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Research Article

Design of a Disturbance Rejection Controller for a Class of Nonholonomic Systems with Uncertainties

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This study investigates the global output feedback stabilization problem for one type of the nonholonomic system with non-vanishing external disturbances. An extended state observer (ESO) is constructed in order to estimate the external disturbance and unmeasurable system states, in which the external disturbance term is seen as a general state. Thus, a new generalized error dynamic system is obtained. Accordingly, a disturbance rejection controller is designed by making use of the backstepping technique. A control law is given to ensure that all the signals in the closed-loop system are globally bounded, while the system states converge to an equilibrium point. The simulation example is proposed to verify that the control algorithm is effective.

1. Introduction

Within recent decades, the control of nonholonomic systems has always been one of the most popular tasks in control fields since such systems can be frequently found in mechanical systems, for example, car-like vehicles, wheeled mobile robots, knife-edge, and so on. In the theoretical analysis of the nonholonomic system model, some nonlinear feedback controllers for these systems were put forward in the literature to ensure that the systems are asymptotically stable or exponentially regulatable, for example, the studies [1–7] and references therein. By using an input/state scaling technique and switching algorithm, a class of feedback control law was obtained for nonholonomic chained systems with uncertainties to realize exponential stabilization [6, 7], and a switching-based state scaling is designed for prescribed-time stabilization of nonholonomic systems with actuator dead-zones [8]. In practical applications, especially in the research of nonholonomic wheeled mobile robot control, the controller design method to realize the robust stabilization of the system is given [6, 9, 10]. Considering the limitations of the hardware and environment of the actual system, the design method of the controller with saturated input is given in [11, 12]. In order to overcome the external disturbances, the robust tracking control for the wheeled mobile robot is proposed based on the ESO [13, 14].

The measurement of full states is usually difficult and sometimes impossible. Moreover, in practical applications, the systems usually contain unknown disturbances, measurement noise, and modeling errors, which are called nonvanishing total disturbances. These disturbances in reality will influence the performance of closed-loop systems. Therefore, it is of great significance to study the output feedback stabilization of nonholonomic systems with nonlinear uncertainties and external disturbances. The output feedback stabilization for nonholonomic systems is more complex and difficult than using the general nonlinear state feedback. The output feedback problem towards asymptotic stability and exponential stability of nonholonomic systems has previously been put forward [15, 16]. In [17–20], the adaptive output feedback global stabilization of a class of nonholonomic systems with parametric uncertainties and strong nonlinear drifts are solved. However, none of the above work considers the existence of disturbance items that do not disappear from the system even though uncertainties or nonlinear drifts exist. This means that the proposed output feedback scheme may be unstable because of the external disturbances. To reject the external disturbances, an output feedback controller has been proposed for nonholonomic systems with nonlinear uncertainties [21] and nonvanishing external disturbances [22–24]. In [23], the external disturbances are considered a generalized system state, and an ESO was constructed. By utilizing the so-called ESO, [25] further investigated the output regulation control problem towards one type of cascade nonlinear systems with the external disturbance, and the output feedback adaptive regulation problem was solved by the time-varying Kalman observer [25]. However, the output regulation controller in [22, 23, 25] requires that the nonlinear uncertainties in the systems are only related to the output of the systems.

The ESO in pioneering work [26] is the key creative advancement towards active disturbance rejection control (ADRC). The ESO has the capability for state observation and real-time estimation of generalized disturbances between the controlled object and the model of the controlled system [27, 28]. By using the ESO, this study addresses robust output feedback adaptive control towards one type of nonholonomic chained form systems that have nonvanishing external disturbances in the input channel and uncertain nonlinearity drift. Different from references [22, 23], in the model studied in this study, the upper bound function of nonlinear uncertainties depend not only on the output variables but also on the system state variables, in which such uncertain nonlinearities meet a linearly growing triangular condition.

The main contribution of this study is that the extended state observer (ESO) and gain scaling technique [29] are constructed. In order overcome unknown system states and the external disturbance, we reconstruct the system state, and the disturbance is regarded as an extended state. The ESO with dynamic gain is put forward, and the disturbance rejection controller based on an observer is developed by designing a variable observer gain to overcome the uncertainty. The controller design is carried out for one type of the nonholonomic system with nonvanishing external disturbances and uncertain nonlinearities satisfying a linearly growing triangular condition. This approach allows the external disturbances to be a larger class of signals.

2. Problem Formulation

In this study, we consider the following nonholonomic system with nonlinear uncertainties and nonvanishing external disturbance:

$$\begin{cases} \dot{x}_0 = u_0 + x_0\phi_0^d(t, x_0), \\ \dot{x}_i = x_{i+1}u_0 + \phi_i^d(t, u_0, x_0, x, \omega_0(t)), \\ \dot{x}_n = u + \phi_n^d(t, u_0, x_0, x, \omega_0(t)) + w(t), \\ y = [x_0, x_1]^T, \end{cases} \tag{1}$$

where $x_0 \in R$, $x = [x_1, \ldots, x_n]^T \in R^n$ are the system states, and the initial values are $x_0(t_0)$, $x(t_0)$, with t_0 as the initial moment of the system; $u = [u_0, u]^T \in R^2$ is the control input, and $y \in R^2$ is the system output. The functions $\phi_i^d(\cdot)$, $i = 1, 2, \ldots, n$ are the uncertainties which represent possible modeling errors and neglected dynamics; $w_0(t)$, $\dot{w}_0(t)$, $w(t) \in R$, and $\dot{w}(t)$ are the uncertainties and bounded, where $w(t) \in R$ is the nonvanishing external disturbance and satisfies that $\dot{w}(t) \in L_2$. The assumptions and lemmas used in this article are listed as follows.

Assumption 1. For every $1 \le i \le n$, the following inequality holds:

$$\begin{aligned} \left|\phi_0^d(t, x_0)\right| &\le \alpha_0(x_0), \\ \left|\phi_i^d(t, u_0, x_0, x, \omega_0(t))\right| &\le \alpha(x_0)\sum_{j=1}^{i}\left|x_j\right|, \end{aligned} \tag{2}$$

where the nonnegative smooth functions $\alpha_0(x_0)$ and $\alpha(x_0)$ are known.

Lemma 1 (see [30]). *For any $x, y \in R$, any scalar $k > 0$, and any positive definite matrix $M \in R^{(n+1)\times(n+1)}$, the following inequality holds:*

$$2x^T y \le k^{-1}x^T \mathrm{Mx} + \mathrm{ky}^T M^{-1} y. \tag{3}$$

Lemma 2 (see [31, 32]). *For any $\mu > 0$, there exist positive real numbers d_1 and d_2, positive definite matrix P, and positive constants a_i, such that the following inequality is satisfied:*

$$\mathrm{PA} + A^T P \le -d_1 I_{n+1}, \ \mathrm{PD} + \mathrm{DP} \ge d_2 I_{n+1}, \tag{4}$$

where I_i is the identity matrix of order i, and A and D are the $(n+1)\times(n+1)$ matrices denoted as

$$\begin{aligned} A &= \begin{bmatrix} -a_1 & & \\ \vdots & I_n & \\ -a_{n+1} & 0 \ \cdots & 0 \end{bmatrix}, \\ D &= \operatorname{diag}\{\mu, 1+\mu, \ldots, n+\mu\}. \end{aligned} \tag{5}$$

3. Controller Design and Stability Analysis

3.1. Output Feedback Controller Design

Lemma 3 (see [33]). *For the first subsystem of (1), if the first control law u_0 is chosen as*

$$u_0 = -\lambda_0 x_0 - x_0\alpha_0(x_0), \lambda_0 > 0, \tag{6}$$

where $(t_0, x_0(t_0))$ is regarded as the initial condition, $x_0(t_0) \ne 0$, then as the corresponding solution, $x_0(t, t_0, x_0(t_0))$ exists and $|x_0(t, t_0, x_0(t_0))| > 0$, $0 \le t_0 < t$. λ_0 is designed as a positive constant parameter. Furthermore, $|u_0(t)| > 0$.

Proof. Substituting (6) into the first formula of system (1), we can obtain

$$\dot{x}_0 = -\lambda_0 x_0 - x_0\alpha_0(x_0) + x_0\phi_0^d(t, x_0). \tag{7}$$

Integrating this nonlinear equation, the solution is $x_0(t) = x_0(0)e^{-\int_0^t \lambda_0+\alpha_0(x_0(\tau))-\phi_0^d(\tau,x_0(\tau))\mathrm{d}\tau}$. This shows that $x_0(t) \neq 0$ at any time if $x_0(0) \neq 0$, and thus, $u_0(t) \neq 0$. Choosing $V_0(x_0) = (1/2)x_0^2$, it can be obtained from (6) that

$$\dot{V}_0 = -\lambda_0 x_0^2 - x_0^2\left(\alpha_0(x_0) - \phi_0^d(t,x_0)\right) \leq -\lambda_0 x_0^2 \leq 0. \quad (8)$$

Thus, $x_0(t)$ asymptotically approaches zero. For any bounded $x_0(t)$ because α_0, ϕ_0^d are smooth functions, there is a positive constant M for $|x_0| \leq 1$, $|\alpha_0| \leq M$, $|\phi_0^d| \leq M$. This yields that

$$\begin{aligned}\dot{V}_0 &= -\lambda_0 x_0^2 - x_0^2\left(\alpha_0(x_0) - \phi_0^d(t,x_0)\right)\\ &\geq -(\lambda_0 + M + M)V_0.\end{aligned} \quad (9)$$

Integrating both sides of this equation, it can be obtained that

$$V_0(t) \geq V_0(0)e^{-(\lambda_0+2M)t}. \quad (10)$$

This means that $x_0(t)$ converges to zero, but $x_0(t) \neq 0$ at any given moment, so that $|u_0(t)| > 0$.

We introduce the following input state scaling:

$$\zeta_i = \frac{x_i}{u_0^{n-i}}, \quad i = 1, \ldots, n. \quad (11)$$

Unknown nonvanishing external disturbance $w(t)$ is treated as a generalized state. To realize symbol consistency, it is defined as

$$\zeta_{n+1} = w(t). \quad (12)$$

In the new state ζ, system (1) is converted to

$$\begin{cases}\dot{\zeta}_i = \zeta_{i+1} + \overline{\phi}_i(t,u_0,x_0,\zeta,\omega_0(t)),\\ \dot{\zeta}_n = \zeta_{n+1} + \overline{\phi}_n(t,u_0,x_0,\zeta,\omega_0(t)) + u,\\ \dot{\zeta}_{n+1} = \dot{w}(t) \triangleq h(t), \quad i = 1,\ldots,n-1.\end{cases} \quad (13)$$

□

Lemma 4. *For any given u_0 in (6), there is a known nonnegative smooth function $\overline{\varphi}_0(x_0)$, such that $|\dot{u}_0/u_0| \leq \overline{\varphi}_0(x_0)$, $t \geq 0$.*

Proof. The following calculation is completed:

$$\begin{aligned}\left|\frac{\dot{u}_0}{u_0}\right| &= \left|\frac{-x_0\left(\lambda_0 + \alpha_0(x_0) - \phi_0^d(t,x_0)\right)}{\lambda_0 + \alpha_0(x_0)}\frac{\partial\alpha_0(x_0)}{\partial x_0} - \lambda_0 - \alpha_0(x_0) + \phi_0^d(t,x_0)\right|\\ &\leq \left|\lambda_0\left(1 + \frac{\lambda_0 x_0}{\lambda_0 + \alpha_0(x_0)}\frac{\partial\alpha_0(x_0)}{\partial x_0}\right)\right| \triangleq \overline{\varphi}_0(x_0).\end{aligned} \quad (14)$$

This completes the proof of the lemma.

We know from Assumption 1 that there is a nonnegative smooth function $\overline{\alpha}(x_0)$:

$$\begin{aligned}\left|\overline{\phi}_i(t,u_0,x_0,\zeta,\omega_0(t))\right| &\leq \frac{\alpha(x_0)\sum_{j=1}^{i}\left|x_j\right|}{\left|u_0^{n-i}\right|} + (n-i)\overline{\alpha}_0(x_0)\left|\zeta_i\right|\\ &\leq (n-i)\overline{\alpha}_0(x_0)\left|\zeta_i\right| + \alpha(x_0)\left(\left|u_0^{i-1}\right|\left|\zeta_1\right| + \left|u_0^{i-2}\right|\left|\zeta_2\right| + \cdots + \left|\zeta_i\right|\right)\\ &\leq \overline{\alpha}(x_0)\sum_{j=1}^{i}\left|\zeta_i\right|.\end{aligned} \quad (15)$$

Considering that $(\zeta_1,\ldots,\zeta_{n+1})$ are unmeasurable signals that cannot be used in feedback control, the dynamic observer for (13) is denoted as follows:

$$\begin{cases}\dot{\hat{\zeta}}_i = \hat{\zeta}_{i+1} + a_i\gamma^i\left(\zeta_1 - \hat{\zeta}_1\right), \quad i = 1,\ldots,n-1,\\ \dot{\hat{\zeta}}_n = u + \hat{\zeta}_{n+1} + a_n\gamma^n\left(\zeta_1 - \hat{\zeta}_1\right),\\ \dot{\hat{\zeta}}_{n+1} = a_{n+1}\gamma^{n+1}\left(\zeta_1 - \hat{\zeta}_1\right) - \gamma c\hat{\zeta}_{n+1},\end{cases} \quad (16)$$

where γ is the dynamic gain, which will be designed later according to the requirements. The observer error dynamics is defined as

$$e_i = \zeta_i - \hat{\zeta}_i \quad (i = 1, \ldots, n+1). \tag{17}$$

We can determine from (13), (16), and (17) that

$$\begin{cases} \dot{e}_i = e_{i+1} + \overline{\phi}_i(t, u_0, x_0, \zeta, \omega_0(t)) - a_i\gamma^i(\zeta_1 - \hat{\zeta}_1), \\ \dot{e}_n = e_{n+1} + \overline{\phi}_n(t, u_0, x_0, \zeta, \omega_0(t)) - a_n\gamma^n(\zeta_1 - \hat{\zeta}_1), \\ \dot{e}_{n+1} = h(t) - a_{n+1}\gamma^{n+1}(\zeta_1 - \hat{\zeta}_1) + \gamma c\hat{\zeta}_{n+1}. \end{cases} \tag{18}$$

Introducing dynamic gain scaling,

$$\varepsilon_i = \frac{e_i}{\gamma^{i-1+\mu}},\ z_i = \frac{\hat{\zeta}_i}{\gamma^{i-1+\mu}}, \quad i = 1, \ldots, n+1, \tag{19}$$

and defining $\varepsilon = [\varepsilon_1, \ldots, \varepsilon_{n+1}]^T$, $z = [z_1, \ldots, z_{n+1}]^T$, $\Phi(\cdot) = [\overline{\phi}_1(\cdot)/\gamma^{\mu}, \ldots, \overline{\phi}_n(\cdot)/\gamma^{n-1+\mu}, 0]^T$, we arrive at

$$\begin{cases} \dot{\varepsilon}_i = \gamma\varepsilon_{i+1} + \Phi_i(t, u_0, x_0, \zeta, \omega_0(t)) - \gamma a_i\varepsilon_1 - \dfrac{\dot{\gamma}}{\gamma}(i-1+\mu)\varepsilon_i, \\ \dot{\varepsilon}_n = \gamma\varepsilon_{n+1} + \Phi_n(t, u_0, x_0, \zeta, \omega_0(t)) - \gamma a_n\varepsilon_1 - \dfrac{\dot{\gamma}}{\gamma}(n-1+\mu)\varepsilon_n, \\ \dot{\varepsilon}_{n+1} = \dfrac{h(t)}{\gamma^{n+\mu}} - \gamma a_{n+1}\varepsilon_1 - \dfrac{\dot{\gamma}}{\gamma}(n+\mu)\varepsilon_{n+1} + \gamma c z_{n+1}, \end{cases} \tag{20}$$

$$\begin{cases} \dot{z}_i = \gamma z_{i+1} + \gamma a_i\varepsilon_1 - \dfrac{\dot{\gamma}}{\gamma}(i-1+\mu)z_i, \\ \dot{z}_n = \dfrac{u}{\gamma^{n-1+\mu}} + \gamma z_{n+1} + \gamma a_n\varepsilon_1 - \dfrac{\dot{\gamma}}{\gamma}(n-1+\mu)z_n, \\ \dot{z}_{n+1} = \gamma a_{n+1}\varepsilon_1 - \dfrac{\dot{\gamma}}{\gamma}(n+\mu)z_{n+1} - \gamma c z_{n+1}. \end{cases} \tag{21}$$

Denoting $a = [a_1, \ldots, a_{n+1}]^T$, $b = [0, \ldots, 1]^T_{1\times(n+1)}$, $F = [0, \ldots, \gamma c z_{n+1}]^T_{1\times(n+1)}$, and $\overline{h}(t) = h(t)/\gamma^{n+\mu}$, we have

$$\dot{\varepsilon} = \gamma A\varepsilon + \Phi(t, u_0, x_0, \zeta, \omega_0(t)) - \frac{\dot{\gamma}}{\gamma}D\varepsilon + b\overline{h}(t) + F. \tag{22}$$

Now, using Assumption 1 and (15), it follows that

$$\left|\Phi_i(t, u_0, x_0, \zeta, \omega_0(t))\right| \le \frac{\overline{\alpha}(x_0)}{\gamma^{i-1+\mu}}\sum_{j=1}^{i}\left|\zeta_j\right|. \tag{23}$$

Choosing $V_\varepsilon = \varepsilon^T P\varepsilon$, it is then obtained that

$$\begin{aligned} \dot{V}_\varepsilon &= \dot{\varepsilon}^T P\varepsilon + \varepsilon^T P\dot{\varepsilon} = \left(\gamma A\varepsilon + \Phi - \frac{\dot{\gamma}}{\gamma}D\varepsilon + b\overline{h}(t) + F\right)^T P\varepsilon \\ &\quad + \varepsilon^T P\left(\Phi + \gamma A\varepsilon - \frac{\dot{\gamma}}{\gamma}D\varepsilon + b\overline{h}(t) + F\right) = \gamma\varepsilon^T\left(A^T P + \mathrm{PA}\right)\varepsilon \\ &\quad + 2\varepsilon^T P\Phi - \frac{\dot{\gamma}}{\gamma}\varepsilon^T(\mathrm{DP} + \mathrm{PD})\varepsilon + 2\varepsilon^T \mathrm{Pb}\overline{h}(t) + 2\varepsilon^T PF \\ &\le 2\varepsilon^T P\Phi - \gamma d_1\varepsilon^T\varepsilon + 2\varepsilon^T \mathrm{Pb}\overline{h}(t) + 2\varepsilon^T \mathrm{PF} - \frac{\dot{\gamma}}{\gamma}d_2\varepsilon^T\varepsilon. \end{aligned} \tag{24}$$

Introducing the following transformation,

$$\begin{cases} \hat{x}_1 = z_1, \\ \hat{x}_i = z_i - \alpha_{i-1}, \alpha_{i-1} = -g_{i-1}\hat{x}_{i-1}, \quad i = 2, \ldots, n, \\ \hat{x}_{n+1} = z_{n+1}, \end{cases} \tag{25}$$

where $g_{i-1} > 0$ is a constant number that will be given later, because $\varepsilon_i = e_i/\gamma^{i-1+\mu}$, and $e_i = \zeta_i - \hat{\zeta}_i$, we have

$$\zeta_i = \gamma^{i-1+\mu}(\varepsilon_i + z_i) = \gamma^{i-1+\mu}\left(\varepsilon_i + \hat{x}_i - g_{i-1}\hat{x}_{i-1}\right). \tag{26}$$

Now, using Lemma 1, it follows that inequality,

$$\begin{aligned} 2\varepsilon^T P\Phi &\le 2\|\varepsilon\|\|P\|\overline{\alpha}(x_0)(\|\varepsilon\| + \|z\|) \\ &= 2\overline{\alpha}(x_0)\|P\|\|\varepsilon\|^2 + 2\overline{\alpha}(x_0)\|P\|\|\varepsilon\|\|z\| \\ &\le \overline{\varphi}_1(x_0)\|\varepsilon\|^2 + \overline{\varphi}_2(x_0)\|z\|^2 \\ &\le \varphi_1(x_0)\|\varepsilon\|^2 + \varphi_2(x_0)\|\hat{x}\|^2, \end{aligned} \tag{27}$$

holds, where $\varphi_1(x_0)$ and $\varphi_2(x_0)$ are the nonnegative smooth functions. On the other hand, by using Young's inequality, one has

$$\begin{aligned} &2\varepsilon^T \mathrm{Pb}\overline{h} \le \gamma L_1\|\varepsilon\|^2 + \gamma\overline{h}^2, \\ &2\varepsilon^T \mathrm{PF} \le \gamma\|P\|c_1 z_{n+1}^2 + \gamma\frac{1}{c_1}\|P\|\|\varepsilon\|^2. \end{aligned} \tag{28}$$

Correspondingly, we can obtain that

$$\begin{aligned} \dot{V}_\varepsilon &\le -\gamma d_1\varepsilon^T\varepsilon - \frac{\dot{\gamma}}{\gamma}d_2\varepsilon^T\varepsilon + \gamma L_1\|\varepsilon\|^2 + \gamma\|P\|c_1 z_{n+1}^2 \\ &\quad + \gamma\frac{1}{c_1}\|P\|\|\varepsilon\|^2 + \varphi_1(x_0)\|\varepsilon\|^2 + \varphi_2(x_0)\|\hat{x}\|^2 + \gamma\overline{h}^2. \end{aligned} \tag{29}$$

Step 1. Choosing $V_1 = 1/2z_1^2 = 1/2\hat{x}_1^2$, we have

$$\begin{aligned}\dot{V}_1 &= \hat{x}_1\left(\gamma z_2 + \gamma a_1\varepsilon_1 - \frac{\dot{\gamma}}{\gamma}\mu z_1\right)\\ &= \gamma\hat{x}_1(z_2 - \alpha_1 + \alpha_1) + \gamma a_1\hat{x}_1\varepsilon_1 - \frac{\dot{\gamma}}{\gamma}\mu\hat{x}_1 z_1\\ &= \gamma\hat{x}_1\hat{x}_2 + \gamma\hat{x}_1\alpha_1 + \gamma a_1\hat{x}_1\varepsilon_1 - \frac{\dot{\gamma}}{\gamma}\mu\hat{x}_1^2.\end{aligned} \tag{30}$$

By using Young's inequality, we have

$$\gamma a_1\hat{x}_1\varepsilon_1 \le \gamma M_1\varepsilon_1^2 + \gamma\frac{a_1^2}{4M_1}\hat{x}_1^2 = \gamma M_1\varepsilon_1^2 + \gamma\beta_1\hat{x}_1^2, \tag{31}$$

Where $M > 0, \beta_1 = a_1^2/4M_1$ are the constants. Choosing $\alpha_1 = -g_1\hat{x}_1 = -(n+\beta_1)\hat{x}_1, \delta_1 = \mu$, we then obtain

$$\dot{V}_1 \le -\gamma n\hat{x}_1^2 + \gamma M_1\varepsilon_1^2 + \gamma\hat{x}_1\hat{x}_2 - \frac{\dot{\gamma}}{\gamma}\delta_1\hat{x}_1^2. \tag{32}$$

Step 2. Choosing $V_2 = p_1V_1 + 1/2\hat{x}_2^2$, where p_1 is a designed positive constant,

$$\begin{aligned}\dot{V}_2 &\le p_1\left(-\gamma n\hat{x}_1^2 + \gamma M_1\varepsilon_1^2 + \gamma\hat{x}_1\hat{x}_2 - \frac{\dot{\gamma}}{\gamma}\delta_1\hat{x}_1^2\right) + \hat{x}_2(\dot{z}_2 - \dot{\alpha}_1)\\ &= p_1\left(-\gamma n\hat{x}_1^2 + \gamma M_1\varepsilon_1^2 + \gamma\hat{x}_1\hat{x}_2 - \frac{\dot{\gamma}}{\gamma}\delta_1\hat{x}_1^2\right) + \hat{x}_2\left[\gamma z_3 + \gamma a_2\varepsilon_1 - \frac{\dot{\gamma}}{\gamma}(1+\mu)z_2\right] - \hat{x}_2\frac{\partial\alpha_1}{\partial z_1}\dot{z}_1\\ &= -\gamma p_1 n\hat{x}_1^2 + \gamma p_1 M_1\varepsilon_1^2 + \gamma p_1\hat{x}_1\hat{x}_2 - \frac{\dot{\gamma}}{\gamma}p_1\delta_1\hat{x}_1^2 + \gamma\hat{x}_2\hat{x}_3 + \gamma\hat{x}_2\alpha_2\\ &\quad + \gamma a_2\hat{x}_2\varepsilon_1 - \frac{\dot{\gamma}}{\gamma}(1+\mu)\hat{x}_2 z_2 - \hat{x}_2\frac{\partial\alpha_1}{\partial z_1}\dot{z}_1.\end{aligned} \tag{33}$$

From Lemma 1, we can derive the following inequalities:

$$\begin{aligned}\gamma p_1\hat{x}_1\hat{x}_2 &\le \frac{\gamma}{3}p_1\hat{x}_1^2 + 3\gamma\frac{p_1}{4}\hat{x}_2^2 = \frac{\gamma}{3}p_1\hat{x}_1^2 + \gamma\beta_{21}\hat{x}_2^2\\ \gamma a_2\hat{x}_2\varepsilon_1 &\le \gamma\theta_{21}\varepsilon_1^2 + \gamma\frac{a_2^2}{4\theta_{21}}\hat{x}_2^2 = \gamma\theta_{21}\varepsilon_1^2 + \gamma\beta_{22}\hat{x}_2^2,\\ -\frac{\dot{\gamma}}{\gamma}(1+\mu)\hat{x}_2 z_2 &= -\frac{\dot{\gamma}}{\gamma}(1+\mu)\hat{x}_2\left(\hat{x}_2 - g_1\hat{x}_1\right)\\ &= -\frac{\dot{\gamma}}{\gamma}(1+\mu)\hat{x}_2^2 + \frac{\dot{\gamma}}{\gamma}(1+\mu)\hat{x}_2\cdot g_1\hat{x}_1\\ &\le -\frac{\dot{\gamma}}{\gamma}\frac{1+\mu}{2}\hat{x}_2^2 + \frac{\dot{\gamma}}{\gamma}\frac{1+\mu}{2}g_1^2\hat{x}_1^2\\ &= -\frac{\dot{\gamma}}{\gamma}\frac{1+\mu}{2}\hat{x}_2^2 + \frac{\dot{\gamma}}{\gamma}N_1(l_2, g_1)\hat{x}_1^2,\end{aligned} \tag{34}$$

where $\beta_{21} = 3p_1/4, l_1 = \mu$, $l_2 = 1+\mu$, $N_1(l_2, g_1) = l_2/2g_1^2$, $\theta_{21} > 0$, $\beta_{22} = a_2^2/4\theta_{21}$ are the constants. Using the relations $\alpha_1 = -g_1\hat{x}_1$ and $\dot{z}_1 = \gamma z_2 + \gamma a_1\varepsilon_1 - \dot{\gamma}/\gamma\mu z_1$, we have

$$-\hat{x}_2\frac{\partial\alpha_1}{\partial z_1}\dot{z}_1 = -\hat{x}_2\frac{\partial\alpha_1}{\partial z_1}\left(\gamma z_2 + \gamma a_1\varepsilon_1 - \frac{\dot{\gamma}}{\gamma}\mu z_1\right). \tag{35}$$

Defining $\hat{x}_0 = 0$, $g_0 = 1$, it is derived that

$$\begin{aligned}\gamma z_2-\frac{\dot{\gamma}}{\gamma}l_1z_1&=\gamma\left(\hat{x}_2-g_1\hat{x}_1\right)-\frac{\dot{\gamma}}{\gamma}l_1\hat{x}_1\\&=\gamma\left(\hat{x}_2-g_1\hat{x}_1\right)-\frac{\dot{\gamma}}{\gamma}l_1\left(\hat{x}_1-g_0\hat{x}_0\right)\\&\leq\gamma\left(\left|\hat{x}_2\right|+\left|g_1\right|\left|\hat{x}_1\right|\right)+\frac{\dot{\gamma}}{\gamma}l_1\left(\left|\hat{x}_1\right|+\left|g_0\right|\left|\hat{x}_0\right|\right)\\&\leq A\left(g_0,g_1\right)\left[\gamma\left(\left|\hat{x}_2\right|+\left|\hat{x}_1\right|\right)+\frac{\dot{\gamma}}{\gamma}l_1\left(\left|\hat{x}_1\right|+\left|\hat{x}_0\right|\right)\right],\end{aligned}\tag{36}$$

where $A(g_0,g_1)=\max\{1,|g_0|,|g_1|\}$. Since $|\partial\alpha_1/\partial z_1|\leq g_1$, this implies that

$$\begin{aligned}-\hat{x}_2\frac{\partial\alpha_1}{\partial z_1}\gamma a_1\varepsilon_1&\leq\left|\hat{x}_2\right|g_1\gamma a_1\varepsilon_1\leq\gamma\theta_{22}\varepsilon_1^2+\gamma\frac{g_1^2a_1^2}{4\theta_{22}}\hat{x}_2^2\\&=\gamma\theta_{22}\varepsilon_1^2+\gamma\beta_{23}\hat{x}_2^2,\\-\hat{x}_2\frac{\partial\alpha_1}{\partial z_1}\left(\gamma z_2-\frac{\dot{\gamma}}{\gamma}l_1z_1\right)&\leq\left|\hat{x}_2\right|g_1A\left(g_0,g_1\right)\\&\quad\left[\gamma\left(\left|\hat{x}_2\right|+\left|\hat{x}_1\right|\right)+\frac{\dot{\gamma}}{\gamma}l_1\left(\left|\hat{x}_1\right|+\left|\hat{x}_0\right|\right)\right].\end{aligned}\tag{37}$$

Letting $B_{21}=g_1A(g_0,g_1)$, it can be concluded that

$$\begin{aligned}\gamma\left|\hat{x}_2\right|g_1A\left(g_0,g_1\right)\left(\left|\hat{x}_2\right|+\left|\hat{x}_1\right|\right)&=\gamma B_{21}\hat{x}_2^2+\gamma B_{21}\left|\hat{x}_2\right|\left|\hat{x}_1\right|\\&\leq\frac{\gamma}{3}p_1\hat{x}_1^2+\gamma\left(B_{21}+\frac{3B_{21}^2}{4p_1}\right)\hat{x}_2^2\\&=\frac{\gamma}{3}p_1\hat{x}_1^2+\gamma\beta_{24}\hat{x}_2^2,\\\left|\hat{x}_2\right|B_{21}\frac{\dot{\gamma}}{\gamma}l_1\left(\left|\hat{x}_1\right|+\left|\hat{x}_0\right|\right)&=\left|\hat{x}_2\right|B_{21}\frac{\dot{\gamma}}{\gamma}l_1\left|\hat{x}_1\right|\\&\leq\frac{\dot{\gamma}}{\gamma}B_{21}^2l_1\hat{x}_1^2+\frac{\dot{\gamma}}{\gamma}\frac{l_1}{4}\hat{x}_2^2\\&=\frac{\dot{\gamma}}{\gamma}B_2\hat{x}_1^2+\frac{\dot{\gamma}}{\gamma}\frac{l_1}{4}\hat{x}_2^2,\end{aligned}\tag{38}$$

where $\theta_{22}>0,\beta_{23}=g_1^2a_1^2/4\theta_{22}$, $\beta_{24}=B_{21}+3B_{21}^2/4p_1$, $B_2=B_{21}^2l_1$ are the constants. Furthermore, we have

$$\begin{aligned}\dot{V}_2\leq&-\gamma p_1n\hat{x}_1^2-\frac{\dot{\gamma}}{\gamma}p_1\delta_1\hat{x}_1^2+\frac{\gamma}{3}p_1\hat{x}_1^2+\frac{\dot{\gamma}}{\gamma}N_1\left(l_2,g_1\right)\hat{x}_1^2\\&+\frac{\gamma}{3}p_1\hat{x}_1^2+\frac{\dot{\gamma}}{\gamma}B_2\hat{x}_1^2+\gamma p_1M_1\varepsilon_1^2+\gamma M_{21}\varepsilon_1^2\\&+\gamma M_{22}\varepsilon_1^2+\gamma\beta_{21}\hat{x}_2^2+\gamma\beta_{22}\hat{x}_2^2+\gamma\beta_{23}\hat{x}_2^2+\gamma\beta_{24}\hat{x}_2^2\\&-\frac{\dot{\gamma}}{\gamma}\frac{1+\mu}{2}\hat{x}_2^2+\frac{\dot{\gamma}}{\gamma}\frac{l_1}{4}\hat{x}_2^2+\gamma\hat{x}_2\hat{x}_3+\gamma\hat{x}_2\alpha_2.\end{aligned}\tag{39}$$

Let us denote

$$\begin{cases}\beta_2=\beta_{21}+\beta_{22}+\beta_{23}+\beta_{24},\\M_2=p_1M_1+\theta_{21}+\theta_{22},\end{cases}\tag{40}$$

thus, it is obtained that

$$\begin{aligned}\dot{V}_2\leq&-\gamma(n-1)p_1\sum_{j=1}^{2}\hat{x}_j^2+\gamma M_2\varepsilon_1^2+\frac{\dot{\gamma}}{\gamma}\left(N_1\left(l_2,g_1\right)+B_2\right)\hat{x}_1^2\\&+\gamma\hat{x}_2\hat{x}_3+\gamma\hat{x}_2\alpha_2-\frac{\dot{\gamma}}{\gamma}p_1\delta_1\hat{x}_1^2-\frac{\dot{\gamma}}{\gamma}\frac{l_2}{4}\hat{x}_2^2+\gamma\left(\beta_2+(n-1)p_1\right)\hat{x}_2^2.\end{aligned}\tag{41}$$

For simplicity, let $p_1'=p_1\delta_1-N_1-B_2>0$ and $\delta_2=\min\{l_2/4,p_1'\},\alpha_2=-g_2\hat{x}_2=-(\beta_2+(n-1)p_1)\hat{x}_2$; then,

$$\dot{V}_2\leq-\gamma(n-1)p_1\sum_{j=1}^{2}\hat{x}_j^2+\gamma M_2\varepsilon_1^2+\gamma\hat{x}_2\hat{x}_3-\frac{\dot{\gamma}}{\gamma}\delta_2\sum_{j=1}^{2}\hat{x}_j^2.\tag{42}$$

Step i $(2<i\leq n-1)$. Assume that in Step $i-1$, we have

$$\dot{V}_{i-1}\leq-\gamma(n-i+2)p_1,\ldots,p_{i-2}\sum_{j=1}^{i-1}\hat{x}_j^2+\gamma M_{i-1}\varepsilon_1^2+\gamma\hat{x}_{i-1}\hat{x}_i-\frac{\dot{\gamma}}{\gamma}\delta_{i-1}\sum_{j=1}^{i-1}\hat{x}_j^2.\tag{43}$$

Letting the i^{th} candidate Lyapunov function be $V_i = p_{i-1}V_{i-1} + 1/2\hat{x}_i^2$ and defining $\hat{x}_i = z_i - \alpha_{i-1}$, where $p_2, \ldots, p_{i-2}$ are the designed positive constants,

$$\begin{aligned}
\dot{V}_i &= p_{i-1}\dot{V}_{i-1} + \hat{x}_i\left(\dot{z}_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial z_j}\dot{z}_j\right) = p_{i-1}\dot{V}_{i-1} + \hat{x}_i\left[\gamma z_{i+1} + \gamma a_i\varepsilon_1 - \frac{\dot{\gamma}}{\gamma}(i-1+\mu)z_i\right] - \hat{x}_i\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial z_j}\dot{z}_j \\
&\leq -\gamma(n-i+2)p_1, \ldots, p_{i-1}\sum_{j=1}^{i-1}\hat{x}_j^2 + \gamma p_{i-1}M_{i-1}\varepsilon_1^2 - \frac{\dot{\gamma}}{\gamma}p_{i-1}\delta_{i-1}\sum_{j=1}^{i-1}\hat{x}_j^2 \\
&\quad + \gamma p_{i-1}\hat{x}_{i-1}\hat{x}_i + \gamma\hat{x}_i z_{i+1} + \gamma a_i\hat{x}_i\varepsilon_1 - \hat{x}_i\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial z_j}\dot{z}_j - \frac{\dot{\gamma}}{\gamma}(i-1+\mu)\hat{x}_i z_i.
\end{aligned} \tag{44}$$

By applying Young's inequality, one has

$$\begin{aligned}
\gamma p_{i-1}\hat{x}_{i-1}\hat{x}_i &\leq \frac{\gamma}{3}p_1, \ldots, p_{i-1}\hat{x}_{i-1}^2 + 3\gamma\frac{p_{i-1}}{4p_1, \ldots, p_{i-2}}\hat{x}_2^2 = \frac{\gamma}{3}p_{i-1}\hat{x}_{i-1}^2 + \gamma\beta_{i1}\hat{x}_i^2 \\
\gamma a_i\hat{x}_i\varepsilon_1 &\leq \gamma M_{i1}\varepsilon_1^2 + \gamma\frac{a_i^2}{4M_{i1}}\hat{x}_i^2 = \gamma M_{i1}\varepsilon_1^2 + \gamma\beta_{i2}\hat{x}_i^2 - \frac{\dot{\gamma}}{\gamma}(i-1+\mu)\hat{x}_i z_i \\
&= -\frac{\dot{\gamma}}{\gamma}(i-1+\mu)\hat{x}_i\left(\hat{x}_i - g_{i-1}\hat{x}_{i-1}\right) \\
&= -\frac{\dot{\gamma}}{\gamma}(i-1+\mu)\hat{x}_i^2 + \frac{\dot{\gamma}}{\gamma}(i-1+\mu)\hat{x}_i \cdot g_{i-1}\hat{x}_{i-1} \\
&\leq -\frac{\dot{\gamma}}{\gamma}\frac{i-1+\mu}{2}\hat{x}_i^2 + \frac{\dot{\gamma}}{\gamma}\frac{i-1+\mu}{2}g_{i-1}^2\hat{x}_{i-1}^2 \\
&= -\frac{\dot{\gamma}}{\gamma}\frac{l_i}{2}\hat{x}_i^2 + \frac{\dot{\gamma}}{\gamma}N_{i-1}(l_i, g_{i-1})\hat{x}_{i-1}^2,
\end{aligned} \tag{45}$$

where $\beta_{i1} = 3p_{i-1}/4p_1, \ldots, p_{i-2}$, $l_i = i-1+\mu$, $N_{i-1}(l_i, g_{i-1}) = l_i/2g_{i-1}^2$, $M_{i1} > 0$, $\beta_{i2} = a_i^2/4M_{i1}$ are the constants. Using the relations

$$\begin{aligned}
\alpha_{i-1} &= -g_{i-1}\hat{x}_{i-1} = -g_{i-1}\left(z_{i-1} + g_{i-2}\hat{x}_{i-2}\right) \\
&= -\sum_{j=1}^{i-1}\prod_{s=j}^{i-1}g_s z_j = -\sum_{j=1}^{i-1}\left(\prod_{s=j}^{i-1}g_s\right)z_j,
\end{aligned} \tag{46}$$

and $\dot{z}_j = \gamma z_{j+1} + \gamma a_j\varepsilon_1 - \dot{\gamma}/\gamma(j-1+\mu)z_j$, it follows that

$$-\hat{x}_i\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial z_j}\dot{z}_j = -\hat{x}_i\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial z_j}\left(\gamma z_{j+1} + \gamma a_j\varepsilon_1 - \frac{\dot{\gamma}}{\gamma}(j-1+\mu)z_j\right), \tag{47}$$

$$\begin{aligned}\gamma z_{j+1}-\frac{\dot{\gamma}}{\gamma}l_j z_j &= \gamma\left(\hat{x}_{j+1}-g_j\hat{x}_j\right)-\frac{\dot{\gamma}}{\gamma}l_j z_j\\ &= \gamma\left(\hat{x}_{j+1}-g_j\hat{x}_j\right)-\frac{\dot{\gamma}}{\gamma}l_j\left(\hat{x}_j-g_{j-1}\hat{x}_{j-1}\right)\\ &\leq \gamma\left(\left|\hat{x}_{j+1}\right|+\left|g_j\right|\left|\hat{x}_j\right|\right)+\frac{\dot{\gamma}}{\gamma}l_j\left(\left|\hat{x}_j\right|+\left|g_{j-1}\right|\left|\hat{x}_{j-1}\right|\right)\\ &\leq A\left(g_{j-1},g_j\right)\left[\gamma\left(\left|\hat{x}_{j+1}\right|+\left|\hat{x}_j\right|\right)+\frac{\dot{\gamma}}{\gamma}l_j\left(\left|\hat{x}_j\right|+\left|\hat{x}_j\right|\right)\right].\end{aligned} \tag{48}$$

Since $|\partial\alpha_{i-1}/\partial z_j|\leq g_j,\ldots,g_{i-1}$, this implies that

$$\begin{aligned}-\hat{x}_i\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial z_j}\gamma a_j\varepsilon_1 &\leq \left|\hat{x}_i\right|\sum_{j=1}^{i-1}g_j,\ldots,g_{i-1}\gamma a_j\varepsilon_1\leq \gamma M_{i2}\varepsilon_1^2+\gamma\frac{\sum_{j=1}^{i-1}\left(g_j,\ldots,g_{i-1}a_j\right)^2}{4M_{i2}}\hat{x}_i^2\\ &= \gamma M_{i2}\varepsilon_1^2+\gamma\beta_{i3}\hat{x}_i^2\\ -\hat{x}_i\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial z_j}\left(\gamma z_{j+1}-\frac{\dot{\gamma}}{\gamma}l_j z_j\right) &\leq \left|\hat{x}_i\right|\sum_{j=1}^{i-1}g_j,\ldots,g_{i-1}A\left(g_{j-1},g_j\right)\left[\gamma\left(\left|\hat{x}_j\right|+\left|\hat{x}_{j+1}\right|\right)+\frac{\dot{\gamma}}{\gamma}l_j\left(\left|\hat{x}_j\right|+\left|\hat{x}_{j+1}\right|\right)\right].\end{aligned} \tag{49}$$

Defining $B_{ij}=g_j,\ldots,g_{i-1}A\left(g_{j-1},g_j\right)$,

$$\begin{aligned}\gamma\left|\hat{x}_i\right|\sum_{j=1}^{i-1}B_{ij}\left(\left|\hat{x}_{j+1}\right|+\left|\hat{x}_j\right|\right) &= \gamma\left|\hat{x}_i\right|\sum_{j=1}^{i-1}B_{ij}\left|\hat{x}_{j+1}\right|+\gamma\left|\hat{x}_i\right|\sum_{j=1}^{i-1}B_{ij}\left|\hat{x}_j\right|\\ &= \gamma B_{i,i-1}\hat{x}_i^2+\gamma\left|\hat{x}_i\right|\sum_{j=2}^{i-1}B_{i,j-1}\left|\hat{x}_j\right|+\gamma\left|\hat{x}_i\right|\sum_{j=1}^{i-1}B_{ij}\left|\hat{x}_j\right|\\ &\leq \gamma B_{i,i-1}\hat{x}_i^2+\gamma\left|\hat{x}_i\right|\sum_{j=1}^{i-1}B_{i,j-1}\left|\hat{x}_j\right|+\gamma\left|\hat{x}_i\right|\sum_{j=1}^{i-1}B_{ij}\left|\hat{x}_j\right|\\ &= \gamma B_{i,i-1}\hat{x}_i^2+\gamma\left|\hat{x}_i\right|\sum_{j=1}^{i-1}\overline{B}_{i,j}\left|\hat{x}_j\right|\\ &\leq \gamma\left(B_{i,i-1}+\frac{3\sum_{j=1}^{i-1}\overline{B}_{ij}^2}{4p_1,\ldots,p_{i-1}}\right)\hat{x}_i^2+\frac{\gamma}{3}p_1,\ldots,p_{i-1}\sum_{j=1}^{i-1}\hat{x}_j^2\\ &= \gamma\beta_{i4}\hat{x}_i^2+\frac{\gamma}{3}p_1,\ldots,p_{i-1}\sum_{j=1}^{i-1}\hat{x}_j^2,\\ \left|\hat{x}_i\right|\sum_{j=1}^{i-1}B_{ij}\left[\frac{\dot{\gamma}}{\gamma}l_j\left(\left|\hat{x}_j\right|+\left|\hat{x}_{j-1}\right|\right)\right] &= \frac{\dot{\gamma}}{\gamma}\left|\hat{x}_i\right|\sum_{j=1}^{i-1}B_{ij}l_j\left|\hat{x}_j\right|+\frac{\dot{\gamma}}{\gamma}\left|\hat{x}_i\right|\sum_{j=1}^{i-1}B_{ij}l_j\left|\hat{x}_{j-1}\right|\\ &\leq \frac{\dot{\gamma}}{\gamma}\left|\hat{x}_i\right|\sum_{j=1}^{i-1}\widetilde{B}_{ij}\left|\hat{x}_j\right|\leq\frac{\dot{\gamma}}{\gamma}l_{i-1}\sum_{j=1}^{i-1}\widetilde{B}_{ij}^2\hat{x}_j^2+\frac{\dot{\gamma}}{\gamma}\frac{l_{i-1}}{4}\hat{x}_i^2=\frac{\dot{\gamma}}{\gamma}B_i\sum_{j=1}^{i-1}\hat{x}_j^2+\frac{\dot{\gamma}}{\gamma}\frac{l_{i-1}}{4}\hat{x}_i^2,\end{aligned} \tag{50}$$

where $\overline{B}_{ij}=B_{ij}+B_{i,j-1}$, $\widetilde{B}_{ij}=B_{ij}l_j+B_{i,j-1}l_{j-1}$, $M_{i2}>0$, $\beta_{i3}=\sum_{j=1}^{i-1}(g_j,\ldots,g_{i-1}a_j)^2/4M_{i2}$, $\beta_{i4}=B_{i,i-1}+3\sum_{j=1}^{i-1}\overline{B}_{ij}^2/4p_1,\ldots,p_{i-1}$, $B_i=l_{i-1}\sum_{j=1}^{i-1}\overline{B}_{ij}^2$ are the constants.

Denoting

$$\begin{cases}\beta_i=\beta_{i1}+\beta_{i2}+\beta_{i3}+\beta_{i4},\\ M_i=p_{i-1}M_{i-1}+M_{i1}+M_{i2},\end{cases} \tag{51}$$

we have

$$\begin{aligned}\dot{V}_i \leq & -2\gamma p_1, \ldots, p_{i-1} \sum_{j=1}^{i} \hat{x}_j^2 + \gamma M_i \varepsilon_1^2 + \gamma \hat{x}_i \hat{x}_{i+1} + \gamma \hat{x}_i \alpha_i \\ & -\frac{\dot{\gamma}}{\gamma} p_{i-1} \delta_{i-1} \sum_{j=1}^{i} \hat{x}_j^2 + \frac{\dot{\gamma}}{\gamma} N_{i-1} \hat{x}_{i-1}^2 + \frac{\dot{\gamma}}{\gamma} B_i \sum_{j=1}^{i-1} \hat{x}_j^2 \\ & -\frac{\dot{\gamma}}{\gamma} \frac{l_i}{4} \hat{x}_i^2 + \gamma \left(\beta_i + 2p_1, \ldots, p_{i-1}\right) \hat{x}_i^2.\end{aligned} \tag{52}$$

Choosing $p'_{i-1} = p_{i-1}\delta_{i-1} - N_{i-1} - B_i > 0, \delta_i = \min \{l_i/4, p'_{i-1}\}$ and using the formula

$$\alpha_i = -g_i \hat{x}_i = -\left(\beta_i + (n-i+1) p_1, \ldots, p_{i-1}\right) \hat{x}_i, \tag{53}$$

we again obtain

$$\begin{aligned}\dot{V}_i \leq & -\gamma (n-i+1) p_1, \ldots, p_{i-1} \sum_{j=1}^{i} \hat{x}_j^2 + \gamma M_i \varepsilon_1^2 \\ & + \gamma \hat{x}_i \hat{x}_{i+1} - \frac{\dot{\gamma}}{\gamma} \delta_i \sum_{j=1}^{i} \hat{x}_j^2.\end{aligned} \tag{54}$$

Step n: for the last step, we choose $V_n = p_{n-1} V_{n-1} + 1/2 \hat{x}_n^2 + 1/2 z_{n+1}^2$, where p_{n-1} is a designed positive constant, and by using (30), (32), and (43), we have

$$\begin{aligned}\dot{V}_n = & \, p_{n-1} \dot{V}_{n-1} + \hat{x}_n \left(\dot{z}_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial z_j} \dot{z}_j \right) \\ & + z_{n+1} \left(\gamma a_{n+1} \varepsilon_1 - \frac{\dot{\gamma}}{\gamma} (n+\mu) z_{n+1} - \gamma c z_{n+1} \right) \\ \leq & -2\gamma p_1, \ldots, p_{n-1} \sum_{j=1}^{n-1} \hat{x}_j^2 \\ & + \gamma p_{n-1} \hat{x}_{n-1} \hat{x}_n - \frac{\dot{\gamma}}{\gamma} p_{n-1} \delta_{n-1} \sum_{j=1}^{n-1} \hat{x}_j^2 + \gamma a_n \hat{x}_n \varepsilon_1 \\ & -\frac{\dot{\gamma}}{\gamma} (n-1+\mu) \hat{x}_n z_n + \hat{x}_n \frac{u}{\gamma^{n-1+\mu}} \\ & + z_{n+1} \left(\gamma a_{n+1} \varepsilon_1 - \frac{\dot{\gamma}}{\gamma} (n+\mu) z_{n+1} - \gamma c z_{n+1} \right) \\ & - \hat{x}_n \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial z_j} \dot{z}_j.\end{aligned} \tag{55}$$

This is similar to step i, in that

$$\begin{aligned}\gamma p_{n-1} \hat{x}_{n-1} \hat{x}_n & \leq \frac{\gamma}{3} p_1, \ldots, p_{n-1} \hat{x}_{n-1}^2 + 3\gamma \frac{p_{n-1}}{4 p_1, \ldots, p_{n-2}} \hat{x}_n^2 \\ & = \frac{\gamma}{3} p_{n-1} \hat{x}_{n-1}^2 + \gamma \beta_{n1} \hat{x}_n^2 \\ \gamma a_n \hat{x}_n \varepsilon_1 & \leq \gamma M_{n1} \varepsilon_1^2 + \gamma \frac{a_n^2}{4 M_{n1}} \hat{x}_n^2 \\ & = \gamma M_{n1} \varepsilon_1^2 + \gamma \beta_{n2} \hat{x}_n^2 - \frac{\dot{\gamma}}{\gamma} (n-1+\mu) \hat{x}_n z_n \\ & = -\frac{\dot{\gamma}}{\gamma} (n-1+\mu) \hat{x}_n \left(\hat{x}_n - g_{n-1} \hat{x}_{n-1} \right) \\ & = -\frac{\dot{\gamma}}{\gamma} (n-1+\mu) \hat{x}_n^2 + \frac{\dot{\gamma}}{\gamma} (n-1+\mu) \hat{x}_n \cdot g_{n-1} \hat{x}_{n-1} \\ & \leq -\frac{\dot{\gamma}}{\gamma} \frac{n-1+\mu}{2} \hat{x}_n^2 + \frac{\dot{\gamma}}{\gamma} \frac{n-1+\mu}{2} g_{n-1}^2 \hat{x}_{n-1}^2 \\ & = -\frac{\dot{\gamma}}{\gamma} \frac{l_n}{2} \hat{x}_n^2 + \frac{\dot{\gamma}}{\gamma} N_{n-1} \left(l_n, g_{n-1}\right) \hat{x}_{n-1}^2,\end{aligned} \tag{56}$$

where $\beta_{n1} = 3p_{n-1}/4p_1, \ldots, p_{n-2}$, $l_n = n-1+\mu$, $N_{n-1}(l_n, g_{n-1}) = l_n/2g_{n-1}^2$, $M_{n1} > 0$, $\beta_{n2} = a_n^2/4M_{n1}$ are the constants. According to the relations

$$\begin{aligned}\alpha_{n-1} & = -g_{n-1} \hat{x}_{n-1} = -g_{n-1} \left(z_{n-1} + g_{n-2} \hat{x}_{n-2} \right) \\ & = -\sum_{j=1}^{n-1} \prod_{s=j}^{n-1} g_s z_j = -\sum_{j=1}^{i-1} \left(\prod_{s=j}^{n-1} g_s \right) z_j\end{aligned} \tag{57}$$

and $\dot{z}_j = \gamma z_{j+1} + \gamma a_j \varepsilon_1 - \dot{\gamma}/\gamma (j-1+\mu) z_j$, we have

$$
\begin{aligned}
&-\hat{x}_n \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial z_j} \dot{z}_j = -\hat{x}_n \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial z_j} \left(\gamma z_{j+1} + \gamma a_j \varepsilon_1 - \frac{\dot{\gamma}}{\gamma} (j - 1 + \mu) z_j \right) \\
&\gamma z_{j+1} - \frac{\dot{\gamma}}{\gamma} l_j z_j = \gamma \left(\hat{x}_{j+1} - g_j \hat{x}_j \right) - \frac{\dot{\gamma}}{\gamma} l_j z_j \\
&\quad = \gamma \left(\hat{x}_{j+1} - g_j \hat{x}_j \right) - \frac{\dot{\gamma}}{\gamma} l_j \hat{x}_j - g_{j-1} \hat{x}_{j-1} \\
&\quad \leq \gamma \left(|\hat{x}_{j+1}| + |g_j| |\hat{x}_j| \right) + \frac{\dot{\gamma}}{\gamma} l_j \left(|\hat{x}_j| + |g_{j-1}| |\hat{x}_{j-1}| \right) \\
&\quad \leq A\left(g_{j-1}, g_j \right) \left[\gamma \left(|\hat{x}_{j+1}| + |\hat{x}_j| \right) + \frac{\dot{\gamma}}{\gamma} l_j \left(|\hat{x}_j| + |\hat{x}_{j-1}| \right) \right].
\end{aligned} \tag{58}
$$

Because $|\partial \alpha_{n-1} / \partial z_j| \leq g_j, \ldots, g_{n-1}$, it is obtained

$$
\begin{aligned}
&-\hat{x}_n \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial z_j} \gamma a_j \varepsilon_1 \leq \left| \hat{x}_n \right| \sum_{j=1}^{n-1} g_j, \ldots g_{n-1} \gamma a_j \varepsilon_1 \\
&\quad \leq \gamma M_{n2} \varepsilon_1^2 + \gamma \frac{\sum_{j=1}^{n-1} \left(g_j, \ldots, g_{n-1} a_j \right)^2}{4 M_{n2}} \hat{x}_n^2 \\
&\quad = \gamma M_{n2} \varepsilon_1^2 + \gamma \beta_{n3} \hat{x}_n^2 \\
&\quad\quad - \hat{x}_n \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial z_j} \gamma a_j \varepsilon_1 \leq \left| \hat{x}_n \right| \sum_{j=1}^{n-1} g_j, \ldots, g_{n-1} \gamma a_j \varepsilon_1 \\
&\quad \leq \gamma M_{n2} \varepsilon_1^2 + \gamma \frac{\sum_{j=1}^{n-1} \left(g_j, \ldots, g_{n-1} a_j \right)^2}{4 M_{n2}} \hat{x}_n^2 \\
&\quad = \gamma M_{n2} \varepsilon_1^2 + \gamma \beta_{n3} \hat{x}_n^2 \\
&-\hat{x}_n \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial z_j} \left(\gamma z_{j+1} - \frac{\dot{\gamma}}{\gamma} l_j z_j \right) \leq \left| \hat{x}_n \right| \sum_{j=1}^{n-1} g_j, \ldots, g_{n-1} A\left(g_{j-1}, g_j \right) \left[\gamma \left(\left| \hat{x}_{j+1} \right| + \left| \hat{x}_j \right| \right) + \frac{\dot{\gamma}}{\gamma} l_j \left(\left| \hat{x}_j \right| + \left| \hat{x}_{j-1} \right| \right) \right].
\end{aligned} \tag{59}
$$

Defining $B_{nj} = g_j, \ldots, g_{n-1} A(g_{j-1}, g_j)$,

$$
\begin{aligned}
&\gamma \left| \hat{x}_n \right| \sum_{j=1}^{n-1} B_{nj} \left(\left| \hat{x}_{j+1} \right| + \left| \hat{x}_j \right| \right) = \gamma \left| \hat{x}_n \right| \sum_{j=1}^{n-1} B_{nj} \left| \hat{x}_{j+1} \right| + \gamma \left| \hat{x}_n \right| \sum_{j=1}^{n-1} B_{nj} \left| \hat{x}_j \right| \\
&\quad \leq \gamma B_{n,n-1} \hat{x}_n^2 + \gamma \left| \hat{x}_n \right| \sum_{j=1}^{n-1} B_{n,j-1} \left| \hat{x}_j \right| + \gamma \left| \hat{x}_n \right| \sum_{j=1}^{n-1} B_{nj} \left| \hat{x}_j \right| \\
&\quad = \gamma B_{n,n-1} \hat{x}_n^2 + \gamma \left| \hat{x}_n \right| \sum_{j=1}^{n-1} \overline{B}_{n,j} \left| \hat{x}_j \right| \\
&\quad \leq \gamma \left(B_{n,n-1} + \frac{3 \sum_{j=1}^{n-1} \overline{B}_{nj}^2}{4 p_1, \ldots, p_{n-1}} \right) \hat{x}_n^2 + \frac{\gamma}{3} p_1, \ldots, p_{n-1} \sum_{j=1}^{n-1} \hat{x}_j^2
\end{aligned}
$$

$$\begin{aligned}
&= \gamma\beta_{n4}\hat{x}_n^2 + \frac{\gamma}{3}p_1,\ldots,p_{n-1}\sum_{j=1}^{n-1}\hat{x}_j^2 \\
&\left|\hat{x}_n\right|\sum_{j=1}^{n-1}B_{nj}\left[\frac{\dot{\gamma}}{\gamma}l_j\left(\left|\hat{x}_j\right| + \left|\hat{x}_{j-1}\right|\right)\right] \\
&= \frac{\dot{\gamma}}{\gamma}\left|\hat{x}_n\right|\sum_{j=1}^{n-1}B_{nj}l_j\left|\hat{x}_j\right| + \frac{\dot{\gamma}}{\gamma}\left|\hat{x}_n\right|\sum_{j=1}^{n-1}B_{nj}l_j\left|\hat{x}_{j-1}\right| \\
&\leq \frac{\dot{\gamma}}{\gamma}\left|\hat{x}_n\right|\sum_{j=1}^{n-1}\widetilde{B}_{nj}\left|\hat{x}_j\right| \\
&\leq \frac{\dot{\gamma}}{\gamma}l_{n-1}\sum_{j=1}^{n-1}\widetilde{B}_{nj}^2\hat{x}_j^2 + \frac{\dot{\gamma}}{\gamma}\frac{l_{n-1}}{4}\hat{x}_n^2 \\
&= \frac{\dot{\gamma}}{\gamma}B_n\sum_{j=1}^{n-1}\hat{x}_j^2 + \frac{\dot{\gamma}}{\gamma}\frac{l_{n-1}}{4}\hat{x}_n^2,
\end{aligned} \tag{60}$$

where $\overline{B}_{nj} = B_{nj} + B_{n,j-1}$, $\widetilde{B}_{nj} = B_{nj}l_j + B_{n,j-1}l_{j-1}$, $M_{n2} > 0$, $\beta_{n3} = \sum_{j=1}^{n-1}(g_j,\ldots,g_{n-1}a_j)^2/4M_{n2}$, $\beta_{n4} = B_{n,n-1} + 3\sum_{j=1}^{n-1}\overline{B}_{nj}^2/4p_1,\ldots,p_{n-1}$, $B_n = l_{n-1}\sum_{j=1}^{n-1}\overline{B}_{nj}^2$ are the constants.

Denoting

$$\begin{cases} \beta_n = \beta_{n1} + \beta_{n2} + \beta_{n3} + \beta_{n4}, \\ M_n = p_{n-1}M_{n-1} + M_{n1} + M_{n2}, \end{cases} \tag{61}$$

we have

$$\begin{aligned}
\dot{V}_n \leq & -\gamma p_1,\ldots,p_{n-1}\sum_{j=1}^{n-1}\hat{x}_j^2 + \gamma M_n\varepsilon_1^2 + \gamma\hat{x}_n z_{n+1} - \frac{\dot{\gamma}}{\gamma}\frac{l_n}{4}\hat{x}_n^2 \\
& - \frac{\dot{\gamma}}{\gamma}p_{n-1}\delta_{n-1}\sum_{j=1}^{n-1}\hat{x}_j^2 + \frac{\dot{\gamma}}{\gamma}N_{n-1}\hat{x}_{n-1}^2 + \frac{\dot{\gamma}}{\gamma}B_n\sum_{j=1}^{n-1}\hat{x}_j^2 \\
& + \gamma\beta_n\hat{x}_n^2 + \hat{x}_n\frac{u}{\gamma^{n-1+\mu}} + \gamma a_{n+1}z_{n+1}\varepsilon_1 \\
& - \frac{\dot{\gamma}}{\gamma}(n+\mu)z_{n+1}^2 - \gamma c z_{n+1}^2).
\end{aligned} \tag{62}$$

For simplicity, we define $p_{n-1}' = p_{n-1}\delta_{n-1} - N_{n-1} - B_n > 0$, $\delta_n = \min\{l_n/4, p_{n-1}', n+\mu\}$, and then,

$$\begin{aligned}
\dot{V}_n \leq & -\gamma p_1,\ldots,p_{n-1}\sum_{j=1}^{n}\hat{x}_j^2 + \gamma M_n\varepsilon_1^2 + \gamma\hat{x}_n z_{n+1} \\
& - \frac{\dot{\gamma}}{\gamma}\delta_n\sum_{j=1}^{n}\hat{x}_j^2 + \gamma(\beta_n + p_1,\ldots,p_{n-1})\hat{x}_n^2 + \hat{x}_n\frac{u}{\gamma^{n-1+\mu}} \\
& + \gamma a_{n+1}z_{n+1}\varepsilon_1 - \frac{\dot{\gamma}}{\gamma}(n+\mu)z_{n+1}^2 - \gamma c z_{n+1}^2 \\
\gamma a_{n+1}z_{n+1}\varepsilon_1 \leq & \gamma M_{n+1}\varepsilon_1^2 + \gamma\frac{a_{n+1}^2}{4M_{n+1}}z_{n+1}^2 \\
= & \gamma M_{n+1}\varepsilon_1^2 + \gamma\beta_{n+1}z_{n+1}^2,
\end{aligned} \tag{63}$$

where $M_{n+1} > 0$, $\beta_{n+1} = a_{n+1}^2/4M_{n+1}$ are the constants. Now, let us choose the final control signal as

$$\begin{aligned}
u &= -\gamma^{n+\mu}(\beta_n + p_1,\ldots,p_{n-1})\hat{x}_n - \gamma^{n+\mu}z_{n+1} \\
&= -\gamma^{n+\mu}(\beta_n + p_1,\ldots,p_{n-1})\hat{x}_n - \zeta_{n+1}.
\end{aligned} \tag{64}$$

Finally, it is derived that

$$\begin{aligned}
\dot{V}_n \leq & -\gamma p_1,\ldots,p_{n-1}\sum_{j=1}^{n}\hat{x}_j^2 + \gamma(M_n + M_{n+1})\varepsilon_1^2 \\
& - \frac{\dot{\gamma}}{\gamma}\delta_n\left(\sum_{j=1}^{n}\hat{x}_j^2 + z_{n+1}^2\right) + \gamma(-c + \beta_{n+1})z_{n+1}^2.
\end{aligned} \tag{65}$$

□

3.2. Stability Analysis. Choosing $V = V_n + V_\varepsilon$, we can obtain that

$$\begin{aligned}
\dot{V} \leq & -\gamma p_1,\ldots,p_{n-1}\sum_{j=1}^{n}\hat{x}_j^2 + \gamma(M_n + M_{n+1})\varepsilon_1^2 + \gamma\overline{h}^2 \\
& - \frac{\dot{\gamma}}{\gamma}\delta_n\sum_{j=1}^{n}\left(\hat{x}_j^2 + z_{n+1}^2\right) + \gamma(-c + \beta_{n+1})z_{n+1}^2 \\
& - \gamma d_1\varepsilon^T\varepsilon - \frac{\dot{\gamma}}{\gamma}d_2\varepsilon^T\varepsilon + \gamma L_1\|\varepsilon\|^2 + \gamma\|P\|c_1 z_{n+1}^2 \\
& + \gamma\frac{1}{c_1}\|P\|\|\varepsilon\|^2 + \varphi_1(x_0)\|\varepsilon\|^2 + \varphi_2(x_0)\|\hat{x}\|^2.
\end{aligned} \tag{66}$$

Select parameters

$$\begin{cases} -c + \beta_{n+1} + \|P\|c_1 < -C_a, \\ -d_1 + (M_n + M_{n+1}) + \dfrac{1}{c_1}\|P\| + L_1 < -C_b, \end{cases} \tag{67}$$

where $C_a > 0, C_b > 0$ denote $p_1,\ldots,p_{n-1} = D$, $\delta_{n+1} = \min D, \delta_n, C_a, C_b$, such that

$$\begin{aligned}\dot{V} \leq & -\gamma D \sum_{j=1}^{n} \hat{x}_j^2 - \gamma C_a z_{n+1}^2 - \gamma C_b \|\varepsilon\|^2 - \frac{\dot{\gamma}}{\gamma}\delta_n\left(\sum_{j=1}^{n} \hat{x}_j^2 + z_{n+1}^2\right) \\ & - \frac{\dot{\gamma}}{\gamma} d_2 \varepsilon^T \varepsilon + \varphi_1(x_0)\|\varepsilon\|^2 + \varphi_2(x_0)\|\hat{x}\|^2 + \gamma \overline{h}^2 \\ \leq & - D_0\left(\gamma\|\hat{x}\|^2 + \gamma\|\varepsilon\|^2\right) - \frac{\dot{\gamma}}{\gamma}\delta_{n+1}\|\hat{x}\|^2 - \frac{\dot{\gamma}}{\gamma}\delta_{n+1}\|\varepsilon\|^2 \\ & + \varphi_1(x_0)\|\varepsilon\|^2 + \varphi_2(x_0)\|\hat{x}\|^2 + \gamma \overline{h}^2 \\ \leq & -\frac{D_0}{2}\gamma\left(\|\hat{x}\|^2 + \|\varepsilon\|^2\right) + \gamma \overline{h}^2 - D_0\left(\frac{\gamma}{2} + \frac{\dot{\gamma}}{\gamma}\frac{\delta_{n+1}}{D_0} - \frac{\varphi_2(x_0)}{D_0}\right) \\ & \|\hat{x}\|^2 - D_0\left(\frac{\gamma}{2} + \frac{\dot{\gamma}}{\gamma}\frac{\delta_{n+1}}{D_0} - \frac{\varphi_1(x_0)}{D_0}\right)\|\hat{x}\|^2.\end{aligned} \tag{68}$$

Denoting $\omega(x_0) > 1/D_0 \max\{\varphi_1(x_0), \varphi_2(x_0)\}$ and $\dot{\gamma} = \max\{(-\gamma^2/2 + \omega(x_0))\delta_{n+1}/D_0, 0\}$, then

$$\dot{V} \leq -\frac{D_0}{2}\gamma\left(\|\hat{x}\|^2 + \|\varepsilon\|^2\right) + \gamma\overline{h}^2. \tag{69}$$

Defining $\gamma\overline{h}^2 \leq D = p_1, \ldots, p_{n-1}$, we can obtain that

$$\dot{V} \leq -\frac{D_0}{2}\gamma\left(\|\hat{x}\|^2 + \|\varepsilon\|^2\right) + D. \tag{70}$$

$\hat{x}, \varepsilon$ are bounded. Now, we shall prove by contradiction that $\gamma(t)$ is bounded. Assume $\gamma(t)$ is unbounded in $[t_0, t_f)$. We notice that $\dot{\gamma}(t) \geq 0$, so $\lim_{t\longrightarrow t_f}\gamma(t) = +\infty$, and thus, there exists a finite time $T_1 \in [t_0, t_f)$, such that $\forall t \in [T_1, t_f)$, and we have

$$\dot{\gamma}(t) = \begin{cases} \left(-\dfrac{\gamma^2}{2} + \omega(x_0)\right)\dfrac{\delta_{n+1}}{D_0} \leq \left(-\dfrac{D_0^2}{2} + \omega(x_0)\right)\dfrac{\delta_{n+1}}{D_0}, \\ 0 \leq \left(-\dfrac{D_0^2}{2} + \omega(x_0)\right)\dfrac{\delta_{n+1}}{D_0}. \end{cases} \tag{71}$$

Integrating both sides of the top equation, when $\dot{\gamma} = (-\gamma^2/2 + \omega(x_0))\delta_{n+1}/D_0$, we can obtain

$$\begin{aligned}&\int_{T_1}^{t_f}\left(-\frac{\gamma^2}{2} + \omega(x_0)\right)\frac{\delta_{n+1}}{D_0}d\tau \\ &= \int_{T_1}^{t_f}\dot{\gamma}(\tau)\mathrm{d}\tau = \gamma(t_f) - \gamma(T_1) = +\infty \\ &\int_{T_1}^{t_f}\left(-\frac{\gamma^2}{2} + \omega(x_0)\right)\frac{\delta_{n+1}}{D_0}d\tau \leq \int_{T_1}^{t_f}\left(-\frac{D_0^2}{2} + \omega(x_0)\right)\frac{\delta_{n+1}}{D_0}\mathrm{d}\tau.\end{aligned} \tag{72}$$

Since x_0 is bounded in $[t_0, t_f)$, $\omega(x_0)$ is bounded, $\int_{T_1}^{t_f}(-\gamma^2/2 + \omega(x_0))\delta_{n+1}/D_0 \mathrm{d}\tau < +\infty$, which is a contradiction. Thus, $\gamma(t)$ is bounded in $[t_0, t_f)$. When $\dot{\gamma} = 0$, one has

$$\begin{aligned}&\int_{T_1}^{t_f}\dot{\gamma}(\tau)\mathrm{d}\tau = \gamma(t_f) - \gamma(T_1) = +\infty, \\ &\int_{T_1}^{t_f}\dot{\gamma}(\tau)\mathrm{d}\tau = 0.\end{aligned} \tag{73}$$

These two equations contradict each other. Thus, $\gamma(t)$ is bounded in $[t_0, t_f)$. Integrating $\dot{V} \leq -D_0/2\gamma(\|\hat{x}\|^2 + \|\varepsilon\|^2) + \gamma\overline{h}^2$, we have

$$V(t) - V(0) \leq -\frac{D_0}{2}\int_0^t \gamma\left(\|\hat{x}\|^2 + \|\varepsilon\|^2\right)\mathrm{d}\tau + \int_0^t \gamma\overline{h}^2\mathrm{d}\tau, \tag{74}$$

namely,

$$V(t) \leq V(0) - \frac{D_0}{2}\int_0^t \gamma\left(\|\hat{x}\|^2 + \|\varepsilon\|^2\right)\mathrm{d}\tau + \int_0^t \gamma\overline{h}^2\mathrm{d}\tau. \tag{75}$$

Because $\overline{h}(t) = \dot{\omega}(t)/\gamma^{n+\mu} \in L_2$, $\int_0^t \gamma\overline{h}(\tau)^2\mathrm{d}\tau < +\infty, \forall t > 0$. Because $\varepsilon, \hat{x} \in L_2$, it can be obtained from (19) and (20) that $\dot{\hat{x}}, \dot{\varepsilon}$ is bounded. According to Barbalat's lemma, $\lim_{t\longrightarrow\infty}\varepsilon = 0$, and $\lim_{t\longrightarrow\infty}\hat{x} = 0$.

4. Simulation Results

In this section, we consider a simulation example to prove that the controller design in this study is effective. Consider the following three-dimensional uncertain nonholonomic system with nonvanishing external disturbance:

$$\begin{aligned}\dot{x}_0 &= u_0 + x_0^2 x_0, \\ \dot{x}_1 &= x_2 u_0 + x_0^2|x_1|\cos^2(u_0), \\ \dot{x}_2 &= u + 0.01x_0^2\left(\frac{|x_1|}{|u_0|} + |x_2|\right) + \frac{\sin(t)}{t+1} + 1.\end{aligned} \tag{76}$$

This example shows that the nonlinear uncertainty in third equation is related to the unknown system state x_2, so that the assumption condition in [23] is not satisfied, and thus, the given method in that of article is not available to deal with this model. However, this example satisfies the given Assumption 1. By designing the controller, execution simulation algorithm, and choosing parameters $u_0 = -\lambda_0 x_0 - x_0 \quad (x_0^2 + 1)$, $u_1 = -2\gamma^3 \quad \hat{x}_2 - \gamma^3 z_3, \lambda_0 = \mu = 0.01, a_1 = a_2 = a_3 = 2, g_1 = 10, c = 100, k = 0.01$, we have

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_2 + x_0^2\cos^2(u_0)|\zeta_1| - \frac{9\left(1 + x_0^2\right)}{3 + x_0^2}\zeta_1, \\ \dot{\zeta}_2 &= u_1 + 0.01x_0^2\left(|\zeta_1| + |\zeta_2|\right) + \zeta_3, \\ \dot{\zeta}_3 &= \frac{\cos(t)}{t+1} - \frac{\sin(t)}{(t+1)^2}, \\ \dot{\hat{\zeta}}_1 &= \hat{\zeta}_2 + 2\gamma\left(\zeta_1 - \hat{\zeta}\right),\end{aligned}$$

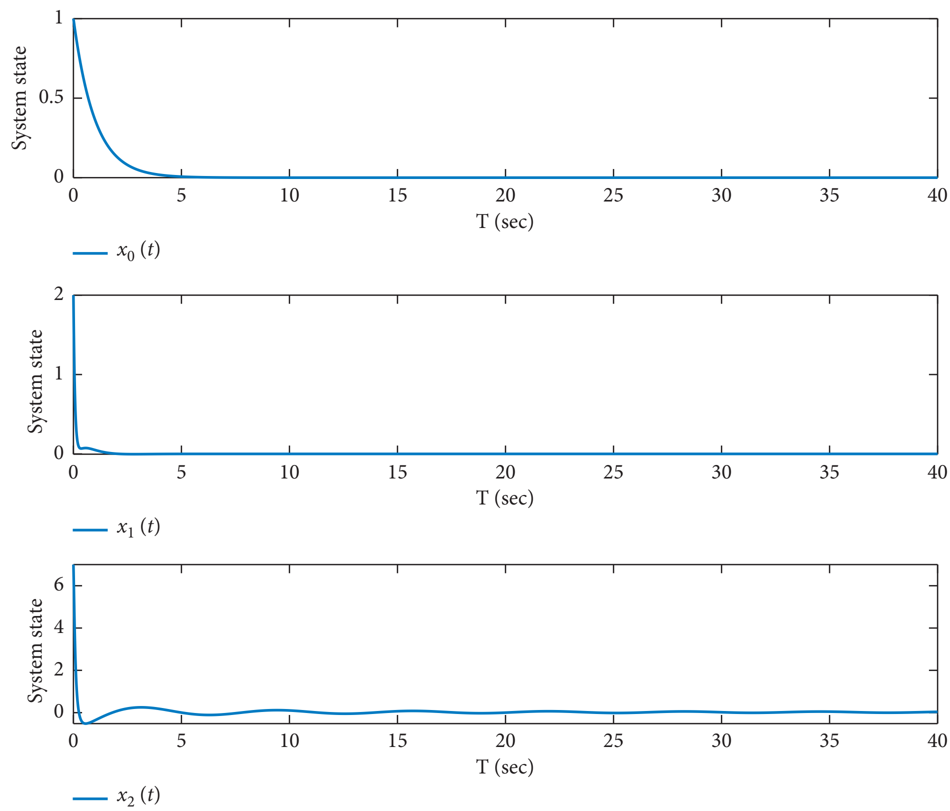

Figure 1: State response curve of the system.

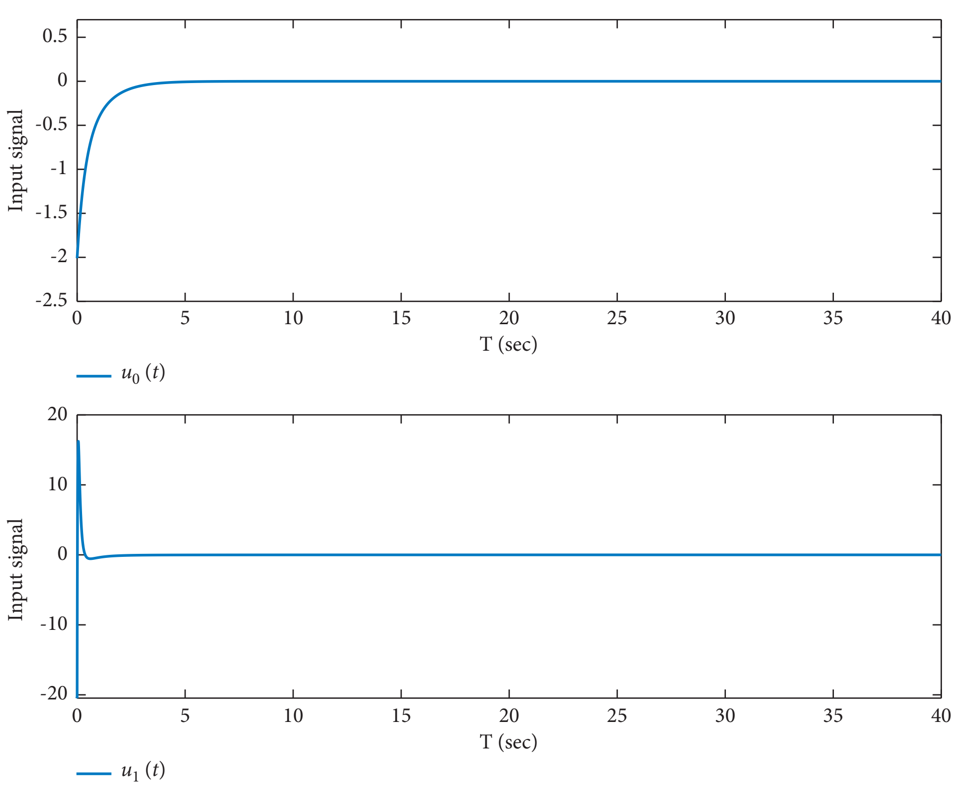

Figure 2: Input response curve of the system.

$$
\begin{aligned}
&\dot{\hat{\zeta}}_2 = u_1 + 2\gamma^2\left(\zeta_1 - \hat{\zeta}\right) + \hat{\zeta}_3,\\
&\dot{\hat{\zeta}}_3 = 2\gamma^3\left(\zeta_1 - \hat{\zeta}\right) - 100\gamma\hat{\zeta}_3,\\
&\dot{z}_1 = \gamma z_2 + 2\gamma^{0.99}\left(\zeta_1 - \hat{\zeta}\right) - 0.01\frac{\dot{\gamma}}{\gamma}z_1,\\
&\dot{z}_2 = \gamma z_3 + 2\gamma^{0.99}\left(\zeta_1 - \hat{\zeta}\right) - 1.01\frac{\dot{\gamma}}{\gamma}z_2 + \frac{u_1}{\gamma^{1.01}},\\
&\dot{z}_3 = 2\gamma^{0.99}\left(\zeta_1 - \hat{\zeta}\right) - 2.01\frac{\dot{\gamma}}{\gamma}z_3 - 100\gamma z_3,\\
&\dot{\hat{x}}_1 = \dot{z}_1,\\
&\dot{\hat{x}}_2 = \dot{z}_2 + 10\dot{\hat{x}}_1,\\
&\dot{\hat{x}}_3 = \dot{\hat{z}}_3,
\end{aligned}
\tag{77}
$$

where the initial states are $x_0 = 1, x_1 = 2, x_2 = 7, \zeta_1 = -1, \zeta_2 = 7, \zeta_3 = 0, \hat{\zeta}_1 = 1, \hat{\zeta}_2 = 1, \hat{\zeta}_3 = 5, z_1 = 1, z_2 = 10, z_3 = 1, \hat{x}_1 = 1, \hat{x}_2 = 10, \hat{x}_3 = 1$, and $\gamma = 15$, and dynamic gain is selected as $\dot{\gamma} = -\gamma^2/2 + 20x_0^2(|\zeta_1| + |\zeta_2|)$. In Figure 1, the simulation results are shown. This study presents an output feedback control scheme that realizes stability control, and the control inputs u_0 and u are bounded, as shown in Figure 2.

5. Conclusion

This study solves the problems of output feedback control for one type of the nonholonomic system with nonvanishing external disturbances and nonlinear uncertainties for which the strong uncertainties are restricted by a generalized lower triangular linearly growing condition. The system is reconstructed by introducing a new extended state observer. The external disturbance is viewed as a general state. An adjustable varying gain scaling transformation and the extended state observer are used to carry out output feedback control and overcome the uncertainties and disturbances. The output of the system and states of the system go to zero, and all signals of the closed-loop system are guaranteed to be bounded. Simulation examples show that the control algorithm is effective. How to reduce the uncertainty and external disturbance assumptions of the model (1) and make the types of the models more extensive will be further considered.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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