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Research Article

Estimates for the Norm of Generalized Maximal Operator on Strong Product of Graphs

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Let $G = G_1 \times G_2 \times \cdots \times G_m$ be the strong product of simple, finite connected graphs, and let $\phi: \mathbb{N} \longrightarrow (0, \infty)$ be an increasing function. We consider the action of generalized maximal operator M_G^{ϕ} on ℓ^p spaces. We determine the exact value of ℓ^p-quasi-norm of M_G^{ϕ} for the case when G is strong product of complete graphs, where $0 < p \leq 1$. However, lower and upper bounds of ℓ^p-norm have been determined when $1 < p < \infty$. Finally, we computed the lower and upper bounds of $\|M_G^{\phi}\|_p$ when G is strong product of arbitrary graphs, where $0 < p \leq 1$.

1. Introduction

We review some of the standard facts on graphs and metric on the graphs. All the graphs considered in this paper are simple, finite, and connected. Let $G(V(G), E(G))$ be a graph, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges of G. The vertices which are at distance one from any vertex $x \in V(G)$ are called neighbors of x. The set of neighbors of $x \in V(G)$ is denoted by $N_G(x)$. The degree of any vertex $x \in V(G)$ is the cardinality of the set $N_G(x)$ and is denoted by $d_G(x)$. The distance between two vertices x and y denoted by $d(x, y)$ is the length of the shortest path between x and y. For more details on graph theory, we refer the readers to [1–3]. The metric (graph metric) $d_G: V(G) \times V(G) \longrightarrow \mathbb{R}$ on graph G is defined as

$$d_G(x, y) = \text{distance between } x \text{ and } y, \tag{1}$$

where $x, y \in V(G)$. This metric space (G, d_G) is called geodesic metric space. For any function $f: V(G) \longrightarrow \mathbb{R}$, the Hardy–Littlewood maximal operator $M_G^x: \ell^p \longrightarrow \ell^p$ [4–7] is defined as

$$M_G^x f(q) = \sup_{r \geq 0} \frac{1}{|B(q, r)|} \sum_{w \in B(q,r)} |f(w)|, \tag{2}$$

where $B(q, r) = \{s \in V(G): d_G(q, s) \leq r\}$ is the ball with center $q \in V(G)$ and radius r on a graph G. It contains all the vertices of G which are at distance atmost r from the vertex q. It is clear from the definition that if $r = 0$, then $|B(q, r)| = 1$, and if $r \geq 1$, then $|B(q, r)| \geq 2$. The values of metric function d_G are natural numbers and radius $r \geq 0$; therefore, equation (2) can be written as

$$M_G^x f(q) = \max_{r \in \mathbb{N}} \frac{1}{|B(q, r)|} \sum_{w \in B(q,r)} |f(w)|. \tag{3}$$

The fractional maximal operator [8] on graphs is defined as

$$M_G^{1-(t/n)} f(q) = \max_{r\in\mathbb{N}} \frac{1}{|B(q,r)|^{1-(t/n)}} \sum_{w\in B(q,r)} |f(w)|, \quad (4)$$

where $0 \le t \le n$. If $t = 0$, then equation (4) reduces to equation (3). For $0 < p < \infty$, the ℓ^p norm of the Hardy–Littlewood maximal operator is defined as

$$\|M_G^x\|_{\ell^p} := \sup_{f\neq 0} \frac{\|M_G^x f\|_{\ell^p}}{\|f\|_{\ell^p}}, \quad (5)$$

where $\|f\|_{\ell^p} = (\sum_{s\in V(G)} |f(s)|^p)^{(1/p)}$.

For every function $f\colon V(G) \longrightarrow \mathbb{R}$, the generalized maximal operator $M_G^\phi\colon \ell^p \longrightarrow \ell^p$ [9, 10] is defined as

$$M_G^\phi f(q) = \max_{r\in\mathbb{N}} \frac{1}{\phi(|B(q,r)|)} \sum_{w\in B(q,r)} |f(w)|, \quad (6)$$

where $\phi\colon \mathbb{N} \longrightarrow (0,\infty)$ is an increasing function. Note that if we take $\phi(x) = x$ in equation (6), then we get the classical Hardy–Littlewood maximal operator M_G^x, and if we take $\phi(x) = x^{1-(t/n)}$ in equation (5), then we get equation (4).

Let K_n be complete graph on n vertices. For any vertex $q \in V(K_n)$, the ball $B(q,r)$ with center q and radius r is defined as

$$B(q,r) = \begin{cases} \{q\}, & \text{for } r = 0, \\ V(K_n), & \text{for } r \ge 1. \end{cases} \quad (7)$$

Therefore, the generalized maximal operator on complete graph K_n takes the form

$$M_{K_n}^\phi f(q) = \max\left\{\frac{1}{\phi(1)}|f(q)|, \frac{1}{\phi(n)} \sum_{v\in V(K_n)} |f(v)|\right\}. \quad (8)$$

For any vertex $i \in V(G)$, the Kronecker delta function is defined as

$$\delta_q(i) = \begin{cases} 1, & q = i, \\ 0, & q \neq i. \end{cases} \quad (9)$$

Soria and Tradacete [6] estimated the norm of maximal operator M_G^x in the following form.

Proposition 1 (see [6])

(i) If $0 < P \le 1$, *then*

$$\|M_{K_n}\|_p = \left(1 + \frac{n-1}{n^p}\right)^{(1/p)}. \quad (10)$$

(ii) If $1 < p < \infty$, *then*

$$\left(1 + \frac{n-1}{n^p}\right)^{(1/p)} \le \|M_{K_n}\|_p \le \left(1 + \frac{n-1}{n}\right)^{(1/p)}. \quad (11)$$

For more details on this topic of research, see [4, 8, 10–13]. The main motivation of this paper is from [4–7, 10].

The paper is structured as follows. Section 2 contains the definitions which are helpful to prove the main results. Section 3 contains the main results; we find the exact value of $\|M_{\mathbf{K}}^\phi\|_p$ for the case $0 < p \le 1$ and give lower and upper bound when $1 < p < \infty$. An example is given to show that these bounds are not optimal. Finally, Section 4 concludes the study.

2. Preliminaries

Let $G_1, G_2, \ldots, G_m$ be m graphs; then, their strong product $G = G_1 \times G_2 \times \cdots \times G_m$ is a graph having vertex set,

$$V(G) = \{(u_1, u_2, \ldots, u_m)\colon u_i \in G_i\ \forall i = 1, 2, \ldots, m\}, \quad (12)$$

and the edge set, which is defined in the following manner; there will be an edge between $(u_1, u_2, \ldots, u_m)$ and $(v_1, v_2, \ldots, v_m)$ in G if

(a) $u_i = v_i$ and $(u_j, v_j) \in E(G_j)$, $j \neq i$

(b) $u_i \neq v_i$ and $(u_i, v_i) \in E(G_i)$, $\forall i$

Example 1. Let K_2 be complete graph on two vertices. The strong product $K = K_2 \times K_2 \times K_2$ of three K_2 graphs is shown in Figure 1.

Let G be the strong product of m graphs. Then, for every function $f\colon V(G) \longrightarrow \mathbb{R}$, we can consider the generalized maximal operator $M_G^\phi\colon \ell^p \longrightarrow \ell^p$ as

$$M_G^\phi f(u_1, u_2, \ldots, u_m) = \max_{r\in\mathbb{N}} \frac{\sum_{v_1\in G_1}\sum_{v_2\in G_2}\cdots\sum_{v_m\in G_m} |f(v_1, v_2, \ldots, v_m)|}{\phi\left(|B_{G_1}| \times |B_{G_2}| \times \cdots \times |B_{G_m}|\right)}, \quad (13)$$

where $B_{G_i} = B(u_i, r)$, $i = 1, 2, \ldots, m$. The norm $\|M_G^\phi\|_p$ of the generalized maximal operator is defined as

$$\|M_G^\phi\|_p = \sup_{f\neq 0} \frac{\|M_G^\phi f\|_p}{\|f\|_p}, \quad (14)$$

where $\|f\|_p = (\sum_{v_1\in G_1}\sum_{v_2\in G_2}\cdots \sum_{v_m\in G_m} |f(v_1, v_2, \ldots, v_m)|^p)^{(1/p)}$.

Let $K = K_{n_1} \times K_{n_2} \times \cdots \times K_{n_m}$ be the strong product of m complete graphs with $n_1, n_2, \ldots, n_m$ vertices, respectively; then,

$$B(u_1,r)\times B(u_2,r)\times\cdots\times B(u_m,r)=\begin{cases}\{(u_1,u_2,\ldots,u_m)\}, & \text{for } r=0,\\ V(K), & \text{for } r\geq 1.\end{cases} \tag{15}$$

For every function $f\colon V(K)\longrightarrow\mathbb{R}$, the generalized maximal operator takes the form

$$M_{\mathbf{K}}^{\phi}f(u_1,u_2,\ldots,u_m)=\max\left\{\begin{array}{c}\dfrac{1}{\phi(1)}\left|f(u_1,u_2,\ldots,u_m)\right|,\\ \dfrac{1}{\phi(n_1\times n_2\times\cdots\times n_m)}\displaystyle\sum_{v_1\in K_{n_1}}\sum_{v_2\in K_{n_2}}\cdots\sum_{v_m\in K_{n_m}}\left|f(v_1,v_2,\ldots,v_m)\right|\end{array}\right\}. \tag{16}$$

Note that the operator M_K^{ϕ} is the smallest in the pointwise ordering among all $M^{\phi}_{G=G_1\times G_2\times\cdots\times G_m}$, where each G_i is a graph with n_i vertices for $i=1,2,\ldots,m$. That is, for every nonnegative function f and every vertex $(u_1,u_2,\ldots,u_m)\in G$, we have that

$$M_K^{\phi}f(u_1,u_2,\ldots,u_m)\leq M_G^{\phi}f(u_1,u_2,\ldots,u_m). \tag{17}$$

In particular, if $0<p<\infty$, then

$$\left\|M_K^{\phi}\right\|_p^p\leq\left\|M_G^{\phi}\right\|_p^p. \tag{18}$$

For any vertex $(u_1,u_2,\ldots,u_m)\in V(K)$, the m Dirac delta function is defined as

$$\Gamma_{(u_1,u_2,\ldots,u_m)}(v_1,v_2,\ldots,v_m)=\delta_{u_1}(v_1).\delta_{u_2}(v_2)\ldots\delta_{u_m}(v_m). \tag{19}$$

It is easy to check that

$$\Gamma_{(u_1,u_2,\ldots,u_m)}(v_1,v_2,\ldots,v_m)=\begin{cases}0, & \text{for } u_j\neq v_j, \text{ for some } j,\\ 1, & \text{for } u_i=v_i,\ \forall i=1,2,\ldots,m.\end{cases} \tag{20}$$

3. Main Results

This section details the steps to find the quasi-norm of M_K^{ϕ}, for the case $0<p\leq 1$, and to find bounds of $\|M_K^{\phi}\|_p$, for the case of $1<p<\infty$. Also, we estimate the bounds of $\|M_G^{\phi}\|_p$ for $0<p\leq 1$. Moreover, some examples are presented to support the results.

Lemma 1. *Let G be the strong product of m graphs, and $\Omega\colon \ell^p\longrightarrow\ell^p$ be a sublinear operator, with $0<p\leq 1$. Then,*

$$\|\Omega\|_p=\max_{(u_1,u_2,\ldots,u_m)\in V(G)}\left\|\Omega\Gamma_{(u_1,u_2,\ldots,u_m)}\right\|_p. \tag{21}$$

Proof. Since $\|\Gamma_{(u_1,u_2,\ldots,u_m)}\|_p=1$, therefore $\|\Omega\|_p\geq\max_{(u_1,u_2,\ldots,u_m)\in V(\mathbf{G})}\|\Omega\Gamma_{(u_1,u_2,\ldots,u_m)}\|_p$. To prove the other inequality, let $h\colon V(G)\longrightarrow\mathbb{R}$, with $\|h\|_p\leq 1$, that is,

$$h=\sum_{u_1\in G_1}\sum_{u_2\in G_2}\cdots\sum_{u_m\in G_m}a_{(u_1,u_2,\ldots,u_m)}\Gamma_{(u_1,u_2,\ldots,u_m)}, \tag{22}$$

with $\sum_{u_1\in G_1}\sum_{u_2\in G_2}\cdots\sum_{u_m\in G_m}|a_{(u_1,u_2,\ldots,u_m)}|^p\leq 1$. Using Hölder's inequality for $0<p\leq 1$, it follows that

$$\begin{aligned}
\|\Omega h\|_p^p&=\sum_{v_1\in G_1}\sum_{v_2\in G_2}\cdots\sum_{v_m\in G_m}\left|\Omega h(v_1,v_2,\ldots,v_m)\right|^p\\
&=\sum_{v_1\in G_1}\sum_{v_2\in G_2}\cdots\sum_{v_m\in G_m}\left|\Omega\left(\sum_{u_1\in G_1}\sum_{u_2\in G_2}\cdots\sum_{u_m\in G_m}a_{(u_1,u_2,\ldots,u_m)}\Gamma_{(u_1,u_2,\ldots,u_m)}(v_1,v_2,\ldots,v_m)\right)\right|^p\\
&\leq\sum_{v_1\in G_1}\sum_{v_2\in G_2}\cdots\sum_{v_m\in G_m}\left|\sum_{u_1\in G_1}\sum_{u_2\in G_2}\cdots\sum_{u_m\in G_m}\left|a_{(u_1,u_2,\ldots,u_m)}\right|\Omega\Gamma_{(u_1,u_2,\ldots,u_m)}(v_1,v_2,\ldots,v_m)\right|^p\\
&\leq\sum_{v_1\in G_1}\sum_{v_2\in G_2}\cdots\sum_{v_m\in G_m}\sum_{u_1\in G_1}\sum_{u_2\in G_2}\cdots\sum_{u_m\in G_m}\left|a_{(u_1,u_2,\ldots,u_m)}\Omega\Gamma_{(u_1,u_2,\ldots,u_m)}(v_1,v_2,\ldots,v_m)\right|^p\\
&=\sum_{u_1\in G_1}\sum_{u_2\in G_2}\cdots\sum_{u_m\in G_m}\left|a_{(u_1,u_2,\ldots,u_m)}\right|^p\sum_{v_1\in G_1}\sum_{v_2\in G_2}\cdots\sum_{v_m\in G_m}\left|\Omega\Gamma_{(u_1,u_2,\ldots,u_m)}(v_1,v_2,\ldots,v_m)\right|^p\\
&=\sum_{u_1\in G_1}\sum_{u_2\in G_2}\cdots\sum_{u_m\in G_m}\left|a_{(u_1,u_2,\ldots,u_m)}\right|^p\left\|\Omega\Gamma_{(u_1,u_2,\ldots,u_m)}\right\|_p^p\\
&\leq\max_{(u_1,u_2,\ldots,u_m)\in V(G)}\left\|\Omega\Gamma_{(u_1,u_2,\ldots,u_m)}\right\|_p^p.
\end{aligned} \tag{23}$$

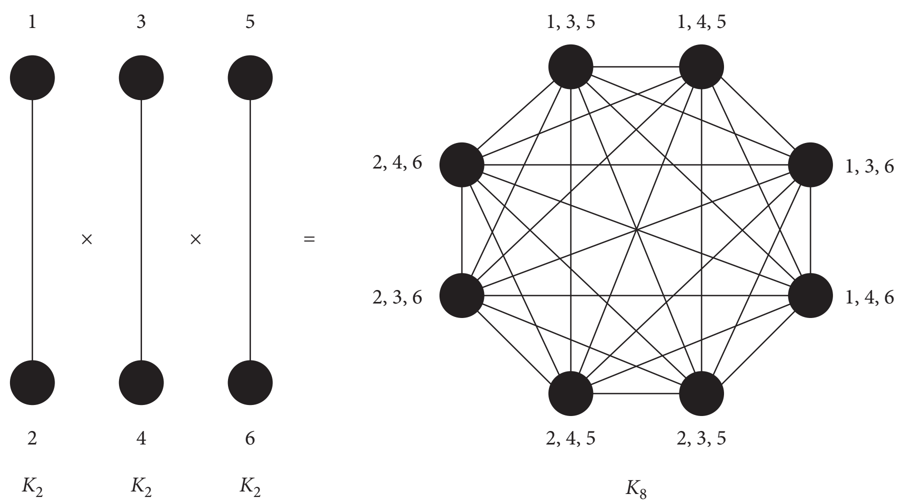

Figure 1: Strong product of three K_2 graphs.

It completes the proof. □

Theorem 1. *If* $0 < p \leq 1$*, then*

$$\left\|M_K^{\phi}\right\|_p = \left(\frac{1}{\phi^p(1)} + \frac{(n_1 \times n_2 \times \cdots \times n_m) - 1}{\phi^p(n_1 \times n_2 \times \cdots \times n_m)}\right)^{(1/p)}, \quad (24)$$

and if $1 < p < \infty$*, then*

$$\begin{aligned}&\left(\frac{1}{\phi^p(1)} + \frac{(n_1 \times n_2 \times \cdots \times n_m) - 1}{\phi^p(n_1 \times n_2 \times \cdots \times n_m)}\right)^{(1/p)} \\ &\leq \left\|M_K^{\phi}\right\|_p \leq \max\left(\frac{(n_1 \times n_2 \times \cdots \times n_m)^p}{\phi^p(n_1 \times n_2 \times \cdots \times n_m)},\right. \\ &\left.\left\{\frac{1}{\phi^p(1)} + \frac{((n_1 \times n_2 \times \cdots \times n_m) - 1)(n_1 \times n_2 \times \cdots \times n_m)^{(p-1)}}{\phi^p(n_1 \times n_2 \times \cdots \times n_m)}\right\}\right)^{(1/p)}.\end{aligned} \quad (25)$$

Proof. Let $f\colon V(K) \longrightarrow \mathbb{R}$ be a function such that $\|f\|_p = 1$. Define m Dirac delta function $\Gamma_{(u_1,u_2,\ldots,u_m)}$, where $u_1 \in V(K_{n_1})$, $u_2 \in V(K_{n_2})$, $\ldots$, $u_m \in V(K_{n_m})$. Then, for $0 < p < \infty$, we have

$$\begin{aligned}\left\|M_K^{\phi}\Gamma_{(u_1,u_2,\ldots,u_m)}\right\|_p &= \left(\left(M_K^{\phi}\Gamma_{(u_1,u_2,\ldots,u_m)}(u_1,u_2,\ldots,u_m)\right)^p + \sum_{v_1 \neq u_1}\sum_{v_2 \neq u_2}\cdots\sum_{v_m \neq u_m}\left(M_K^{\phi}\Gamma_{(u_1,u_2,\ldots,u_m)}(v_1,v_2,\ldots,v_m)\right)^p\right)^{(1/p)} \\ &= \left(\frac{1}{\phi^p(1)} + \frac{(n_1 \times n_2 \times \cdots \times n_m) - 1}{\phi^p(n_1 \times n_2 \times \cdots \times n_m)}\right)^{(1/p)}.\end{aligned} \quad (26)$$

As $\|\Gamma\|_p = 1$, so we have, for $0 < p < \infty$,

$$\left(\frac{1}{\phi^p(1)} + \frac{(n_1 \times n_2 \times \cdots \times n_m) - 1}{\phi^p(n_1 \times n_2 \times \cdots \times n_m)}\right)^{(1/p)} \leq \left\|M_K^{\phi}\right\|_p. \quad (27)$$

For $0 < p \leq 1$, using Lemma 1, we have

$$\left(\frac{1}{\phi^p(1)} + \frac{(n_1 \times n_2 \times \cdots \times n_m) - 1}{\phi^p(n_1 \times n_2 \times \cdots \times n_m)}\right)^{(1/p)} = \left\|M_K^{\phi}\right\|_p. \quad (28)$$

Now, we will prove upper bound for $1 < p < \infty$:

$$\left\|M_K^{\phi} f\right\|_p = \left(\sum_{u_1 \in B_{G_1}} \sum_{u_2 \in B_{G_2}} \cdots \sum_{u_m \in B_{G_m}} \max\left\{ \frac{1}{\phi(1)} |f(u_1, u_2, \ldots, u_m)|, \right.\right.$$
$$\left.\left. \cdot \frac{1}{\phi(n_1 \times n_2 \times \cdots \times n_m)} \sum_{v_1 \in B_{G_1}} \sum_{v_2 \in B_{G_2}} \cdots \sum_{v_m \in B_{G_m}} |f(v_1, v_2, \ldots, v_m)|^p \right\} \right)^{(1/p)}. \tag{29}$$

After applying Hölder's inequality, we have

$$\left\|M_K^{\phi}\right\|_p \leq \sup\left(\sum_{u_1 \in B_{G_1}} \sum_{u_2 \in B_{G_2}} \cdots \sum_{u_m \in B_{G_m}} \max\left\{ \frac{1}{\phi^p(1)} |f(u_1, u_2, \ldots, u_m)|^p, \frac{1}{\phi^p(n_1 \times n_2 \times \cdots \times n_m)} (n_1 \times n_2 \times \cdots \times n_m)^{(p-1)} \right\} \right)^{(1/p)}. \tag{30}$$

If $(|f(u_1, u_2, \ldots, u_m)|^p / \phi^p(1)) \leq ((n_1 \times n_2 \times \cdots \times n_m)^{(p-1)} / \phi^p(n_1 \times n_2 \times \cdots \times n_m))$ for all vertices, then we have

$$\left\|M_K^{\phi}\right\|_p \leq \left(\frac{(n_1 \times n_2 \times \cdots \times n_m)^p}{\phi^p(n_1 \times n_2 \times \cdots \times n_m)} \right)^{(1/p)}. \tag{31}$$

If $(|f(u_1', u_2', \ldots, u_m')|^p / \phi^p(1)) > ((n_1 \times n_2 \times \cdots \times n_m)^{(p-1)} / \phi^p(n_1 \times n_2 \times \cdots \times n_m))$ for some $(u_1', u_2', \ldots, u_m') \in V(K)$, then we have

$$\begin{aligned} \left\|M_K^{\phi}\right\|_p &\leq \left(\frac{1}{\phi^p(1)} |f(u_1', u_2', \ldots, u_m')|^p + \frac{((n_1 \times n_2 \times \cdots \times n_m) - 1)(n_1 \times n_2 \times \cdots \times n_m)^{p-1}}{\phi^p(n_1 \times n_2 \times \cdots \times n_m)} \right)^{(1/p)} \\ &\leq \left(\frac{1}{\phi^p(1)} + \frac{((n_1 \times n_2 \times \cdots \times n_m) - 1)(n_1 \times n_2 \times \cdots \times n_m)^{(p-1)}}{\phi^p(n_1 \times n_2 \times \cdots \times n_m)} \right)^{(1/p)}, \end{aligned} \tag{32}$$

which completes our arguments.

The graph of the result of Theorem 1 is shown in Figure 2, where $\phi(x) = x$, $n_1 \times n_2 \times \cdots \times n_m$ is from 4 to 10, and $p = 2$ and $p = 3$.

3D solution region for Theorem 1 is shown in Figure 3, where $\phi(x) = x$, $n_1 \times n_2 \times \cdots \times n_m$ is from 4 to 12, and p is from 1 to 10.

The graph presented in Figure 3 shows the results of Theorem 1 that are not optimal. It is quite difficult task to calculate the exact value of $\|M_K^{\phi}\|_p$ for the case $1 < p < \infty$. The following example explains the situation. □

Example 2. The estimates we obtained in Theorem 1 for $1 < p < \infty$ is not optimal in general. For example, if we take graph $K_2 \times K_2$ and $\phi(x) = x$. Consider the function f: $\{(1,3), (1,4), (2,3), (2,4)\} \longrightarrow \mathbb{R}$. We suppose that $|f(1,3)| = |f(1,2)| = |f(2,3)|$, $M^x_{K_2 \times K_2} f(1,3) = M^x_{K_2 \times K_2} f(1,4) = M^x_{K_2 \times K_2} f(2,3) = (3|f(1,3)| + |f(1,4)|/4)$, and $M^x_{K_2 \times K_2} f(2,4) = |f(2,4)|$. Then, $|f(2,4)| \geq |f(1,3)|$. If we denote $(|f(1,3)|/|f(2,4)|)$ by λ, then, for every $0 < p < \infty$, we have

$$\begin{aligned} \frac{\left\|M^x_{K_2 \times K_2} f\right\|_p}{\|f\|_p} &= \left(\frac{3(3|f(1,3)| + |f(1,4)|/4)^p + |f(1,4)|^p}{3|f(1,3)|^p + |f(1,4)|^p} \right)^{(1/p)} \\ &= \frac{1}{4} \left(\frac{3(3\lambda + 1)^p + 4^p}{3\lambda^p + 1} \right)^{(1/p)}, \end{aligned} \tag{33}$$

which leads to

$$\left\|M^x_{K_2 \times K_2}\right\|_p = \frac{1}{4} \left(\sup_{0 \leq \lambda \leq 1} \frac{3(3\lambda + 1)^p + 4^p}{3\lambda^p + 1} \right)^{(1/p)}. \tag{34}$$

It is easy to see that, for $1 < p < \infty$, the supremum is attained at the unique root $\lambda_p \in (0,1)$ of the equation

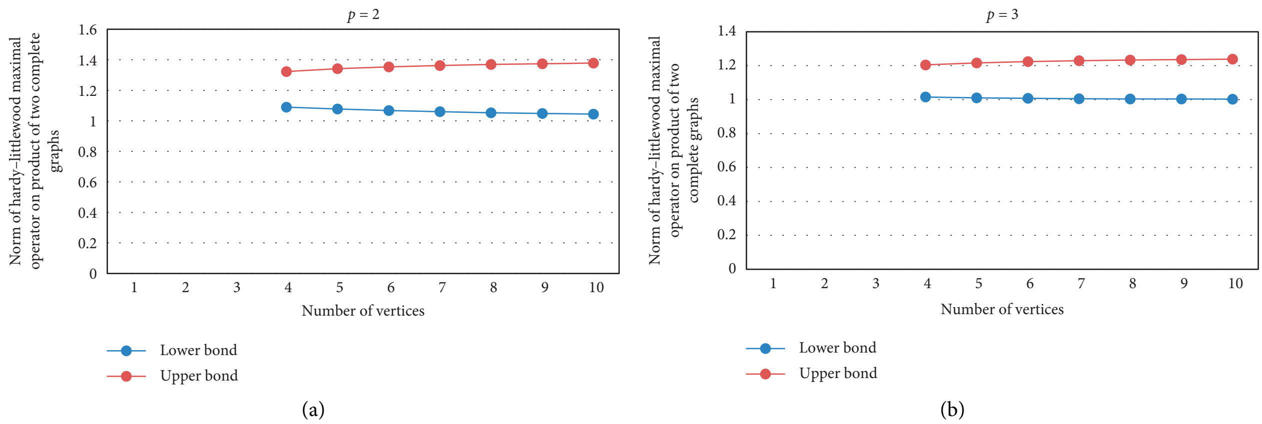

Figure 2: Estimation for $p = 2$ and $p = 3$.

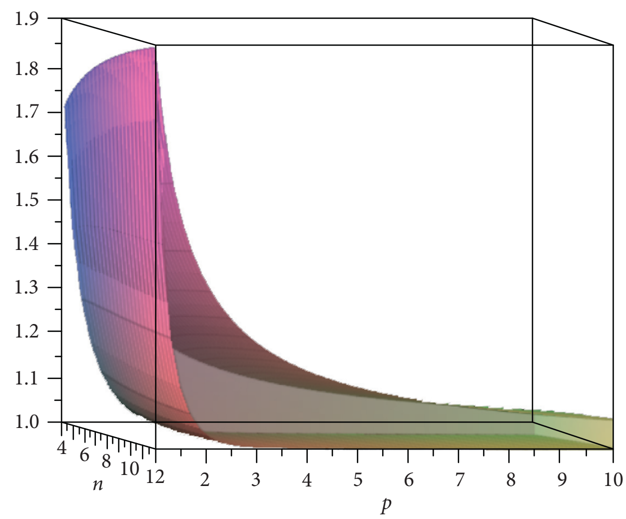

Figure 3: 3D view of estimation for $n = 4 \ldots 12$ and $p = 1 \ldots 10$.

$$(1 + 3\lambda)^{p-1} = \frac{3\lambda^{p-1}(3\lambda + 1)^p + 4^p\lambda^{p-1}}{9\lambda^p + 3}. \tag{35}$$

In particular, if we take $p = 2$, then we get $\lambda = 0.246$, and from equation (34), we get $\|M^x_{K_2\times K_2}\|_2 = 1.151$. If we calculate it from Theorem 1, we get $1.090 \le \|M^x_{K_2\times K_2}\|_2 \le 1.323$. This shows that the estimation in Theorem 1 is not optimal in general for $1 < p < \infty$. Now, in the next theorem, we find the estimates of G.

Theorem 2. *Let G be the strong product of m graphs and $0 < p \le 1$; then, we have*

$$\left(\frac{1}{\phi^p(1)} + \frac{(n_1 \times n_2 \times \cdots \times n_m) - 1}{\phi^p(n_1 \times n_2 \times \cdots \times n_m)}\right)^{(1/p)} \le \left\|M_G^\phi\right\|_p \le \left(\frac{1}{\phi^p(1)} + \frac{(n_1 \times n_2 \times \cdots \times n_m) - 1}{\phi^p(2^m)}\right)^{(1/p)}. \tag{36}$$

Proof. Lower bound is trivial. For the upper bound, let $(u_1, u_2, \ldots, u_m) \in V(G)$ and consider the m Dirac delta function $\Gamma_{(u_1,u_2,\ldots,u_m)}$. Then, we have

$$\left\|M_G^{\phi}\Gamma_{(u_1,u_2,\dots,u_m)}\right\|_p = \left(\left(M_G^{\phi}\Gamma_{(u_1,u_2,\dots,u_m)}(u_1,u_2,\dots,u_m)\right)^p + \sum_{v_1\neq u_1}\sum_{v_2\neq u_2}\cdots\sum_{v_m\neq u_m}\left\{M_G^{\phi}\Gamma_{(u_1,u_2,\dots,u_m)}(v_1,v_2,\dots,v_m)\right\}^p\right)^{(1/p)}$$
$$= \left(\frac{1}{\phi^p(1)} + \sum_{v_1\neq u_1}\sum_{v_2\neq u_2}\cdots\sum_{v_m\neq u_m}\left\{\frac{1}{\phi\left(\left|B_{G_1}\right|\times\left|B_{G_2}\right|\times\cdots\times\left|B_{G_m}\right|\right)}\sum_{w_1}\sum_{w_2}\cdots\sum_{w_m}\Gamma_{(u_1,u_2,\dots,u_m)}(w_1,w_2,\dots,w_m)\right\}^p\right)^{(1/p)}. \tag{37}$$

As each G_i is connected, $|B_{G_i}| \geq 2$ for each i and radius $r \geq 1$. Hence,

$$\left\|M_G^{\phi}\Gamma_{(u_1,u_2,\dots,u_m)}\right\|_p \leq \left(\frac{1}{\phi^p(1)} + \frac{(n_1\times n_2\times\cdots\times n_m)-1}{\phi^p(2^m)}\right)^{(1/p)}. \tag{38}$$

By using Lemma 1, we obtain

$$\left\|M_G^{\phi}\right\|_p \leq \left(\frac{1}{\phi^p(1)} + \frac{(n_1\times n_2\times\cdots n_m)-1}{\phi^p(2^m)}\right)^{(1/p)}. \tag{39}$$

If we take $\phi(x) = x$ and $m = 1$, then Theorems 1 and 2, respectively, yields the same results obtained in [9]. This shows that the results presented in this paper are the generalized form of the results in [6].

We have graph for the result of Theorem 2 in Figure 4, where $p = 0.5$, $\phi(x) = x$, and $n_1 \times n_2 \times \cdots \times n_m$ is from 4 to 10.

Some particular examples to support the result of Theorem 2 are given below. □

Example 3. Let W_5 be a wheel graph on five vertices and consider the strong product $K_2 \times W_5$ of K_2 with W_5. Take $\phi(x) = x$, $f = \Gamma$, and $p = 1$. Then, $\|M_{K_2\times W5}^x\| = 2.100$.

Let $V(K_2) = \{1, 2\}$ and $V(W_5) = \{3, 4, 5, 6, 7\}$, where 7 is the central vertex of W_5. Now, $6 \notin N_{W_5}(3)$ and $5 \notin N_{W_5}(4)$. Then, the strong product $K_2 \times W_5$ has a vertex set

$$V(K_2\times W_5) = \{(1,3),(1,4),(1,5),(1,6),(1,7),(2,3),(2,4),(2,5),(2,6),(2,7)\}. \tag{40}$$

Note that $K_2 \times W_5$ has 37 edges and $d_{(1,7)}(K_2\times W_5) = d_{(2,7)}(K_2\times W_5) = 9$, while all other vertices of this graph have degree 7. Hence,

$$M_{K_2\times W_5}^x\Gamma_{(1,3)} = \begin{cases} 1, & \text{for } \{(1,3)\}, \\ \frac{1}{10}, & \text{for } \{(1,6),(1,7),(2,6),(2,7)\}, \\ \frac{1}{8}, & \text{otherwise,} \end{cases} \tag{41}$$

with $\|M_{K_2\times W_5}^x\Gamma_{(1,3)}\| = 2.025$. It is easy to see that $\|M_{K_2\times W_5}^x\Gamma_{(1,4)}\| = \|M_{K_2\times W_5}^x\Gamma_{(1,5)}\| = \|M_{K_2\times W_5}^x\Gamma_{(1,6)}\| = \|M_{K_2\times W_5}^x\Gamma_{(2,3)}\| = \|M_{K_2\times W_5}^x\Gamma_{(2,4)}\| = \|M_{K_2\times W_5}^x\Gamma_{(2,5)}\| = \|M_{K_2\times W_5}^x\Gamma_{(2,6)}\| = 2.025$.

Also,

$$M_{K_2\times W_5}^x\Gamma_{(1,7)} = \begin{cases} 1, & \text{for } \{(1,7)\}, \\ \frac{1}{10}, & \text{for } \{(2,7)\}, \\ \frac{1}{8}, & \text{otherwise,} \end{cases} \tag{42}$$

with $\|M_{K_2\times W_5}^x\Gamma_{(1,7)}\| = 2.100$, $\|M_{K_2\times W_5}^x\Gamma_{(2,7)}\| = 2.100$, and $\|M_{K_2\times W_5}^x\| = 2.100$.

Example 4. Consider the graph used in Example 3. Take $\phi(x) = x^2$, $f = \Gamma$, and $p = 1$. Then, $\|M_{K_2\times W5}^{x^2}\| = 1.135$. We have

$$M_{K_2\times W_5}^{x^2}\Gamma_{(1,3)} = \begin{cases} 1, & \text{for } \{(1,3)\}, \\ \frac{1}{100}, & \text{for } \{(1,6),(1,7),(2,6),(2,7)\}, \\ \frac{1}{64}, & \text{otherwise,} \end{cases} \tag{43}$$

with $\|M_{K_2\times W_5}^{x^2}\Gamma_{(1,3)}\| = 1.118$ and $\|M_{K_2\times W_5}^{x^2}\Gamma_{(1,4)}\| = \|M_{K_2\times W_5}^{x^2}\Gamma_{(1,5)}\| = \|M_{K_2\times W_5}^{x^2}\Gamma_{(1,6)}\| = \|M_{K_2\times W_5}^{x^2}\Gamma_{(2,3)}\| = \|M_{K_2\times W_5}^{x^2}\Gamma_{(2,4)}\| = \|M_{K_2\times W_5}^{x^2}\Gamma_{(2,5)}\| = \|M_{K_2\times W_5}^{x^2}\Gamma_{(2,6)}\| = 1.118$.

In a similar way, we have

$$M_{K_2\times W_5}^{x^2}\Gamma_{(1,7)} = \begin{cases} 1, & \text{for } \{(1,7)\}, \\ \frac{1}{100}, & \text{for } \{(2,7)\}, \\ \frac{1}{64}, & \text{otherwise,} \end{cases} \tag{44}$$

with $\|M_{K_2\times W_5}^{x^2}\Gamma_{(1,7)}\| = 1.135$ and $\|M_{K_2\times W_5}^{x^2}\Gamma_{(2,7)}\| = 1.135$. This implies that $\|M_{K_2\times W_5}^{x^2}\| = 1.135$.

Example 5. Let S_3 be star graph on three vertices and consider the strong product $K_2 \times K_2 \times S_3$. Take $\phi(x) = x$, $f = \Gamma$, and $p = 1$. Then, $\|M_{K_2\times K_2\times S_3}^x\| = 2.250$.

Let $V(K_2) = \{1, 2\}$, $V(K_2) = \{3, 4\}$, and $V(S_3) = \{5, 6, 7\}$ with 7 as a central vertex of S_3. Then, the strong product $K_2 \times K_2 \times S_3$ is a graph with vertex set

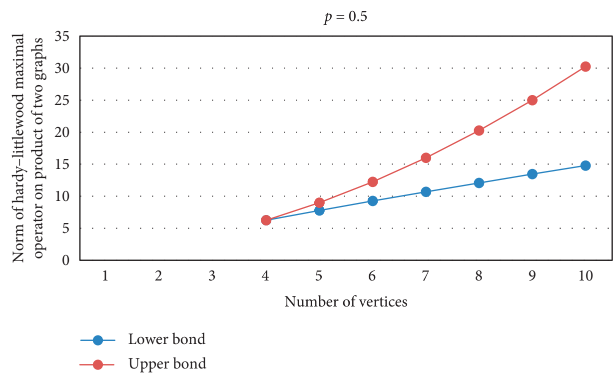

Figure 4: Estimation for $p = 0.5$.

$$V(K_2 \times K_2 \times S_3) = \{(1,3,5), (1,3,6), (1,3,7), (1,4,5), (1,4,6), (1,4,7), (2,3,5), (2,3,6), (2,3,7), (2,4,5), (2,4,6), (2,4,7)\}. \tag{45}$$

Note that there are 50 edges in this graph and $d_{(1,3,7)}(K_2 \times K_2 \times S_3) = d_{(1,4,7)}(K_2 \times K_2 \times S_3) = d_{(2,3,7)}(K_2 \times K_2 \times S_3) = d_{(2,4,7)}(K_2 \times K_2 \times S_3) = 11$, while all the other vertices of the graph have degree 7. We have

$$M^x_{K_2 \times K_2 \times S_3} \Gamma_{(1,3,5)} = \begin{cases} 1, & \text{for } \{(1,3,5)\}, \\ \dfrac{1}{8}, & \text{for } \{(1,4,5), (2,3,5), (2,4,5)\}, \\ \dfrac{1}{12}, & \text{otherwise,} \end{cases} \tag{46}$$

with $\|M^x_{K_2 \times K_2 \times S_3} \Gamma_{(1,3,5)}\| = 2.042$. It is easy to see that $\|M^x_{K_2 \times K_2 \times S_3} \Gamma_{(1,3,6)}\| = \|M^x_{K_2 \times K_2 \times S_3} \Gamma_{(1,4,5)}\| = \|M^x_{K_2 \times K_2 \times S_3} \Gamma_{(1,4,6)}\| = \|M^x_{K_2 \times K_2 \times S_3} \Gamma_{(2,3,5)}\| = \|M^x_{K_2 \times K_2 \times S_3} \Gamma_{(2,3,6)}\| = \|M^x_{K_2 \times K_2 \times S_3} \Gamma_{(2,4,5)}\| = \|M^x_{K_2 \times K_2 \times S_3} \Gamma_{(2,4,6)}\| = 2.042$.

Also,

$$M^x_{K_2 \times K_2 \times S_3} \Gamma_{(1,3,7)} = \begin{cases} 1, & \text{for } \{(1,3,7)\}, \\ \dfrac{1}{12}, & \text{for } \{(1,4,7), (2,3,7), (2,4,7)\}, \\ \dfrac{1}{8}, & \text{otherwise,} \end{cases} \tag{47}$$

with $\|M^x_{K_2 \times K_2 \times S_3} \Gamma_{(1,3,7)}\| = 2.250$. Similarly, $\|M^x_{K_2 \times K_2 \times S_3} \Gamma_{(1,4,7)}\| = \|M^x_{K_2 \times K_2 \times S_3} \Gamma_{(2,3,7)}\| = \|M^x_{K_2 \times K_2 \times S_3} \Gamma_{(2,4,7)}\| = 2.250$. This implies that $\|M^x_{K_2 \times K_2 \times S_3}\| = 2.250$.

Example 6. Consider the graph used in example 4, $\phi(x) = 1 + \ln(x)$, $f = \Gamma$, and $p = 1$; then, $\|M^{1+\ln(x)}_{K_2 \times K_2 \times S_3}\| = 4.459$.

Here, we have

$$M^{1+\ln(x)}_{K_2 \times K_2 \times S_3} \Gamma_{(1,3,5)} = \begin{cases} 1, & \text{for } \{(1,3,5)\}, \\ \dfrac{1}{1+\ln(8)}, & \text{for } \{(1,4,5), (2,3,5), (2,4,5)\}, \\ \dfrac{1}{1+\ln(12)}, & \text{otherwise,} \end{cases} \tag{48}$$

with $\|M^{1+\ln(x)}_{K_2 \times K_2 \times S_3} \Gamma_{(1,3,5)}\| = 4.271$. Similarly, $\|M^{1+\ln(x)}_{K_2 \times K_2 \times S_3} \Gamma_{(1,3,6)}\| = \|M^{1+\ln(x)}_{K_2 \times K_2 \times S_3} \Gamma_{(1,4,5)}\| = \|M^{1+\ln(x)}_{K_2 \times K_2 \times S_3} \Gamma_{(1,4,6)}\| = \|M^{1+\ln(x)}_{K_2 \times K_2 \times S_3} \Gamma_{(2,3,5)}\| = \|M^{1+\ln(x)}_{K_2 \times K_2 \times S_3} \Gamma_{(2,3,6)}\| = \|M^{1+\ln(x)}_{K_2 \times K_2 \times S_3} \Gamma_{(2,4,5)}\| = \|M^{1+\ln(x)}_{K_2 \times K_2 \times S_3} \Gamma_{(2,4,6)}\| = 4.271$.

Now,

$$M^{1+\ln(x)}_{K_2 \times K_2 \times S_3} \Gamma_{(1,3,7)} = \begin{cases} 1, & \text{for } \{(1,3,7)\}, \\ \dfrac{1}{1+\ln(12)}, & \text{for } \{(1,4,7), (2,3,7), (2,4,7)\}, \\ \dfrac{1}{1+\ln(8)}, & \text{otherwise.} \end{cases} \tag{49}$$

So, $\|M^{1+\ln(x)}_{K_2 \times K_2 \times S_3} \Gamma_{(1,3,7)}\| = 4.459$. Similarly, $\|M^{1+\ln(x)}_{K_2 \times K_2 \times S_3} \Gamma_{(1,4,7)}\| = \|M^{1+\ln(x)}_{K_2 \times K_2 \times S_3} \Gamma_{(2,3,7)}\| = \|M^{1+\ln(x)}_{K_2 \times K_2 \times S_3} \Gamma_{(2,4,7)}\| = 4.459. \Rightarrow \|M^{1+\ln(x)}_{K_2 \times K_2 \times S_3}\| = 4.459$.

If we take the same conditions which we used in examples 3–6 in the result of Theorem 2, then we get $1.900 \leq \|M^x_{G_1 \times G_2}\| \leq 3.250$, $1.090 \leq \|M^{x^2}_{G_1 \times G_2}\| \leq 1.563$, $1.917 \leq \|M^x_{G_1 \times G_2 \times G_3}\| \leq 2.375$, and $4.156 \leq \|M^{1+\ln(x)}_{G_1 \times G_2 \times G_3}\| \leq 4.572$. This implies that the examples 3–6 verify the result of Theorem 2.

4. Conclusion

In this paper, we have considered the action of generalized maximal operator on ℓ^p spaces and calculated the quasi-norm $\|M^{\phi}_K\|_p$ for $0 < p \leq 1$. We gave the lower bound and upper bound for the quasi-norm $\|M_K^{\phi}\|_p$, where $1 < p < \infty$. Finally, we have proved that $((1/\phi^p\ (1)) + ((n_1 \times n_2 \times \cdots \times n_m) - 1/\phi^p(n_1 \times n_2 \times \cdots \times n_m)))^{(1/p)}$ and $((1/\phi^p(1)) + ((n_1 \times n_2 \times \cdots \times n_m) - 1/\phi^p(2^m)))^{(1/p)}$ are the lower bound and upper bound, respectively.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] B. Bolloba´s, *Modern Graph Theory, Graduate Texts in Mathematics*, vol. 184, Springer-Verlag, New York, NY, USA.
[2] A. Bondy and U. S. R. Murty, *Graph Theory, Graduate Texts in Mathematics*, vol. 244, Springer-Verlag, New York, NY, USA.
[3] R. Hammack, W. Imrich, and S. Klavžar, *Handbook of Product Graphs*, CRC Press, Boca Raton, FL, USA, 2011.
[4] M. Cowling, S. Meda, and A. G. Setti, "Estimates for functions of the Laplace operator on homogeneous trees," *Transactions of the American Mathematical Society*, vol. 352, no. 9, pp. 4271–4293, 2000.
[5] C. Gonzalez-Riquelme and J. Madrid, "Sharp P-bounds for maximal operators on finite graphs," 2020, http://arxiv.org/abs/2005.03146.
[6] J. Soria and P. Tradacete, "Best constants for the hardy-littlewood maximal operator on finite graphs," *Journal of Mathematical Analysis and Applications*, vol. 436, no. 2, pp. 661–682, 2016.
[7] J. Soria and P. Tradacete, "Geometric properties of infinite graphs and the hardy–littlewood maximal operator," *Journal d'Analyse Mathematique*, vol. 1–25, 2016.
[8] C. Capone, D. Cruz-Uribe, A. Fiorenza, and A. Fiorenza, "The fractional maximal operator and fractional integrals on variable L^p spaces," *Revista Matemática Iberoamericana*, vol. 23, no. 3, pp. 743–770, 2007.
[9] I. Ahmad and W. Nazeer, "Optimal couples of rearrangement invariant spaces for generalized maximal operators," *Journal of Function Spaces*, vol. 2014, Article ID 647123, 5 pages, 2014.
[10] Z. Hussain and S. Talib, "A note on the paper "best constants for the hardy–litllewood maximal operator on finite graphs"" *Hacettepe Journal of Mathematics and Statistics*, vol. 49, no. 2, pp. 498–504, 2020.
[11] D. Aalto and J. Kinnunen, "The discrete maximal operator in metric spaces," *Journal d'Analyse Mathématique*, vol. 111, no. 1, pp. 369–390, 2010.
[12] N. Badr and J. M. Martell, "Weighted norm inequalities on graphs," *Journal of Geometric Analysis*, vol. 22, no. 4, pp. 1173–1210, 2012.
[13] A. Koranyi and M. A. Picardello, "Boundary behaviour of eigenfunctions of the Laplace operator on trees," *Annali della Scuola Normale Superiore. Classe di Scienze*, vol. 13, no. 3, pp. 389–399, 1986.