Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 8759055, 14 pages
https://doi.org/10.1155/2021/8759055

Research Article

A New Exponential-*X* Family: Modeling Extreme Value Data in the Finance Sector

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Received 22 July 2021; Revised 4 October 2021; Accepted 6 October 2021; Published 21 October 2021

Academic Editor: Amer Al-Omari

In this paper, a family of statistical models, namely, a new exponential-*X* family is proposed. A subcase of the introduced family, called the new exponential-Weibull (NE-Weibull) model, is studied. The NE-Weibull model is very competent and possesses heavy-tailed properties. The maximum likelihood estimators of its parameters are derived. The consistency and efficiency of these estimators are assessed in a brief simulation study. Finally, the effectiveness of the NE-Weibull distribution is illustrated by modeling real insurance claims data. The practical analysis shows that the NE-Weibull distribution outclassed other distributions and it can be a better choice for modeling data in the finance sector.

1. Introduction

The extreme value phenomena such as financial returns and other related events can be modeled effectively by extreme value methods. The heavy-tailed (HT) distributions have been proven to be substantial in modeling HT and extreme value data. Researchers have shown a deep concern in the financial sector to study new HT distributions. Among the applicability of the statistical distributions in the applied area, the HT distributions have received much attention for modeling financial phenomena. Most of the data sets in the financial sector possesses HT behavior with long right tail (see Lane [1]; Cooray and Ananda [2]; Wang et al. [3]; Ahn et al. [4]; Jelenković and Tan [5]; Forbes and Wraith [6]; Guo [7]; Punzo et al. [8]; Bhati and Ravi [9]; Punzo [10]; Ke et al. [11]; Dos Reis et al. [12]; and Niu et al. [13]).

Recently, different copulas and new approaches of introducing HT distribution have been studied due to the importance of HT distributions in the financial sector (for example, Bladt et al. [14]; Tikhomirov [15]; Lugosi and Mendelson [16]; and Yousri et al. [17]). For more information about the usefulness of statistical distributions, one can refer to studies by Ramos et al. [18, 19]; He et al. [20]; Alfaer et al. [21]; Afify et al. [22]; and Al Mutairi et al. [23].

A statistical distribution with SF (survival function), say $\overline{M}(x;\Delta) = 1 - M(x;\Delta)$, is said to be a HT model, if its SF verifies

$$\lim_{x\longrightarrow\infty} \exp(\eta x)[1 - M(x;\Delta)] = \infty, \tag{1}$$

for all $\eta > 0$. For more details, see the study by Resnick [24].

An interesting characteristic of the HT distributions is the regularly varying behavior. A statistical distribution is said to possess the regularly varying behavior, if it satisfies

$$\lim_{x\longrightarrow\infty} \frac{1 - M(qx;\Delta)}{1 - M(x;\Delta)} = q^{b}, \tag{2}$$

where $b \in \{0,\infty\}$, and it is also known as an index of regular variation.

The statistical models possessing such property are very prominent models for modeling HT phenomena in the financial sector [25]. Furthermore, actuaries are very interested in looking for new flexible HT models (see Nadarajah and Bakar [26]; and Ahmad et al. [27]).

Alzaatreh et al. [28] proposed a prominent approach called the T-X family method. Let $u(t)$ represent the PDF (probability density function) of T, where T is a RV (random variable) belonging to $[\delta_1, \delta_2]$ and $-\infty \le \delta_1 < \delta_2 < \infty$. Suppose that $O[H(x;\Delta)]$ is a function of $H(x;\Delta)$ of a RV-X and it has the following conditions:

(i) $O[H(x;\Delta)] \in [\delta_1, \delta_2]$

(ii) $O[H(x;\Delta)]$ is a differentiable function as well as monotonically increasing

(iii) $O[H(x;\Delta)] \longrightarrow \delta_1$ for $x \longrightarrow -\infty$ and $O[H(x;\Delta)] \longrightarrow \delta_2$ for $x \longrightarrow \infty$

The CDF (cumulative distribution function) of the T-X distributions is

$$M(x;\mathbf{\Delta}) = \int_{\delta_1}^{O[H(x;\Delta)]} u(t)\mathrm{d}t, \quad x \in \mathbb{R}, \tag{3}$$

where $O[H(x;\Delta)]$ fulfills the aforementioned conditions. The corresponding PDF of equation (3), say $m(x;\Delta)$, reduces to

$$m(x;\mathbf{\Delta}) = \left\{\frac{\partial}{\partial x} O[H(x;\mathbf{\Delta})]\right\} u\{O[H(x;\mathbf{\Delta})]\}. \tag{4}$$

More details about the T-X approach can be found in the work of Ahmad et al. [29]. By implementing the T-X method, the survival distribution family [30] can be obtained via the CDF:

$$M(x;\mathbf{\Delta}) = 1 - \int_{\delta_1}^{O[\overline{H}(x;\Delta)]} u(t)\mathrm{d}t, \quad x \in \mathbb{R}, \tag{5}$$

where $\overline{H}(x;\Delta) = 1 - H(x;\Delta)$, representing the SF of X.

In this study, a new exponential-X (NE-X) family is proposed based on equation (3). Suppose $T \sim \exp(1)$; then, its respective CDF is

$$U(t) = 1 - \exp(-t), \quad t \ge 0. \tag{6}$$

Linking this to equation (6), the PDF is

$$u(t) = \exp(-t). \tag{7}$$

By using $u(t)$ and $O[H(x;\beta,\Delta)] = -\log((1 - H^{\beta}(x;\Delta))/e^{H^{\beta}(x;\Delta)})$ in equation (3), the CDF of the NE-X family is as follows:

$$M(x;\beta,\mathbf{\Delta}) = 1 - \left(\frac{1 - H^{\beta}(x;\mathbf{\Delta})}{e^{H^{\beta}(x;\mathbf{\Delta})}}\right), \quad \beta > 0, x, \mathbf{\Delta} \in \mathbb{R}, \tag{8}$$

where $H(x;\Delta)$ is a baseline CDF with parametric space $\Delta \in \mathbb{R}$. Next, in Propositions 1 and 2, we are going to prove that the expression provided in equation (8) is a CDF.

Proposition 1. *For the $M(x;\beta,\Delta)$ defined in equation (8), $lim_{x\longrightarrow -\infty} M(x;\beta,\Delta) = 0$ and $lim_{x\longrightarrow\infty} M(x;\beta,\Delta) = 1$.*

Proof.

$$\begin{aligned}
\lim_{x\longrightarrow\infty} M(x;\beta,\mathbf{\Delta}) &= \lim_{x\longrightarrow -\infty}\left\{1 - \left(\frac{1 - H^{\beta}(x;\mathbf{\Delta})}{e^{H^{\beta}(x;\mathbf{\Delta})}}\right)\right\} \\
&= 1 - \left(\frac{1 - H^{\beta}(-\infty;\mathbf{\Delta})}{e^{H^{\beta}(-\infty;\mathbf{\Delta})}}\right) \\
&= 1 - \left(\frac{1-0}{e^{0}}\right) = 0, \\
\lim_{x\longrightarrow\infty} M(x;\beta,\mathbf{\Delta}) &= \lim_{x\longrightarrow\infty}\left\{1 - \left(\frac{1 - H^{\beta}(x;\mathbf{\Delta})}{e^{H^{\beta}(x;\mathbf{\Delta})}}\right)\right\} \\
&= 1 - \left(\frac{1 - H^{\beta}(\infty;\mathbf{\Delta})}{e^{H^{\beta}(\infty;\mathbf{\Delta})}}\right) \\
&= 1 - \left(\frac{1-1}{e}\right) = 1.
\end{aligned} \tag{9}$$

□

Proposition 2. *The CDF $M(x;\beta,\Delta)$ is right continuous and differentiable.*

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}x} M(x;\beta,\mathbf{\Delta}) = m(x;\beta,\mathbf{\Delta}). \tag{10}$$

From the results proved in Propositions 1 and 2, we arrive at the conclusion that the function $M(x;\beta,\Delta)$ defined in equation (8) is differentiable, right continuous, and a compact CDF. Moreover, taking into account the differentiability property of $M(x;\beta,\Delta)$ for all $x \in \mathbb{R}$, we have the following theorem. □

Theorem 1. *Let $\kappa_1(x) = e^{H^{\beta}(x;\Delta)} + H^{\beta}(x;\Delta) - 1$ and $\kappa_2(x) = e^{H^{\beta}(x;\Delta)}$. With the condition $\nu_2(x) \neq 0$, if $\kappa_1(x)$ and $\kappa_2(x)$ are differentiable, then the quotient $(\kappa_1(x)/\kappa_2(x))$ is differentiable for all $x \in \mathbb{R}$ and*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\kappa_1(x)}{\kappa_2(x)}\right] = \frac{\kappa_1'(x)\kappa_2(x) - \kappa_1(x)\kappa_2'(x)}{\kappa_2^2(x)}. \tag{11}$$

Proof. See "Proof of Quotient Rule."

Taking into account the right continuity of $M(x;\beta,\Delta)$, we have the following theorem. □

Theorem 2. *If $H(x;\Delta)$ is a right continuous function, so is $M(x;\beta,\Delta)$.*

Proof. Assume that $\lim_{x\longrightarrow a^+} H(x;\Delta) = H(a;\Delta)$ for all $a \in \sup(H(x;\Delta))$. Hence, we have

$$\begin{aligned}
\lim_{x\longrightarrow a^+}\left\{e^{H^{\beta}(x;\mathbf{\Delta})} + H^{\beta}(x;\mathbf{\Delta}) - 1\right\} &= e^{H^{\beta}(a;\mathbf{\Delta})} + H^{\beta}(a;\mathbf{\Delta}) - 1, \\
\lim_{x\longrightarrow a^+} e^{H^{\beta}(x;\mathbf{\Delta})} &= e^{H^{\beta}(a;\mathbf{\Delta})}.
\end{aligned} \tag{12}$$

Since $\sup(M(x;\beta,\Delta)) = \sup(H(x;\Delta))$,

$$\lim_{x \longrightarrow a^+} M(x;\beta,\mathbf{\Delta}) = M(a;\beta,\mathbf{\Delta}), \tag{13}$$

for each $a \in \sup(M(x;\beta,\Delta))$. Therefore, $M(x;\beta,\Delta)$ is right continuous.

As we mentioned earlier that the regularly varying tail behavior (RVTB) is a very crucial property to characterize the HT distributions, now we provide the mathematical treatment of RVTB of the NE-X family. ☐

Theorem 3. *If $\overline{H}(x;\Delta)$ is a regularly varying distribution, so is $\overline{M}(x;\beta,\Delta)$.*

Proof. Suppose $\lim_{x\longrightarrow\infty}((1-H^\beta(bx;\beta,\Delta))/(1-H^\beta(x;\beta,\Delta))) = g(b)$ is finite for all $b>0$. Then, using equation (8), we have

$$\begin{aligned}\lim_{x\longrightarrow\infty}\frac{\overline{M}(bx;\beta,\mathbf{\Delta})}{\overline{M}(x;\beta,\mathbf{\Delta})} &= \lim_{x\longrightarrow\infty}\frac{1-H^\beta(bx;\mathbf{\Delta})}{1-H^\beta(x;\mathbf{\Delta})}\times\frac{e^{H^\beta(x;\mathbf{\Delta})}}{e^{H^\beta(bx;\mathbf{\Delta})}}\\ &= \lim_{x\longrightarrow\infty}\frac{1-H^\beta(bx;\mathbf{\Delta})}{1-H^\beta(x;\mathbf{\Delta})}\times\frac{e^{H^\beta(\infty;\mathbf{\Delta})}}{e^{H^\beta(b.\infty;\mathbf{\Delta})}}.\end{aligned} \tag{14}$$

Since $\lim_{x\longrightarrow\infty}H(x;\Delta)=1$, we have

$$\begin{aligned}\lim_{x\longrightarrow\infty}\frac{\overline{M}(bx;\beta,\mathbf{\Delta})}{\overline{M}(x;\beta,\mathbf{\Delta})} &= \lim_{x\longrightarrow\infty}\frac{1-H^\beta(bx;\mathbf{\Delta})}{1-H^\beta(x;\mathbf{\Delta})}\times\frac{e}{e}\\ &= \lim_{x\longrightarrow\infty}\frac{1-H^\beta(bx;\mathbf{\Delta})}{1-H^\beta(x;\mathbf{\Delta})} = g(b),\end{aligned} \tag{15}$$

for $b>0$; thus, $\overline{M}(x,\beta,\Delta)$ is a regularly varying distribution.

Linking this to equation (8), the PDF represented by $m(x;\beta,\Delta)$ is

$$m(x;\beta,\mathbf{\Delta}) = \frac{\beta h(x;\mathbf{\Delta})H^{\beta-1}(x;\mathbf{\Delta})}{e^{H^\beta(x;\mathbf{\Delta})}}\left\{2-H^\beta(x;\mathbf{\Delta})\right\}, \quad x\in\mathbb{R}. \tag{16}$$

The RV with PDF (16) is represented by $X\sim NE\text{-}X(x;\beta,\Delta)$.

The SF, $\overline{M}(x;\beta,\Delta)$, and HRF (hazard rate function), $w(x;\beta,\Delta)$, of X are

$$\begin{aligned}\overline{M}(x;\beta,\mathbf{\Delta}) &= \left(\frac{1-H^\beta(x;\mathbf{\Delta})}{e^{H^\beta(x;\mathbf{\Delta})}}\right), \quad x,\in\mathbb{R},\\ w(x;\beta,\mathbf{\Delta}) &= \frac{\beta h(x;\mathbf{\Delta})H^{\beta-1}(x;\mathbf{\Delta})}{1-H^\beta(x;\mathbf{\Delta})}\left\{2-H^\beta(x;\mathbf{\Delta})\right\}, \quad x\in\mathbb{R},\end{aligned} \tag{17}$$

respectively. In the next section, we discuss a subcase of the NE-X family, called the new exponential-Weibull (NE-Weibull) distribution. ☐

2. NE-Weibull Distribution

This section deals with the NE-Weibull distribution as a subcase of the NE-X family. Suppose that $X\sim$ Weibull(α,γ) with CDF, $H(x;\Delta)=1-\exp(-\gamma x^\alpha)$, and PDF, $h(x;\Delta)=\alpha\gamma x^{\alpha-1}\exp(-\gamma x^\alpha)$, where $\Delta=(\alpha,\gamma)$. By inserting the CDF of the Weibull model in equation (8), the CDF and PDF of the NE-Weibull distribution take the forms

$$\begin{aligned}M(x;\beta,\mathbf{\Delta}) &= 1-\frac{1-[1-\exp(-\gamma x^\alpha)]^\beta}{\exp\left\{[1-\exp(-\gamma x^\alpha)]^\beta\right\}}, \quad x\geq 0,\alpha,\gamma,\beta>0,\\ m(x;\beta,\mathbf{\Delta}) &= \frac{\alpha\gamma\beta x^{\alpha-1}\exp(-\gamma x^\alpha)[1-\exp(-\gamma x^\alpha)]^{\beta-1}}{\exp\left\{[1-\exp(-\gamma x^\alpha)]^\beta\right\}}\left\{2-[1-\exp(-\gamma x^\alpha)]^\beta\right\}.\end{aligned} \tag{18}$$

Some possible PDF behaviors for the NE-Weibull distribution are shown in Figure 1. From Figure 1, it is obvious that as the values of α and β increase, the NE-Weibull model becomes a HT distribution.

As mentioned above, different PDF shapes of the NE-Weibull model for various values of β, $\gamma=1$, and α are sketched in Figure 1. When $\alpha,\beta<1$, the NE-Weibull distribution behaves similar to the exponential distribution $\exp(\gamma)$. For $\beta>1$, the NE-Weibull distribution captures the characteristics of Wei(α,γ). The NE-Weibull model, however, has certain benefits over the Weibull distribution. For instance, it has heavier tails than the Weibull distribution and provides closer fit to data in the finance sector.

Some possible HRF behaviors for the NE-Weibull distribution are shown in Figure 2. The NE-Weibull distribution provides increasing, unimodal, decreasing, and bathtub HRF shapes.

3. Properties

Here, we provide a concise treatment of the mathematical properties of NE-X distribution. These properties include QF (quantile function), moments, SK (skewness), and KUR (kurtosis). Furthermore, different plots for the SK and KUR are also provided.

3.1. Quantile Function. Suppose X denotes the NE-X family with CDF presented in equation (8); then, the QF of NE-X distributions, $Q(u)$, is

$$x_u = Q(u) = M^{-1}(u) = H^{-1}(t), \tag{19}$$

where t is the solution of $H(x;\Delta)+H^{\beta}(x;\Delta)+\log(1-u)-1$.

3.2. Moments. Now, we introduce the moments of the NE-X distributions which can further be used to obtain other characteristics.

The r^{th} moment of the NE-X family reduces to

$$\mu_r' = \int_{-\infty}^{\infty} x^r m(x;\beta,\Delta)\mathrm{d}x. \tag{20}$$

By putting equation (16) in (20), we obtain

$$\begin{aligned}\mu_r' &= \beta\int_{-\infty}^{\infty} x^r \frac{h(x;\Delta)H^{\beta-1}(x;\Delta)}{\exp\left\{H^{\beta}(x;\Delta)\right\}}\left\{2-H^{\beta}(x;\Delta)\right\}\mathrm{d}x\\ &= \beta\sum_{i=0}^{n} 2^{1-i}(-1)^i\binom{n}{i}\int_{-\infty}^{\infty} x^r\frac{h(x;\Delta)H^{\beta+i\beta-1}(x;\Delta)}{\exp\left\{H^{\beta}(x;\Delta)\right\}}\mathrm{d}x\\ &= \beta\sum_{i=0}^{n}\sum_{j=0}^{\infty} 2^{1-i}(-1)^{i+j}\binom{n}{i}\kappa_{r,i,j,\beta},\end{aligned} \tag{21}$$

where $\kappa_{r,i,j,\beta} = \int_{-\infty}^{\infty} x^r h(x;\Delta)H^{\beta+i\beta+j\beta-1}(x;\Delta)\mathrm{d}x$. For $r = 1,2,3,4$, we obtain the first four moments of the NE-X family.

Using the expressions of the moments, we obtain the mathematical form of the SK and KUR measures. The SK and KUR of the NE-Weibull distribution can be calculated, respectively, via the expressions

$$\begin{aligned}\mathrm{SK} &= \frac{\mu_3}{\mu_2^{3/2}},\\ \mathrm{KUR} &= \frac{\mu_4}{\mu_2^{2}}.\end{aligned} \tag{22}$$

The effects of β, α, and γ on the SK, KUR, variance, and mean of the NE-Weibull distribution are displayed in Figures 3–5.

4. Estimation

Within this section, we derive the maximum likelihood estimators (MLEs) of the NE-Weibull parameters. Consider $x_1, x_2, \ldots, x_n$ as the values of a sample from the NE-Weibull distribution with parameters β and Δ. The log-likelihood (LL) function $\ell(\beta,\Delta)$ of the NE-Weibull distribution takes the form

$$\begin{aligned}\ell(\beta,\Delta) &= k\log\beta + \sum_{i=0}^{k}\log h(x_i;\Delta)\\ &\quad + (\beta-1)\sum_{i=0}^{k}\log H(x_i;\Delta) - \sum_{i=0}^{k} H^{\beta}(x_i;\Delta)\\ &\quad + \sum_{i=0}^{k}\log\left[2-H^{\beta}(x_i;\Delta)\right].\end{aligned} \tag{23}$$

The partial derivatives of the LL function are

$$\begin{aligned}\frac{\partial}{\partial\beta}\ell(\beta,\Delta) &= \frac{n}{\beta} + \sum_{i=0}^{k}\log H(x_i;\Delta) - \sum_{i=0}^{k}\left[\log\left(H^{\beta}(x_i;\Delta)\right)\right]H^{\beta}(x_i;\Delta)\\ &\quad - \sum_{i=0}^{k}\frac{\left[\log\left(H^{\beta}(x_i;\Delta)\right)\right]H^{\beta}(x_i;\Delta)}{2-H^{\beta}(x_i;\xi)},\\ \frac{\partial}{\partial\Delta}\ell(\beta,\Delta) &= (\beta-1)\sum_{i=0}^{k}\frac{\partial H(x_i;\Delta)/\partial\Delta}{H(x_i;\Delta)} - \beta\sum_{i=0}^{k}\frac{H^{\beta-1}(x_i;\Delta)\partial H(x_i;\Delta)}{\partial\Delta}\\ &\quad + \sum_{i=0}^{k}\frac{\partial h(x_i;\Delta)/\partial\Delta}{h(x_i;\Delta)} - \beta\sum_{i=0}^{k}\frac{H^{\beta-1}(x_i;\Delta)\partial H(x_i;\Delta)/\partial\Delta}{2-H^{\beta-1}(x_i;\Delta)}.\end{aligned} \tag{24}$$

Equating $(\partial/\partial\beta)\ell(\beta,\Delta)$ and $(\partial/\partial\Delta)\ell(\beta,\Delta)$ to zero and simultaneously solving them yield the MLEs $(\widehat{\beta},\widehat{\Delta})$ of (β,Δ).

5. Simulation Study

Here, we implement the Monte Carlo simulation approach to address the MLE behavior in estimating the NE-Weibull parameters. The NE-Weibull distribution can be simulated by using equation (8). Let U follow the standard uniform distribution; hence, the quantile function reduces to

$$H(x;\Delta) + (H(x;\Delta))^{\theta} + \log(1-u) - 1. \tag{25}$$

The simulation is done for (i) $\alpha = 0.5, \beta = 1.2, \gamma = 1$, (ii) $\alpha = 1.2, \beta = 0.8, \gamma = 0.9$, and (iii) $\alpha = 0.8, \beta = 0.5, \gamma = 1.1$.

The simulation results are obtained by utilizing the R software with the algorithms (root Solve) and "LBFGS – B" with optim. The results obtained are based on $K = 500$ replications for samples of size k, where $k = 10, 20, \ldots, 500$. Statistical tools such as biases and mean square errors (MSEs) are obtained as assessing tools. These tools are calculated as follows:

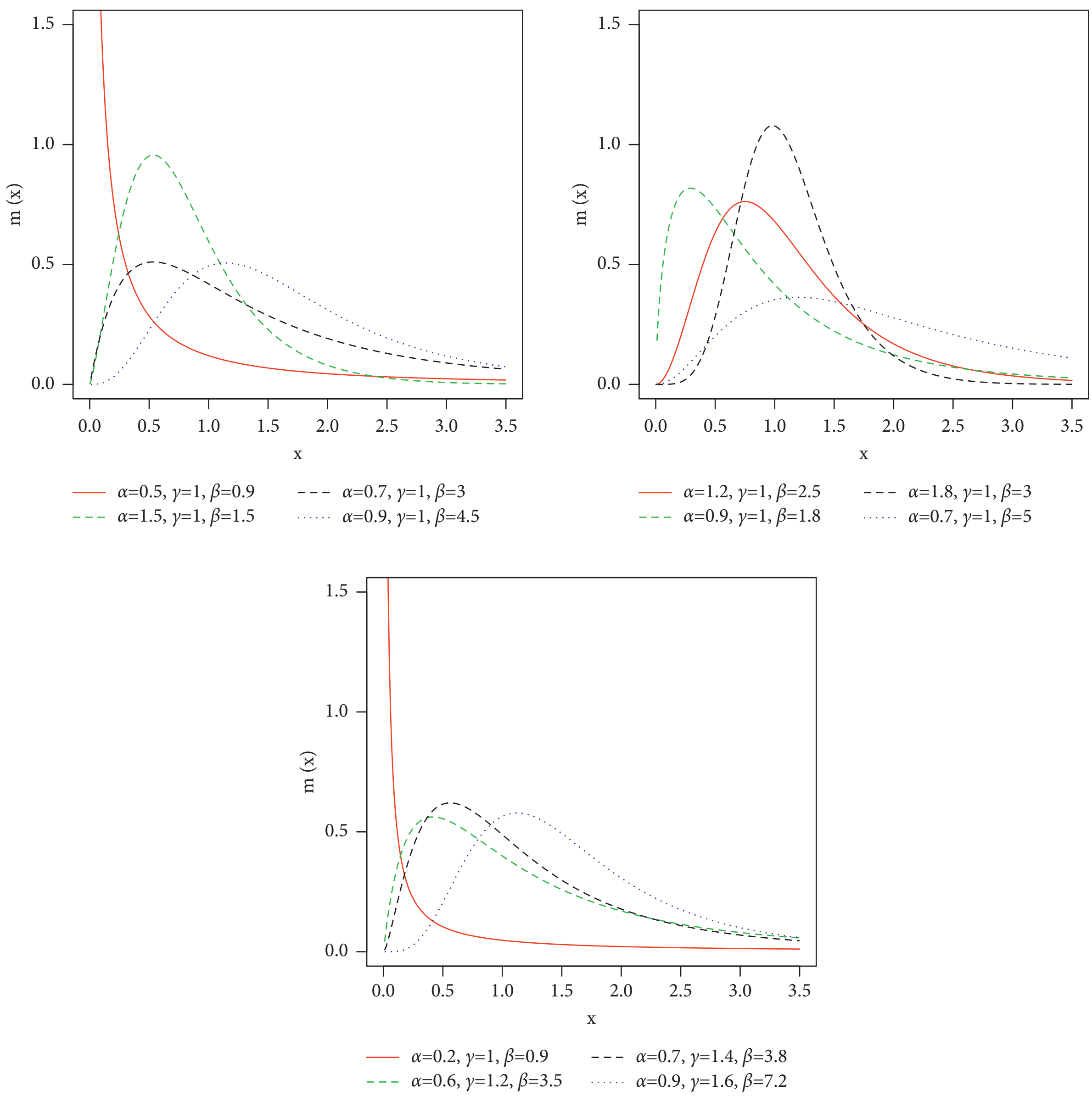

FIGURE 1: Visual display of $m(x;\beta,\Delta)$.

$$\text{bias}(\boldsymbol{\Lambda}) = \frac{1}{500}\sum_{i=1}^{500}(\widehat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda}),$$
$$\text{MSE}(\boldsymbol{\Lambda}) = \frac{1}{500}\sum_{i=1}^{500}(\widehat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda})^2, \tag{26}$$

where $\Lambda = (\beta, \Delta)$.

The summary measures (SMs) of the three simulated sets of data are provided in Table 1, whereas the histograms, box plots, kernel density estimator, and estimated CDF of the NE-Weibull model are provided in Figures 6–8. The simulation results are visually displayed in Figures 9–11.

6. Data Modeling in the Finance Sector

Here, we illustrate the importance of the NE-Weibull distribution in modeling insurance claims data from the financial sector. We also calculated the risk measures such as VaR (value at risk) and TVaR (tail value at risk) for this data.

6.1. Insurance Claims Data. The insurance claims data represents the initial claims of the unemployment insurances per month from 1971 to 2018 [20]. The data set can also be retrieved from https://data.worlddatany-govns8z-xewg.

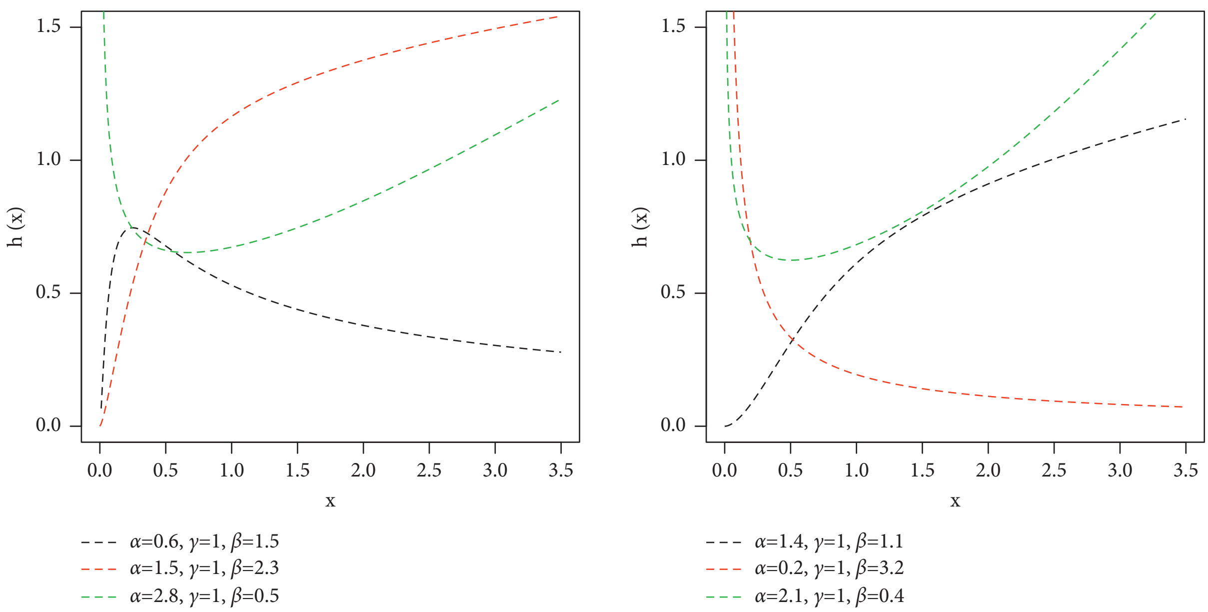

FIGURE 2: Different HRF plots of the NE-Weibull model.

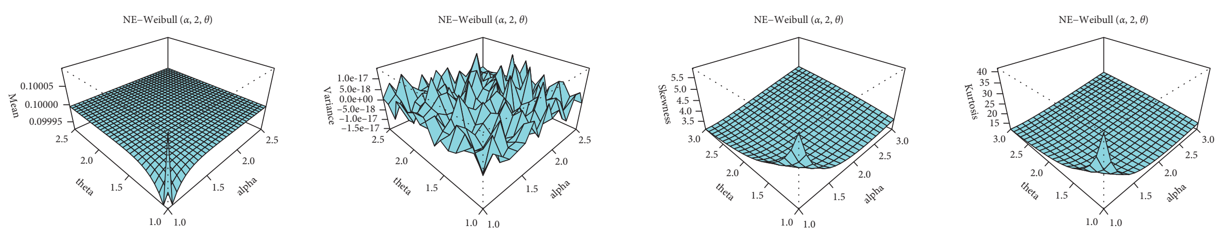

FIGURE 3: Plots of SK, KUR, variance, and mean of the NE-Weibull distribution for $\gamma = 2$ and different values of α and β.

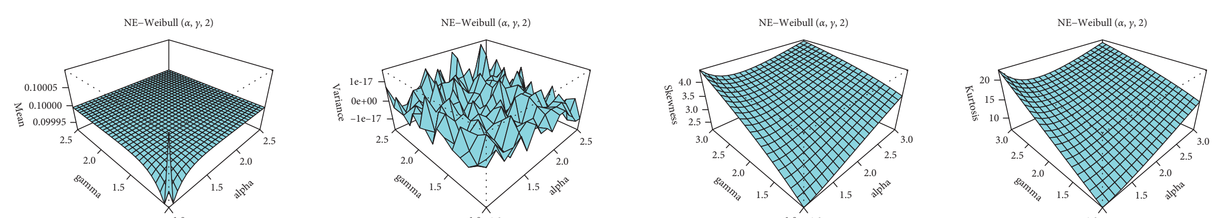

FIGURE 4: Plots of SK, KUR, variance, and mean of the NE-Weibull distribution for $\beta = 2$ and different values of α and γ.

The summary statistics for the insurance claims data are reported in Table 2. Corresponding to this data, the histogram, box plot, and total time test (TTT) plots are provided in Figure 12. Figure 12 shows that the insurance claims data is unimodal, right-skewed, and HT.

The NE-Weibull distribution is applied to fit the financial data, and it is compared with the Weibull, W-Loss (Weibull-Loss), NHT-Weibull (new heavy-tailed Weibull), Ku-Weibull (Kumaraswamy Weibull), and B-Weibull (beta Weibull) distributions. The CDFs of these models are as follows:

(i) Weibull model:

$$H(x) = 1 - \exp\{-\gamma x^{\alpha}\}, \quad x \geq 0. \tag{27}$$

(ii) W-Loss model:

$$H(x) = 1 - \frac{\beta \exp\{-\gamma x^{\alpha}\}}{\beta + \gamma x^{\alpha}}, \quad x \geq 0. \tag{28}$$

(iii) NHT-Weibull model:

$$H(x) = \frac{\beta\left(1 - \exp\{-\gamma x^{\alpha}\}\right)}{\left(\beta - \exp\{-\gamma x^{\alpha}\}\right)^{2}}, \quad x \geq 0. \tag{29}$$

(iv) Ku-Weibull model:

$$H(x) = 1 - \left[1 - \left(1 - \exp\{-\gamma x^{\alpha}\}\right)^{a}\right]^{b}, \quad x \geq 0. \tag{30}$$

(v) B-Weibull model:

$$H(x) = I_{\left(1-e^{-\gamma x^{\alpha}}\right)}(x; a, b), \quad x \geq 0. \tag{31}$$

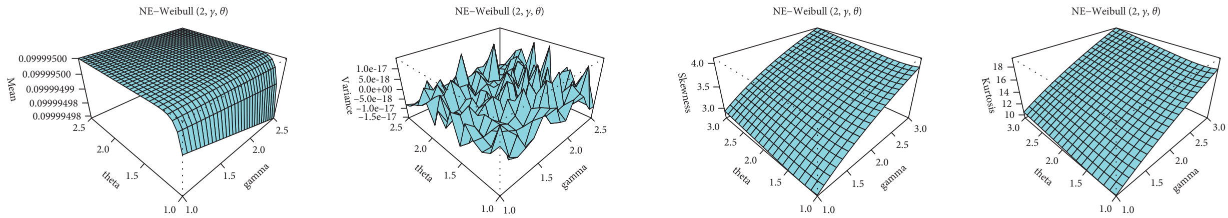

Figure 5: Plots of SK, KUR, variance, and mean of the NE-Weibull distribution for $\alpha = 2$ and different values of β and γ.

Table 1: SMs of the simulated data sets.

Simulated data	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
Data set 1	0.0000	0.0390	0.1926	1.1191	0.9717	63.2960
Data set 2	0.0000	0.1388	0.3475	0.5469	0.7188	3.4976
Data set 3	0.0000	0.0070	0.0707	0.3364	0.3125	9.1907

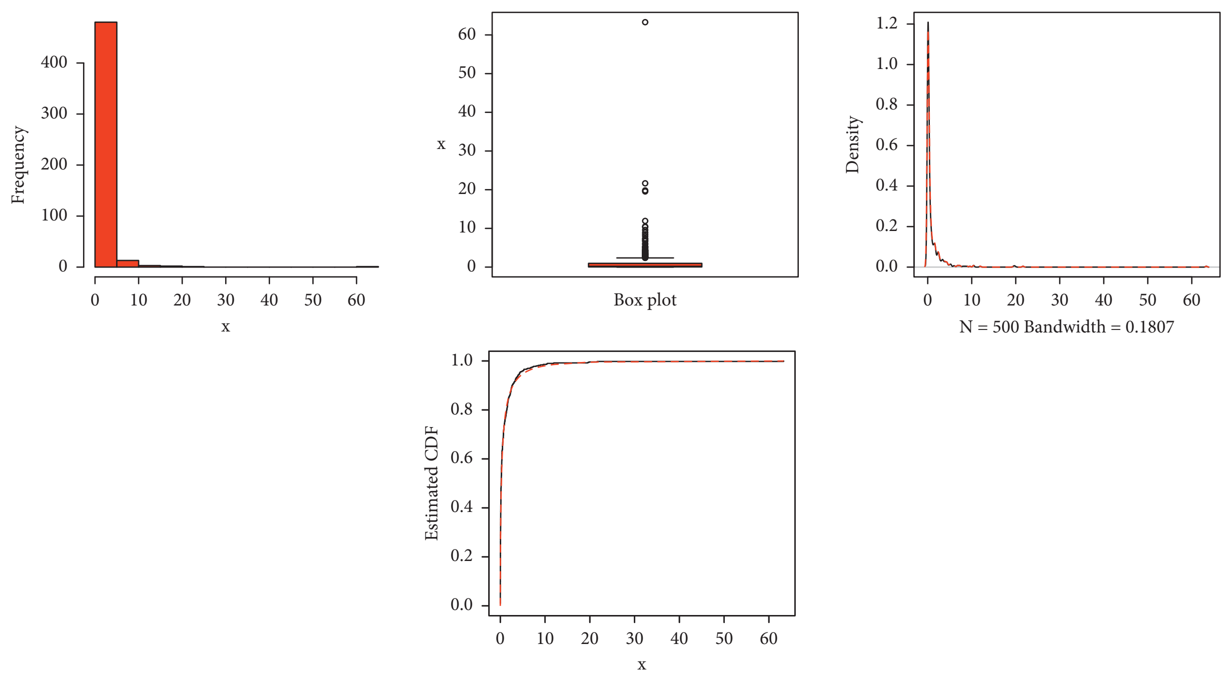

Figure 6: Kernel density estimator, box plot, histogram, and fitted CDF of the NE-Weibull distribution for simulated data set 1.

To decide about the best fitting of the applied distributions, certain criteria are taken into account. These criteria are as follows:

(i) The AIC is

$$2p - 2\ell(\Theta). \tag{32}$$

(ii) The BIC is

$$p\log(k) - 2\ell(\Theta). \tag{33}$$

(iii) The HQIC is

$$2p\log(\log(k)) - 2\ell(\Theta). \tag{34}$$

(iv) The CAIC is

$$\frac{2pk}{k-p-1} - 2\ell(\Theta). \tag{35}$$

Here, $\ell(\Theta)$ denotes the LL function, Θ is a parametric space, p represents the model parameters, and k is the number of selected samples. In addition to the criteria measures, three goodness-of-fit tests with its corresponding p values are also considered. These tests are given by the following:

(i) The Anderson–Darling (AD) test statistic is

$$\mathrm{AD} = -k - \frac{1}{n}\sum_{j=1}^{k}(2j-1)\left[\log H\left(x_j\right) + \log\left\{1 - H\left(x_{k-j+1}\right)\right\}\right], \tag{36}$$

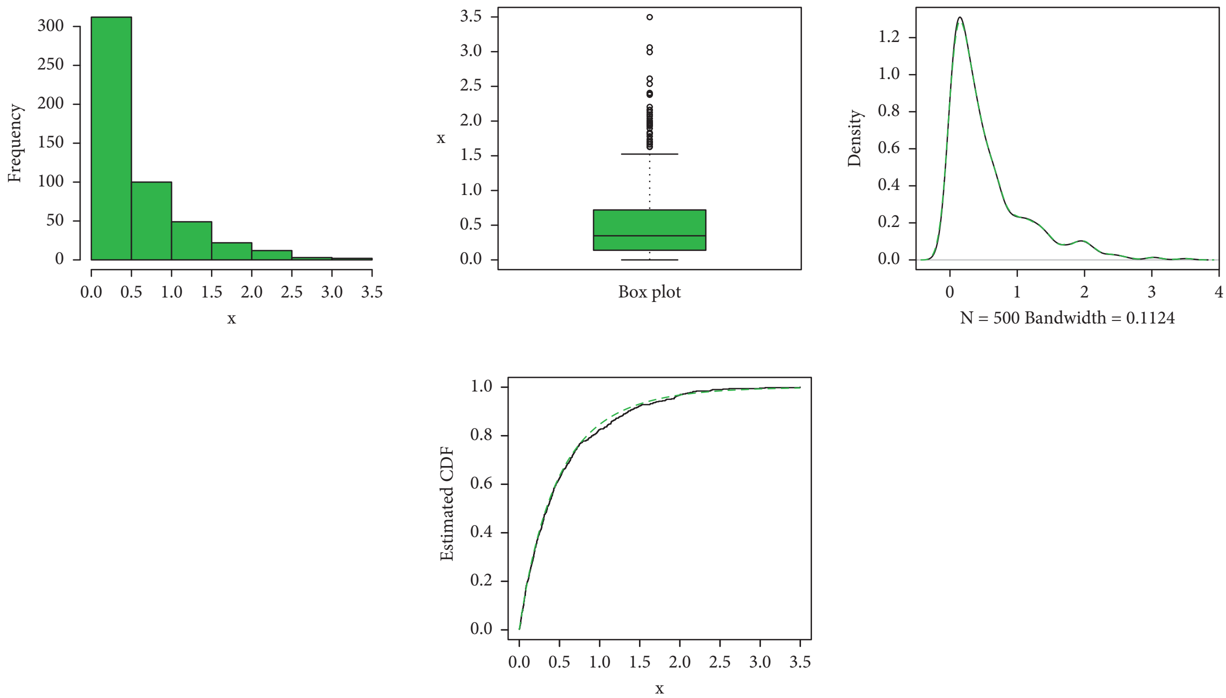

Figure 7: Kernel density estimator, box plot, histogram, and fitted CDF of the NE-Weibull distribution for simulated data set 2.

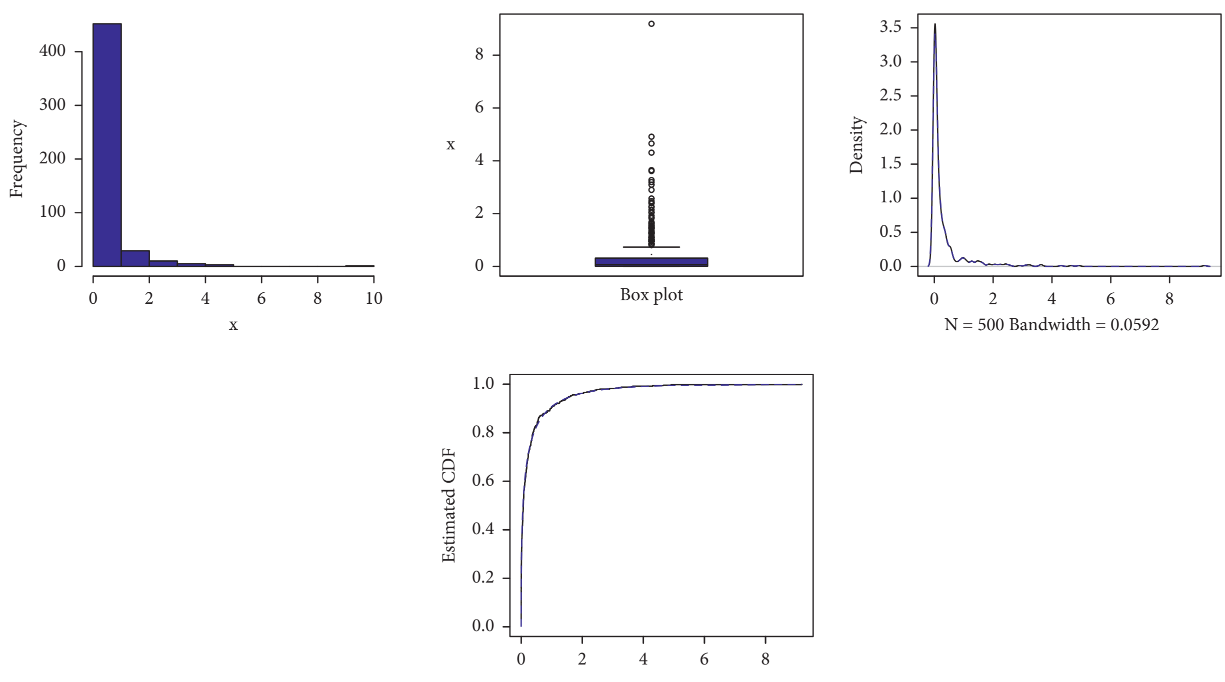

Figure 8: Kernel density estimator, box plot, histogram, and fitted CDF of the NE-Weibull distribution for simulated data set 3.

where x_j is the j^{th} observation in k samples, calculated after sorting the data in the ascending order.

(ii) The Cramer–von Mises (CM) test statistic is

$$\mathrm{CM} = \frac{1}{12k} + \sum_{j=1}^{k} \left[\frac{2j-1}{2k} - H\left(x_j\right)\right]^2. \tag{37}$$

(iii) The Kolmogorov–Smirnov (KS) test statistic is

$$\mathrm{KS} = \sup_x \left[H_n(x) - H(x)\right], \tag{38}$$

where $\sup_x$ represents the supremum of the set of distances and $H_n(x)$ indicates the empirical CDF.

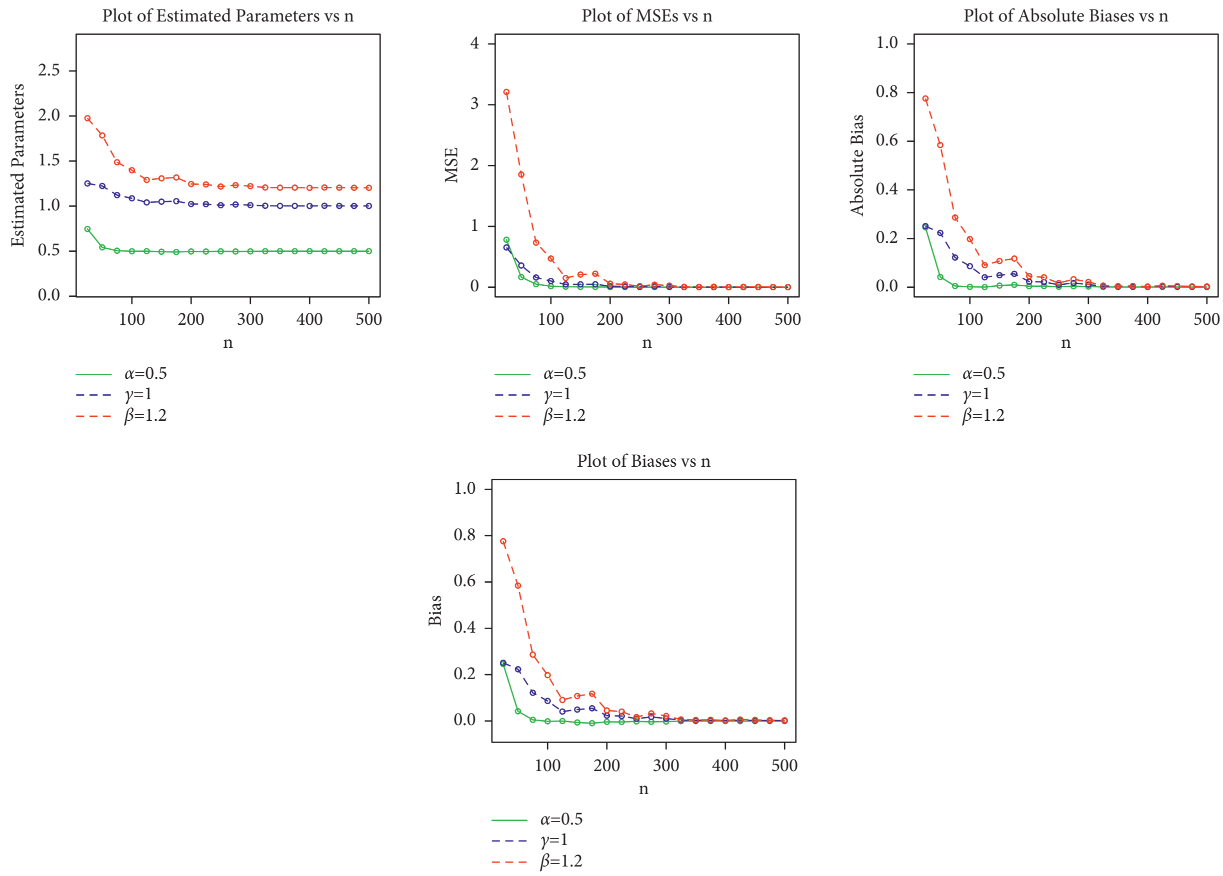

Figure 9: Graphical display of different measures for simulated data set 1.

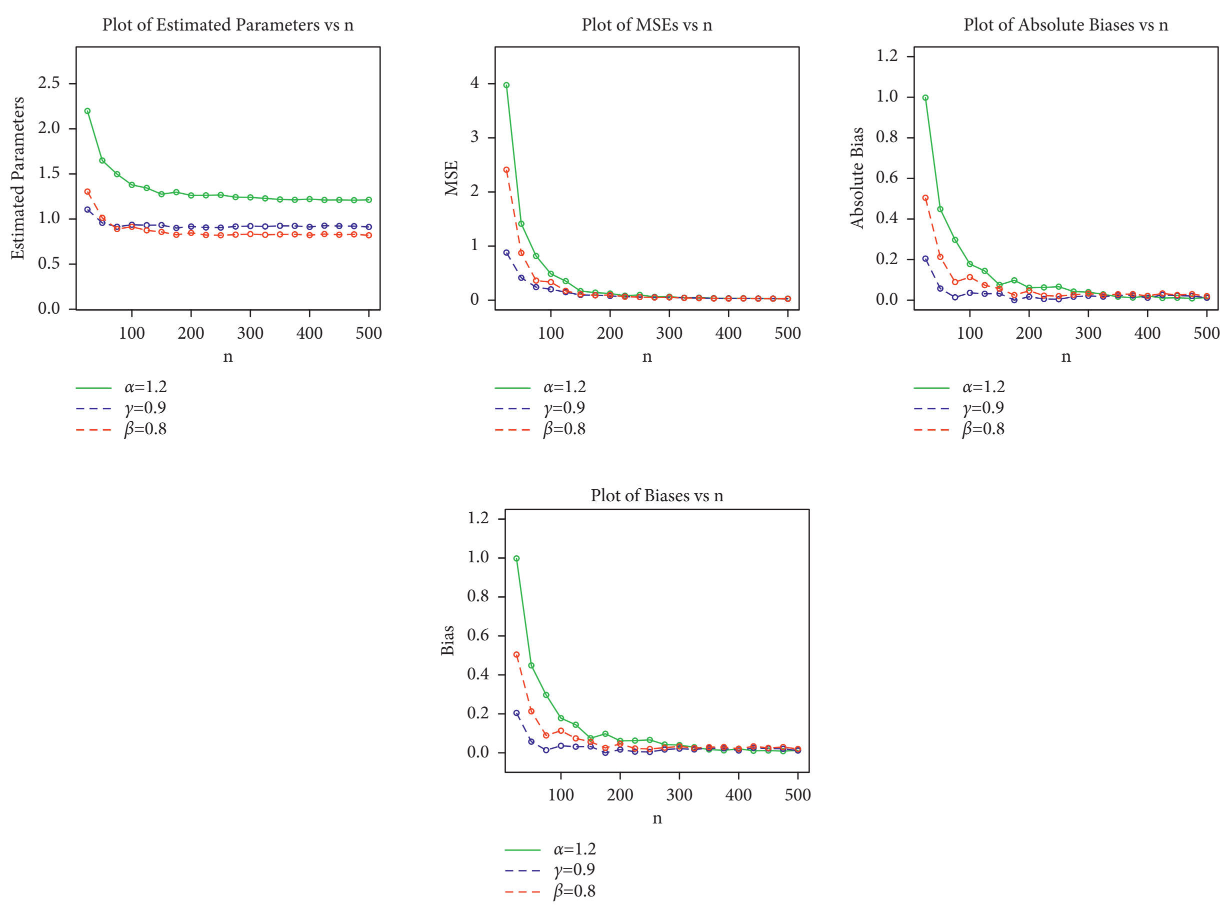

Figure 10: Graphical display of different measures for simulated data set 2.

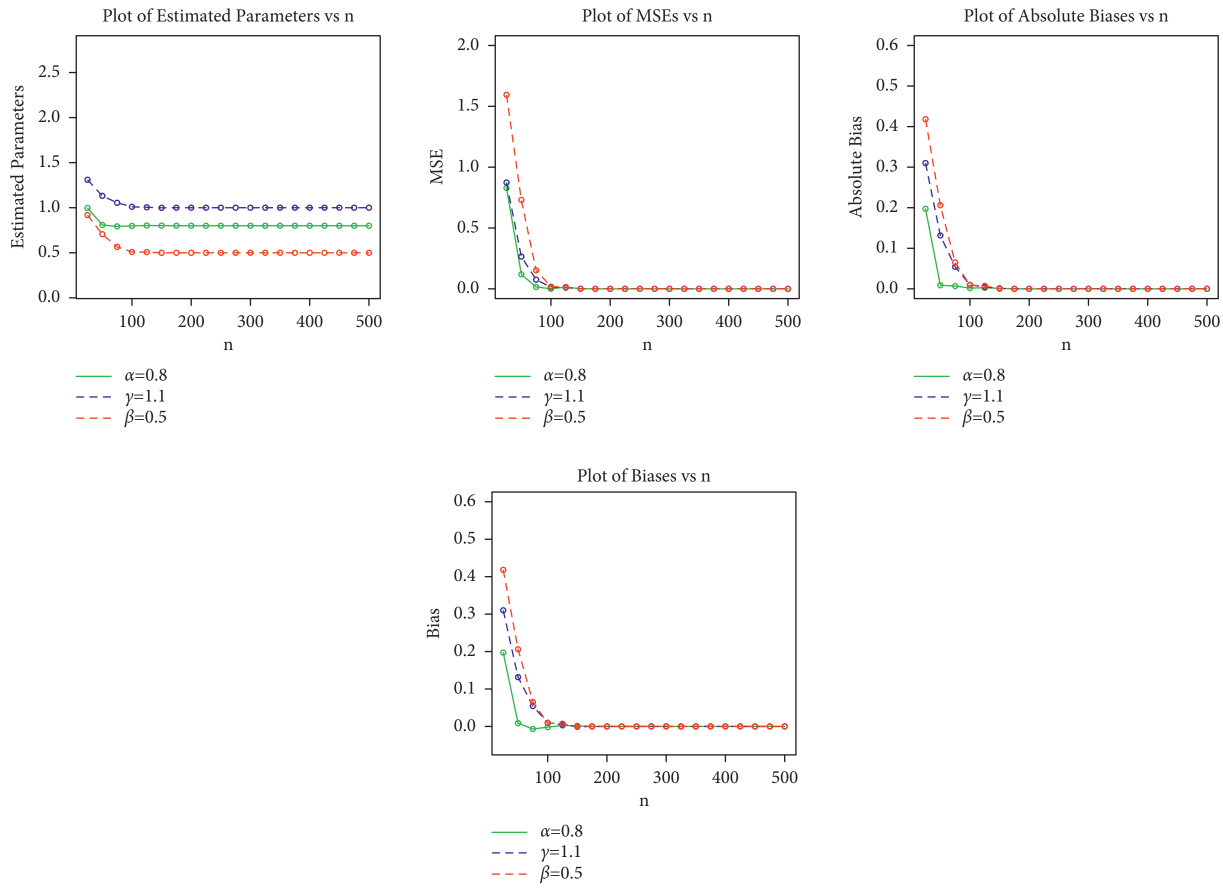

Figure 11: Graphical display of different measures for simulated data set 3.

Table 2: SMs of the insurance claims data.

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
49.26	82.71	100.02	109.18	129.74	308.35

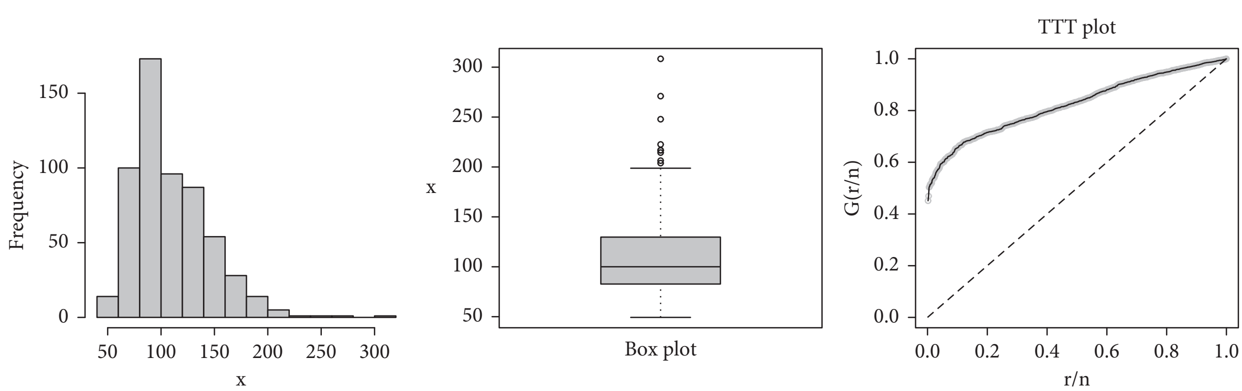

Figure 12: Histogram, TTT, and box plots for the insurance claims data.

The LL function is optimized, and the goodness-of-fit measures are calculated using the algorithm method "BFGS" via optim() R-function [31]. The MLEs of the fitted models with standard errors are listed in Table 3. Tables 4 and 5 show the goodness-of-fit values for the considered models.

Table 3: Estimated parameters for the insurance claims data.

Distribution	α	γ	β	a	b
NE-Weibull	1.283 (0.012)	0.005 (0.001)	8.689 (0.610)		
Weibull	1.486 (1.096)	0.001 (0.095)			
W-Loss	1.564 (1.075)	0.041 (0.076)	12.765 (0.963)		
NHT-Weibull	1.316 (0.964)	0.025 (0.049)	10.498 (0.875)	31.713 (3.791)	
Ku-Weibull	0.984 (0.076)	0.052 (0.017)		64.933 (0.063)	0.657 (0.167)
B-Weibull	1.090 (0.058)	0.031 (0.000)		40.704 (10.392)	0.592 (0.139)

Table 4: Discrimination criteria of the competitive models for the insurance claims data.

Distribution	AIC	BIC	CAIC	HQIC
NE-Weibull	5519.460	5532.523	5519.502	5524.555
Weibull	6169.574	6178.290	6169.634	6172.975
W-Loss	5585.738	5596.708	5585.984	5590.767
NHT-Weibull	5546.007	5564.756	5546.083	5553.865
Ku-Weibull	5601.734	5619.152	5601.805	5608.527
B-Weibull	5602.555	5602.625	5619.972	5609.348

Table 5: Goodness-of-fit criteria of the competing distributions for insurance claims data.

Distribution	CM	AD	KS	p value
NE-Weibull	0.146	0.674	0.016	0.668
Weibull	0.974	2.094	0.896	0.129
W-Loss	0.196	0.895	0.027	0.538
NHT-Weibull	0.175	0.795	0.023	0.618
Ku-Weibull	0.200	1.037	0.037	0.383
B-Weibull	0.221	1.121	0.035	0.471

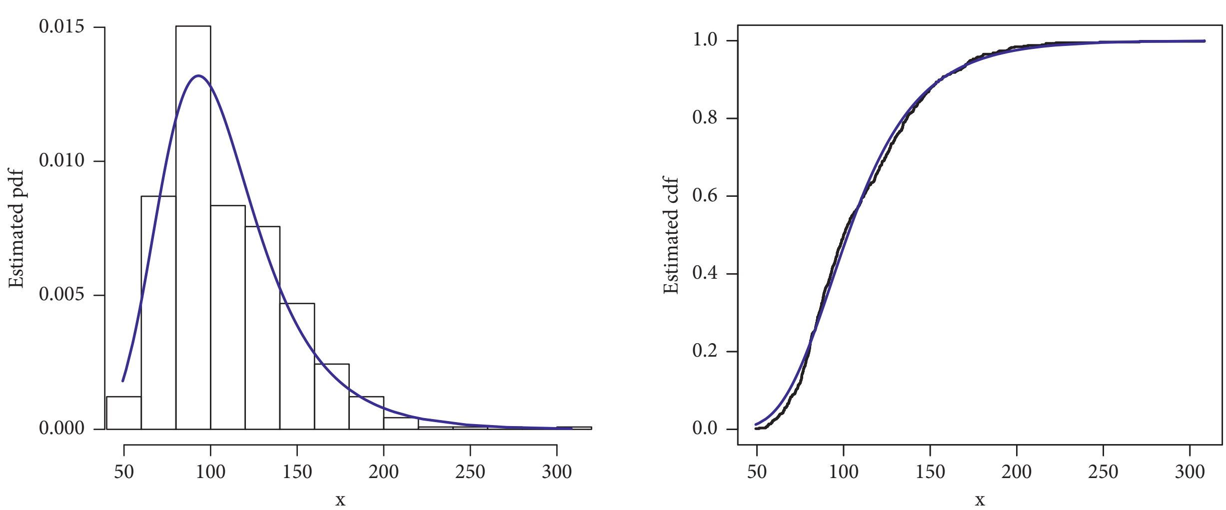

Figure 13: Estimated density and distribution functions of the NE-Weibull distribution.

The fitted PDF and CDF of the NE-Weibull model are provided in Figure 13, showing that the right-skewed HT data can be adequately fitted by the NE-Weibull distribution. The Kaplan–Meier survival (KMS) and PP (probability-probability) plots of the NE-Weibull distribution are sketched in Figure 14.

6.2. VaR and TVaR for Insurance Claims Data. Now, we calculate the VaR and TVaR measures for the Weibull and proposed NE-Weibull distributions based on the MLEs and the data provided in Section 6.1. The results of the VaR and TVaR of the NE-Weibull and Weibull distributions are presented in Table 6. We also display these results graphically in Figure 15.

A model having higher values for the TVaR and VaR is considered a HT distribution. The numerical results in Table 6 and their graph in Figure 15 show that the NE-Weibull is a HT model.

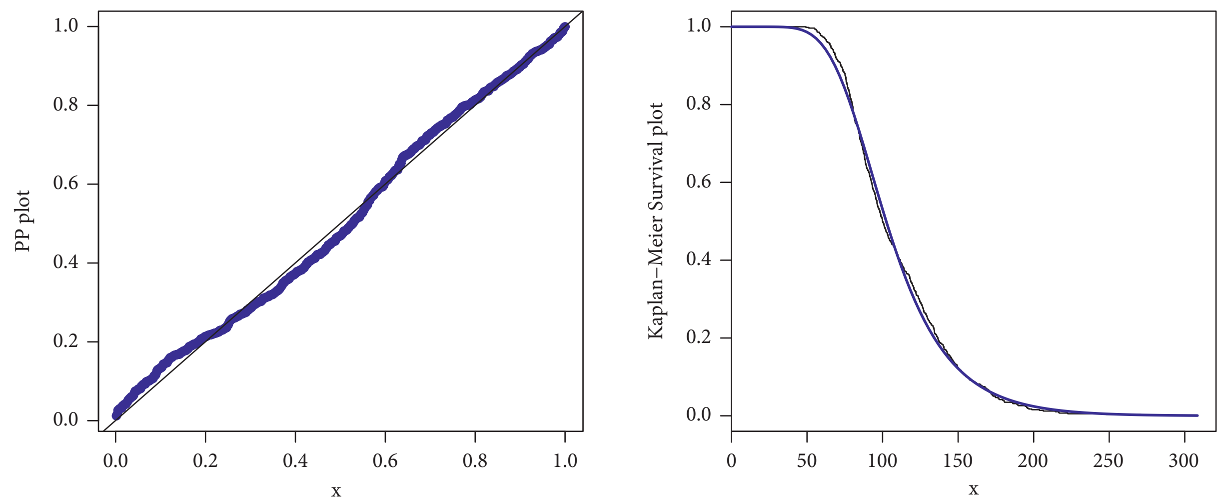

Figure 14: KMS and PP plots of the NE-Weibull distribution.

Table 6: VaR and TVaR results for the NE-Weibull and Weibull distributions.

Distribution	Parameters	Level of significance	VaR	TVaR
NE-Weibull	$\alpha = 1.283$ $\gamma = 0.005$ $\beta = 8.689$	0.700	145.8537	153.9659
		0.750	153.5376	179.2175
		0.800	168.5360	198.7654
		0.850	192.3638	218.7189
		0.900	209.8965	231.9876
		0.950	224.0768	249.0786
		0.975	239.7853	268.0298
		0.999	253.6358	287.0948
Weibull	$\alpha = 1.486$ $\gamma = 0.001$	0.700	70.3462	89.7568
		0.750	77.7643	101.8539
		0.800	83.8743	115.7943
		0.850	98.3245	135.8764
		0.900	120.5643	156.9875
		0.950	142.6750	179.9876
		0.975	176.7656	201.9875
		0.999	202.6752	230.9786

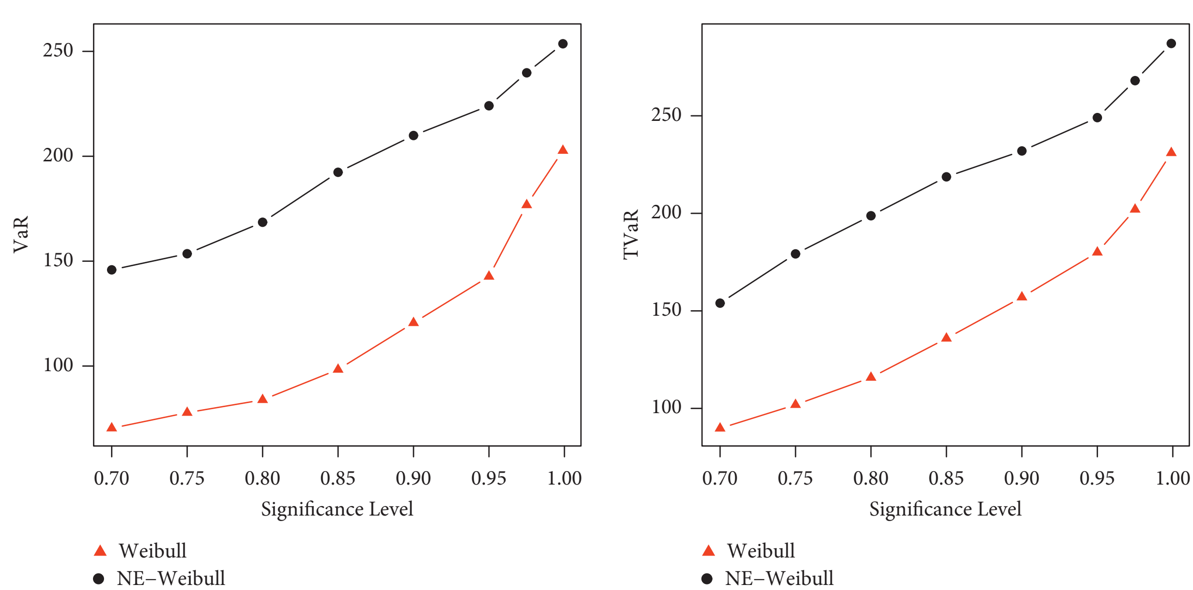

Figure 15: Graphical presentation for the values of the VaR and TVaR.

7. Final Comments

During the last couple of years, the study and modeling of extreme value data has gained increased interest in numerous areas, particularly in the hydrology and finance sector. Recent studies have shown the importance and potential of the HT distributions in various areas for analyzing real data sets, especially in the finance sector. In this article, a new interesting HT extension of the Weibull distribution, called the NE-Weibull distribution, is proposed using the NE-X approach. The MLEs of the NE-Weibull parameters β and Δ are obtained. To show the applicability of the NE-Weibull model, a real-life insurance claims data from the financial sector is considered. Based on certain tests, it is showed that the NE-Weibull model offers an adequate fit to the insurance claims data than the Weibull, W-Loss, NHT-Weibull, Ku-Weibull, and B-Weibull. On the other hand, the numerical results for the VaR and TVaR measures showed that the NE-Weibull model is a HT distribution as compared to the classical Weibull distribution.

Data Availability

To replicate the findings of this paper, the data is available from the corresponding author upon request.

Disclosure

This article was drafted from the PhD work of the first author (Zubair Ahmad).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this work.

Authors' Contributions

Conceptualization and methodology were carried out by Zubair Ahmad and Eisa Mahmoudi. The original draft was prepared by Zubair Ahmad and Morad Alizadeh. The original draft was edited by Eisa Mahmoudi, Rasool Roozegar, and Ahmed Z. Afify. Formal analysis was performed by Zubair Ahmad. Software was collected by Zubair Ahmad and Morad Alizadeh. The study was supervised by Eisa Mahmoudi, Morad Alizadeh, and Rasool Roozegar. Investigation was conducted by Zubair Ahmad. The manuscript was reviewed and edited by Zubair Ahmad, Eisa Mahmoudi, Morad Alizadeh, Rasool Roozegar, and Ahmed Z. Afify.

Acknowledgments

This work was supported by the Department of Statistics, Yazd University, Yazd, Iran.

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