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Research Article

Obtaining Solutions of a Vakhnenko Lattice System by *N*-Fold Darboux Transformation

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Using a suitable gauge transformation matrix, we present a N-fold Darboux transformation for a Vakhnenko lattice system. This transformation preserves the form of Lax pair of the Vakhnenko lattice system. Applying the obtained Darboux transformation, we arrive at an exact solution of the Vakhnenko lattice system.

1. Introduction

Since the beginning of this century, the integrable lattice systems (or lattice soliton systems) have received considerable attention. Many important integrable lattice systems have been studied from the perspective of Mathematics and Physics, for instance, the Ablowitz-Ladik lattice [1], the Toda lattice [2], and the relativistic Toda lattice [3], [4–11].

In the soliton theory, the Darboux transformation is a very effective method for solving soliton equations [12, 13]. Later, it also applies to solving integrable lattice equations [7–9, 14, 15]. In reference [10], a Vakhnenko lattice system is introduced:

$$\begin{cases} p_{n,t} = p_n q_n x_{n-1} - p_n q_{n+1} x_n - r_n x_{n-1} + r_n x_n - q_n y_n + q_{n+1} y_n, \\ q_{n,t} = q_n r_{n-1} y_n - q_n q_{n+1} x_n + q_{n+1} z_n - p_n q_n - r_{n-1} + r_n, \\ r_{n,t} = r_n r_{n-1} y_n - q_{n+1} r_n x_n - p_n r_{n-1} + r_n z_n - q_n + q_{n+1} \\ x_{n,t} = q_n x_n x_{n-1} - r_n x_n y_{n+1} - x_{n-1} z_n + p_n x_n - y_n + y_{n+1}, \\ y_{n,t} = q_n x_{n-1} y_n - r_n y_n y_{n+1} + p_n y_{n+1} - y_n z_n - x_{n-1} + x_n, \\ z_{n,t} = r_{n-1} y_n z_n - r_n y_{n+1} z_n - r_{n-1} x_n + r_n x_n - q_n y_n + q_n y_{n+1}, \end{cases} \tag{1}$$

where $p_n = p(n,t), q_n = q(n,t), r_n = r(n,t), x_n = x(n,t)$, $y_n = y(n,t), z_n = z(n,t)$, n is a discrete variable, $n \in N$, and t is a continuous variable, $t \in R$. Equation (1) can be rewritten as a discrete zero curvature equation

$$U_{n,t} = E(V_n)U_n - U_nV_n = V_{n+1}U_n - U_nV_n, \quad (2)$$

of a discrete spatial spectral problem

$$E\varphi_n = U_n(u_n,\lambda)\varphi_n, \quad U_n(u_n,\lambda) = \begin{pmatrix} \lambda^2 + p_n & \lambda q_n + \frac{1}{\lambda}r_n \\ \lambda x_n + \frac{1}{\lambda}y_n & z_n + \frac{1}{\lambda^2} \end{pmatrix}, \quad (3)$$

and a corresponding continuous time evolution equation

$$\varphi_{n,t} = V_n(u_n,\lambda)\varphi_n, \quad V_n(u_n,\lambda) = \begin{pmatrix} \lambda^2 - q_nx_{n-1} & q_n\lambda + r_{n-1}\frac{1}{\lambda} \\ \lambda q_{n-1} + y_n\frac{1}{\lambda} & \frac{1}{\lambda^2} - r_{n-1}y_n \end{pmatrix}. \quad (4)$$

Here, for a lattice function $f_n = f(n)$, the shift operator E and the inverse of E are defined as follows.

$$\begin{aligned} Ef_n &= f_{n+1}, \\ E^{-1}f_n &= f_{n-1}, \quad n \in Z. \end{aligned} \quad (5)$$

In equations (3) and (4), $(\varphi_n^{(1)}, \varphi_n^{(2)})^T$ is the eigenfunction vector, $(p_n, q_n, r_n, x_n, y_n, z_n)^T$ is a potential vector, (1) is a very meaningful lattice system, and many important lattice systems can be reduced from it, such as the nonlinear self-dual network equation, the two coupled discrete nonlinear Schrödinger equation, and the relativistic Volterra lattice [9]. In reference [9], the authors discussed the Darboux transformation of (1), but their results are incorrect [11]. In reference [10], although the author gave some properties of the Darboux transformation of (1), but his approach is different from ours. In reference [11], we derived a 1-fold Darboux transformation of (1). In addition, let $p_n = z_n = w_n$, $q_n = u_n, r_n = v_n, x_n = v_n, y_n = u_n$, and the equation (1) is reduced to a three-component differential-difference system.

$$\begin{cases} u_{n,t} = (u_n)^2 v_{n-1} - u_nu_{n+1}v_n + u_{n+1}w_n - u_nv_n - v_{n-1} + v_n, \\ v_{n,t} = u_nv_{n-1}v_n - u_{n+1}(v_n)^2 - w_nv_{n-1} + v_nw_n - u_n + u_{n+1}, \\ w_{n,t} = u_nv_{n-1}w_n - u_{n+1}v_nw_n - v_{n-1}v_n + (v_n)^2 - (u_n)^2 + u_nu_{n+1}. \end{cases} \quad (6)$$

In reference [14], its N-fold Darboux transformation is presented. Furthermore, if we set $p_n = z_n = I_nV_n, q_n = I_n$, $r_n = V_n, x_n = I_n, y_n = V_n$, equation (1) becomes the nonlinear self-dual network equation.

$$\begin{cases} I_{n,t} = \left(1 + (I_n)^2\right)\left((V_{n-1}) - V_n\right), \\ V_{n,t} = \left(1 + (V_n)^2\right)\left(I_n - I_{n+1}\right). \end{cases} \quad (7)$$

In reference [15], the author derived its N-fold Darboux transformation. In this letter, for arbitrary positive integer N, we will present a N-fold Darboux transformation for the Vakhnenko lattice system (1). Finally, an exact solution of (1) is derived.

2. *N*-Fold Darboux Transformation

For any positive integer N, we introduce the following matrix:

$$\Pi_n^{(N)}(\lambda) = \begin{pmatrix} \lambda^{2N} + \sum_{j=0}^{N-1} A_n^{(2j)}\lambda^{2j} + \sum_{j=1}^{N} A_n^{(-2j)}\lambda^{-2j} & \sum_{j=1}^{N} B_n^{(2j-1)}\lambda^{2j-1} + \sum_{j=1}^{N} C_n^{(-2j+1)}\lambda^{-2j+1} \\ \sum_{j=1}^{N} F_n^{(2j-1)}\lambda^{2j-1} + \sum_{j=1}^{N} G_n^{(-2j+1)}\lambda^{-2j+1} & \sum_{j=1}^{N} H_n^{(2j)}\lambda^{2j} + \sum_{j=0}^{N-1} H_n^{(-2j)}\lambda^{-2j} + \lambda^{-2N} \end{pmatrix}. \quad (8)$$

Here, $A_n^{(2j)}\,(-N \le j \le N-1), B_n^{(2j-1)}\,(-N \le j \le N)$, $C_n^{(2j-1)}\,(-N \le j \le N), F_n^{(2j-1)}\,(-N \le j \le N)$, $G_n^{(2j)}\,(-N \le j \le N), H_n^{(2j)}\,(-N+1 \le j \le N)$ are $8N$ undetermined constants.

Next, we consider the gauge transformation [7, 8].

$$\widehat{\phi}_n = \Pi_n^{(N)}(\lambda)\phi_n. \quad (9)$$

By the transformation (9), the Lax pairs (3) and (4) become

$$E\widehat{\phi}_n = U_n^{(\mathrm{New})}\widehat{\phi}_n, \quad (10)$$

$$\widehat{\phi}_{n,t} = V_n^{(\mathrm{New})}\widehat{\phi}_n, \quad (11)$$

$$U_n^{(\mathrm{New})} = \Pi_{n+1}^{(N)}(\lambda)U_n\left(\Pi_n^{(N)}(\lambda)\right)^{-1}, \quad (12)$$

$$V_n^{(\mathrm{New})} = \left(\Pi_{n,t}^{(N)}(\lambda) + \Pi_n^{(N)}(\lambda)V_n\right)\left(\Pi_n^{(N)}(\lambda)\right)^{-1}. \quad (13)$$

Equations (10) and (11) constitute a new Lax pair. Let us denote

$$\left(\xi_n^{(1)}(\lambda), \xi_n^{(2)}(\lambda)\right)^T, \quad \cdot \left(\eta_n^{(1)}(\lambda), \eta_n^{(2)}(\lambda)\right)^T, \tag{14}$$

are two real linear independent solutions of equations (3) and (4), and $\lambda_i\,(1 \le i \le 4N)$ are $4N$ distinct eigenvalues of spectral problem (1).

Proposition 1. *The matrix $U_n^{(New)}$ has the same form as U_n in equation (3), and the transformation formula is presented by*

$$\begin{cases} P_n = A_{n+1}^{(2N-2)} - A_n^{(2N-2)} + p_n + x_n B_n^{(2N-1)} - F_n^{(2N-1)}\left(\dfrac{q_n - B_n^{(2N-1)}}{H_n^{(2N)}}\right), \\ Q_n = \dfrac{q_n - B_n^{(2N-1)}}{H_n^{(2N)}}, \\ R_n = r_n A_{n+1}^{(-2N)} + C_{n+1}^{(-2N+1)}, \\ X_n = x_n H_{n+1}^{(2N)} + F_{n+1}^{(2N-1)}, \\ Y_n = \dfrac{y_n - G_n^{(-2N+1)}}{A_n^{(-2N)}}, \\ Z_n = z_n + H_{n+1}^{(-2N+2)} - H_n^{(-2N+2)} + r_n G_{n+1}^{(+2N-1)} - C_n^{(-2N-1)}\left(\dfrac{y_n - G_n^{(-2N+1)}}{A_n^{(-2N)}}\right). \end{cases} \tag{15}$$

Namely,

$$U_n^{(\mathrm{New})} = \begin{pmatrix} \lambda^2 + P_n & \lambda Q_n + \dfrac{1}{\lambda} R_n \\ \lambda X_n + \dfrac{1}{\lambda} Y_n & Z_n + \dfrac{1}{\lambda^2} \end{pmatrix}. \tag{16}$$

In the above matrix, $P_n, Q_n, R_n, X_n, Y_n, Z_n$ are determined by (15), and they are all independent λ. Obviously, (15) transform the old potentials $(p_n, q_n, r_n, x_n, y_n, z_n)^T$ of (1) into the new potentials $(P_n, Q_n, R_n, X_n, Y_n, Z_n)^T$ of (9).

Proof. We consider the following linear system:

$$\begin{cases} \displaystyle\sum_{j=0}^{N-1} A_n^{(2j)} \lambda_i^{2j} + \sum_{j=1}^{N} A_n^{(-2j)} \lambda_i^{-2j} \\ \displaystyle +\left(\sum_{j=0}^{N-1} B_n^{(2j+1)} \lambda_i^{2j+1} + \sum_{j=0}^{N-1} C_n^{(-2j-1)} \lambda_i^{-2j-1}\right)\rho_i(n) = -\lambda_i^{2N}, \\ \displaystyle\sum_{j=0}^{N-1} F_n^{(2j+1)} \lambda_i^{2j+1} + \sum_{j=0}^{N-1} G_n^{(-2j-1)} \lambda_i^{-2j-1} \\ \displaystyle +\left(\sum_{j=1}^{N} H_n^{(2j)} \lambda_i^{2j} + \sum_{j=0}^{N-1} H_n^{(-2j)} \lambda_i^{-2j}\right)\rho_i(n) = -\rho_i(n)\lambda_i^{-2N}, \end{cases} \quad i = 1, \ldots 4N. \tag{17}$$

Here,

$$\rho_i[n] = \frac{\eta_n^{(2)}(\lambda_i) + \zeta_i \xi_n^{(2)}(\lambda_i)}{\eta_n^{(1)}(\lambda_i) + \zeta_i \xi_n^{(1)}(\lambda_i)}, \quad i = 1, 2, \ldots, 4N, \tag{18}$$

where $\lambda_i\,(1 \le i \le 4N)$, $\zeta_i\,(1 \le i \le 4N)$ are suitably chosen, such that all the determinants of coefficients for the equation and (17) are nonzero. By solving the linear system (17), we get $A_n^{(2j)}\,(-N \le j \le N-1)$, $B^{(2j+1)}\,(0 \le N-1)$,

$C_n^{(-2j-1)}\,(0\le j\le N-1), F_n^{(2j+1)}\,(0\le j\le N-1), G_n^{(-2j}-1)\,(0\le j\le N-1), H_n^{(2j)}\,(-N+1\le j\le N)$.

From equations (3) and (18), we have

$$\rho_i[n+1]=\frac{x_n\lambda_i+(1/\lambda_i)y_n+z_n\rho_i(n)+\left(1/\lambda_i^2\right)\rho_i(n)}{\lambda_i^2+p_n+q_n\lambda_i\rho_i(n)+(1/\lambda_i)r_n\rho_i(n)},\quad i=1,2. \tag{19}$$

For convenience, we set

$$\Pi_n^{(N)}(\lambda)=\begin{pmatrix}\Theta_n^{(a)} & \Theta_n^{(b)}\\ \Theta_n^{(c)} & \Theta_n^{(d)}\end{pmatrix}, \tag{20}$$

where

$$\begin{aligned}
\Theta_n^{(a)}&=\lambda^{2N}+\sum_{j=1}^{N-1}A_n^{(2j)}\lambda^{2j}+\sum_{j=1}^{N}A_n^{(-2j)}\lambda^{-2j},\\
\Theta_n^{(b)}&=\sum_{j=0}^{N-1}B_n^{(2j+1)}\lambda^{2j+1}+\sum_{j=0}^{N-1}C_n^{(-2j-1)}\lambda^{-2j-1},\\
\Theta_n^{(c)}&=\sum_{j=0}^{N-1}F_n^{(2j+1)}\lambda^{2j+1}+\sum_{j=0}^{N-1}G_n^{(-2j-1)}\lambda^{-2j-1},\\
\Theta_n^{(d)}&=\sum_{j=1}^{N-1}A_n^{(2j)}\lambda^{2j}+\sum_{j=1}^{N}A_n^{(-2j)}\lambda^{-2j}+\lambda^{-2N}.
\end{aligned} \tag{21}$$

Then, we obtain the following equation:

$$\Pi_{n+1}^{(N)}U_n\left(\Pi_n^{(N)}\right)^*=\begin{pmatrix}\Omega_{11}(\lambda,n) & \Omega_{12}(\lambda,n)\\ \Omega_{21}(\lambda,n) & \Omega_{22}(\lambda,n)\end{pmatrix}, \tag{22}$$

where $(\Pi_n^{(N)})^*$ is the adjoint matrix of $\Pi_n^{(N)}$ and

$$\begin{aligned}
\Omega_{11}(\lambda,n)&=\Theta_{n+1}^{(a)}\Theta_n^{(d)}\lambda^2+\left(x_n\Theta_{n+1}^{(b)}\Theta_n^{(d)}-q_n\Theta_{n+1}^{(a)}\Theta_n^{(c)}\right)\\
&\quad\cdot\lambda+\left(p_n\Theta_{n+1}^{(a)}\Theta_n^{(d)}-z_n\Theta_{n+1}^{(b)}\Theta_n^{(c)}\right)+\left(y_n\Theta_{n+1}^{(b)}\Theta_n^{(d)}-r_n\Theta_{n+1}^{(a)}\Theta_n^{(c)}\right)\\
&\quad\cdot\lambda^{-1}-\Theta_{n+1}^{(b)}\Theta_n^{(c)}\lambda^{-2},\\
\Omega_{12}(\lambda,n)&=-\Theta_{n+1}^{(a)}\Theta_n^{(b)}\lambda^2+\left(q_n\Theta_{n+1}^{(b)}\Theta_n^{(b)}-x_n\Theta_{n+1}^{(a)}\Theta_n^{(a)}\right)\\
&\quad\cdot\lambda+\left(z_n\Theta_{n+1}^{(b)}\Theta_n^{(a)}-p_n\Theta_{n+1}^{(a)}\Theta_n^{(b)}\right)+\left(r_n\Theta_n^{(a)}\Theta_{n+1}^{(a)}-y_n\Theta_{n+1}^{(b)}\Theta_n^{(b)}\right)\\
&\quad\cdot\lambda^{-1}+\Theta_{n+1}^{(b)}\Theta_n^{(a)}\lambda^{-2},\\
\Omega_{21}(\lambda,n)&=\Theta_{n+1}^{(c)}\Theta_n^{(d)}\lambda^2+\left(x_n\Theta_{n+1}^{(d)}\Theta_n^{(d)}-q_n\Theta_{n+1}^{(c)}\Theta_n^{(c)}\right)\\
&\quad\cdot\lambda+\left(p_n\Theta_{n+1}^{(c)}\Theta_n^{(d)}-z_n\Theta_{n+1}^{(d)}\Theta_n^{(c)}\right)+\left(p_n\Theta_{n+1}^{(c)}\Theta_n^{(d)}-z_n\Theta_{n+1}^{(d)}\Theta_n^{(c)}\right)\\
&\quad\cdot\lambda^{-1}-\Theta_{n+1}^{(b)}\Theta_{n+1}^{(c)}\lambda^{-2},\\
\Omega_{22}(\lambda,n)&=-\Theta_n^{(a)}\Theta_n^{(d)}\lambda^2+\left(r_n\Theta_{n+1}^{(c)}\Theta_n^{(a)}-y_n\Theta_{n+1}^{(d)}\Theta_n^{(b)}\right)\lambda\\
&\quad+\left(z_n\Theta_{n+1}^{(d)}\Theta_n^{(a)}-p_n\Theta_n^{(b)}\Theta_{n+1}^{(c)}\right)+\left(r_n\Theta_{n+1}^{(c)}\Theta_n^{(a)}-y_n\Theta_{n+1}^{(d)}\Theta_n^{(b)}\right)\\
&\quad\cdot\lambda^{-1}+\Theta_{n+1}^{(d)}\Theta_n^{(a)}\lambda^{-2},
\end{aligned} \tag{23}$$

In the light of (17), we can get

$$\mathrm{Det}\left[\Pi_n^{(N)}(\lambda)\right]=H_n^{(2N)}\prod_{j=1}^{4N}\left(\lambda^2-\lambda_j^2\right). \tag{24}$$

Analyzing the coefficients of powers of λ in (22), we obtain that

$$\Pi_{n+1}^{(N)}(\lambda)U_n=\alpha_n\Pi_n^{(N)}(\lambda). \tag{25}$$

Here,

$$\alpha_n=\begin{pmatrix}\alpha_n^{(a,2)}\lambda^2+\alpha_n^{(a,0)} & \alpha_n^{(b,1)}\lambda+\dfrac{\alpha_n^{(b,-1)}}{\lambda}\\ \alpha_n^{(c,1)}\lambda+\dfrac{\alpha_n^{(c,-1)}}{\lambda} & \alpha_n^{(d,0)}+\dfrac{\alpha_n^{(d,-2)}}{\lambda^2}\end{pmatrix}, \tag{26}$$

where $\alpha_n^{(a,2)},\alpha_n^{(a,0)},\alpha_n^{(b,1)},\alpha_n^{(b,-1)},\alpha_n^{(c,1)},\alpha_n^{(c,-1)},\alpha_n^{(d,0)},\alpha_n^{(d,-2)}$ are all independent of λ. By comparing the coefficients of λ in (25), we have

$$
\begin{aligned}
\alpha_n^{(a,2)} &= 0,\\
\alpha_n^{(a,2)} &= A_{n+1}^{(2N-2)} - A_n^{(2N-2)} + p_n\\
+x_n B_n^{(2N-1)} - F_n^{(2N-1)}\left(\frac{q_n - B_n^{(2N-1)}}{H_n^{(2N)}}\right) &= P_n,\\
\alpha_n^{(b,1)} &= \frac{q_n - B_n^{(2N-1)}}{H_n^{(2N)}} = Q_n,\\
\alpha_n^{(b,-1)} &= r_n A_{n+1}^{(-2N)} + C_{n+1}^{(-2N+1)} = R_n,\\
\alpha_n^{(c,1)} &= x_n H_{n+1}^{(2N)} + F_{n+1}^{(2N-1)} = X_n,\\
\alpha_n^{(c,-1)} &= \frac{y_n - G_n^{(-2N+1)}}{A_n^{(-2N)}} = Y_n,\\
\alpha_n^{(d,-2)} &= 1,\\
\alpha_n^{(d,0)} &= z_n + H_{n+1}^{(-2N+2)} - H_n^{(-2N+2)} + r_n G_{n+1}^{(-2N+1)}\\
-C_n^{(-2N+1)}\left(\frac{y_n - G_n^{(-2N+1)}}{A_n^{(-2N)}}\right) &= Z_n.
\end{aligned}
\tag{27}
$$

The proposition is proved. □

Proposition 2. *Under the transformation (15), the matrix* $V_n^{(New)}$ *has the same form as* V_n *in equation (2). In other words,*

$$
V^{(\mathrm{New})} = \begin{pmatrix} \lambda^2 - Q_n X_{n-1} & Q_n\lambda + R_{n-1}\dfrac{1}{\lambda} \\ \lambda Q_{n-1} + Y_n\dfrac{1}{\lambda} & \dfrac{1}{\lambda^2} - R_{n-1}Y_n \end{pmatrix}. \tag{28}
$$

Proof. We consider

$$
\left(\Pi_{n,t}^{(N)} + \Pi_n^{(N)} V_n\right)\left(\Pi_n^{(N)}\right)^* = \begin{pmatrix} \Gamma_{11}(\lambda,n) & \Gamma_{12}(\lambda,n) \\ \Gamma_{21}(\lambda,n) & \Gamma_{22}(\lambda,n) \end{pmatrix}, \tag{29}
$$

where

$$
\begin{aligned}
\Gamma_{11}(\lambda,n) &= \Theta_n^{(a)}\Theta_n^{(d)}\lambda^2 + x_{n-1}\Theta_n^{(b)}\Theta_n^{(d)}\lambda + \Theta_{n,t}^{(a)}\Theta_n^{(d)}\\
&\quad - x_{n-1}q_n\Theta_n^{(a)}\Theta_n^{(d)} + q_n\Theta_n^{(b)}\Theta_n^{(d)}\lambda^{-1},\\
\Gamma_{12}(\lambda,n) &= -q_n\Theta_n^{(a)}\Theta_n^{(b)}\lambda - \Theta_n^{(b)}\Theta_{n,t}^{(b)} + r_{n-1}y_n\left(\Theta_n^{(b)}\right)^2\\
&\quad - r_{n-1}\Theta_n^{(a)}\Theta_n^{(b)}\lambda^{-1} - \left(\Theta_n^{(b)}\right)^2\lambda^{-2},\\
\Gamma_{21}(\lambda,n) &= -\left(\Theta_n^{(c)}\right)^2\lambda^2 - q_n\Theta_n^{(a)}\Theta_n^{(c)}\lambda\\
&\quad - \Theta_n^{(c)}\Theta_{n,t}^{(c)} + x_{n-1}q_n\left(\Theta_n^{(c)}\right)^2 - y_n\Theta_n^{(c)}\Theta_n^{(d)}\lambda^{-1},\\
\Gamma_{22}(\lambda,n) &= q_n\Theta_n^{(a)}\Theta_n^{(c)}\lambda - \Theta_n^{(a)}\Theta_{n,t}^{(d)}\\
&\quad - r_{n-1}y_n\Theta_n^{(a)}\Theta_n^{(d)} + r_{n-1}\Theta_n^{(a)}\Theta_n^{(c)}\lambda^{-1} + \Theta_n^{(a)}\Theta_n^{(d)}\lambda^{-2}.
\end{aligned}
\tag{30}
$$

Owing to $\mathrm{Det}[\Pi_n^{(N)}] = H_n^{(2N)} \prod_{i=1}^{N} (\lambda^2 - \lambda_i^2)$ and paying attention to the coefficients of powers of λ in (29), we find

$$\Pi_{n,t}^{(N)} + \Pi_n^{(N)} V_n = \beta_n \Pi_n^{(N)}. \tag{31}$$

In equation (31),

$$\beta_n = \begin{pmatrix} \beta_{11}^{(2)}\lambda^2 + \beta_{11}^{(0)} & \beta_{12}^{(1)}\lambda^1 + \beta_{12}^{(-1)}\lambda^{-1} \\ \beta_{21}^{(1)}\lambda + \beta_{21}^{(-1)}\lambda^{-1} & \beta_{22}^{(0)} + \beta_{22}^{(-2)}\lambda^{-2} \end{pmatrix}, \tag{32}$$

where $\beta_{11}^{(0)}, \beta_{11}^{(2)}, \beta_{12}^{(-1)}, \beta_{12}^{(1)}, \beta_{21}^{(-1)}, \beta_{12}^{(1)}, \beta_{22}^{(0)}, \beta_{22}^{(-2)}$ are all independent of spectral parameter λ. Comparing the coefficients of λ^{N+i} $(-2 \le i \le 2)$ in equation (31), we arrive at

$$\begin{aligned}
\beta_{11}^{(2)} &= 1, \\
\beta_{11}^{(0)} &= -\frac{q_n - B_n^{(2N-1)}}{H_n^{(2N)}} \left(x_{n-1} H_n^{(2N)} + F_n^{(2N-1)} \right) = -Q_n X_{n-1}, \\
\beta_{12}^{(1)} &= \frac{q_n - B_n^{(2N-1)}}{H_n^{(2N)}} = Q_n, \\
\beta_{12}^{(-1)} &= \left(x_{n-1} H_n^{(2N)} + F_n^{(2N-1)} \right) R_{n-1}, \\
\beta_{21}^{(1)} &= \frac{q_{n-1} - B_{n-1}^{(2N-1)}}{H_{n-1}^{(2N)}} \\
\beta_{12}^{(-1)} &= \frac{y_n - G_n^{(-2N+1)}}{A_n^{(-2N)}} = Y_n, \\
\beta_{22}^{(-2)} &= 1, \\
\beta_{22}^{(0)} &= -\left(r_{n-1} A_n^{(-2N)} + C_n^{(-2N+1)} \right) \left(\frac{y_n - G_n^{(-2N+1)}}{A_n^{(-2N)}} \right) = -R_{n-1} Y_n.
\end{aligned} \tag{33}$$

The proposition is proved.

In summary, we get the following theorem. □

Theorem 1. *The equations (9) and (15) constitute a Darboux transformation of (1), that is, from an old solution* $(p_n, q_n, r_n, x_n, y_n, z_n)^T$ *of (1), through transformation (15), a new solution* $(P_n, Q_n, R_n, X_n, Y_n, Z_n)^T$ *of (1) is derived.*

3. An Exact Solution

In what follows, we will derive a solution of equation (1) by the Darboux transformation (15). For simplicity, we consider the case of $N = 1$.

First, we select the seed solution of the lattice system (1), namely, the simple special solution, $(p_n, q_n, r_n, x_n, y_n, z_n)^T = (1, 0, 0, 0, 0, 1)^T$. Substituting this solution into the corresponding Lax pair, we get

$$E\phi_n = \begin{pmatrix} \lambda^2 + 1 & 0 \\ 0 & 1 + \dfrac{1}{\lambda^2} \end{pmatrix} \phi_n, \quad \phi_{n_t} = \begin{pmatrix} \dfrac{-\lambda^2}{2} & -\lambda \\ \lambda & \dfrac{\lambda^2}{2} + 1 \end{pmatrix} \phi_n. \tag{34}$$

Solving the above two equations, we get two real linear independent solutions:

$$\begin{aligned}
\xi_n(\lambda) &= \exp\left(\lambda^2 t\right) \begin{pmatrix} \left(1 + \lambda^2\right)^{n-1} \\ 2\left(1 + \dfrac{1}{\lambda^2}\right)^{n-1} \end{pmatrix}, \\
\eta_n(\lambda) &= \exp\left(\lambda^2 t\right) \begin{pmatrix} 2\left(1 + \lambda^2\right)^{n-1} \\ \left(1 + \dfrac{1}{\lambda^2}\right)^{n-1} \end{pmatrix}.
\end{aligned} \tag{35}$$

Then, we have

$$\begin{aligned}
\rho_i[n] &= \frac{\xi_n^{(2)}(\lambda_i) + \gamma_i \eta_n^{(2)}(\lambda_i)}{\xi_n^{(1)}(\lambda_i) + \gamma_i \eta_n^{(1)}(\lambda_i)} \\
&= \lambda_i^{2n-2} \frac{1 + 2\gamma_i}{2 + \gamma_i}, \quad i = 1, 2, 3, 4.
\end{aligned} \tag{36}$$

Here, λ_i $(1 \le i \le 4)$ are four different arbitrary constants. When $N = 1$,

$$\Pi_n^{(1)}(\lambda) = \begin{pmatrix} \lambda^2 + A_n^{(0)} + A_n^{(-2)}\lambda^{-2} & B_n^{(1)}\lambda + C_n^{(-1)}\lambda^{-1} \\ F_n^{(1)}\lambda + G_n^{(-1)}\lambda^{-1} & H_n^{(2)}\lambda^2 + H_n^{(0)} + \lambda^{-2} \end{pmatrix}. \tag{37}$$

According to Proposition 1, we can get the following Darboux transformation:

$$\widehat{\phi}_n = \Pi_n^{(1)}(\lambda)\phi_n,$$

$$\cdot \begin{cases} P_n = A_{n+1}^{(0)} - A_n^{(0)} + p_n + x_n B_n^{(1)} - F_n^{(1)}\left(\dfrac{q_n - B_n^{(1)}}{H_n^{(2)}}\right), \\ Q_n = \dfrac{q_n - B_n^{(1)}}{H_n^{(2)}}, \\ R_n = r_n A_{n+1}^{(-2)} + C_{n+1}^{(-1)}, \\ X_n = x_n H_{n+1}^{(2)} + F_{n+1}^{(1)}, \\ Y_n = \dfrac{y_n - G_n^{(-1)}}{A_n^{(-2)}}, \\ Z_n = z_n + H_{n+1}^{(0)} - H_n^{(0)} + r_n G_{n+1}^{(-1)} - C_n^{(-1)}\left(\dfrac{y_n - G_n^{(-1)}}{A_n^{(-2)}}\right). \end{cases} \tag{38}$$

Here,

$$A_n^{(0)} = -\frac{\operatorname{Det}\begin{pmatrix} \lambda_1^2 & \lambda_1^{-2} & \lambda_1\rho_1[n] & \lambda_1^{-1}\rho_1[n] \\ \lambda_2^2 & \lambda_2^{-2} & \lambda_2\rho_2[n] & \lambda_2^{-1}\rho_2[n] \\ \lambda_3^2 & \lambda_3^{-2} & \lambda_3\rho_3[n] & \lambda_3^{-1}\rho_3[n] \\ \lambda_4^2 & \lambda_4^{-2} & \lambda_4\rho_4[n] & \lambda_4^{-1}\rho_1[n] \end{pmatrix}}{\operatorname{Det}\begin{pmatrix} 1 & \lambda_1^{-2} & \lambda_1\rho_1[n] & \lambda_1^{-1}\rho_1[n] \\ 1 & \lambda_2^{-2} & \lambda_2\rho_2[n] & \lambda_2^{-1}\rho_2[n] \\ 1 & \lambda_3^{-2} & \lambda_3\rho_3[n] & \lambda_3^{-1}\rho_3[n] \\ 1 & \lambda_4^{-2} & \lambda_4\rho_4[n] & \lambda_4^{-1}\rho_1[n] \end{pmatrix}},$$

$$A_n^{(-2)} = -\frac{\operatorname{Det}\begin{pmatrix} 1 & \lambda_1^2 & \lambda_1\rho_1[n] & \lambda_1^{-1}\rho_1[n] \\ 1 & \lambda_2^2 & \lambda_2\rho_2[n] & \lambda_2^{-1}\rho_2[n] \\ 1 & \lambda_3^2 & \lambda_3\rho_3[n] & \lambda_3^{-1}\rho_3[n] \\ 1 & \lambda_4^2 & \lambda_4\rho_4[n] & \lambda_4^{-1}\rho_1[n] \end{pmatrix}}{\operatorname{Det}\begin{pmatrix} 1 & \lambda_1^{-2} & \lambda_1\rho_1[n] & \lambda_1^{-1}\rho_1[n] \\ 1 & \lambda_2^{-2} & \lambda_2\rho_2[n] & \lambda_2^{-1}\rho_2[n] \\ 1 & \lambda_3^{-2} & \lambda_3\rho_3[n] & \lambda_3^{-1}\rho_3[n] \\ 1 & \lambda_4^{-2} & \lambda_4\rho_4[n] & \lambda_4^{-1}\rho_1[n] \end{pmatrix}},$$

$$B_n^{(1)} = -\frac{\operatorname{Det}\begin{pmatrix} 1 & \lambda_1^{-2} & \lambda_1^2 & \lambda_1^{-1}\rho_1[n] \\ 1 & \lambda_2^{-2} & \lambda_2^2 & \lambda_2^{-1}\rho_2[n] \\ 1 & \lambda_3^{-2} & \lambda_3^2 & \lambda_3^{-1}\rho_3[n] \\ 1 & \lambda_4^{-2} & \lambda_4^2 & \lambda_4^{-1}\rho_1[n] \end{pmatrix}}{\operatorname{Det}\begin{pmatrix} 1 & \lambda_1^{-2} & \lambda_1\rho_1[n] & \lambda_1^{-1}\rho_1[n] \\ 1 & \lambda_2^{-2} & \lambda_2\rho_2[n] & \lambda_2^{-1}\rho_2[n] \\ 1 & \lambda_3^{-2} & \lambda_3\rho_3[n] & \lambda_3^{-1}\rho_3[n] \\ 1 & \lambda_4^{-2} & \lambda_4\rho_4[n] & \lambda_4^{-1}\rho_1[n] \end{pmatrix}},$$

$$C_n^{(-1)} = -\frac{\operatorname{Det}\begin{pmatrix} 1 & \lambda_1^{-2} & \lambda_1\rho_1[n] & \lambda_1^2 \\ 1 & \lambda_2^{-2} & \lambda_2\rho_2[n] & \lambda_2 \\ 1 & \lambda_3^{-2} & \lambda_3\rho_3[n] & \lambda_3^2 \\ 1 & \lambda_4^{-2} & \lambda_4\rho_1[n] & \lambda_4^2 \end{pmatrix}}{\operatorname{Det}\begin{pmatrix} 1 & \lambda_1^{-2} & \lambda_1\rho_1[n] & \lambda_1^{-1}\rho_1[n] \\ 1 & \lambda_2^{-2} & \lambda_2\rho_2[n] & \lambda_2^{-1}\rho_2[n] \\ 1 & \lambda_3^{-2} & \lambda_3\rho_3[n] & \lambda_3^{-1}\rho_3[n] \\ 1 & \lambda_4^{-2} & \lambda_4\rho_4[n] & \lambda_4^{-1}\rho_1[n] \end{pmatrix}},$$

$$F_n^{(1)} = \frac{\operatorname{Det}\begin{pmatrix} \lambda_1^{-1} & \lambda_1^{-2}\rho_1[n] & \lambda_1^2\rho_1[n] & \rho_1[n] \\ \lambda_2^{-1} & \lambda_2^{-2}\rho_2[n] & \lambda_2^2\rho_2[n] & \rho_2[n] \\ \lambda_3^{-1} & \lambda_3^{-2}\rho_3[n] & \lambda_3^2\rho_3[n] & \rho_3[n] \\ \lambda_4^{-1} & \lambda_4^{-2}\rho_4[n] & \lambda_4^2\rho_4[n] & \rho_4[n] \end{pmatrix}}{\operatorname{Det}\begin{pmatrix} \lambda_1 & \lambda_1^{-1} & \lambda_1^2\rho_1[n] & \rho_1[n] \\ \lambda_2 & \lambda_2^{-1} & \lambda_2\rho_2[n] & \rho_2[n] \\ \lambda_3 & \lambda_3^{-1} & \lambda_3^2\rho_3[n] & \rho_3[n] \\ \lambda_4 & \lambda_4^{-1} & \lambda_4^2\rho_4[n] & \rho_1[n] \end{pmatrix}},$$

$$G_n^{(-1)} = \frac{\operatorname{Det}\begin{pmatrix} \lambda_1^{-2}\rho_1[n] & \lambda_1 & \lambda_1^2\rho_1[n] & \rho_1[n] \\ \lambda_2^{-2}\rho_2[n] & \lambda_2 & \lambda_2\rho_2[n] & \rho_2[n] \\ \lambda_3^{-2}\rho_3[n] & \lambda_3 & \lambda_3^2\rho_3[n] & \rho_3[n] \\ \lambda_4^{-2}\rho_4[n] & \lambda_4 & \lambda_4^2\rho_4[n] & \rho_1[n] \end{pmatrix}}{\operatorname{Det}\begin{pmatrix} \lambda_1 & \lambda_1^{-1} & \lambda_1^2\rho_1[n] & \rho_1[n] \\ \lambda_2 & \lambda_2^{-1} & \lambda_2\rho_2[n] & \rho_2[n] \\ \lambda_3 & \lambda_3^{-1} & \lambda_3^2\rho_3[n] & \rho_3[n] \\ \lambda_4 & \lambda_4^{-1} & \lambda_4^2\rho_4[n] & \rho_1[n] \end{pmatrix}},$$

$$
H_n^{(2)} = -\frac{\operatorname{Det}\begin{pmatrix} \lambda_1 & \lambda_1^{-1} & \lambda_1^{-2}\rho_1[n] & \rho_1[n] \\ \lambda_2 & \lambda_2^{-1} & \lambda_2^{-2}\rho_2[n] & \rho_2[n] \\ \lambda_3 & \lambda_3^{-1} & \lambda_3^{-2}\rho_3[n] & \rho_3[n] \\ \lambda_4 & \lambda_4^{-1} & \lambda_4^{-2}\rho_4[n] & \rho_4[n] \end{pmatrix}}{\operatorname{Det}\begin{pmatrix} \lambda_1 & \lambda_1^{-1} & \lambda_1^{2}\rho_1[n] & \rho_1[n] \\ \lambda_2 & \lambda_2^{-1} & \lambda_2\rho_2[n] & \rho_2[n] \\ \lambda_3 & \lambda_3^{-1} & \lambda_3^{2}\rho_3[n] & \rho_3[n] \\ \lambda_4 & \lambda_4^{-1} & \lambda_4^{2}\rho_4[n] & \rho_1[n] \end{pmatrix}},
$$

$$
H_n^{(0)} = -\frac{\operatorname{Det}\begin{pmatrix} \lambda_1 & \lambda_1^{-1} & \lambda_1^{2}\rho_1[n] & \lambda_1^{-2}\rho_1[n] \\ \lambda_2 & \lambda_2^{-1} & \lambda_2\rho_2[n] & \lambda_2^{-2}\rho_2[n] \\ \lambda_3 & \lambda_3^{-1} & \lambda_3^{2}\rho_3[n] & \lambda_3^{-2}\rho_3[n] \\ \lambda_4 & \lambda_4^{-1} & \lambda_4^{2}\rho_4[n] & \lambda_4^{-2}\rho_4[n] \end{pmatrix}}{\operatorname{Det}\begin{pmatrix} \lambda_1 & \lambda_1^{-1} & \lambda_1^{2}\rho_1[n] & \rho_1[n] \\ \lambda_2 & \lambda_2^{-1} & \lambda_2\rho_2[n] & \rho_2[n] \\ \lambda_3 & \lambda_3^{-1} & \lambda_3^{2}\rho_3[n] & \rho_3[n] \\ \lambda_4 & \lambda_4^{-1} & \lambda_4^{2}\rho_4[n] & \rho_1[n] \end{pmatrix}}. \tag{39}
$$

Thus, we can obtain a solution of equation (1) as follows:

$$
\begin{cases} P_n = 1 + A_{n+1}^{(0)} - A_n^{(0)} + F_n^{(-1)}\left(\dfrac{B_n^{(1)}}{H_n^{(2)}}\right), \\ Q_n = -\dfrac{B_n^{(1)}}{H_n^{(2)}}, \\ R_n = C_{n+1}^{(-1)}, \\ X_n = F_{n+1}^{(1)}, \\ Y_n = -\dfrac{G_n^{(-1)}}{A_n^{(-2)}}, \\ Z_n = 1 + H_{n+1}^{(0)} - H_n^{(0)} + C_n^{(-1)}\left(\dfrac{G_n^{(-1)}}{A_n^{(-2)}}\right). \end{cases} \tag{40}
$$

In equation (40),

$$
A_{n+1}^{(0)} = \frac{\operatorname{Det}\begin{pmatrix} \lambda_1^{-2} & \lambda_1^{2} & \lambda_1\rho_1[n+1] & \lambda_1^{-1}\rho_1[n+1] \\ \lambda_2^{-2} & \lambda_2^{2} & \lambda_2\rho_2[n+1] & \lambda_2^{-1}\rho_2[n+1] \\ \lambda_3^{-2} & \lambda_3^{2} & \lambda_3\rho_3[n+1] & \lambda_3^{-1}\rho_3[n+1] \\ \lambda_4^{-2} & \lambda_4^{2} & \lambda_4\rho_4[n+1] & \lambda_4^{-1}\rho_1[n+1] \end{pmatrix}}{\operatorname{Det}\begin{pmatrix} 1 & \lambda_1^{-2} & \lambda_1\rho_1[n+1] & \lambda_1^{-1}\rho_1[n+1] \\ 1 & \lambda_2^{-2} & \lambda_2\rho_2[n+1] & \lambda_2^{-1}\rho_2[n+1] \\ 1 & \lambda_3^{-2} & \lambda_3\rho_3[n+1] & \lambda_3^{-1}\rho_3[n+1] \\ 1 & \lambda_4^{-2} & \lambda_4\rho_4[n+1] & \lambda_4^{-1}\rho_1[n+1] \end{pmatrix}},
$$

$$
C_{n+1}^{(-1)} = \frac{\operatorname{Det}\begin{pmatrix} \lambda_1^{-2} & 1 & \lambda_1\rho_1[n+1] & \lambda_1^{2} \\ \lambda_2^{-2} & 1 & \lambda_2\rho_2[n+1] & \lambda_2 \\ \lambda_3^{-2} & 1 & \lambda_3\rho_3[n+1] & \lambda_3^{2} \\ \lambda_4^{-2} & 1 & \lambda_4\rho_1[n+1] & \lambda_4^{2} \end{pmatrix}}{\operatorname{Det}\begin{pmatrix} 1 & \lambda_1^{-2} & \lambda_1\rho_1[n+1] & \lambda_1^{-1}\rho_1[n+1] \\ 1 & \lambda_2^{-2} & \lambda_2\rho_2[n+1] & \lambda_2^{-1}\rho_2[n+1] \\ 1 & \lambda_3^{-2} & \lambda_3\rho_3[n+1] & \lambda_3^{-1}\rho_3[n+1] \\ 1 & \lambda_4^{-2} & \lambda_4\rho_4[n+1] & \lambda_4^{-1}\rho_1[n+1] \end{pmatrix}},
$$

$$
F_{n+1}^{(1)} = \frac{\operatorname{Det}\begin{pmatrix} \lambda_1^{-1} & \lambda_1^{-2}\rho_1[n] & \lambda_1^{2}\rho_1[n+1] & \rho_1[n+1] \\ \lambda_2^{-1} & \lambda_2^{-2}\rho_2[n] & \lambda_2^{2}\rho_2[n+1] & \rho_2[n+1] \\ \lambda_3^{-1} & \lambda_3^{-2}\rho_3[n] & \lambda_3^{2}\rho_3[n+1] & \rho_3[n+1] \\ \lambda_4^{-1} & \lambda_4^{-2}\rho_4[n] & \lambda_4^{2}\rho_4[n+1] & \rho_4[n+1] \end{pmatrix}}{\operatorname{Det}\begin{pmatrix} \lambda_1 & \lambda_1^{-1} & \lambda_1^{2}\rho_1[n+1] & \rho_1[n+1] \\ \lambda_2 & \lambda_2^{-1} & \lambda_2\rho_2[n+1] & \rho_2[n+1] \\ \lambda_3 & \lambda_3^{-1} & \lambda_3^{2}\rho_3[n+1] & \rho_3[n+1] \\ \lambda_4 & \lambda_4^{-1} & \lambda_4^{2}\rho_4[n+1] & \rho_1[n+1] \end{pmatrix}},
$$

$$
H_{n+1}^{(0)} = \frac{\operatorname{Det}\begin{pmatrix} \lambda_1^{-1} & \lambda_1 & \lambda_1^{2}\rho_1[n+1] & \lambda_1^{-2}\rho_1[n] \\ \lambda_2^{-1} & \lambda_2 & \lambda_2\rho_2[n+1] & \lambda_2^{-2}\rho_2[n] \\ \lambda_3^{-1} & \lambda_3 & \lambda_3^{2}\rho_3[n+1] & \lambda_3^{-2}\rho_3[n] \\ \lambda_4^{-1} & \lambda_4 & \lambda_4^{2}\rho_4[n+1] & \lambda_4^{-2}\rho_4[n] \end{pmatrix}}{\operatorname{Det}\begin{pmatrix} \lambda_1 & \lambda_1^{-1} & \lambda_1^{2}\rho_1[n+1] & \rho_1[n+1] \\ \lambda_2 & \lambda_2^{-1} & \lambda_2\rho_2[n+1] & \rho_2[n+1] \\ \lambda_3 & \lambda_3^{-1} & \lambda_3^{2}\rho_3[n+1] & \rho_3[n+1] \\ \lambda_4 & \lambda_4^{-1} & \lambda_4^{2}\rho_4[n+1] & \rho_1[n+1] \end{pmatrix}}. \tag{41}
$$

Here,

$$\rho_i[n+1] = \frac{\rho_i[n] + \left(1/\lambda_i^2\right)\rho_i[n]}{1+\lambda_i^2}, \quad i = 1,2,3,4. \tag{42}$$

4. Conclusion

In this work, by means of a gauge transformation of Lax pair, we established a N-fold Darboux transformation for a Vakhnenko lattice system. Under this transformation, the structure of the Lax pair remains unchanged. Finally, as an application of this transformation, an exact solution of the Vakhnenko lattice system (1) is given. Starting from the exact solution (40), we apply the Darboux transformation (38) once again; then, another new solution of equation (1) is derived. This process can be performed continually. So, we can get many exact solutions for the lattice system (1).

Data Availability

The data used to support the findings of this study are available from the corresponding author reasonable request.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

Ning Zhang completed computing. Xi-Xiang Xu proposed the problem, drafted the manuscript, read, and approved the final manuscript.

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