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Research Article

Blow-Up of Solutions for Wave Equation Involving the Fractional Laplacian with Nonlinear Source

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In this paper, we study the blow-up of solutions for wave equation involving the fractional Laplacian with nonlinear source.

1. Introduction and Brief History of Fractional Integrodifferentiation

Let $\Omega \subset \mathbb{R}^n, n \geq 1$ be an open domain with Lipschitz boundary $\partial\Omega$. In this article, we consider the hyperbolic initial-boundary value problem involving the fractional Laplacian; for $w = w(x,t)$, we consider the wave equation with power nonlinearity:

$$\begin{cases} \partial_t^2 w + (-\Delta)^r w + (-\Delta)^r \partial_t w = w|w|^{p-2}, & x \in \Omega, t > 0, \\ w = 0, & x \in \partial\Omega, t > 0, \\ w(x,0) = w_0(x), \partial_t w(x,0) = w_1(x), & x \in \Omega, \end{cases} \tag{1}$$

where $(-\Delta)^r$ is the fractional Laplacian such that $r \in (0,1)$. The exponent p satisfies

$$2 < p \leq \frac{2n}{n-2r} = 2_r^*, \quad n > 2r. \tag{2}$$

The fractional integrodifferentiation operation is a generalization of the differentiation operations. The idea of fractional differentiation as a generalization of the concept of the derivative to the noninteger value of a arose almost simultaneously with the very concept of differentiation. The first mention of this idea occurs in the correspondence of G. W. Leibniz and Marquis de l'Hospital in 1695 (see [1]). The idea of fractional integrodifferentiation was further developed in the works of L. Euler, who in 1738 noticed that an expression can be given meaning even for noninteger values (see [2]). An explicit calculation formula was given in the treatise by S. Lacroix in 1820 (see [3]). Also in 1812, P.S. Laplace put forward the idea of the possibility of differentiating noninteger order for some functions. The first definition of the derivative of noninteger order was given by J. Fourier in 1822. In its modern form, fractional integrodifferentiation was formed in the works of N.H. Abel and J. Liouville. In 1823, in connection with the problem of tautochrone—a curve, when sliding along which, under the influence of gravitational forces, a body reaches its lowest point in the same time, regardless of its initial position. The idea of considering fractional differentiation as an operation inverse to fractional integration was first proposed by Holmgren in 1865 (see [4–6]). A year later, Grunwald, who was not familiar with Holmgren's work, came to the same idea of Letnikov in 1868 (see [7–13]).

In [14], an efficient novel technique, namely, the q-homotopy analysis transform method (q-HATM), is applied to find the solution for the time-fractional Kaup–Kupershmidt (KK) equation and the study of fractional Emden–Fowler (FEF) equations by utilizing a new adequate procedure; specifically, the q-homotopy analysis transform method (q-HATM) is considered in [15].

Fractional wave systems with continuous nonlinearities are possessed by a large number of researchers. In [16], the authors considered initial-boundary value problem of degenerate Kirchhoff-type

$$\partial_t^2 w + [w]_r^{2(\theta-1)} (-\Delta)^r w = |w|^{p-1} w, \text{ in } \Omega \times \mathbb{R}_+, \tag{3}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$ is a bounded domain with Lipschitz boundary, $\theta \in [1, 2^*]$, and $[w]_r$ is the Gagliardo seminorm of w defined by

$$[w]_r = \left(\int_\Omega \int_\Omega \frac{|w(x) - w(z)|^2}{|x - z|^{n+2r}} \mathrm{d}x\mathrm{d}z \right)^{1/2}. \tag{4}$$

The authors obtained, under appropriate conditions, the global existence in time and finite blow-up of solutions for (3) owing to the Galerkin method combined with the potential wells. They also showed the global existence of solutions under critical initial conditions. In [17], the authors studied the following damped degenerate Kirchhoff equation:

$$\begin{aligned} &\partial_t^2 w + [w]_r^{2(\theta-1)} (-\Delta)^r w + \left|\partial_t w\right|^{\alpha-1} \partial_t w \\ &\quad + w = |w|^{p-2} w, \text{ in } \Omega \times \mathbb{R}_+, \end{aligned} \tag{5}$$

where $2 < \alpha < 2\theta < p < 2^* < r$. The global existence, behavior of solutions, and blow-up in time for (4) are obtained, under appropriate assumptions. In [18], the IBVP of Kirchhoff wave equation is considered. Under some sufficient conditions, the blow-up in finite time is shown by using a modified concavity method; for more details, see [19–27].

$$\partial_t^2 w + [w]_r^{2(\theta-1)} (-\Delta)^r w = |w|^{p-2} w, \; in\, \Omega \times \mathbb{R}_+. \tag{6}$$

We highlight here the novelty of the problem:

(1) It is interesting to note that simultaneously with the theoretical developments of classical nonlinear wave operation, practical applications of fractional integrodifferentiation operation can also be found

(2) It is shown that when the nonlinear source dominates the fractional Laplacian in (2), this ensures the global nonexistence in time (blow-up) of solutions

(3) Our results extend many recent results in the literature

2. Auxiliary Results and Function Spaces

The fractional Laplacian $(-\Delta)^r w$ of the function w is given by

$$(-\Delta)^r w(x) = C \int_{\mathbb{R}^n} \frac{w(x) - w(z)}{|x - z|^{n+2r}} dz, \quad \forall x \in \mathbb{R}^n, \tag{7}$$

where

$$C^{-1} = \int_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{|\zeta|^{n+2r}} \mathrm{d}\zeta. \tag{8}$$

We define the fractional-order Sobolev space by

$$W^{r,2}(\Omega) = \left\{ v \in L^2(\Omega) \colon \int_\Omega \int_\Omega \frac{|v(x) - v(z)|^2}{|x - z|^{n+2r}} \mathrm{d}x\mathrm{d}z < \infty \right\}, \tag{9}$$

equipped with the norm

$$\|w\|_{W^{r,2}(\Omega)} = \left(\int_\Omega |w|^2 \mathrm{d}x + \int_\Omega \int_\Omega \frac{|v(x) - v(z)|^2}{|x - z|^{n+2r}} \mathrm{d}x\mathrm{d}z \right)^{1/2}. \tag{10}$$

Let

$$W_0^{r,2}(\Omega) = \left\{ w \in W^{r,2}(\Omega) \colon w = 0, x \in \partial\Omega \right\}, \tag{11}$$

be a closed linear subspace of $W^{r,2}(\Omega)$, and its norm is given by

$$\|w\|_{W_0^{r,2}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|v(x) - v(z)|^2}{|x - z|^{n+2r}} \mathrm{d}x\mathrm{d}z \right)^{1/2}. \tag{12}$$

The space $W_0^{r,2}(\Omega)$ is a Hilbert space with inner product

$$\langle w, u \rangle_{W_0^{r,2}(\Omega)} = \int_\Omega \int_\Omega \frac{(w(x) - w(z))(u(x) - u(z))}{|x - z|^{n+2r}} \mathrm{d}x\mathrm{d}z. \tag{13}$$

3. The Potential Wells

For simplicity, in this section, we consider problem (1) in stationary case. In fact, if we replace w in this section by $w(t)$ for any $t \in [0, T)$, all the facts are still valid. We define

$$\mathcal{J}(w) = \frac{1}{2} \|w\|_{W_0^{r,2}(\Omega)}^2 - \frac{1}{p} \|w\|_p^p. \tag{14}$$

We denote

$$\mathcal{I}(w) = \|w\|_{W_0^{r,2}(\Omega)}^2 - \|w\|_p^p. \tag{15}$$

We introduce now the stable set as follows:

$$\mathcal{W} = \left\{ w \in W_0^{r,2}(\Omega) \colon \mathcal{I}(w) > 0, \mathcal{J}(w) < d \right\} \cup \{0\}, \tag{16}$$

where the mountain pass level d is defined as

$$d = \inf_{w \in W_0^{r,2}(\Omega) \setminus \{0\}} \left\{ \sup_{\mu \geq 0} \mathcal{J}(\mu w) \right\}. \tag{17}$$

We introduce the so-called *Nehari manifold*:

$$\mathcal{N} = \left\{ w \in W_0^{r,2}(\Omega) \setminus \{0\} \colon \mathcal{I}(w) = 0 \right\}. \tag{18}$$

Then potential depth d is characterized by

$$d = \inf_{w \in \mathcal{N}} \mathcal{J}(w), \tag{19}$$

which implies that

$$\operatorname{dist}(0, \mathcal{N}) = \min_{w \in \mathcal{N}} \|w\|_{W_0^{r,2}(\Omega)}. \tag{20}$$

We will prove the invariance of the set $\mathcal{W}$.

For the reader's convenience, we recall the main embedding results for the fractional Sobolev spaces; see [28] for details.

Lemma 1. *Let Ω be bounded domain. Then*

(1) The embedding $W_0^{r,2}(\Omega)\hookrightarrow L^p(\Omega)$ is compact for any $p \in [1, 2_r^*)$

(2) The embedding $W_0^{r,2}(\Omega)\hookrightarrow L^{2_r^*}(\Omega)$ is continuous

Lemma 2

(1) For any $s \in [1, 2_r^*]$, there exists a positive constant $C_0 = C_0(n, s, r)$ such that for any $u \in W_0^{r,2}(\Omega)$

$$\|w\|_{L^s(\Omega)} \leq C_0 \int_\Omega \int_\Omega \frac{|u(x) - u(z)|^2}{|x - y|^{n+2r}} \mathrm{d}x\mathrm{d}y. \quad (21)$$

(2) For any $s \in [1, 2_r^*]$ and any bounded sequence $(u_j)_j$ in $W_0^{r,2}(\Omega)$, there exists u in $L^s(\mathbb{R}^n)$, with $u = 0$ a.e. in $\mathbb{R}^n - \Omega$, such that up to a subsequence, still denoted by $(u_j)_j$

$$u_j \longrightarrow u \,\text{strongly in}\, L^s(\Omega) as j \longrightarrow \infty. \quad (22)$$

Definition 1. A function $w = w(x,t)$ is said to be a global (weak) solution of problem (1), if

$$\begin{aligned} &w \in L^\infty\left(0,\infty, W_0^{r,2}(\Omega)\right), w_t \in L^\infty\left(0,\infty, L^2(\Omega)\right),\\ &w_o \in L^\infty\left(0,\infty, W_0^{r,2}(\Omega)\right), w_t \in L^\infty\left(0,\infty, L^2(\Omega)\right)\\ &w_1 \in L^\infty\left(0,\infty, L^2(\Omega)\right), \end{aligned} \quad (23)$$

and for any $\phi \in L^\infty(0,\infty, W_0^{r,2}(\Omega))$, $t \in \mathbb{R}_+^*$,

$$\begin{aligned} &(w_t(.,t),\phi(.,t)) + \frac{1}{2}\int_0^t \left(w_t(.,t),\phi(.,t)_{W_0^{r,2}(\Omega)}\right)\mathrm{d}\tau + \int_0^t \left(w_t(.,t),\phi(.,t)_{W_0^{r,2}(\Omega)}\right)\mathrm{d}\tau\\ &= (w_1,\phi(.,0)) + \int_0^t \left(w(.,\tau)|w(.,\tau)|^{p-2},\phi(.,\tau)\right)\mathrm{d}\tau. \end{aligned} \quad (24)$$

If a (weak) global solution w belongs to $C(0,\infty; W_0^{r,2}(\Omega))$, we say that u is a strong global solution of problem (1).

The energy $\mathscr{E}$ of solution at time t to (1) is given by

$$\mathscr{E}(t) = \frac{1}{2}\left\|\partial_t w(t)\right\|_2^2 + \mathscr{J}(w). \quad (25)$$

Lemma 3. *Let $w(x,t)$ be a weak solution of problem (1). If $w_0 \in \mathscr{W}$, $w_1 \in L^2(\Omega)$, then $\mathscr{E}(t) \leq \mathscr{E}(0)$.*

4. Blow-Up Result

In this section, we prove the blow-up result to problem (1)

Lemma 4. *Let $w(x,t)$ is the weak solution of problem (1). If $w_0 \in \mathscr{W}$ and $w_1 \in L^2(\Omega)$ satisfying that*

$$\|w\|^2 \geq \frac{2p}{p-2} K\mathscr{E}(0), \quad (26)$$

$$(\partial_t w, w)_{W_0^{r,2}(\Omega)} < 0, \quad (27)$$

$$w_0 \in \mathscr{M}, \quad (28)$$

$$\int_\Omega w_0 w_1 \mathrm{d}x > 0, \quad (29)$$

then any solution of (1) belongs to $\mathscr{M}$.

Proof. We claim that $w \in \mathscr{M}$ for $t \in [0,T)$; by contradiction, we suppose that $t_0 \in (0,T)$ is the first time such that

$$I(w(t_0)) = 0, \quad (30)$$

$$\mathscr{I}(w(t)) < 0 \text{ for } t \in [0,t_0). \quad (31)$$

We first introduce an auxiliary function,

$$M(t) = \|w\|^2, \quad (32)$$

and directly

$$M'(t) = (\partial_t w, w) + (w, \partial_t w) = 2(\partial_t w, w), \quad (33)$$

$$M''(t) = 2\left(\partial_t^2 w, w\right) + 2\left\|\partial_t w\right\|^2. \quad (34)$$

Multiplying (1) 1 by w and then by integration over $\mathbb{R}^n$, we have

$$\begin{aligned} &(\partial)t^2 w, w + (w,w)W^{r,2}0(\Omega) + (\partial t)w, wW^{r,2}0(\Omega)\\ &= \int \Omega w^{p-2} w\mathrm{d}x, \end{aligned} \quad (35)$$

so that

$$\left(\partial t^2 w, w\right) = -\|w\|_{W_0^{r,2}(\Omega)} - (\partial_t w, w)_{W_0^{r,2}(\Omega)} + \int_\Omega w|w|^{p-2} w\mathrm{d}x. \quad (36)$$

Substituting (27) into (36), we obtain

$$M''(t) = 2\|\partial_t w\|^{r,2}_{W}0(\Omega) - 2w_{W^{r,2}}0(\Omega) - 2\partial_t w, w)_{W^{r,2}}0(\Omega) + 2\int_\Omega w|w^{p-2}w\mathrm{d}x. \tag{37}$$

By (27), we have

$$M''(t) \geq 2\|\partial_t w\|^2 - 2\mathcal{J}(w). \tag{38}$$

By (31), we have $M''(t) > 0$ for any $t \in [0, t_0)$; then, $M'(t)$ is strictly increasing on $[0, t_0)$. Thus

$$M'(t) > M'(0) > 0 \text{ for } t \in [0, t_0). \tag{39}$$

We have $M(t)$ is also strictly increasing on $[0, t_0)$. We have

$$M(t) > M(0) \geq \frac{2p}{p-2} K\mathcal{E}(0) \text{ for all } t \in [0, t_0). \tag{40}$$

From the continuity of w at $t = t_0$, it follows that

$$M(t_0) = \|w(t_0)\|^2 > 2\frac{p}{p-2}K\mathcal{E}(0). \tag{41}$$

On the other hand,

$$\begin{aligned}\mathcal{E}(0) \geq \mathcal{E}(t) &= \frac{1}{2}\|\partial_t w\|_2^2 + \frac{1}{2}\|w\|^2_{W_0^{r,2}(\Omega)} - \frac{1}{p}\|w\|_p^p \\ &= \frac{1}{2}\|\partial_t w\|_2^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\|w\|^2_{W_0^{r,2}(\Omega)} + \frac{1}{p}\mathcal{J}(w).\end{aligned} \tag{42}$$

Together with (30) and Lemma 2, we get

$$\begin{aligned}\mathrm{E}(0) &\geq \frac{1}{2}\|\partial_t w(., t_0)\|_2^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\|w(., t_0)\|^2_{W_0^{r,2}(\Omega)} \\ &\geq \frac{1}{2}\|\partial_t w(., t_0)\|_2^2 + \frac{p-2}{2p}K^{-1}\|w(., t_0)\|_2^2 \\ &\geq \frac{p-2}{2p}K^{-1}\|w(., t_0)\|_2^2,\end{aligned} \tag{43}$$

which contradicts (41). This completes the proof. □

We are now ready to prove the finite time blow-up of solution to (1) when $\mathcal{E}(0) > 0$.

Definition 2. We say that the function $w(x, t)$ blows up in finite time if there exists $t^* \in (0, \infty)$ such that

$$\|w(x, t)\|_{L^2(\Omega)} \longrightarrow \infty \, as \, t \longrightarrow t^*. \tag{44}$$

Theorem 1. *Let $w_0 \in W_0^{r,2}(\Omega)$ and $w_1 \in L^2(\Omega)$. Assume that $w_0 \in \mathcal{M}$, $\mathcal{E}(0) > 0$, and $\int_\Omega w_0 w_1 dx > 0$, then any solution of (1) blows-up in finite time.*

Proof. We have $w \in \mathcal{M}$; arguing by contradiction, we suppose that w is weak global solution , for any $t \in [0, \infty)$. From (34) and Cauchy–Schwarts inequality, we get

$$M'^2(t) = 4(w, \partial_t)^2 \leq \|w\|^2\|\partial_t w\|^2, \quad t \in [0, \infty), \tag{45}$$

which together with (36) implies that

$$\begin{aligned}&M''(t)M(t) - (1+\alpha)(M'(t))^2 \\ &\geq \left(2\|\partial_t w\|^2 - 2\|w\|^2_{{}_0^{r,2}(\Omega)} + 2\int_\Omega w|w|^{p-2}wdx\right)M(t) - 4(1+\alpha)(w, \partial_t w)^2 \\ &\geq \left(2\|\partial_t w\|^2 - 2\|w\|^2_{{}_0^{r,2}(\Omega)} + 2\int_\Omega w|w|^{p-2}wdx\right)M(t) - 4(1+\alpha)\|w\|^2\|\partial_t w\|^2 \\ &= \left(-2\|w\|^2_{{}_0^{r,2}(\Omega)} + 2\int_\Omega w|w|^{p-2}wdx - 2(1+2\alpha)\|\partial_t w\|^2\right)M(t) = A(t)M(t),\end{aligned} \tag{46}$$

where $\alpha > 0$. We notice that

$$\begin{aligned}A(t) &:= -2\|w\|^2_{W_0^{r,2}(\Omega)} + 2\|w\|_p^p - 2(1+2\alpha)\|\partial_t w\|^2 \\ &\geq -2(1+2\alpha)\|\partial_t w\|^2 - 2\|w\|^2_{W_0^{r,2}(\Omega)} + p\|\partial_t w\|^2 + p\|w\|^2_{W_0^{r,2}(\Omega)} - 2p\mathrm{E}(0) \\ &= -(4\alpha - p + 2)\|\partial_t w\|^2 + (p-2)\|w\|^2_{W_0^{r,2}(\Omega)} - 2p\mathrm{E}(0) \\ &\geq -(4\alpha - p + 2)\|\partial_t w\|^2 + (p-2)K^{-1}\|w\|_2^2 - 2p\mathrm{E}(0),\end{aligned} \tag{47}$$

for $t \in [t, \infty)$. Set $\alpha = (p-2)/4 > 0$, and then $(4\alpha - p + 2)\|\partial_t w\|^2 = 0$. So from (47), we get

$$A(t) \geq K^{-1}(p-2)\|w\|^2 - 2p\mathscr{E}(0) \geq 0. \tag{48}$$

At this point, by (33)–(48), we obtain

$$M''(t)M(t) - (1+\alpha)\left(M'(t)\right)^2 > 0, \quad t \in [0, \infty), \tag{49}$$

where $\alpha > 0$. This implies that

$$\left(M^{-\alpha}\right)' = -\alpha M^{-\alpha-1}M'(t) < 0, \tag{50}$$

$$\left(M^{-\alpha}\right)'' = -\alpha M^{-\alpha-2}\left(M''(t)M(t) - (1+\alpha)\left(M'(t)\right)^2\right) < 0, \tag{51}$$

for all $t \in [0, \infty)$, which means that the function $M^{-\alpha}$ is concave. Obviously, $M(0) > 0$; then, there must exist a $T > 0$ such that

$$\lim_{t \longrightarrow TM^{-\alpha}} (t) = 0, \tag{52}$$

so that

$$\lim_{t \longrightarrow T^-} M(t) = \infty \quad i.e., \lim_{t \longrightarrow T^-} \|w\|^2 = \infty. \tag{53}$$

Thus, the proof is completed. □

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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