + \frac{\mathbf{L}_k}{\Gamma(\alpha)} \left(\int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} \sigma(\ell) d\ell + \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} \sigma(h(\ell)) d\ell \right), \quad \mathcal{J} \in J.
\tag{19}
$$

In the continuation, we show that T is a Picard operator. By (ii), we have

$$|T\sigma(\mathcal{J}) - T\vartheta(\mathcal{J})|$$

$$\leq \frac{\mathbf{L}_k}{\Gamma(\alpha)} \left(\int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} [|\sigma(\ell) - \vartheta(\ell)| d\ell + |\sigma(h(\ell)) - \vartheta(h(\ell))|] d\ell \right) \tag{20}$$

$$\leq \frac{2\mathbf{L}_k q^\alpha}{\Gamma(\alpha+1)} \|\sigma - \vartheta\|_C,$$

for all $\mathcal{J} \in J$ and for every $\sigma, \vartheta \in C(J, \mathbb{R}_+)$, which implies that

$$\|T(\sigma) - T(\vartheta)\|_C \leq \frac{2\mathbf{L}_k q^\alpha}{\Gamma(\alpha+1)} \|\sigma - \vartheta\|_C. \tag{21}$$

Consequently, T w.r.t. the Chebyshev norm $\|.\|_C$ is a contraction mapping on $C(J, \mathbb{R}_+)$. Thus, utilizing the Banach contraction principle, we obtain that T is a Picard operator and $E_T = \{\sigma^*\}$. Now, for,

$$\sigma^*(\mathcal{J}) = \varepsilon F(\aleph, \beta; \wp; \mathcal{J}^\alpha) + \frac{\mathbf{L}_k}{\Gamma(\alpha)} \left(\int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} \sigma^*(\ell) d\ell + \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} \sigma^*(h(\ell)) d\ell \right). \tag{22}$$

In the sequel, we show that the solution σ^* is increasing. Suppose that and denote $m := \min_{\ell \in J} [\sigma^*(\ell) + \sigma^*(h(\ell))] \in \mathbb{R}_+$, and we have

$$\sigma^*(\mathcal{J}_2) - \sigma^*(\mathcal{J}_1) = \varepsilon \left[F(\aleph, \beta; \wp; \mathcal{J}_2^\alpha) - F(\aleph, \beta, \wp; \mathcal{J}_1^\alpha) \right]$$

$$\frac{\mathbf{L}_k}{\Gamma(\alpha)} \int_0^{\mathcal{J}_1} \left[(\mathcal{J}_2 - \ell)^{\alpha-1} - (\mathcal{J}_1 - \ell)^{\alpha-1} \right] [\sigma^*(\ell) + \sigma^*(h(\ell))] d\ell$$

$$+ \frac{\mathbf{L}_k}{\Gamma(\alpha)} \int_{\mathcal{J}_1}^{\mathcal{J}_2} (\mathcal{J}_2 - \ell)^{\alpha-1} [\sigma^*(\ell) + \sigma^*(h(\ell))] d\ell \geq \varepsilon \left[F(\aleph, \beta; \wp; \mathcal{J}_2^\alpha) - F(\aleph, \beta; \wp; \mathcal{J}_1^\alpha) \right] \tag{23}$$

$$+ \frac{m\mathbf{L}_k}{\Gamma(\alpha)} \int_0^{\mathcal{J}_1} \left[(\mathcal{J}_2 - \ell)^{\alpha-1} - (\mathcal{J}_1 - \ell)^{\alpha-1} \right] d\ell + \frac{m\mathbf{L}_k}{\Gamma(\alpha)} \int_{\mathcal{J}_1}^{\mathcal{J}_2} (\mathcal{J}_2 - \ell)^{\alpha-1} d\ell$$

$$= \varepsilon \left[F(\aleph, \beta; \wp; \mathcal{J}_2^\alpha) - F(\aleph, \beta; \wp; \mathcal{J}_1^\alpha) \right] + \frac{m\mathbf{L}_k}{\Gamma(\alpha+1)} (\mathcal{J}_2^\alpha - \mathcal{J}_1^\alpha) > 0.$$

Then, σ^* is increasing. From $h(\mathcal{J}) \leq \mathcal{J}$, we obtain $\sigma^*(h(\mathcal{J})) \leq \sigma^*(\mathcal{J})$,

$$\sigma^*(\mathcal{J}) \leq \varepsilon F(\aleph, \beta; \wp; \mathcal{J}^\alpha) + \frac{2\mathbf{L}_k}{\Gamma(\alpha)} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} \sigma^*(\ell) d\ell. \tag{24}$$

Now, utilizing Lemma 2, Remark 2, and $c_F := F(\aleph, \beta; \wp; 2\mathbf{L}_k q^\alpha)$, we obtain

$$\sigma^*(\mathcal{J}) \leq c_F \varepsilon F(\aleph, \beta; \wp; \mathcal{J}^\alpha), \quad \mathcal{J} \in J. \tag{25}$$

Since T is a Picard operator and increasing, by Lemma 1 and from (18), we conclude that $\sigma \leq \sigma^*$. If we consider $\sigma = |w - \rho|$, then

$$|w(\mathcal{J}) - \rho(\mathcal{J})| \leq c_F \varepsilon F(\aleph, \beta; w p; \mathcal{J}^\alpha), \quad \mathcal{J} \in J. \tag{26}$$

Therefore, equation (1) is Gauss hypergeometric stable. □

In the following, we give an example to illustrate the previous theorem.

Example 1. Consider a differential equation for $\alpha = 1/2$ and for any $\mathscr{J} \in (0, 1]$ ($q = 1$) as follows:

$$^{c}D_{\mathscr{J}}^{1/2}\rho(\mathscr{J}) = 0.01\rho^{5/2}\sqrt{\pi}(1 + \mathscr{J})^{-(3/2)}, \quad \mathscr{J} \in (0, 1]. \quad (27)$$

Based on what has been said in the above theorem, we have

$$\rho(\mathscr{J}) = w(0) + \frac{0.01}{\Gamma(1/2)}\int_{0}^{\mathscr{J}}(\mathscr{J} - \ell)^{-(1/2)}\rho(\mathscr{J})^{5/2}\sqrt{\pi}(1 + \ell)^{-(3/2)}\mathrm{d}\ell. \quad (28)$$

According to equation (5) and using the fractional-order differential equation, the following inequality is established for the differentiable function w:

$$\left|^{c}D_{\mathscr{J}}^{1/2}w(\mathscr{J}) - k\big(\mathscr{J}, w(\mathscr{J}), w\big((1 + \mathscr{J})^{-(3/2)}\big)\big)\right| \leq \varepsilon F\big(\aleph, \beta; \wp; \mathscr{J}^{1/2}\big). \quad (29)$$

The functions k and h used in the above inequality are considered as follows:

$$h(\mathfrak{I}) = (1 + \mathfrak{I})^{-(3/2)},$$
$$k(\mathfrak{I}, \rho(\mathfrak{I}), \rho(h(\mathfrak{I}))) = 0.01\rho^{5/2}\sqrt{\pi}(1 + \mathfrak{I})^{-(3/2)}. \quad (30)$$

To be Ulam–Hyers-hypergeometric stable, we need a positive constant $\mathbf{L}_{k}$ and a coefficient of Lipschitz $2\mathbb{L}_{k}q^{\alpha}/\Gamma(\alpha + 1)$. Then, we consider $\mathbf{L}_{k} = 2/5$ and as a result $2\mathbb{L}_{k}q^{\alpha}/\Gamma(\alpha + 1) = 8/5\sqrt{\pi} \approx 0.9 < 1$.

According to Theorem 1, all the conditions for the existence of the solution and the Gauss hypergeometric stability are established. Thus, the coefficient $c_{F} = F(\aleph, \beta; \wp; (4/5))$ exists to us so that

$$|w(\mathscr{J}) - \rho(\mathscr{J})| \leq c_{F}\varepsilon F\big(\aleph, \beta; \wp; \mathscr{J}^{1/2}\big), \quad \mathscr{J} \in (0, 1]. \quad (31)$$

Based on equation (28), we have a nonlinear equation whose diagram for the arbitrary values in the interval $(0, 1]$ is shown in Figure 1.

In the next theorem, we prove the Gauss hypergeometric stability utilizing Bielecki's norm $\|.\|_{B}$ ($\|\rho\|_{B}$: = $\max_{\mathscr{J} \in I}|\rho(\mathscr{J})|\exp(-\eta\mathscr{J}), \eta > 0, I \subset \mathbb{R}_{+}$).

Theorem 2. *Assume that the functions $k \in C(J \times \mathbb{R}^{2}, \mathbb{R})$, $h \in C(\overline{J}, \overline{J})$ exist, such that $h(\mathscr{J}) \leq \mathscr{J}$. Assume the following are satisfied:*

   (i) There exists $\mathbf{L}_{k} > 0$ such that $|k(\mathscr{J}, u_{1}, u_{2}) - k(\mathscr{J}, v_{1}, v_{2})| \leq \mathbf{L}_{k}\sum_{i=1}^{2}|u_{i} - v_{i}|$ for all $\mathscr{J} \in J, u_{i}, v_{i} \in \mathbb{R}, i = 1, 2$

   (ii) $(2\mathbf{L}_{k}q^{\alpha}\exp(\eta q)/\Gamma(\alpha)\sqrt{2(2\alpha - 1)\eta}) < 1$, for some $\alpha \in (1/2, 1)$ and $\eta > 0$

Therefore, equation (1) has a unique solution in $C(C, \mathbb{R})$ and is Gauss hypergeometric stable.

Proof. Here, we prove that $\Lambda_{k}: Y \longrightarrow Y$ is defined in Theorem 1 with Bielecki's norm $\|.\|_{B}$, which is a contraction operator. Also, we ignore from saying the similar arguments that are expressed in Theorem 1. Then, for all $\mathscr{J} \in J$, we have

$$\left|\Lambda_{k}\rho(\mathscr{J}) - \Lambda_{k}w(\mathscr{J})\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_{0}^{\mathscr{J}}(\mathscr{J} - \ell)^{\alpha-1}|k(\ell, \rho(\ell), \rho(h(\ell))) - k(\ell, w(\ell), w(h(\ell)))|\mathrm{d}\ell$$

$$\leq \frac{\mathbf{L}_{k}}{\Gamma(\alpha)}\int_{0}^{\mathscr{J}}(\mathscr{J} - \ell)^{\alpha-1}\exp(\eta\ell)$$

$$\left(\max_{0 \leq \ell \leq q}|\rho(\ell) - w(\ell)|\exp(-\eta\ell) + \max_{0 \leq \ell \leq q}|\rho(h(\ell)) - w(h(\ell))|\exp(-\eta\ell)\right)\mathrm{d}\ell \quad (32)$$

$$\leq \frac{2\mathbf{L}_{k}}{\Gamma(\alpha)}\|\rho - w\|_{B}\int_{0}^{\mathscr{J}}(\mathscr{J} - \ell)^{\alpha-1}\exp(\eta\ell)\mathrm{d}\ell.$$

Now, we utilize Holder's inequality, for $\alpha \in (1/2, 1)$. Then,

$$\int_{0}^{\mathscr{J}}(\mathscr{J} - \ell)^{\alpha-1}\exp(\eta\ell) \leq \left(\int_{0}^{\mathscr{J}}(\mathscr{J} - \ell)^{2(\alpha-1)}\mathrm{d}\ell\right)^{1/2}\left(\int_{0}^{\mathscr{J}}\exp(2\eta\ell)\mathrm{d}l\right)^{1/2}$$

$$\leq \frac{1}{\sqrt{2\eta}}\frac{q^{\alpha}}{\sqrt{2\alpha - 1}}\exp(\eta q). \quad (33)$$
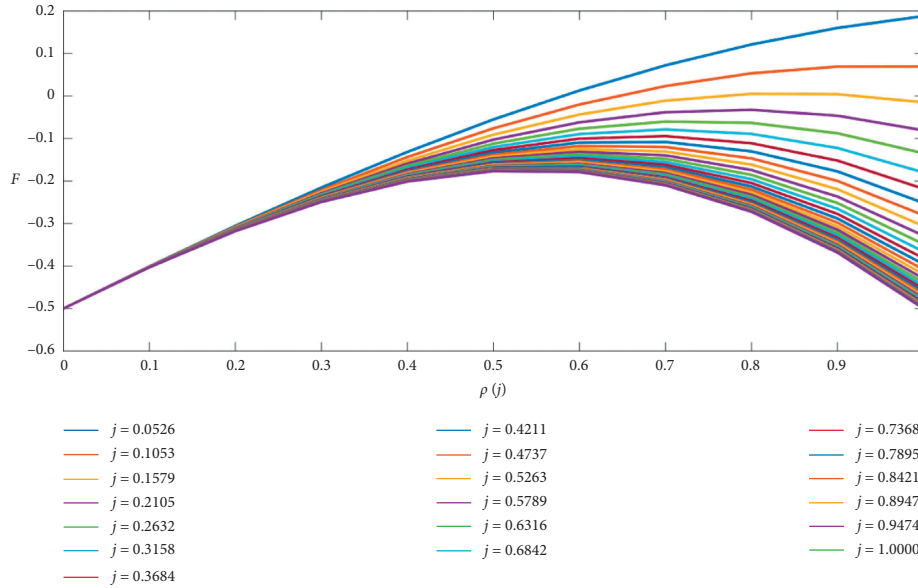
FIGURE 1: Fraction integral equation diagram. F is the nonlinear equation obtained from equation (28).

Therefore,

$$\left\| \Lambda_k(\rho) - \Lambda_k(w) \right\|_B \le \frac{2\mathbf{L}_k q^\alpha \exp(\eta q)}{\Gamma(\alpha)\sqrt{2(2\alpha-1)\eta}} \|\rho - w\|_B. \quad (34)$$

Consequently, we have

$$\left\| \Lambda_k(\rho) - \Lambda_k(w) \right\|_B \le \frac{2\mathbf{L}_k q^\alpha \exp(\eta q)}{\Gamma(\alpha)\sqrt{2(2\alpha-1)\eta}} \|\rho - w\|_B, \quad \rho, w \in C(J, \mathbb{R}).$$

$$(35)$$

Therefore, Λ_k on Y is a contraction mapping w.r.t. Bielecki's norm $\|.\|_B$. Now, we apply the Banach contraction principle to prove (ii). To prove that T in Theorem 1 is a Picard operator, we show that T is a contraction. For all $\mathcal{J} \in J$ and $\alpha \in (1/2, 1)$ and for every $\sigma, \vartheta \in C(J, \mathbb{R}_+)$, we have

$$|T\sigma(\mathcal{J}) - T\vartheta(\mathcal{J})| \le \frac{2\mathbf{L}_k q^\alpha \exp(\eta q)}{\Gamma(\alpha)\sqrt{2(2\alpha-1)\eta}} \|\rho - w\|_B. \quad (36)$$

Then,

$$\|T(\sigma) - T(\vartheta)\|_B \le \frac{2\mathbf{L}_k q^\alpha \exp(\eta q)}{\Gamma(\alpha)\sqrt{2(2\alpha-1)\eta}} \|\rho - w\|_B. \quad (37)$$

Consequently, T is a contraction mapping on $C(J, \mathbb{R}_+)$ with Bielecki's norm $\|.\|_B$. The Gauss hypergeometric stability proof is exactly the same as Theorem 1. $\qquad\square$

Example 2. Consider a fractional-order differential equation as

$$^c D_{\mathcal{J}}^{2/3} \rho(\mathcal{J}) = \frac{1}{27}\Gamma\left(\frac{2}{3}\right)(\mathcal{J} - h), \quad \mathcal{J} \in \left(0, \frac{1}{4}\right], \quad (38)$$

for $\alpha = 2/3$ and for any $\mathcal{J} \in (0, 1/4]$. Based on what has been said in Theorem 2, we have

$$\rho(\mathcal{J}) = w(0) + \frac{1}{27\Gamma(2/3)} \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{-(9/10)} \Gamma\left(\frac{2}{3}\right)(\mathcal{J} - 0.01)d\ell.$$

$$(39)$$

Let $\mathbf{L}_k = 1/2$ and $\eta = 1/2$. Consequently,

$$\frac{2\mathbf{L}_k q^\alpha \exp(\eta q)}{\Gamma(\alpha)\sqrt{2(2\alpha-1)\eta}} = \frac{(1/4)^{2/3}\exp(1/8)}{\Gamma(2/3)\sqrt{1/3}} < 0.58 < 1. \quad (40)$$

According to equation (5) and using the fractional-order differential equation, for the differentiable function w, we have

$$\left| ^c D_{\mathcal{J}}^{2/3} w(\mathcal{J}) - k(\mathcal{J}, w(\mathcal{J}), w(\mathcal{J} - h)) \right| \le \varepsilon F\left(\aleph, \beta; \wp; \mathcal{J}^{2/3}\right).$$

$$(41)$$

All the conditions for the existence of the solution and the Gauss hypergeometric stability are established by Theorem 2. Therefore, the coefficient $c_F = F(\aleph, \beta; \wp; (1/4)^{2/3})$ exists, and we have

$$|w(\mathcal{J}) - \rho(\mathcal{J})| \le c_F \varepsilon F\left(\aleph, \beta; \wp; \mathcal{J}^{2/3}\right), \quad \mathcal{J} \in \left(0, \frac{1}{4}\right]. \quad (42)$$

According to equation (39) for the function ρ, we have Figure 2 for every $\mathcal{J} \in (0, 1/4)$.

Remark 6. In Theorem 2, we examined our problem for $\alpha \in (1/2, 1)$ by utilizing Holder's inequality. At present, we re-examine the problem for $\alpha \in (0, 1)$.

Let ; therefore, we have

$$\left| \Lambda_k \rho(\mathcal{J}) - \Lambda_k w(\mathcal{J}) \right| \le \frac{2\mathbf{L}_k}{\Gamma(\alpha)} \|\rho - w\|_B \int_0^{\mathcal{J}} (\mathcal{J} - \ell)^{\alpha-1} \exp(\eta\ell)d\ell$$

$$\le \frac{2\mathbf{L}_k q^\alpha \exp(\eta q)}{\Gamma(\alpha+1)} \|\rho - w\|_B.$$
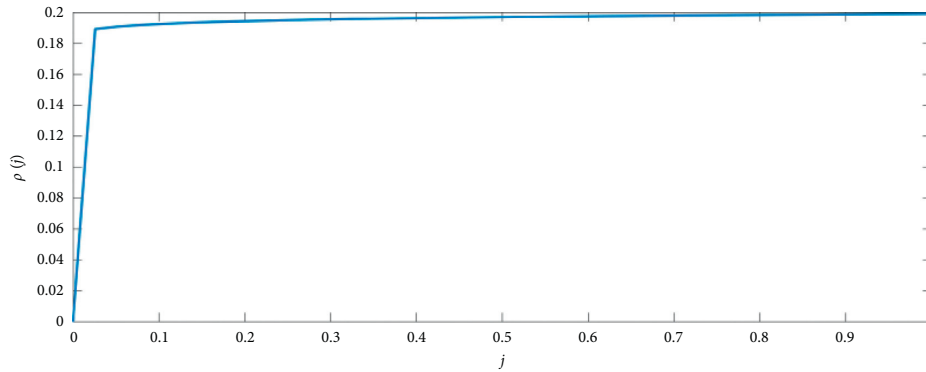
$$(43)$$

Figure 2: Fractional integral equation diagram.

For some $\alpha \in (0, 1)$ and $\eta > 0$, we have $2\mathbf{L}_k q^\alpha \exp(\eta q)/\Gamma(\alpha + 1) < 1$. Now, we be able demonstrate that Λ_k is a contraction via Bielecki's norm $\|.\|_B$ on Y as well.

4. Conclusion

In this paper, we applied the Picard method to investigate existence, uniqueness, and Gauss hypergeometric stability of fractional-order differential equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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