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Research Article

Hadamard and Fejér–Hadamard Inequalities for $(\alpha, h-m)-p$-Convex Functions via Riemann–Liouville Fractional Integrals

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In this paper, we introduce $(\alpha, h-m)-p$-convex function and some related functions. By applying this generalized definition, new versions of Hadamard and Fejér–Hadamard fractional integral inequalities for Riemann–Liouville fractional integrals are given. The presented results hold at the same time for different types of convexities.

1. Introduction and Preliminary Results

Convex functions play an important role in mathematical inequalities. Many inequalities for convex and related functions have been studied in recent decades and consequently published in well-reputed journals (see [1–15]). The utilization of fractional integral operators for establishing the generalized versions of classical inequalities has become a fashion in modern study of mathematical inequalities. In this regard, the Hadamard inequality is studied extensively for fractional integral operators.

The aim of this paper is to present Hadamard-type inequalities and their weighted versions called Fejér–Hadamard-type inequalities for Riemann–Liouville fractional integral operators of generalized convex functions. These fractional inequalities will hold simultaneously for convex, p-convex, harmonically convex, m-convex, h-convex, (α, m)-convex, $(h-m)$-convex, (p, h)-convex, and (s, m)-convex functions.

First, we give the definition of Riemann–Liouville fractional integrals (see [16]).

Definition 1. The left and right sided Riemann–Liouville fractional integrals for a function $f \in L_1[a, b]$ of order $\tau \in \mathbb{R}\,(\tau > 0)$ are defined by

$$I_{a^+}^{\tau} f(x) = \frac{1}{\Gamma(\tau)} \int_a^x (x-t)^{\tau-1} f(t)\mathrm{d}t, \quad x > a, \tag{1}$$

$$I_{b^-}^{\tau} f(x) = \frac{1}{\Gamma(\tau)} \int_x^b (t-x)^{\tau-1} f(t)\mathrm{d}t, \quad x < b, \tag{2}$$

where $\Gamma(\tau) = \int_0^{\infty} e^{-t} t^{\tau-1}\mathrm{d}t$.

Next, we give definitions of convex and related functions which are useful for this study.

Definition 2 (see [17]). A function $f\colon [a, b] \longrightarrow \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \le t f(x) + (1-t) f(y), \quad \forall x, y \in [a, b], t \in [0, 1] \tag{3}$$

holds. If the inequality in (3) is reversed, then f is called concave function.

Definition 3 (see [18, 19]). Let $I \subset (0,\infty)$ be an interval and $p \in \mathbb{R}\setminus\{0\}$. Then, a function $f\colon I \longrightarrow \mathbb{R}$ is said to be p-convex, if

$$f\left(\left(ta^p + (1-t)b^p\right)^{1/p}\right) \le tf(a) + (1-t)f(b) \tag{4}$$

holds for $a, b \in I$ and $t \in [0,1]$. If the inequality in (4) is reversed, then f is called p-concave function.

Definition 4 (see [3]). Let $h\colon J \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a positive function. Let $I \subset (0,\infty)$ be an interval and $p \in \mathbb{R}\setminus\{0\}$. A function $f\colon I \longrightarrow \mathbb{R}$ is said to be (p,h)-convex, if

$$f\left(\left(ta^p + (1-t)b^p\right)^{1/p}\right) \le h(t)f(a) + h(1-t)f(b) \tag{5}$$

holds for $a, b \in I$ and $t \in [0,1]$. If the inequality in (5) is reversed, then f is called (p,h)-concave function.

The following definition unifies several kinds of convex functions for example $(h-m)$-convex, (s,m)-convex, and (α,m)-convex functions in a single inequality.

Definition 5 (see [20]). Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h\colon J \longrightarrow \mathbb{R}$ be a non-negative function. A function $f\colon [0,b] \longrightarrow \mathbb{R}$ is called $(\alpha, h-m)$-convex function, if f is non-negative and for all $x, y \in [0,b], t \in (0,1)$ and $(\alpha,m) \in [0,1]^2$, one has

$$f(tx + m(1-t)y) \le h(t^\alpha)f(x) + mh(1-t^\alpha)f(y). \tag{6}$$

In the upcoming section, we define a new generalized notion which will be called $(\alpha, h-m)-p$-convex function. It will generate all kinds of convex functions discussed in aforementioned premise. By using this new definition and Riemann–Liouville fractional integrals, we will prove two versions of the Hadamard inequality. In Section 3, we give the weighted versions of the Hadamard inequalities presented in Section 2. Also, results for $(h-m)-p$-convex, $(\alpha,m)-p$-convex, and $(\alpha,h)-p$-convex functions are explicitly given in the form of corollaries.

2. Hadamard-Type Inequalities for $(\alpha, h-m)$-p-Convex Function

We define $(\alpha, h-m)$-p-convex function as follows.

Definition 6. Let $J\subseteq\mathbb{R}$ be an interval containing $(0,1)$ and let $h\colon J \longrightarrow \mathbb{R}$ be a non-negative function. Let $I \subset (0,\infty)$ be an interval and $p \in \mathbb{R}\setminus\{0\}$. A function $f\colon I \longrightarrow \mathbb{R}$ is said to be $(\alpha, h-m)$-p-convex, if

$$f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right) \le h(t^\alpha)f(a) + mh(1-t^\alpha)f(b) \tag{7}$$

holds provided $(ta^p + m(1-t)b^p)^{1/p} \in I$ for $t \in [0,1]$ and $(\alpha,m) \in [0,1]^2$.

For different choices of α, h, m, and p, the outcomes of Definition 6 are given in the remark as follows.

Remark 1

(i) Fixing $p=1$ in Definition 6, it gives $(\alpha, h-m)$-convexity (see 20, [Definition 4.5]).
(ii) Fixing $p=-1$ and $h(t)=t$ in Definition 6, it gives (α,m)-HA-convexity (see 4, Definition 2.1]).
(iii) Fixing $\alpha=m=1$ in Definition 6, it gives (p,h)-convex function (see [3]).
(iv) Fixing $\alpha=1=p$ and $h(t)=t^s$ in Definition 6, it gives (s,m)-convex function (see [21]).
(v) Fixing $\alpha=1=p$ and $h(t)=t$ in Definition 6, it gives m-convex function (see [17]).

We define $(h-m)-p$-convex function by setting $\alpha=1$ in Definition 6 as follows.

Definition 7. Let $J\subseteq\mathbb{R}$ be an interval containing $(0,1)$ and let $h\colon J \longrightarrow \mathbb{R}$ be a non-negative function. Let $I \subset (0,\infty)$ be an interval and $p \in \mathbb{R}\setminus\{0\}$. A function $f\colon I \longrightarrow \mathbb{R}$ is said to be $(h-m)-p$-convex, if

$$f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right) \le h(t)f(a) + mh(1-t)f(b) \tag{8}$$

holds provided $(ta^p + m(1-t)b^p)^{1/p} \in I$ for $t \in [0,1]$ and $(\alpha,m) \in [0,1]^2$.

We define $(\alpha,m)-p$-convex function by setting $h(t)=t$ in Definition 6 as follows.

Definition 8. Let $I \subset (0,\infty)$ be an interval and $p \in \mathbb{R}\setminus\{0\}$. A function $f\colon I \longrightarrow \mathbb{R}$ is said to be $(\alpha,m)-p$-convex, if

$$f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right) \le t^\alpha f(a) + m(1-t^\alpha)f(b) \tag{9}$$

holds provided $(ta^p + m(1-t)b^p)^{1/p} \in I$ for $t \in [0,1]$ and $(\alpha,m) \in [0,1]^2$.

We define $(\alpha,h)-p$-convex function by setting $m=1$ in Definition 6 as follows.

Definition 9. Let $J\subseteq\mathbb{R}$ be an interval containing $(0,1)$ and let $h\colon J \longrightarrow \mathbb{R}$ be a non-negative function. Let $I \subset (0,\infty)$ be an interval and $p \in \mathbb{R}\setminus\{0\}$. A function $f\colon I \longrightarrow \mathbb{R}$ is said to be $(\alpha,h)-p$-convex, if

$$f\left(\left(ta^p + (1-t)b^p\right)^{1/p}\right) \le h(t^\alpha)f(a) + h(1-t^\alpha)f(b) \tag{10}$$

holds for $a, b \in I$, $t \in [0,1]$, $\alpha \in [0,1]$.

We define $(\alpha, h-m)$-HA-convex function by setting $p=-1$ in Definition 6 as follows.

Definition 10. Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h\colon J \longrightarrow \mathbb{R}$ be a non-negative function. Let $I \subset (0,\infty)$ be an interval and $p \in \mathbb{R}\setminus\{0\}$. A function $f\colon I \longrightarrow \mathbb{R}$ is said to be $(\alpha, h-m)$-HA-convex, if

$$f\left(\frac{ab}{am(1-t)+bt}\right) \le h(t^\alpha) f(a) + mh(1-t^\alpha) f(b) \quad (11)$$

holds for a, b, $\in\in I$, $t \in (0,1)$ and $(\alpha, m) \in [0,1]^2$.

We define $(s,m)-p$-convex function by setting $\alpha = 1$ and $h(t) = t^s$ in Definition 6 as follows.

Definition 11. Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h\colon J \longrightarrow \mathbb{R}$ be a non-negative function. Let $I \subset (0,\infty)$ be an interval. A function $f\colon I \longrightarrow \mathbb{R}$ is said to be $(s,m)-p$-convex, if

$$f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right) \le t^s f(a) + m(1-t)^s f(b) \quad (12)$$

holds provided $(ta^p + m(1-t)b^p)^{1/p} \in I$ for $t \in [0,1]$ and $(s,m) \in [0,1]^2$.

We define $(s,m)-p$-Godunova–Levin function by setting $\alpha = 1$ and $h(t) = t^{-s}$ in Definition 6 as follows.

Definition 12. Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h\colon J \longrightarrow \mathbb{R}$ be a non-negative function. Let $I \subset (0,\infty)$ be an interval and $p \in \mathbb{R}\setminus\{0\}$. A function $f\colon I \longrightarrow \mathbb{R}$ is said to be $(s,m)-p$-Godunova–Levin, if

$$f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right) \le \frac{f(a)}{t^s} + \frac{mf(b)}{(1-t)^s} \quad (13)$$

holds provided $(ta^p + m(1-t)b^p)^{1/p} \in I$ for $t \in [0,1]$ and $(s,m) \in [0,1]^2$.

In the next results, it is assumed that functions f, g, and h are integrable and all involved integrals are finite.

Theorem 1. *Let $f\colon I \longrightarrow \mathbb{R}$ be a positive $(\alpha, h-m)-p$-convex function as given in Definition 6 with $(tb^p + (1-t)(a^p/m))^{1/p} \in I, m \neq 0, a^p < mb^p$. Then, the following inequality for fractional integral operators (1) and (2) holds:*

$$\begin{aligned} f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right) &\le \frac{\Gamma(\tau+1)}{(mb^p - a^p)^\tau}\left(h\left(\frac{1}{2^\alpha}\right)\left(I^\tau_{a^{p}+} f\circ\xi\right)(mb^p) + m^{\tau+1} h\left(\frac{2^\alpha - 1}{2^\alpha}\right)\left(I^\tau_{b^{p}-} f\circ\xi\right)\left(\frac{a^p}{m}\right)\right) \\ &\le \tau\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(a) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(b)\right)\int_0^1 t^{\tau-1} h(t^\alpha)\mathrm{d}t \right. \\ &\quad \left. + m\tau\left(h\left(\frac{1}{2^\alpha}\right)f(b) + mh\left(1 - \frac{1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1} h(1-t^\alpha)\mathrm{d}t\right\}, \quad \xi(t) = t^{1/p}. \end{aligned} \quad (14)$$

Proof. By $(\alpha, h-m)-p$-convexity of f, one can have

$$f\left(\left(\frac{x^p + my^p}{2}\right)^{1/p}\right) \le h\left(\frac{1}{2^\alpha}\right) f(x) + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right) f(y). \quad (15)$$

By setting $x = (ta^p + m(1-t)b^p)^{1/p}$ and $y = (tb^p + (1-t)(a^p/m))^{1/p}$ in (15), we get

$$f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right) \le h\left(\frac{1}{2^\alpha}\right) f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) f\left(\left(tb^p + \frac{a^p}{m}(1-t)\right)^{1/p}\right). \quad (16)$$

Multiplying both sides of (16) by $t^{\tau-1}$ and integrating over $[0,1]$, we get

$$\frac{1}{\tau} f\left(\left(\frac{a^p + b^p}{2}\right)^{1/p}\right) \le h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1} f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)\mathrm{d}t + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)\int_0^1 t^{\tau-1} f\left(\left(tb^p + \frac{a^p}{m}(1-t)\right)^{1/p}\right)\mathrm{d}t. \quad (17)$$

Setting $ta^p + m(1-t)b^p = x$, that is, $t = (mb^p - x)/(mb^p - a^p)$ and $tb^p + (a^p/m)(1-t) = y$, that is, $t = y - (a^p/m)/b^p - (a^p/m)$, on the right hand side of (17) and after simplification, one can get first inequality of (14).

Again by using the $(\alpha, h-m) - p$-convexity of f, the right hand side of (16) leads to the following inequality:

$$\begin{aligned}
&h\left(\frac{1}{2^\alpha}\right)f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right) + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)f\left(\left(tb^p + \frac{a^p}{m}(1-t)\right)^{1/p}\right)\\
&\leq \left(h\left(\frac{1}{2^\alpha}\right)f(a) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(b)\right)h(t^\alpha) + m\left(h\left(\frac{1}{2^\alpha}\right)f(b) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\right)h(1-t^\alpha).
\end{aligned} \tag{18}$$

Multiplying $t^{\tau-1}$ on both sides of (18) and integrating over $[0,1]$, we have

$$\begin{aligned}
&h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1} f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)\mathrm{d}t + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)\int_0^1 t^{\tau-1} f\left(\left(tb^p + \frac{a^p}{m}(1-t)\right)^{1/p}\right)\mathrm{d}t\\
&\leq \left(h\left(\frac{1}{2^\alpha}\right)f(a) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(b)\right)\int_0^1 t^{\tau-1}h(t^\alpha)\mathrm{d}t + m\left(h\left(\frac{1}{2^\alpha}\right)f(b) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1}h(1-t^\alpha)\mathrm{d}t.
\end{aligned} \tag{19}$$

Setting $ta^p + m(1-t)b^p = x$, that is, $t = (mb^p - x)/(mb^p - a^p)$ and $(1-t)(a^p/m) + tb^p = y$, that is, $t = (y - (a^p/m))/(b^p - (a^p/m))$, in (19) and after calculation, one can get the second inequality of (14). □

The inequality established in (14) gives the Hadamard inequality for $(\alpha, h-m) - p$-convex function by fixing $\tau = 1$. The readers can also obtain the Hadamard inequality for $(h-m) - p$-convex, $(\alpha, m) - p$-convex, $(\alpha, h) - p$-convex, and all deducible functions by fixing parameters and the function h of their choice. Also, inequality (14) provides fractional versions of the Hadamard inequality for Riemann–Liouville fractional integrals of functions stated in the introduction section.

Further consequences of the above theorem are stated in the following.

Remark 2

(i) On fixing $\alpha = m = 1$, $h(t) = t$, and $p = 1$, [[14], Theorem 2] is obtained.

(ii) On fixing $\alpha = m = 1$, $p = 1$, $h(t) = t$, and $\tau = 1$, classical Hadamard inequality [22] is obtained.

(iii) On fixing $\alpha = m = 1$, $h(t) = t$, and $p = -1$, [[23], Theorem 2.4] is obtained.

Now, we give another variant of the Hadamard inequality as follows.

Theorem 2. *Let the assumptions of Theorem 1 hold. Then, we have the following inequality:*

$$\begin{aligned}
f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right) &\leq \Gamma(\tau+1)\left(\frac{2}{mb^p - a^p}\right)^\tau\\
&\times\left(h\left(\frac{1}{2^\alpha}\right)\left(I^\tau_{(a^p+mb^p/2)^+} f\circ\xi\right)(mb^p) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(I^\tau_{(a^p+mb^p/2)^-} f\circ\xi\right)\left(\frac{a^p}{m}\right)\right)\\
&\leq \tau\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(a) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(b)\right)\int_0^1 t^{\tau-1}h\left(\left(\frac{t}{2}\right)^\alpha\right)\mathrm{d}t\right.\\
&\left.+ m\left(h\left(\frac{1}{2^\alpha}\right)f(b) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1}h\left(\left(\frac{2-t}{2}\right)^\alpha\right)\mathrm{d}t\right\},\quad \xi(t) = t^{1/p}.
\end{aligned} \tag{20}$$

Proof. By setting $x = (t/2a^p + m(2-t/2)b^p)^{1/p}$ and $y = ((t/2)b^p + (a^p/m)(2-t/2))^{1/p}$ in (15), we get

$$f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right) \le h\left(\frac{1}{2^\alpha}\right) f\left(\left(\frac{t}{2}a^p + m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) f\left(\left(\frac{t}{2}b^p + \left(\frac{2-t}{2}\right)\frac{a^p}{m}\right)^{1/p}\right). \tag{21}$$

Multiplying both sides of (21) by $t^{\tau-1}$ and integrating over $[0,1]$, we get

$$\frac{1}{\tau} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right) \le h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1} f\left(\left(\frac{t}{2}a^p + m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)dt + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 t^{\tau-1} f\left(\left(\frac{t}{2}b^p + \frac{a^p}{m}\left(\frac{2-t}{2}\right)\right)^{1/p}\right)dt. \tag{22}$$

Setting $(t/2)a^p + m((2-t)/2)b^p = x$, that is, $(t/2) = ((mb^p - x)/mb^p - a^p)$ and $((2-t)/2)(a^p/m) + (t/2)b^p = y$, that is, $t/2 = (y - (a^p/m))/(mb^p - (a^p/m))$, on the right hand side of (22) and after simplification, one can get first inequality of (20).

Again by using $(\alpha, h-m)-p$-convexity of f, the right hand side of (21) leads to the following inequality:

$$\begin{aligned} &h\left(\frac{1}{2^\alpha}\right) f\left(\left(\frac{t}{2}a^p + m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) f\left(\left(\frac{t}{2}b^p + \left(\frac{2-t}{2}\right)\frac{a^p}{m}\right)^{1/p}\right) \\ &\le \left(h\left(\frac{1}{2^\alpha}\right) f(a) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) f(b)\right) h\left(\left(\frac{t}{2}\right)^\alpha\right) + m\left(h\left(\frac{1}{2^\alpha}\right) f(b) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) f\left(\frac{a}{m}\right)\right) h\left(1-\left(\frac{t}{2}\right)^\alpha\right). \end{aligned} \tag{23}$$

Multiplying $t^{\tau-1}$ on both sides of (23) and integrating over $[0,1]$, we have

$$\begin{aligned} &h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1} f\left(\left(\frac{t}{2}a^p + m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)dt + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 t^{\tau-1} f\left(\left(\frac{t}{2}b^p + \frac{a^p}{m}\left(\frac{2-t}{2}\right)\right)^{1/p}\right)dt \\ &\le \left(h\left(\frac{1}{2^\alpha}\right) f(a) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) f(b)\right)\int_0^1 t^{\tau-1} h\left(\left(\frac{t}{2}\right)^\alpha\right)dt + m\left(h\left(\frac{1}{2^\alpha}\right) f(b) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1} h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt. \end{aligned} \tag{24}$$

Setting $(t/2)a^p + m((2-t)/2)b^p = x$, that is, $t/2 = (mb^p - x)/(mb^p - x)$ and $((2-t)/2)a^p + (t/2)b^p = y$, that is, $t/2 = (y - (a^p/m))/(b^p - (a^p/m))$, in (24) and after calculation, one can get second inequality of (20). □

The above version of the Hadamard inequality gives the corresponding inequalities for $(h-m)-p$-convex, $(\alpha, m)-p$-convex, $(\alpha, h)-p$-convex, and all deducible functions by fixing parameters and the function h.

Further consequences of the above theorem are stated in the following.

Remark 3

(i) On fixing $\alpha = 1 = m$, $p > 0$, and $h(t) = t$ in Theorem 2, [[24], Theorem 2.1 (i)] is obtained.

(ii) On fixing $\alpha = 1 = m$, $p < 0$, and $h(t) = t$ in Theorem 2, [[24], Theorem 2.1 (ii)] is obtained.

(iii) On fixing $\alpha = 1 = m$, $p = 1$, and $h(t) = t$ in Theorem 2, [[24], Corollary 2.1] is obtained.

Remark 4. From Theorems 1 and 2, one can deduce results for convex, p-convex, m-convex, h-convex, (α, m)-convex, $(h - m)$-convex, (s, m)-convex, and (p, h)-convex functions.

3. Fejér–Hadamard-Type Inequalities

In this section, we present the Fejér–Hadamard-type inequalities for $(\alpha, h - m) - p$-convex functions by applying definition of Riemann–Liouville fractional integral operators. We also give results for new definitions obtained in Section 2.

Theorem 3. *Let $f: I \longrightarrow \mathbb{R}$ be a positive $(\alpha, h - m) - p$-convex function as given in Definition 6 and $f(a^p + mb^p - x) = f(x)$. If $g: I \longrightarrow \mathbb{R}$ is a positive function, then the following inequality for fractional integral operators (1) and (2) holds:*

$$
\begin{aligned}
& f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right)\left(I^{\tau}_{a^{p}+} g \circ \xi\right)(mb^p) \le h\left(\frac{1}{2^\alpha}\right)\left(I^{\tau}_{a^{p}+} fg \circ \xi\right)(mb^p) + m^{\mu+1} h\left(\frac{2^\alpha - 1}{2^\alpha}\right)\left(I^{\tau}_{b^{p}-} fg \circ \xi\right)\left(\frac{a^p}{m}\right) \\
& \le \frac{(mb^p - a^p)^\tau}{\Gamma(\tau)}\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(a) + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)f(b)\right)\int_0^1 t^{\tau-1} g\left(\left(ta^p + m(1-t)b^p\right)^{\frac{1}{p}}\right)h(t^\alpha)dt\right. \\
& \left. + m\left(h\left(\frac{1}{2^\alpha}\right)f(b) + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1} g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)h(1 - t^\alpha)dt\right\} \\
& \xi(t) = t^{1/p}, fg \circ \xi = (f \circ \xi)(g \circ \xi).
\end{aligned}
\tag{25}
$$

Proof. Multiplying (16) by $t^{\tau-1} g((ta^p + m(1-t)b^p)^{1/p})$ and integrating over $[0, 1]$, we get

$$
\begin{aligned}
& f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right)\int_0^1 t^{\tau-1} g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)dt \\
& \le h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1} f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right) g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)dt \\
& + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)\int_0^1 t^{\tau-1} f\left(\left(tb^p + (1-t)\frac{a^p}{m}\right)^{1/p}\right) g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)dt.
\end{aligned}
\tag{26}
$$

Setting $ta^p + m(1-t)b^p = x$, that is, $(1-t)a^p + mtb^p = a^p + mb^p - x$, in (26) and utilizing condition $f(x) = f(a^p + mb^p - x)$, by fractional integral operators (1) and (2), one can get first inequality of (25).

Now, multiplying $t^{\tau-1} g((ta^p + m(1-t)b^p)^{1/p})$ in (18) and integrating over $[0, 1]$, we have

$$
\begin{aligned}
& h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1} f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right) g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)dt \\
& + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)\int_0^1 t^{\tau-1} f\left(\left(tb^p + \frac{a^p}{m}(1-t)\right)^{1/p}\right) g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)dt \\
& \le \left(h\left(\frac{1}{2^\alpha}\right)f(a) + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)f(b)\right)\int_0^1 t^{\tau-1} g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)h(t^\alpha)dt \\
& + m\left(h\left(\frac{1}{2^\alpha}\right)f(b) + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1} g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)h(1 - t^\alpha)dt.
\end{aligned}
\tag{27}
$$

Again setting $ta^p + m(1-t)b^p = x$, that is, $(1-t)a^p + mtb^p = a^p + mb^p - x$, in (27) and utilizing condition $f(x) = f(a^p + mb^p - x)$, by using definitions of fractional integral operators (1) and (2), one can get second inequality of (25). □

The consequences of the above theorem are stated in the following.

Remark 5

(i) On fixing $\alpha = m = 1$, $h(t) = t$, $g(x) = 1$, and $p = 1$, [[14], Theorem 2] is obtained.

(ii) On fixing $\alpha = m = 1$, $p = 1$, $h(t) = t$, $g(x) = 1$, and $\tau = 1$, the classical Hadamard inequality [25] is obtained.

(iii) On fixing $\alpha = m = 1$, $p = 1$, $h(t) = t$, and $\tau = 1$, the Fejér–Hadamard inequality [26] is obtained.

Now, we give another variant of the Fejér–Hadamard inequality as follows.

Theorem 4. *Let the assumptions of Theorem 3 hold. Then, we have the following inequality:*

$$\begin{aligned}
&f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I^{\tau}_{((a^p+mb^p)/2)^+}g\circ\xi\right)(mb^p) \le h\left(\frac{1}{2^\alpha}\right)\left(I^{\tau}_{((a^p+mb^p)/2)^+}fg\circ\xi\right)(mb^p)\\
&+m^{\mu+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(I^{\tau}_{((a^p+mb^p)/2)^-}fg\circ\xi\right)\left(\frac{a^p}{m}\right)\\
&\le\frac{1}{\Gamma(\tau)}\left(\frac{mb^p-a^p}{2}\right)^{\tau}\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(a)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(b)\right)\times\int_0^1 t^{\tau-1}g\left(\left(\frac{t}{2}a^p+m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)h\left(\left(\frac{t}{2}\right)^{\alpha}\right)\mathrm{d}t\right.\\
&\left.+m\left(h\left(\frac{1}{2^\alpha}\right)f(b)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1}g\left(\left(\frac{t}{2}a^p+m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)\mathrm{d}t\right\}
\end{aligned}\tag{28}$$

$\xi(t) = t^{1/p}$, $fg\circ\xi = (f\circ\xi)(g\circ\xi)$.

Proof. Multiplying (21) by $t^{\tau-1}g(((t/2)a^p + m((2-t)/2)b^p)^{1/p})$ and integrating over $[0,1]$, we get

$$\begin{aligned}
&f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\int_0^1 t^{\tau-1}g\left(\left(\frac{t}{2}a^p+m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)\mathrm{d}t\\
&\le h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1}f\left(\left(\frac{t}{2}a^p+m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)g\left(\left(\frac{t}{2}a^p+m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)\mathrm{d}t\\
&+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 t^{\tau-1}f\left(\left(\frac{t}{2}b^p+\frac{a^p}{m}\left(\frac{2-t}{2}\right)\right)^{1/p}\right)g\left(\left(\frac{t}{2}a^p+m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)\mathrm{d}t.
\end{aligned}\tag{29}$$

Setting $(t/2)a^p + m((2-t)/2)b^p = x$, that is, $m((2-t)/2)a^p + (t/2)b^p = a^p + mb^p - x$, in (29) and utilizing condition $f(x) = f(a^p + mb^p - x)$, by fractional integral operators (1) and (2), one can get first inequality of (28).

Multiplying $t^{\tau-1}g(((t/2)a^p + m((2-t)/2)b^p)^{1/p})$ on both sides of (23) and integrating over $[0,1]$, we have

$$\begin{aligned}
&h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1} f\left(\left(\frac{t}{2}a^p + m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right) g\left(\left(\frac{t}{2}a^p + m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)dt\\
&+ mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 t^{\tau-1} f\left(\left(\frac{t}{2}b^p + \left(\frac{2-t}{2}\right)\frac{a^p}{m}\right)^{1/p}\right) g\left(\left(\frac{t}{2}a^p + m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)dt\\
&\leq \left(h\left(\frac{1}{2^\alpha}\right)f(a) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(b)\right)\int_0^1 t^{\tau-1} g\left(\left(\frac{t}{2}a^p + m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)h\left(\left(\frac{t}{2}\right)^\alpha\right)dt\\
&+ m\left(h\left(\frac{1}{2^\alpha}\right)f(b) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1} g\left(\left(\frac{t}{2}a^p + m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)h\left(\frac{2^\alpha - t^\alpha}{2^\alpha}\right)dt.
\end{aligned} \tag{30}$$

Again setting $(t/2)a^p + m((2-t)/2)b^p = x$, that is, $m((2-t)/2)a^p + (t/2)b^p = a^p + mb^p - x$, in (30) and utilizing condition $f(x) = f(a^p + mb^p - x)$, by fractional integral operators (1) and (2), one can get second inequality of (28). □

The consequences of the above theorem are stated in the following.

Remark 6

(i) On fixing $\alpha = 1 = m$, $p > 0$, $g(x) = 1$, and $h(t) = t$ in Theorem 4, [[24], Theorem 2.1 (i)] is obtained.

(ii) On fixing $\alpha = 1 = m$, $p < 0$, $g(x) = 1$, and $h(t) = t$ in Theorem 4, [[24], Theorem 2.1 (ii)] is obtained.

(iii) On fixing $\alpha = 1 = m$, $p = 1$, $g(x) = 1$, and $h(t) = t$ in Theorem 4, [[24], Corollary 2.1] is obtained.

Remark 7. From Theorems 3 and 4, one can deduce results for convex, p-convex, m-convex, h-convex, (α, m)-convex, $(h-m)$-convex, and (p, h)-convex functions.

3.1. Results for $(h-m)-p$-Convex Functions. By setting $\alpha = 1$ in Theorems 1–4, the results for $(h-m)-p$-convex functions are obtained as follows.

Theorem 5. *Under the assumptions of Theorem 1, the following inequality holds for $(h-m)-p$-convex functions:*

$$\begin{aligned}
\frac{1}{h}\left(\frac{1}{2}\right)f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right) &\leq \frac{\Gamma(\tau+1)}{(mb^p - a^p)^\tau}\left\{\left(I^\tau_{a^{p+}} f\circ\xi\right)(mb^p) + m^{\mu+1}\left(I^\tau_{b^{p-}} f\circ\xi\right)\left(\frac{a^p}{m}\right)\right\}\\
&\leq \tau\left\{(f(a) + mf(b))\int_0^1 t^{\tau-1}h(t)dt + m\left(f(b) + mf\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1}h(1-t)dt\right\}.
\end{aligned} \tag{31}$$

Theorem 6. *Under the assumptions of Theorem 2, the following inequality holds for $(h-m)-p$-convex functions:*

$$\begin{aligned}
\frac{1}{h(1/2)f}\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right) &\leq \Gamma(\tau+1)\left(\frac{2}{mb^p - a^p}\right)^\tau\left\{\left(I^\tau_{((a^p+mb^p)/2)^+} f\circ\xi\right)(mb^p) + m^{\mu+1}\left(I^\tau_{((a^p+mb^p)/2)^-} f\circ\xi\right)\left(\frac{a^p}{m}\right)\right\}\\
&\leq \tau\left\{(f(a) + mf(b))\int_0^1 t^{\tau-1}h\left(\frac{t}{2}\right)dt + m\left(f(b) + mf\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1}h\left(\frac{2-t}{2}\right)dt\right\}.
\end{aligned} \tag{32}$$

Theorem 7. *Under the assumptions of Theorem 3, the following inequality holds for $(h-m)-p$-convex functions:*

$$\frac{1}{h(1/2)}f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I^{\tau}_{a^{p+}}g\circ\xi\right)(mb^p)\le\left(I^{\tau}_{a^{p+}}fg\circ\xi\right)(mb^p)+m\left(I^{\tau}_{b^{p-}}fg\circ\xi\right)\left(\frac{a^p}{m}\right)$$
$$\le\frac{(mb^p-a^p)^{\tau}}{\Gamma(\tau)}\left\{(f(a)+mf(b))\int_0^1 t^{\tau-1}g\left(\left(ta^p+m(1-t)b^p\right)^{1/p}\right)h(t)\mathrm{d}t\right.$$
$$\left.+m\left(f(b)+mf\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1}g\left(\left(ta^p+m(1-t)b^p\right)^{1/p}\right)h(1-t)\mathrm{d}t\right\}. \tag{33}$$

Theorem 8. *Under the assumptions of Theorem 4, the following inequality holds for $(h-m)-p$-convex functions:*

$$\frac{1}{h(1/2)}f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I^{\tau}_{((a^p+mb^p)/2)^+}g\circ\xi\right)(mb^p)\le\left(I^{\tau}_{((a^p+mb^p)/2)^+}fg\circ\xi\right)(b^p)+m\left(I^{\tau}_{((a^p+b^p)/2)^-}fg\circ\xi\right)\left(\frac{a^p}{m}\right)$$
$$\le\frac{1}{\Gamma(\tau)}\left(\frac{mb^p-a^p}{2}\right)^{\tau}\left\{(f(a)+mf(b))\int_0^1 t^{\tau-1}g\left(\left(\frac{t}{2}a^p+\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)h\left(\frac{t}{2}\right)\mathrm{d}t\right.$$
$$\left.+m\left(f(b)+mf\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\tau-1}g\left(\left(\frac{t}{2}a^p+m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)h\left(\frac{2-t}{2}\right)\mathrm{d}t\right\}. \tag{34}$$

3.2. Results for $(\alpha,m)-p$-Convex Functions. By setting $h(t)=t$ in Theorems 1, 2, 3, and 4, the results for $(\alpha,h)-p$-convex functions are obtained as follows.

Theorem 9. *Under the assumptions of Theorem 1, the following inequality holds for $(\alpha,m)-p$-convex functions:*

$$2^{\alpha}f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\le\frac{\Gamma(\tau+1)}{(mb^p-a^p)^{\tau}}\left(\left(I^{\tau}_{a^{p+}}f\circ\xi\right)(mb^p)+m^{\mu+1}(2^{\alpha}-1)\left(I^{\tau}_{b^{p-}}f\circ\xi\right)\left(\frac{a^p}{m}\right)\right)$$
$$\le\frac{\tau}{\tau+\alpha}(f(a)+m(2^{\alpha}-1)f(b))+\frac{m\alpha}{\tau+\alpha}\left(f(b)+m(2^{\alpha}-1)f\left(\frac{a}{m^2}\right)\right). \tag{35}$$

Theorem 10. *Under the assumptions of Theorem 2, the following inequality holds for $(\alpha,m)-p$-convex functions:*

$$2^{\alpha}f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\le\Gamma(\tau+1)\left(\frac{2}{mb^p-a^p}\right)^{\tau}\left(\left(I^{\tau}_{((a^p+mb^p)/2)^+}f\circ\xi\right)(mb^p)+m(2^{\alpha}-1)\left(I^{\tau}_{((a^p+mb^p)/2)^-}f\circ\xi\right)\left(\frac{a^p}{m}\right)\right),$$
$$\le\frac{\tau}{2^{\alpha}(\tau+\alpha)}(f(a)+m(2^{\alpha}-1)f(b))+m\left(1-\frac{\tau}{2^{\alpha}(\tau+\alpha)}\right)\left(f(b)+m(2^{\alpha}-1)f\left(\frac{a}{m^2}\right)\right). \tag{36}$$

Theorem 11. *Under the assumptions of Theorem 3, the following inequality holds for $(\alpha,m)-p$-convex functions:*

$$2^{\alpha} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I^{\tau}_{a^{p+}} g\circ\xi\right)(mb^p) \le \left(I^{\tau}_{a^{p+}} fg\circ\xi\right)(mb^p) + m(2^{\alpha}-1)\left(I^{\tau}_{b^{p-}} fg\circ\xi\right)\left(\frac{a^p}{m}\right)$$
$$\le \frac{(mb^p-a^p)^{\tau}}{\Gamma(\tau)}\left\{\left(f(a)+m(2^{\alpha}-1)f(b)\right)\int_0^1 t^{\tau+\alpha-1} g\left(\left(ta^p+m(1-t)b^p\right)^{1/p}\right)dt\right.$$
$$\left.+m\left(f(b)+m(2^{\alpha}-1)f\left(\frac{a}{m^2}\right)\right)\int_0^1\left(t^{\tau-1}-t^{\tau+\alpha-1}\right) g\left(\left(ta^p+m(1-t)b^p\right)^{1/p}\right)dt\right\}. \tag{37}$$

Theorem 12. *Under the assumptions of Theorem 4, the following inequality holds for* $(\alpha, m)-p$*-convex functions:*

$$2^{\alpha} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I^{\tau}_{((a^p+mb^p)/2)^+} g\circ\xi\right)(mb^p) \le \left(I^{\tau}_{((a^p+mb^p)/2)^+} fg\circ\xi\right)(mb^p) + m(2^{\alpha}-1)\left(I^{\tau}_{((a^p+mb^p)/2)^-} fg\circ\xi\right)\left(\frac{a^p}{m}\right)$$
$$\le \frac{1}{\Gamma(\tau)}\left(\frac{mb^p-a^p}{2}\right)^{\tau}\left\{\frac{1}{2^{\alpha}}\left(f(a)+m(2^{\alpha}-1)f(b)\right)\int_0^1 t^{\tau+\alpha-1} g\left(\left(\frac{t}{2}a^p+\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)dt\right.$$
$$\left.+m\left(f(b)+m(2^{\alpha}-1)f\left(\frac{a}{m^2}\right)\right)\int_0^1\left(t^{\tau-1}-\frac{t^{\tau+\alpha-1}}{2^{\alpha}}\right) g\left(\left(\frac{t}{2}a^p+m\left(\frac{2-t}{2}\right)b^p\right)^{1/p}\right)dt\right\}. \tag{38}$$

3.3. Results for $(\alpha, h)-p$*-Convex Functions.* By setting $m = 1$ in Theorems 1, 2, 3, and 4, the results for $(\alpha, h)-p$-convex functions are obtained as follows.

Theorem 13. *Under the assumptions of Theorem 1, the following inequality holds for* $(\alpha, h)-p$*-convex functions:*

$$f\left(\left(\frac{a^p+b^p}{2}\right)^{1/p}\right) \le \frac{\Gamma(\tau+1)}{(b^p-a^p)^{\tau}}\left(h\left(\frac{1}{2^{\alpha}}\right)\left(I^{\tau}_{a^{p+}} f\circ\xi\right)(b^p) + h\left(1-\frac{1}{2^{\alpha}}\right)\left(I^{\tau}_{b^{p-}} f\circ\xi\right)(a^p)\right),$$
$$\le \tau\left\{\left(h\left(\frac{1}{2^{\alpha}}\right)f(a)+h\left(1-\frac{1}{2^{\alpha}}\right)f(b)\right)\int_0^1 t^{\tau-1}h(t^{\alpha})dt\right.$$
$$\left.+\left(h\left(\frac{1}{2^{\alpha}}\right)f(b)+h\left(1-\frac{1}{2^{\alpha}}\right)f(a)\right)\int_0^1 t^{\tau-1}h(1-t^{\alpha})dt\right\},\quad \xi(t)=t^{1/p}. \tag{39}$$

Theorem 14. *Under the assumptions of Theorem 2, the following inequality holds for* $(\alpha, h)-p$*-convex functions:*

$$f\left(\left(\frac{a^p+b^p}{2}\right)^{1/p}\right) \le \Gamma(\tau+1)\left(\frac{2}{b^p-a^p}\right)^{\tau}\left(h\left(\frac{1}{2^{\alpha}}\right)\left(I^{\tau}_{((a^p+b^p)/2)^+} f\circ\xi\right)(b^p) + h\left(1-\frac{1}{2^{\alpha}}\right)\left(I^{\tau}_{((a^p+b^p)/2)^-} f\circ\xi\right)(a^p)\right),$$
$$\le \tau\left\{\left(h\left(\frac{1}{2^{\alpha}}\right)f(a)+h\left(1-\frac{1}{2^{\alpha}}\right)f(b)\right)\int_0^1 t^{\tau-1}h\left(\left(\frac{t}{2}\right)^{\alpha}\right)dt\right.$$
$$\left.+\left(h\left(\frac{1}{2^{\alpha}}\right)f(b)+h\left(1-\frac{1}{2^{\alpha}}\right)f(a)\right)\int_0^1 t^{\tau-1}h\left(1-\left(\frac{t}{2}\right)^{\alpha}\right)dt\right\},\quad \xi(t)=t^{1/p}. \tag{40}$$

Theorem 15. *Under the assumptions of Theorem 3, the following inequality holds for* $(\alpha, h) - p$*-convex functions:*

$$f\left(\left(\frac{a^p+b^p}{2}\right)^{1/p}\right)\left(I^{\tau}_{a^{p+}} g\circ\xi\right)(b^p) \le h\left(\frac{1}{2^\alpha}\right)\left(I^{\tau}_{a^{p+}} fg\circ\xi\right)(b^p) + h\left(1-\frac{1}{2^\alpha}\right)\left(I^{\tau}_{b^{p-}} fg\circ\xi\right)(a^p)$$
$$\le \frac{(b^p-a^p)^\tau}{\Gamma(\tau)}\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(a) + h\left(1-\frac{1}{2^\alpha}\right)f(b)\right)\int_0^1 t^{\tau-1} g\left(\left(ta^p+(1-t)b^p\right)^{1/p}\right)h(t^\alpha)\mathrm{d}t\right.$$
$$\left.+\left(h\left(\frac{1}{2^\alpha}\right)f(b) + h\left(1-\frac{1}{2^\alpha}\right)f(a)\right)\int_0^1 t^{\tau-1} g\left(\left(ta^p+(1-t)b^p\right)^{1/p}\right)h(1-t^\alpha)\mathrm{d}t\right\},$$
$$\xi(t) = t^{1/p}, fg\circ\xi = (f\circ\xi)(g\circ\xi). \tag{41}$$

Theorem 16. *Under the assumptions of Theorem 4, the following inequality holds for* $(\alpha, h) - p$*-convex functions:*

$$f\left(\left(\frac{a^p+b^p}{2}\right)^{1/p}\right)\left(I^{\tau}_{(a^p+b^p/2)^+} g\circ\xi\right)(b^p) \le h\left(\frac{1}{2^\alpha}\right)\left(I^{\tau}_{(a^p+b^p/2)^+} fg\circ\xi\right)(b^p) + h\left(1-\frac{1}{2^\alpha}\right)\left(I^{\tau}_{(a^p+b^p/2)^-} fg\circ\xi\right)(a^p)$$
$$\le \frac{1}{\Gamma(\tau)}\left(\frac{b^p-a^p}{2}\right)^\tau\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(a) + h\left(1-\frac{1}{2^\alpha}\right)f(b)\right)\int_0^1 t^{\tau-1} g\left(\left(\frac{t}{2}a^p+\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)\times h\left(\left(\frac{t}{2}\right)^\alpha\right)\mathrm{d}t\right.$$
$$+\left(h\left(\frac{1}{2^\alpha}\right)f(b) + \left(1-\frac{1}{2^\alpha}\right)f(a)\right)$$
$$\left.\times\int_0^1 t^{\tau-1} g\left(\left(\frac{t}{2}a^p+\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)h\left(1-\left(\frac{t}{2}\right)^\alpha\right)\mathrm{d}t\right\},\quad \xi(t) = t^{1/p}, fg\circ\xi = (f\circ\xi)(g\circ\xi). \tag{42}$$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Wenyan Jia proved the main results, Muhammad Yussouf wrote the paper, Ghulam Farid supervised this work, and Khuram Ali Khan verified the results.

References

[1] B. Ahmad, A. Alsaedi, M. Kirane, and B. T. Torebek, "Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals," *Journal of Computational and Applied Mathematics*, vol. 353, pp. 120–129, 2019.

[2] F. Chen and S. Wu, "Fejer and Hermite-Hadamard type inequalities for harmonically convex functions," *Journal of Applied Mathematics*, vol. 2014, Article ID 386206, 6 pages, 2014.

[3] Z. B. Fang and R. Shi, "On the (p, h)-convex function and some integral inequalities," *Journal of Inequalities and Applications*, no. 2014, p. 45, 2014.

[4] C.-Y. He, Y. Wang, B.-Y. Xi, and F. Qi, "Hermite--Hadamard type inequalities for (alpha,m)-HA and strongly (alpha,m)-HA convex functions," *The Journal of Nonlinear Science and Applications*, vol. 10, no. 01, pp. 205–214, 2017.

[5] N. Mehreen and M. Anwar, "Hermite-Hadamard-Fejér type inequalities for co-ordinated harmonically convex functions via Katugampola fractional integrals," *Engineering and Applied Science Letters*, vol. 4, no. 2, pp. 12–28, 2021.

[6] M. E. Omaba, L. O. Omenyi, and L. O. Omenyi, "Generalized fractional Hadamard type inequalities for Q_s-class functions of the second kind," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 270–278, 2021.

[7] S. Mehmood, G. Farid, K. A. Khan, and M. Yussouf, "New Hadamard and Fejér-Hadamard fractional inequalities for exponentially m-convex function," *Engineering and Applied Science Letters*, vol. 3, no. 1, pp. 45–55, 2020.

[8] S. Mehmood, G. Farid, K. A. Khan, and M. Yussouf, "New fractional Hadamard and Fejér-Hadamard fractional

inequalities associated with exponentially (h-m)-convex functions," *Engineering and Applied Science Letters*, vol. 3, no. 2, pp. 9–18, 2020.
[9] M. Tariq and S. I. Butt, "Some Ostrowski type integral inequalities via generalized harmonic convex functions," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 200–208, 2021.
[10] N. Mehreen and M. Anwar, "Hermite-hadamard type inequalities via exponentially (p, h)-convex functions," *IEEE Access*, vol. 8, pp. 37589–37595, 2020.
[11] A. Adu-Sackey, F. T. Oduro, F. T. Oduro, and G. O. Fosu, "Inequalities approach in determination of convergence of recurrence sequences," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 65–72, 2021.
[12] X. Qiang, G. Farid, M. Yussouf, K. A. Khan, and A. U. Rehman, "New generalized fractional versions of Hadamard and Fejér inequalities for harmonically convex functions," *Journal of Inequalities and Applications*, p. 191, 2020.
[13] B. Meftah and A. Souahi, "Cebyšev inequalities for co-ordinated QC-convex and (s, QC)-convex," *Engineering and Applied Science Letters*, vol. 4, no. 1, pp. 14–20, 2021.
[14] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013.
[15] X. Yang, G. Farid, W. Nazeer, M. Yussouf, Y. M. Chu, and C. Dong, "Fractional generalized Hadamard and the Fejér-Hadamard type inequalities for m-convex functions," *AIMS Mathematics*, vol. 5, no. 6, pp. 6325–6340, 2020.
[16] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier, New York, NY, USA, 2006.
[17] G. H. Toader, "Some generalization of convexity," in *Proceedings of the Colloquium on Approximation and Optimization*, pp. 329–338, Cluj Napoca(Romania), October 1984.
[18] I. Iscan, "Ostrowski type inequalities for p-convex functions," *New Trends in Mathematical Science*, vol. 4, no. 3, p. 140, 2016.
[19] I. Işcan, "Hermite-Hadamard type inequalities for p-convex functions," *International Journal of Analysis and Applications*, vol. 11, no. 2, pp. 137–145, 2016.
[20] G. Farid, A. U. Rehman, and Q. U. Ain, "k-fractional integral inequalities of Hadamard type for $(h-m)$-convex functions," *Computational Methods for Differential Equations*, vol. 8, no. 1, pp. 119–140, 2020.
[21] H. Hudzik and L. Maligranda, "Some remarks on s-convex functions," *Aequationes Mathematicae*, vol. 48, no. 1, pp. 100–111, 1994.
[22] J. Hadamard, "Etude sur les proprietes des fonctions entieres e.t en particulier dune fonction consideree par Riemann," *Journal de Mathematiques Pures et Appliquees*, vol. 58, pp. 171–215, 1893.
[23] İ. İşcan, "Hermite-Hadamard type inequalities for harmonically convex functions," *Hacettepe Journal of Mathematics and Statistics*, vol. 43, no. 6, pp. 935–942, 2014.
[24] M. Kunt and I. Işcan, "Hermite-Hadamard type inequalities for p-convex functions via fractional integrals," *Moroccan Journal of Pure and Applied Analysis*, vol. 3, no. 1, pp. 22–34, 2017.
[25] C. Hermite, "Sur deux limites d'une intgrale dfinie," *Mathesis*, vol. 3, p. 82, 1883.
[26] L. Fejér, "Überdie fourierreihen II," *Math Naturwiss Anz Ungar Akad Wiss*, vol. 24, pp. 369–390, 1906.