Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 9966248, 12 pages
https://doi.org/10.1155/2021/9966248

Research Article

Characterizing Inequalities for Biwarped Product Submanifolds of Sasakian Space Forms

Meraj Ali Khan [1] and Ibrahim Al-dayel [2]

[1] *Department of Mathematics, University of Tabuk, Tabuk, Saudi Arabia*
[2] *Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University, Riyadh 11566, Saudi Arabia*

Correspondence should be addressed to Meraj Ali Khan; meraj79@gmail.com

Received 31 March 2021; Revised 4 July 2021; Accepted 9 August 2021; Published 20 August 2021

Academic Editor: Francesco Lolli

The biwarped product submanifolds generalize the class of product submanifolds and are particular case of multiply warped product submanifolds. The present paper studies the biwarped product submanifolds of the type $S_T \times_{\psi_1} S_\perp \times_{\psi_2} S_\theta$ in Sasakian space forms $\overline{S}(c)$, where S_T, $S_\perp$, and S_θ are the invariant, anti-invariant, and pointwise slant submanifolds of $\overline{S}(c)$. Some characterizing inequalities for the existence of such type of submanifolds are proved; besides these inequalities, we also estimated the norm of the second fundamental form.

1. Introduction

Because of its expected applications in material science and relativistic theory, the investigation of warped product manifolds has obtained a conspicuous subject in the field of differential geometry; for example, warped products give numerous major solutions for Einstein field equations [1]. The theory of warped product manifolds is being used to demonstrate space-time close to the black holes [2]. The warped product $P \times_r S^2(1)$ represents Schwartzschild space-time, with base $P = R \times R^+$, $r > 0$, and fibre $S^2(1)$ that is sphere with radius one. However, the Schwartzschild space-time will transform into a black hole under some instances [3].

In the paper [4], some of the inherent properties of warped product manifolds were investigated. Chen (see [5]) undertook the very first extrinsic study of warped product manifolds in the almost complex setting while acquiring certain existence results for CR-submanifolds to be CR-warped product submanifolds in Kaehler manifolds. Hasegawa and Mihai [6], on the other hand, analyzed contact CR-warped product submanifolds in almost contact environments. Many other people have investigated warped product manifolds in contact geometry, yielding an assortment of existence outcomes for instance (see [7–10]).

Another general class of warped product semislant submanifolds and contact CR-warped product submanifolds is the warped product pointwise semislant submanifold. The analysis was then continued by I. Mihai and S. Uddin in the framework of Sasakian manifolds, and few ideal inequalities relating to the second fundamental form and warping function were obtained. In the papers (see [11–13]), warped product pointwise semislant submanifolds for almost contact and almost complex manifolds were investigated.

One more generalized class of product manifolds is biwarped product manifolds, which are a subclass of multiply warped product manifolds. Chen and Dillen [14] looked at multiply warped product submanifolds immersed in Kaehler manifolds and found the remarkable optimum inequalities for them. Biwarped product submanifolds have recently been investigated (cf., see [15, 16]). Also, there is a recent paper [17], which initiates the study of inequalities for biwarped product submanifolds of nearly trans-Sasakian manifolds.

In this manuscript, authors established some inequalities for the squared norm of the second fundamental forms.

These inequalities generalize several results available in the literature. Since Sasakian manifolds are particular cases of the nearly trans-Sasakian manifolds, therefore, a natural question arises that the inequalities obtained in the present paper may be particular case of inequalities obtained in [17]. Although in the paper [17] authors studied whole norm of second fundamental form, in the present study, the inequalities for second fundamental form are obtained by taking the restriction on the distribution μ, which is a part of the normal distribution. Therefore, our main results are different from the results obtained in [17] except some initial results.

Basically, in this manuscript, we look at biwarped product submanifolds of Sasakian space forms and determine some interesting inequalities. In terms of warping functions and slant functions, we estimate the norm of the second fundamental form. As a result, the equity case is taken into account.

The article is structured as follows. Second section is contributed to fundamental concepts, formulae, and results that are essential for the paper's next analysis. We prove our key findings in Section 3 by looking into the nature of biwarped product submanifolds in Sasakian space forms.

Throughout the text, we used some abbreviations like Biwarped Product ≡ BW-P, Sasakian space form ≡ S-S-F, totally Geodesic ≡ T-G, and totally umbilical ≡ T-U.

2. Preliminaries

A $(2n+1)$–dimensional C^∞–manifold $\overline{S}$ is said to have an almost contact structure if on $\overline{S}$ there exist a tensor field ϕ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying the following properties [18]:

$$\begin{aligned} \phi^2 &= -I + \eta\otimes\xi, \\ \phi\xi &= 0, \\ \eta^\circ\phi &= 0, \\ \eta(\xi) &= 1. \end{aligned} \tag{1}$$

The manifold $\overline{S}$ with the structure (ϕ,ξ,η) is called almost contact metric manifold. There exists a Riemannian metric g on an almost contact metric manifold $\overline{S}$, satisfying the following:

$$\eta(E_1) = g(E_1,\xi), g(\phi E_1,\phi E_2) = g(E_1,E_2) - \eta(E_1)\eta(E_2), \tag{2}$$

for all $E_1, E_2 \in T\overline{S}$ where $T\overline{S}$ is the tangent bundle of $\overline{S}$.

An almost contact metric manifold $\overline{S}(\phi,\xi,\eta,g)$ is said to be Sasakian manifold if it satisfies the following relation [18]:

$$\left(\overline{\nabla}_{E_1}\phi\right)F = g(E_1,E_2)\xi - \eta(E_2)E_1, \tag{3}$$

for any $E_1, E_2 \in T\overline{S}$, where $\overline{\nabla}$ denotes the Riemannian connection of the metric g. More details of almost contact metric manifold can be seen in [18]. For a Sasakian manifold, we have

$$\overline{\nabla}_{E_1}\xi = -\phi E_1. \tag{4}$$

A Sasakian manifold $\overline{S}$ is said to be a Sasakian space form if it has constant ϕ-holomorphic sectional curvature c and is denoted by $\overline{S}(c)$. The curvature tensor $\overline{R}$ of S-S-F $\overline{S}(c)$ is given by

$$\begin{aligned} \overline{R}(E_1,E_2)E_3 = {}& \frac{c-3}{4}\{g(E_2,E_3)E_1 - g(E_1,E_3)E_2\} + \frac{c-1}{4} \\ & \left\{ \begin{array}{c} g(E_1,\phi E_3)\phi E_2 - g(E_2,\phi E_3)\phi E_1 + 2g(E_1,\phi E_2)\phi E_3 + \eta(E_1)\eta(E_3)E_2 \\ -\eta(E_2)\eta(E_3)E_1 + g(E_1,E_3)\eta(E_2)\xi - g(E_2,E_3)\eta(E_1)\xi \end{array} \right\}, \end{aligned} \tag{5}$$

for all vector fields E_1, E_2, and E_3 on $\overline{S}$.

Let S be a submanifold of an A-C-M manifold $\overline{S}$ with induced metric g. The Riemannian connection $\overline{\nabla}$ of $\overline{S}$ induces canonically the connections ∇ and $\nabla^\perp$ on the tangent bundle TS and the normal bundle $T^\perp S$ of S, respectively, and then the Gauss and Weingarten formulae are governed by

$$\overline{\nabla}_{E_1}E_2 = \nabla_{E_1}E_2 + \sigma(E_1,E_2), \tag{6}$$

$$\overline{\nabla}_{E_1}V = -A_V E_1 + \nabla^\perp_{E_1}V, \tag{7}$$

for each $E_1, E_2 \in TS$ and $V \in T^\perp S$, where σ and A_V are the second fundamental form and the shape operator, respectively, for the immersion of S into $\overline{S}$. They are related as

$$g(\sigma(E_1,E_2),V) = g(A_V E_1,E_2), \tag{8}$$

where g is the Riemannian metric on $\overline{S}$ as well as the induced metric on S.

For a submanifold $S \longrightarrow \overline{S}$, the equation of Codazzi is provided by

$$\begin{aligned} \left(\overline{R}(E_1,E_2)E_3\right)^\perp = {}& \nabla^\perp_{E_1}\sigma(E_2,E_3) - \nabla^\perp_{E_2}\sigma(E_1,E_3) \\ & + \sigma\left(\nabla_{E_3}E_1,E_3\right) - \sigma\left(\nabla_{E_1}E_2,E_3\right) \\ & + \sigma\left(E_1,\nabla_{E_2}E_3\right) - \sigma\left(E_2,\nabla_{E_1}E_3\right), \end{aligned} \tag{9}$$

where $(\overline{R}(E_1,E_2)E_3)^\perp$ is the normal component of the curvature tensor $\overline{R}(E_1,E_2)E_3$.

If TE_1 and NE_1 represent the tangential and normal part of ϕE_1, respectively, for any $E_1 \in TS$, one can write

$$\phi E_1 = TE_1 + NE_1. \tag{10}$$

Similarly, for any $V \in T^{\perp}S$, we write

$$\phi V = tV + nV, \tag{11}$$

where tV and nV are the tangential and normal parts of ϕV, respectively. Thus, T (resp. n) is 1-1 tensor field on TS (respectively, $T^{\perp}S$), and t (respectively, n) is a tangential (respectively, normal) valued 1-form on $T^{\perp}S$ (respectively, TS). The covariant derivatives of the tensor fields ϕ, T, and N are defined as

$$\left(\overline{\nabla}_{E_1}\phi\right)E_2 = \overline{\nabla}_{E_1}\phi E_2 - \phi\overline{\nabla}_{E_1}E_2, \tag{12}$$

$$\begin{aligned}\left(\overline{\nabla}_{E_1}T\right)E_2 &= \nabla_{E_1}TE_2 - T\nabla_{E_1}E_2,\\ \left(\overline{\nabla}_{E_1}N\right)E_2 &= \nabla^{\perp}_{E_1}NE_2 - N\nabla_{E_1}E_2.\end{aligned} \tag{13}$$

From equations (3), (6), (7), (10), and (11), we have

$$\begin{aligned}\left(\overline{\nabla}_{E_1}T\right)E_2 &= A_{NE_2}E_1 + t\sigma(E_1,E_2) - g(E_1,E_2)\xi - \eta(E_2)E_1,\\ \left(\overline{\nabla}_{E_1}N\right)E_2 &= n\sigma(E_1,E_2) - \sigma(E_1,TE_2).\end{aligned} \tag{14}$$

The mean curvature vector Π of S is defined as

$$\Pi = \frac{1}{k}\sum_{i=1}^{k}\sigma(u_i,u_i), \tag{15}$$

where k is the dimension of S and $\{u_1,u_2,\ldots,u_k\}$ is a local orthonormal basis of S. The squared norm of the second fundamental form σ is defined as

$$\|\sigma\|^2 = \sum_{i,j=1}^{k} g\left(\sigma\left(u_i,u_j\right),\sigma\left(u_i,u_j\right)\right). \tag{16}$$

A submanifold S of $\overline{S}$ is said to be a T-G submanifold if $\sigma(E_1,E_2)=0$ and T-U submanifold if $\sigma(E_1,E_2) = g(E_1,E_2)\Pi$, for each $E_1,E_2 \in TS$.

The notion of slant submanifolds in contact geometry was first defined by Lotta [19]. Later, these submanifolds were studied by Cabrerizo et al. [20]. Now, we have following definition of slant submanifolds.

Definition 1. A submanifold M of an almost contact metric manifold $\overline{M}$ is said to be slant submanifold if for any $x \in M$ and $X \in T_xM - \langle\xi\rangle$, the angle between X and ϕX is constant. The constant angle $\theta \in [0,\pi/2]$ is then called slant angle of M in $\overline{M}$. If $\theta = 0$, the submanifold is invariant submanifold, and if $\theta = \pi/2$, then it is anti-invariant submanifold. If $\theta \neq 0, \pi/2$, it is proper slant submanifold.

Etayo [21] presented the idea of pointwise slant submanifolds as a generalization of slant submanifolds in the context of almost Hermitian manifolds. Further, Chen and Garay [22] looked into pointwise slant submanifolds for almost Hermitian manifolds and came up with some important results. However, Park [23] expanded the definition of pointwise slant submanifolds in almost contact metric manifolds, which was an important development in this direction. Nevertheless, for almost contact metric manifolds, Uddin and Alkhalidi [24] revised the concept of pointwise slant submanifolds. More specifically, a submanifolds S of an almost contact metric manifold $\overline{S}$ are claimed to be pointwise slant submanifold if for any $E \in T_pS$ in the sense that ξ is tangential to S, and the angle $\theta(E)$ between ϕE and $T_xS - \{0\}$ is independent of the choice of nonzero vector field $E \in T_pS - \{0\}$. In this case, θ is viewed as the slant function of the pointwise slant submanifold, which is a function on S. We now have the following descriptive theorem.

Theorem 1 (see [24]). *Let S be a submanifold of an A-C-M manifold $\overline{S}$ such that $\xi \in TS$. Then, S is pointwise slant iff*

$$T^2 = \cos^2\theta(-I + \eta\otimes\xi), \tag{17}$$

where θ is the real valued function on TS.

As a result, the above formula has the following implications:

$$g(TE_1,TE_2) = \cos^2\theta[g(E_1,E_2) - \eta(E_1)\eta(E_2)], \tag{18}$$

$$g(NE_1,NE_2) = \sin^2\theta[g(E_1,E_2) - \eta(E_1)\eta(E_2)], \tag{19}$$

$\forall E_1,E_2 \in TS$.

One can conceive the warped product of manifolds as a generalization of the product manifolds, which are explained as follows.

Consider two Riemannian manifolds (S_1,g_1) and (S_2,g_2) with corresponding Riemannian metrics g_1 and g_2 and ψ: $S_1 \longrightarrow R$ be a positive differentiable function. If x and y are projection maps such that x: $S_1\times S_2 \longrightarrow S_1$ and y: $S_1\times S_2 \longrightarrow S_2$, which are defined as $x(m,n)=m$ and $y(m,n)=n\forall(m,n)\in S_1\times S_2$, then $S=S_1\times S_2$ is called warped product manifold if the Riemannian structure on S satisfies

$$g(E_1,E_2) = g_1(x_*\overline{E}_1,x_*\overline{E}_2) + (\psi^{\circ}x)^2 g_2(y_*\overline{E}_1,y_*\overline{E}_2), \tag{20}$$

for all $\overline{E}_1,\overline{E}_2 \in TS$. The function ψ represents the warping function of $S_1\times S_2$. We can generalize this definition to multiply W-P manifolds as follows.

Let $\{S_i\}_{i=1,2,\ldots,k}$ be Riemannian manifolds with respective Riemannian metrics $\{g_i\}_{i=1,2,\ldots,k}$, and let $\{\psi_i\}_{i=2,3,\ldots,k}$ be positive valued functions on S_1. Then, the product manifold $S=S_1\times S_2\times\cdots S_k$ equipped with Riemannian metric g given by

$$g = \pi_{1*}(g_1) + \sum_{i=2}^{k}(\psi_i{}^{\circ}\pi_1)^2\pi_{i*}(g_i), \tag{21}$$

is said to multiply W-P manifold denoted by $S = S_1\times_{\psi_2}S_2\times\cdots\times_{\psi_k}S_k$ where π_i are the projection maps of S onto $_i$, respectively, and π_{i*} are their respective tangent maps for $i=1,2,\ldots,k$. The functions ψ_i are known as the warping functions [14]. If the warping functions are constants, the warped product is simply a Riemannian product, known as a trivial multiply warped product.

The analysis of multiply warped product manifolds has recently gained attention in both complex and almost contact settings [14, 25]. We may define biwarped product manifolds as a special case of multiply warped product manifolds by using $i = 3$ in the above description. For the BW-P manifold $\overline{S} = S_0 \times_{\psi_1} S_1 \times_{\psi_2} S_2$ with the Levi-Civita connection, $\overline{\nabla}$ and ∇^i denote the Levi-Civita connection of S_i for $i = 0, 1, 2$. Some formulae relating to covariant derivatives for a BW-P manifold are given in the following lemma.

Lemma 1 (see [26]). *Let $\overline{S} = S_0 \times_{\psi_1} S_1 \times_{\psi_2} S_2$ be a BW-P manifold. Then, we have*

$$\overline{\nabla}_E F = \nabla^0_E F, \tag{22}$$

$$\overline{\nabla}_E G = \overline{\nabla}_G E = E(\ln \psi_i) G, \tag{23}$$

for $E, F \in TS_0$ and $G \in TS_i$, $i = 1, 2$.

$\nabla\psi$ is the gradient of ψ and is defined as

$$g(\nabla\psi, E) = E\psi, \tag{24}$$

$\forall\, E \in TS$. Let S be an m−dimensional Riemannian manifold with the Riemannian metric g, and let $\{u_1, u_2, \ldots u_m\}$ be an orthogonal basis of TS. As a consequence of (24), we have

$$\|\nabla\psi\|^2 = \sum_{i=1}^{m} \left(u_i(\psi)^2\right). \tag{25}$$

The Laplacian of ψ is defined by

$$\Delta\psi = \sum_{i=1}^{m} \left\{\left(\nabla_{u_i} u_i\right)\psi - u_i u_i \psi\right\}. \tag{26}$$

Hopf's lemma is now described.

Lemma 2 (see [27]). *Let S be an n-dimensional connected compact Riemannian manifold. If ψ is a differentiable function on S such that $\Delta\psi \geq 0$ everywhere on S (or $\Delta\psi \leq 0$ everywhere on S), then ψ is a constant function.*

3. Main Results

In the present section, first we trace the existence of BW-P submanifolds $S = S_1 \times_{\psi_1} S_2 \times_{\psi_2} S_3$ for any Riemannian submanifolds S_1, S_2, and S_3 in Sasakian manifolds with warping functions ψ_1 and ψ_2 and then we demonstrate our key findings. Hasegawa and Mihai [6] set up the following result.

Theorem 2. *Let $\overline{S}$ be a $(2m+1)$−dimensional Sasakian manifold. Then, there do not exist W-P submanifolds $S = S_\perp \times_\psi S_T$ such that $S_\perp$ is an anti-invariant submanifold tangent to ξ and S_T an invariant submanifold of $\overline{S}$.*

We draw the conclusion based on the above result; that is, if S_T, $S_\perp$, and S_θ are invariant, anti-invariant, and pointwise proper slant submanifolds, then BW-P submanifolds of the forms $S_\perp \times_{\psi_1} S_T \times_{\psi_2} S_\theta$ and $S_\theta \times_{\psi_1} S_\perp \times_{\psi_2} S_T$ in a Sasakian manifold do not exist. From [23], we have the following observation.

Theorem 3. *Let $\overline{S}$ be a $(2m+1)$− dimensional Sasakian manifold. Then, there do not exist W-P submanifolds $S = S_\theta \times_\psi S_T$ tangential to S such that S_θ is pointwise proper slant submanifold and S_T is invariant submanifold of $\overline{S}$, respectively.*

It can be deduced by Theorem 2 that BW-P submanifolds of the types $S_\theta \times_{\psi_1} S_T \times_{\psi_2} S_\perp$ and $S_\perp \times_{\psi_1} S_\theta \times_{\psi_2} S_T$ in a Sasakian manifold are trivial.

Park identified the existence of the warped product pointwise semislant submanifolds of Sasakian manifolds of the form $S_T \times_\psi S_\theta$ in his paper [23], with warping function ψ, where S_T and S_θ are the holomorphic and pointwise slant submanifolds of $\overline{S}$, as well as proving the next lemma.

Lemma 3. *Let $S = S_T \times_\psi S_\theta$ be a W-P pointwise semislant submanifold of a Sasakian manifold $\overline{S}$ such that $\xi \in TS_T$, where S_T and S_θ are invariant and pointwise slant submanifolds of $\overline{S}$, respectively. Then,*

$$\begin{aligned} g(\sigma(E,G), NTH) = {} & \cos^2\theta E \ln\psi g(G,H) \\ & - \phi E \ln\psi g(G,TH) - \eta(E) g(G,TH), \end{aligned} \tag{27}$$

for any $E \in TS_T$ and $G, H \in TS_\theta$.

Consider the biwarped product submanifolds of the type $S_T \times_{\psi_1} S_\perp \times_{\psi_2} S_\theta$ of a Sasakian manifold $(\overline{S}, \phi, \xi, \eta)$ with warping functions ψ_1 and ψ_2 such that S_T, $S_\perp$, and S_θ are the invariant, anti-invariant, and pointwise slant submanifolds of $\overline{S}$ correspondingly. To address the question of which factor of the biwarped product submanifold is parallel to ξ, we have the following.for all $E \in TS_T$ and $G \in TS_\theta$.where $\{u_0 = \xi, u_1, u_2, \ldots, u_p, \phi u_1, \phi u_2, \ldots, \phi u_p\}$ and $\{u^1, u^2, \ldots, u^q, \sec\theta T u^1, \ldots, \sec\theta T u^q\}$ are the basis of the orthonormal vector fields on TS_T and TS_θ, respectively.- Proof. Choosing unit vector fields $E \in TS_T$, $F \in TS_\perp$, and $G \in TS_\theta$ and using (5) and (19), we have

$$\overline{R}(E, \phi E, G, NG) = -\frac{c-1}{2}\sin^2\theta \|E\|^2 \|G\|^2, \tag{49}$$

$$\overline{R}(E, \phi E, F, \phi F) = -\frac{c-1}{2}\|E\|^2 \|F\|^2. \tag{50}$$

Theorem 4. *Let $\overline{S}$ be a Sasakian manifold. If $S_T \times_{\psi_1} S_\perp \times_{\psi_2} S_\theta$ is a biwarped product submanifold of $\overline{S}$ such that S_T, $S_\perp$, and S_θ are the invariant, anti-invariant, and pointwise slant submanifolds of $\overline{S}$, respectively, then we have the following:*

(i) If $\xi \in TS_\perp$, ψ_1 is constant

(ii) If $\xi \in TS_\theta$, ψ_2 is constant

Proof. The proof of this theorem can be deduced directly from Proposition 1 in [17] for $\beta = 0$. □

Remark 1. Proposition 1 in [17] was proved for slant distributions, and the same proof is also valid for pointwise slant distributions.

As a consequence of the above, we can point out that there are no any nontrivial BW-P submanifolds of the type $S_T \times_{\psi_1} S_\perp \times_{\psi_2} N_\theta$ of a Sasakian manifold if the vector field ξ is tangential to $S_\perp$ or S_θ.

Now, let $S = S_T \times_{\psi_1} S_\perp \times_{\psi_2} S_\theta$ be a BW-P submanifold of a Sasakian manifold $\bar{S}$ and consider the vector field ξ tangent to be S_T. If D is invariant distribution, $D^\perp$ is anti-invariant and D^θ is pointwise slant distribution with the slant function θ. The following decomposition refers to the tangent bundle TM:

$$TS = D \oplus D^\perp \oplus D^\theta \oplus \langle \xi \rangle. \tag{28}$$

The normal bundle $T^\perp S$ is decompounded as

$$T^\perp S = \phi D^\perp \oplus ND^\theta \oplus \mu, \tag{29}$$

where μ is the invariant orthogonal complementary distribution of $\phi D^\perp \oplus ND^\theta$ in $T^\perp S$.

The second fundamental form σ can be written as a consequence of the above direct decomposition.

$$\sigma(E_1, E_2) = \sigma_{\phi D^\perp}(E_1, E_2) + \sigma_{ND^\theta}(E_1, E_2) + \sigma_\mu(E_1, E_2), \tag{30}$$

for $E_1, E_2 \in TS$, where $\sigma_{\phi D^\perp}(E_1, E_2)$, $\sigma_{ND^\theta}(E_1, E_2)$, and $\sigma_\mu(E_1, E_2)$ are the components of $\sigma(E_1, E_2)$ in the normal sub-bundles $\phi D^\perp$, ND^θ and μ, respectively. Moreover if $\{F_1, F_2, \ldots, F_q\}$ be a local orthonormal frame of vector fields of D^θ, then

$$\sigma_{ND^\theta}(E_1, E_2) = \sum_{r=1}^{q} \sigma^r(E_1, E_2) NF_r, \tag{31}$$

where

$$\sigma^r(E_1, E_2) = \csc^2 \theta g(\sigma(E_1, E_2), NF_r). \tag{32}$$

We create an example of a BW-P submanifold of the form $S = S_T \times_{\psi_1} S_\perp \times_{\psi_2} S_\theta$ in Sasakian manifold with $\xi \in S_T$.

Example 1. It is well known that $(R^{2m+1}, \phi_0, \xi, \eta, g)$ denotes a Sasakian manifold with its standard Sasakian structure given by

$$\begin{aligned}
\eta &= \frac{1}{2}\left(dz - \sum_{i=1}^{m} y^i dx^i\right), \\
\xi &= 2\frac{\partial}{\partial z}, \\
g &= \eta \otimes \eta + \frac{1}{4}\left(\sum_{i=1}^{m}\left(dx^i \otimes dx^i + dy^i \otimes dy^i\right)\right), \\
\phi_0\left(\sum_{i=1}^{m}\left(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i} + Z\frac{\partial}{\partial z}\right)\right) &= \sum_{i=1}^{m}\left(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i}\right) + \sum_{i=1}^{m} Y_i y^i \frac{\partial}{\partial z}.
\end{aligned} \tag{33}$$

Consider the submanifold as follows:

$$S = \left\{2\left(u, 0, we^t, 0, 0, v, 0, se^t \cos\theta, se^t \sin\theta, \sin t, f\right) \in R^{11}\right\}. \tag{34}$$

And consider a frame $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of orthogonal vector fields tangent to M as

$$\begin{aligned}
u_1 &= 2\left(\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z}\right), \\
u_2 &= 2\frac{\partial}{\partial y^1}, \\
u_3 &= 2e^t\frac{\partial}{\partial x^3}, \\
u_4 &= 2e^t \cos\theta\frac{\partial}{\partial y^3} + 2e^t \sin\theta\frac{\partial}{\partial y^4}, \\
u_5 &= 2\sin t\frac{\partial}{y^5}, \\
u_6 &= 2\frac{\partial}{\partial z} = \xi.
\end{aligned} \tag{35}$$

It is then simple to note that $D = \text{span}\{u_1, u_2, u_6\}$, $D^\perp = \text{span}\{u_5\}$, and $D^\theta = \text{span}\{u_3, u_4\}$ defined as the invariant, anti-invariant, and pointwise slant distributions with the slant function $\theta \in (0, \pi/2)$ on the Sasakian manifold R^{11}. If we denote the integral manifold of D, $D^\perp$, and D^θ by S_T, $S_\perp$, and S_θ correspondingly, then the metric g on S is given by

$$g = g_{S_T} + \sin^2 t g_{S_\perp} + e^{2t} g_{S_\theta}. \tag{36}$$

Then, $S = S_T\times_{\psi_1}S_\perp\times_{\psi_2}S_\theta$ is a BW-P submanifold with the warping functions $\psi_1 = \sin t$, $\psi_2(t) = e^t$.

First, we demonstrate some preliminary findings.

Lemma 4. *Let $S_T\times_{\psi_1}S_\perp\times_{\psi_2}S_\theta$ be a BW-P submanifold of a Sasakian manifold $\overline{S}$. Then,*

(i) $\xi \ln \psi_1 = 0$ *and* $\xi \ln \psi_2 = 0$

(ii) $g(\sigma(\phi E, G), NG) = E \ln \psi \|G\|^2$

(iii) $g(\sigma(\phi E, H), JH) = E \ln \psi \|H\|^2$

(iv) $g(\sigma(\phi E, G), \phi\sigma(E, G)) = \|\sigma_\mu(E, G)\|^2 + \cos^2\theta (E\ln\psi)^2\|G\|^2$

(v) $g(\sigma(\phi E, H), \phi\sigma(E, H)) = \|\sigma_\mu(E, H)\|^2$

for all $E \in TN_T$, $H \in TN_\perp$, and $G \in TN_\theta$, where σ_μ is the μ component of the second fundamental form σ.

Proof. The part (i) can be deduced from Proposition 2 of [17]. Moreover, the parts (ii) and (iii) can be concluded from equations (30) and (20) in [17], respectively.

Using (6) and (3), we can prove part (iv) as

$$\sigma(\phi E, G) = -\eta(E)G + \phi\sigma(E, G) + \phi\nabla_G E - \nabla_G \phi E. \tag{37}$$

By applying (23), the above equation can now be written as

$$\sigma(\phi E, G) = -\eta(E)G + \phi\sigma(E, G) + E \ln \psi_2 \phi G - \phi E \ln \psi_2 G. \tag{38}$$

Comparing the normal parts,

$$\sigma(\phi E, G) = \phi\sigma_\mu(E, G) + E \ln \psi_2 NG. \tag{39}$$

On taking Riemannian product with $\phi\sigma(E, G)$, we find

$$g(\sigma(\phi E, G), \phi\sigma(E, G)) = \left\|\sigma_\mu(E, G)\right\|^2 + E \ln \psi_2 g(\phi\sigma(E, G), NG). \tag{40}$$

Calculating the last term of (40) by using (3) and (6) and (23),

$$g(\phi\sigma(E, G), NG) = g(\sigma(\phi E, G), NG) - \sin^2\theta E \ln \psi_2 \|G\|^2. \tag{41}$$

Utilizing part (ii), we get

$$g(\phi\sigma(E, G), NG) = \cos^2\theta E \ln \psi_2 \|G\|^2. \tag{42}$$

Using the above equation in (40), we get the required result. Part (v) of the lemma can also be verified in a similar way. □

Lemma 5. *Let $S = S_T\times_{\psi_1}S_\perp\times_{\psi_2}S_\theta$ be a BW-P submanifold of a Sasakian manifold $\overline{S}$. Then,*

$$\begin{aligned} g(\sigma(E, TG), NG) &= -g(\sigma(E, G), NTG) \\ &= -\cos^2\theta E \ln \psi \|G\|^2, \end{aligned} \tag{43}$$

Proof. The proof of the present lemma can be concluded from equation (33) in [17] for $\beta = 0$. □

Lemma 6. *On a BW-P submanifold $S = S_T\times_{\psi_1}S_\perp\times_{\psi_2}S_\theta$ of a Sasakian manifold $\overline{S}$, we have*

$$\sum_{i=1}^{p}\left[\begin{array}{c}\sum_{j,k=1}^{2q} g\left(\sigma\left(\phi u_i, u^k\right), Nu^j\right) g\left(\sigma\left(u_i, Tu^k\right), Nu^j\right) - \\ g\left(\sigma\left(u_i, u^k\right), Nu^j\right) g\left(\sigma\left(\phi u_i, Tu^k\right), Nu^j\right)\end{array}\right] = -4q \cos^2\theta \left\|\nabla \ln \psi_2\right\|^2, \tag{44}$$

Proof. First, we will modify the left-hand term as follows:

$$\begin{aligned} &\sum_{i=1}^{p}\left[\sum_{j,k=1}^{2q} g\left(\sigma\left(\phi u_i, u^k\right), Nu^j\right) g\left(\sigma\left(u_i, Tu^k\right), Nu^j\right)\right] \\ &= \sum_{i=1}^{p}\left[\sum_{j=1}^{2q} g\left(\sigma\left(\phi u_i, u^j\right), Nu^j\right) g\left(\sigma\left(u_i, Tu^j\right), Nu^j\right)\right. \\ &\quad + \sum_{j\neq k=1}^{2q} g\left(\sigma\left(\phi u_i, u^k\right), Nu^j\right) g\left(\sigma\left(u_i, Tu^k\right), Nu^j\right) \end{aligned}$$

$$=\sum_{i=1}^{p}\left[\begin{array}{c}\sum_{j=1}^{2q} g\left(\sigma\left(\phi u_i, u^j\right), Nu^j\right) g\left(\sigma\left(u_i, Tu^j\right), Nu^j\right)\\ +\sum_{j=1}^{q} g\left(\sigma\left(\phi u_i, u^j\right), Nu^{j+q}\right) g\left(\sigma\left(u_i, Tu^j\right), Nu^{j+q}\right)\\ +\sum_{j=1}^{q} g\left(\sigma\left(\phi u_i, u^{j+q}\right), Nu^j\right) g\left(\sigma\left(u_i, Tu^{j+q}\right), Nu^j\right)\end{array}\right]$$
$$=\sum_{i=1}^{p}\left[\begin{array}{c}\sum_{j=1}^{2q} g\left(\sigma\left(\phi u_i, u^j\right), Fu^j\right) g\left(\sigma\left(u_i, Pu^j\right), Fu^j\right)\\ +\sec^2\theta\sum_{j=1}^{q} g\left(\sigma\left(\phi u_i, u^j\right), NTu^j\right) g\left(\sigma\left(u_i, Tu^j\right), NTu^j\right)\\ -\sum_{j=1}^{q} g\left(\sigma\left(\phi u_i, Tu^j\right), Nu^j\right) g\left(\sigma\left(u_i, u^j\right), Nu^j\right).\end{array}\right]. \tag{45}$$

On using part (ii) of Lemmas 4 and 5 and combining (25), we find

$$\begin{aligned}&\sum_{i=1}^{p}\left[\sum_{j,k=1}^{2q} g\left(\sigma\left(\phi u_i, u^k\right), Nu^j\right) g\left(\sigma\left(u_i, Tu^k\right), Nu^j\right)\right]\\ &=\sum_{i=1}^{p}\left[-2q\cos^2\theta\left(u_i \ln\psi_2\right)^2-2q\cos^2\theta\left(\phi u_i \ln\psi_2\right)^2-2q\phi u_i \ln\psi_2\eta\left(u_i\right)\right]\\ &=-2q\cos^2\theta\left\|\nabla\ln\psi_2\right\|^2.\end{aligned} \tag{46}$$

Replacing u_i by ϕu_i in the above equation, we get

$$\begin{aligned}&\sum_{i=1}^{p}\left[\sum_{j,k=1}^{2q} g\left(\sigma\left(u_i, u^k\right), Fu^j\right) g\left(\sigma\left(\phi u_i, Tu^k\right), Nu^j\right)\right]\\ &\quad=2q\cos^2\theta\left\|\nabla\ln\psi_2\right\|^2.\end{aligned} \tag{47}$$

We get the required result while subtracting the above two findings.

The following characterization is now established. □

Theorem 5. *Let $S=S_T\times_{\psi_1}S_{\perp}\times_{\psi_2}S_{\theta}$ be a BW-P submanifold of a Sasakian space form $\overline{S}(c)$ such that S_T is a compact submanifold. The following characterization is now established. If the following inequalities hold, S is a Riemannian product submanifold.*

$$\begin{aligned}&\sum_{i=1}^{2p}\left[\sum_{j=1}^{2q}\left\|\sigma_{\mu}\left(u_i, u^j\right)\right\|^2+\sum_{k=1}^{r}\left\|\sigma_{\mu}\left(u_i, f^k\right)\right\|^2\right]\leq p(c-1)\left(q\sin^2\theta+\frac{r}{2}\right)-\\ &\quad-2q\left(\cos^2\theta+2\cot^2\theta\right)\left\|\nabla\ln\psi_2\right\|^2\sum_{i=1}^{p}\sum_{j=1}^{2q} g\left(\sigma_{\mu}\left(\phi u_i, u^j\right), \sigma_{\mu}\left(u_i, Tu^j\right)\right)\geq 0,\end{aligned} \tag{48}$$

where σ_{μ} represents the projection of σ in μ and $(2p+1)$, $2q$, and r are the dimensions of S_T, S_{θ}, and $S_{\perp}$ subsequently while $\{f^1, f^2, \ldots, f^r\}$ is a local orthonormal basis of $TS_{\perp}$.

Then, again by the Codazzi equation,

$$\begin{aligned}\overline{R}(E,\phi E,G,NG)&=g\left(\nabla_E^{\perp}\sigma(\phi E,G),NG\right)-g\left(\nabla_{\phi X}^{\perp}\sigma(E,G),NG\right)\\ &\quad+g\left(\sigma\left(E,\nabla_{\phi E}G\right),NG\right)-g\left(\sigma\left(\phi E,\nabla_E G\right),NG\right)\\ &\quad-g\left(\sigma\left(\nabla_E\phi N,G\right),NG\right)+g\left(\sigma\left(\nabla_{\phi E}E,G\right),NG\right).\end{aligned} \tag{51}$$

The estimations of the terms engaged with (51) are presently processed. Foremost, we have

$$g\left(\nabla_E^{\perp}\sigma(\phi E, G), NG\right) = Eg\left(\sigma(\phi E, G), NG\right) - g\left(\sigma(\phi E, G), \nabla_E^{\perp} NG\right). \tag{52}$$

Applying the part (ii) of Lemma 4 in the last equation, we find

$$g\left(\nabla_E^{\perp}\sigma(\phi E, G), NG\right) = E^2 \ln \psi_2 \|G\|^2 + 2\left(E \ln \psi_2\right)^2 \|G\|^2 - g\left(\sigma(\phi E, G), \nabla_E^{\perp} NG\right). \tag{53}$$

Calculating the last term of (53) and using (10), we have

$$g\left(\sigma(\phi E, G), \nabla_E^{\perp} NG\right) = g\left(\sigma(\phi E, G), \overline{\nabla}_E(\phi G - TG)\right). \tag{54}$$

By the use of (6) and (12), the previous equation changes to

$$g\left(\sigma(\phi E, G), \nabla_E^{\perp} NG\right) = g\left(\sigma(\phi E, G), (\overline{\nabla}_E\phi)G + \phi\overline{\nabla}_E G\right) - g\left(\sigma(\phi E, G), \sigma(E, TG)\right). \tag{55}$$

By the application of (3), (6), and (23) and part (ii) and (iii) of Lemma 4, we get

$$g\left(\sigma(\phi E, G), \nabla_E^{\perp} NG\right) = \left(E \ln \psi_2\right)^2\left(1 + \cos^2\theta\right)\|G\|^2 + \left\|\sigma_\mu(E, G)\right\|^2 - g\left(\sigma(\phi E, G), \sigma(E, TG)\right). \tag{56}$$

Making use of (56) in (53), we find

$$g\left(\nabla_E^{\perp}\sigma(\phi E, G), NG\right) = E^2 \ln \psi_2 \|G\|^2 + \left(E \ln \psi_2\right)^2 \sin^2\theta \|G\|^2 - \left\|\sigma_\mu(E, G)\right\|^2 + g\left(\sigma(\phi E, G), \sigma(E, TG)\right). \tag{57}$$

In similar fashion, we are able to write

$$g\left(\nabla_{\phi E}^{\perp}\sigma(E, G), NG\right) = -(\phi E)^2 \ln \psi_2 \|G\|^2 - \left(\phi E \ln \psi_2\right)^2 \sin^2\theta \|G\|^2 + \left\|\sigma_\mu(\phi E, G)\right\|^2 + g\left(\sigma(E, G), \sigma(\phi E, TG)\right). \tag{58}$$

We have the following from part (ii) of Lemma 4:

$$g\left(A_{NG}G, \phi E\right) = E \ln \psi_2 \|G\|^2. \tag{59}$$

Changing out E by $\nabla_E E$ (applying the totally geodesicness of S_T, $\nabla_E E \in TS_T$) in the previous equation, we obtain

$$g\left(A_{NG}G, \phi\nabla_E E\right) = \nabla_E E \ln \psi_2 \|G\|^2. \tag{60}$$

By (6), the equation above has the following form:

$$g\left(A_{NG}G, \phi\overline{\nabla}_E E - \phi\sigma(E, E)\right) = \nabla_E E \ln \psi_2 \|G\|^2. \tag{61}$$

It is simple to verify that $\sigma(E, F) \in \mu$, for all E, F in TS_T by using the fact that the first factor S_T is totally geodesic in S. Substituting this and (12) in the last equation, we obtain

$$g\left(\sigma(\nabla_E \phi E, G), NG\right) = \nabla_E E \ln \psi_2 \|G\|^2. \tag{62}$$

Adopting similar steps, we can put

$$g\left(\sigma\left(\nabla_{\phi E} E, G\right), NG\right) = -\nabla_{\phi E}\phi E \ln \psi_2 \|G\|^2. \tag{63}$$

By part (ii) of Lemma 4 and (23), we get

$$g\left(\sigma(\phi X, \nabla_E G), NG\right) = \left(E \ln \psi_2\right)^2 \|G\|^2, \tag{64}$$

$$g\left(\sigma\left(E, \nabla_{\phi E} G\right), NG\right) = -\left(\phi E \ln \psi_2\right)^2 \|G\|^2. \tag{65}$$

Substituting values of (49) and (57)–(65) in (51), we obtain

$$\begin{aligned} -\frac{c-1}{2}\sin^2\theta \|E\|^2 \|G\|^2 = {} & E^2 \ln \psi_2 \|G\|^2 + (\phi E)^2 \ln \psi_2 \|G\|^2 \\ & - \left(E \ln \psi_2\right)^2 \cos^2\theta \|G\|^2 - \left(\phi E \ln \psi_2\right)^2 \cos^2\theta \|G\|^2 \\ & - \left\|\sigma_\mu(E, G)\right\|^2 - \left\|\sigma_\mu(\phi E, G)\right\|^2 - \nabla_E E \ln \psi_2 \|G\|^2 - \nabla_{\phi E}\phi E \ln \psi_2 \|G\|^2 \\ & + g\left(\sigma(\phi E, G), \sigma(E, TG)\right) - g\left(\sigma(E, G), \sigma(\phi E, TG)\right). \end{aligned} \tag{66}$$

On using (30), (32), (6), and (3), the previous equation becomes

$$
\begin{aligned}
-\frac{c-1}{2}\sin^2\theta\|E\|^2\|G\|^2 = {} & E^2 \ln\psi_2\|G\|^2 + (\phi E)^2 \ln\psi_2\|G\|^2 \\
& - (E\ln\psi_2)^2\cos^2\theta\|G\|^2 - (\phi E\ln\psi_2)^2\cos^2\theta\|G\|^2 \\
& - \left\|\sigma_\mu(E,G)\right\|^2 - \left\|\sigma_\mu(\phi E,G)\right\|^2 \\
& - \nabla_E E\ln\psi_2\|G\|^2 - \nabla_{\phi E}\phi E\ln\psi_2\|G\|^2 \\
& + \csc^2\theta\sum_{j=1}^{2q}\begin{bmatrix} g\left(\sigma(\phi E,G),NF_j\right)g\left(\sigma(E,TG),NF_j\right) \\ -g\left(\sigma(\phi E,G),NF_j\right)g\left(\sigma(\phi E,TG),NF_j\right)\end{bmatrix}\left\|F_j\right\|^2 \\
& + 2g\left(\sigma_\mu(\phi E,G),\sigma_\mu(E,TG)\right).
\end{aligned}
\tag{67}
$$

Let $\{u_0=\xi,u_1,u_2,\ldots,\ u_p,u_{p+1}=\phi u_1,u_{p+2}=\ \phi u_2,\ldots,u_{2p}=\phi u_p\}$ be the orthonormal frame on TS_T and $\{u^1,u^2,\ldots,u^q,\ \sec\theta Tu^1,\ \sec\theta Tu^2,\ldots,\ \sec\theta Tu^q\}$ be an orthonormal frame on TS_θ. Taking sum of the above equation with the indices $i=1,2,\ldots,p$ and $j=1,2,\ldots 2q$ and making use of (25) and (26) and part (iii) of Lemma 4, we get

$$
\begin{aligned}
2q\Delta(\ln\psi_2) = {} & pq(c-1)\sin^2\theta - 2q\cos^2\theta\left\|\nabla\ln\psi_2\right\|^2 - \sum_{i=1}^{2p}\sum_{j=1}^{2q}\left\|\sigma_\mu\left(u_i,u^j\right)\right\|^2 \\
& - 4q\cot^2\theta\left\|\nabla\ln\psi_2\right\|^2 + 2\sum_{i=1}^{p}\sum_{j=1}^{2q} g\left(\sigma_\mu\left(\phi u_i,u^j\right),\sigma_\mu\left(u_i,Tu^j\right)\right).
\end{aligned}
\tag{68}
$$

In the similar way, for $E\in TS_T$ and $F\in TS_\perp$ again using the Codazzi equation, we can prove the following:

$$
\begin{aligned}
-\frac{c-1}{2}\|E\|^2\|F\|^2 = {} & E^2\ln\psi_1\|F\|^2 + (\phi E)^2\ln\psi_1\|F\|^2 \\
& - \left\|\sigma_\mu(E,F)\right\|^2 - \left\|\sigma_\mu(\phi E,F)\right\|^2 - \nabla_E E\ln\psi_1\|F\|^2 - \nabla_{\phi E}\phi E\ln\psi_1\|F\|^2.
\end{aligned}
\tag{69}
$$

Let $\{f^1,f^2,\ldots,f^r\}$ be an orthonormal frame of $TS_\perp$. Taking sum by using $i=1,2,\ldots,p$ and $l=1,2,\ldots,r$ and simultaneously using (24) and (25), the above equation yields the following:

$$
r\Delta(\ln\psi_1) = \frac{(c-1).p.r}{2} - \sum_{i=1}^{2p}\sum_{l=1}^{r}\left\|\sigma_\mu\left(u_i,f^l\right)\right\|^2. \tag{70}
$$

From (69),

$$
\begin{aligned}
& \sum_{i=1}^{2p}\sum_{j=1}^{2q}\left\|\sigma_\mu\left(u_i,u^j\right)\right\|^2 \le pq(c-1)\sin^2\theta - 2q\left(\cos^2\theta + 2\cot^2\theta\right)\left\|\nabla\ln\psi_2\right\|^2 \\
& \sum_{i=1}^{p}\sum_{j=1}^{2q} g\left(\sigma_\mu\left(\phi u_i,u^j\right),\sigma_\mu\left(u_i,Pu^j\right)\right) \ge 0.
\end{aligned}
\tag{71}
$$

This means $\Delta \ln \psi_2 \geq 0$, so by application of Hopf's Lemma, $\ln \psi_2$ is constant that indicates ψ_2 is constant. Moreover, in (70), if

$$\sum_{i=1}^{2p}\sum_{l=1}^{r}\left\|\sigma_\mu\left(u_i, f^l\right)\right\|^2 \leq \frac{(c-1).p.r}{2}, \tag{72}$$

then $\Delta \ln \psi_1 \geq 0$, so by Hopf's lemma, $\ln \psi_1$ is constant that implies that the W–F ψ_1 is constant. We get the necessary result when these two statements are combined.

The squared norm of the second fundamental form is obtained using the warping functions and the slant function in the following theorem.

Theorem 6. *Let $\overline{S}(c)$ be a $(2n+1)-$dimensional S-C-F and $S_T\times_{\psi_1}S_\perp\times_{\psi_2}S_\theta$ be an $m-$dimensional BW-P submanifold such that S_T is a $2p-$dimensional invariant submanifold, $S_\perp$ is a $r-$dimensional anti-invariant submanifold, and S_θ be a $2q-$dimensional proper pointwise slant submanifold of $\overline{S}(c)$. If*

$$\sum_{i=1}^{p}\sum_{j=1}^{2q} g\left(\sigma\left(\phi u_i, u^j\right), \sigma\left(u_i, Pu^j\right)\right) \geq 0, \tag{73}$$

then

(i) The squared norm of the second fundamental form σ satisfies

$$\begin{aligned}\|\sigma\|^2 \geq{}& p(c-1)\left(q\sin^2\theta + \frac{r}{2}\right)\\ &+ 2q\sin^2\theta\left\|\nabla \ln \psi_2\right\|^2 + r\left\|\nabla \ln \psi_1\right\|^2\\ &- 2q\Delta(\ln \psi_2) - r\Delta(\ln \psi_1).\end{aligned} \tag{74}$$

(ii) The equality sign of (74) satisfies identically if and only if

(i) S_T is T-G invariant submanifold of $\overline{S}(c)$. Hence, S_T is a S–C-F.

(ii) $S_\perp$ and S_θ are T-U submanifolds of $\overline{S}(c)$.

(iii) $\sum_{i=1}^{p}\sum_{j=1}^{2q} g(\sigma(\phi u_i, u^j), \sigma(u_i, Pu^j)) = 0$.

Proof. From (69), we have

$$\sum_{i=1}^{2p}\sum_{j=1}^{2q}\left\|\sigma_\mu\left(u_i, u^j\right)\right\|^2 \geq pq(c-1)\sin^2\theta - 2q\left(\cos^2\theta + 2\cot^2\theta\right)\|\nabla \ln \psi\|^2 - 2q\Delta(\ln \psi). \tag{75}$$

For the orthonormal frames $\{u_0=\xi,\ u_1, u_2, \ldots, u_p, u_{p+1}=\phi u_1, u_{p+2}=\phi u_2, \ldots, u_{2p}=\phi u_p\}$ and $\{u^1, u^2, \ldots, u^q,\ \sec\theta Tu^1,\ \sec\theta Tu^2, \ldots,\ \sec\theta Tu^q\}$, in view of formulae (31) and (32) and part (ii) of Lemma 4, we get

$$\begin{aligned}\sum_{i=0}^{2p}\sum_{j=1}^{2q}\left\|\sigma_{ND_\theta}\left(u_i, u^j\right)\right\|^2 &= \sum_{i=0}^{2p}\sum_{j,k=1}^{2q}\csc^2\theta g\left(\sigma\left(u_i, u^j\right), Nu^k\right)^2\\ &= \csc^2\theta\sum_{i=0}^{2p}\left[\sum_{j=1}^{2q} g\left(\sigma\left(u_i, u^j\right), Nu^j\right)^2 + \sum_{j\neq k=1}^{2q} g\left(\sigma\left(u_i, u^j\right), Nu^k\right)^2\right]\\ &= \csc^2\theta\sum_{i=0}^{2p}\left[2q\left(u_i \ln \psi\right)^2 + \sec^2\theta\sum_{j=1}^{q}\left\{g\left(\sigma\left(u_i, u^j\right), NTu^j\right)^2 + g\left(\sigma\left(u_i, Tu^j\right), Nu^j\right)^2\right\}\right].\end{aligned} \tag{76}$$

Further, using Lemma 5 and (25), the above equation is reduced to

$$\begin{aligned}\sum_{i=0}^{2p}\sum_{j=1}^{2q}\left\|\sigma_{FD_\theta}\left(u_i, u^j\right)\right\|^2 ={}& 2q\csc^2\theta\|\nabla \ln \psi\|^2\\ &+ 2q\cot^2\theta\|\nabla \ln \psi\|^2.\end{aligned} \tag{77}$$

Now, for any $E \in TN_T$ and $F \in TN_\perp$, from part (iii) of Lemma 1, we have

$$\begin{aligned} g(\sigma(\phi E, F), \phi F) &= E \ln \psi_1\|F\|^2,\\ g(\sigma(\xi, F), \phi F) &= 0.\end{aligned} \tag{78}$$

By the above equations for the frame $\{u_0=\xi, u_1, u_2, \ldots, u_p, u_{p+1}=\phi u_1, u_{p+2}=\phi u_2, \ldots, u_{2p}=\phi u_p\}$ and $\{f^1, f^2, \ldots, f^r\}$, it is simple to conclude that

$$\sum_{i=1}^{2p}\sum_{l=1}^{r}\left\|h_{\phi D^\perp}\left(u_i, f^l\right)\right\|^2 = r\left\|\nabla \ln \psi_1\right\|^2. \tag{79}$$

Moreover, from (3) and (23), we get

$$g(\sigma(E,G),\phi F)=0,$$
$$g(\sigma(E,F),NG)=0, \tag{80}$$

for all $E\in TS_T$, $G\in TS_\theta$, and $F\in TS_\perp$. We can deduce the following from these two findings:

$$\sum_{i=1}^{2p}\sum_{j=1}^{2q}\left\|\sigma_{\phi D^\perp}\left(u_i,u^j\right)\right\|^2=0, \tag{81}$$

$$\sum_{i=1}^{2p}\sum_{k=1}^{r}\left\|\sigma_{\phi D^\theta}\left(u_i,f^k\right)\right\|^2=0. \tag{82}$$

From (75), (77), (79), (81), and (82), we get the required inequality.

To prove the part (ii), let σ' and σ'' be the second fundamental forms for the immersion of S_θ and $S_\perp$ in S, respectively. Then, for any $G,K\in TS_\theta$ and $E\in TS_T$, using the Gauss formula, we have

$$g(\sigma'(G,K),E)=g(\nabla_G K,E)=-E\ \ln\ \psi_2 g(G,K). \tag{83}$$

By (24), we obtain

$$g(\sigma'(G,K),E)=-g(G,K)g(\nabla\ \ln\ \psi_2,E), \tag{84}$$

or

$$\sigma'(G,K)=-g(G,K)\nabla\ \ln\ \psi_2. \tag{85}$$

Accordingly, for any $F_1,F_2\in TS_\perp$ and $E\in TS_T$, we have

$$\sigma''(F_1,F_2)=-g(F_1,F_2)\nabla\ \ln\ \psi_1. \tag{86}$$

If the equality sign of (74) holds identically, then we have

$$\sigma(D,D)=0,$$
$$\sigma\left(D^\perp,D^\perp\right)=0, \tag{87}$$
$$\sigma\left(D^\theta,D^\theta\right)=0,$$

$$g\left(\sigma_\mu\left(\phi\ D,D^\theta\right),\sigma_\mu\left(D,TD^\theta\right)\right)=0. \tag{88}$$

The first condition of (87) suggests that S_T is T-G submanifold in S. Then, again it is not difficult to see that $g(\sigma(E_1,\phi E_2),\mathrm{NG})=0$ and $g(\sigma(E,\phi E_2),\phi F)=0$, for all $E_1,E_2\in \mathrm{TS}_T$, $G\in \mathrm{TS}_\theta$, and $F\in \mathrm{TS}_\perp$ It follows that S_T is T-G in $\bar{S}(c)$ and hence is a S-C-F. The second condition of (87) with (86) implies that $S_\perp$ is T-U. Besides, the third condition of (87) along with (85) suggests that S_θ is a T-U submanifold. This demonstrates the proof. □

4. Conclusion

In this paper, by utilizing Hopf's Lemma, we acquired the describing inequalities for the existence of biwarped product submanifolds of Sasakian space forms. Besides, we additionally worked out an assessment for the squared norm of the second fundamental form in terms of the warping function and slant function. To fortify our study, we gave a nontrivial example of a biwarped product submanifold in a Sasakian manifold.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

References

[1] J. K. Beem, P. Ehrlich, and T. G. Powell, *Warped Product Manifolds in Relativity. Selected Studies*, North-Holland, Amsterdam, NY, USA, 1982.

[2] S. W. Hawkings and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge, UK, 1973.

[3] B. O'Neill, *Semi-Riemannian Geometry with Application to Relativity*, Academic Press, Cambridge, MA, USA, 1983.

[4] R. L. Bishop and B. O'Neill, "Manifolds of negative curvature," *Transactions of the American Mathematical Society*, vol. 145, pp. 1–49, 1965.

[5] B.-Y. Chen, "Geometry of warped product CR-submanifolds in kaehler manifolds," *Monatshefte fur Mathematik*, vol. 133, no. 3, pp. 177–195, 2001.

[6] I. Hasegawa and I. Mihai, "Contact CR-warped product submanifolds in Sasakian manifolds," *Geometriae Dedicata*, vol. 102, no. 1, pp. 143–150, 2003.

[7] F. R. Al-Solamy and M. A. Khan, "Semi-slant warped product submanifolds of a kenmotsu manifold," *Mathematical Problems in Engineering*, vol. 2012, Article ID 708191, 10 pages, 2012.

[8] M. Atceken, "Warped product semi-slant submanifolds in Kenmotsu manifold," *Turkish Journal of Mathematics*, vol. 34, pp. 425–433, 2010.

[9] V. A. Khan, K. A. Khan, and S. Uddin, "A note on warped product submanifold of Kenmotsu manifolds," *Mathematica Slovaka*, vol. 61, no. 1, pp. 1–14, 2011.

[10] S. Uddin, V. A. Khan, and K. A. Khan, "Warped product submanifolds of a Kenmotsu manifolds," *Turkish Journal of Mathematics*, vol. 36, pp. 319–330, 2012.

[11] B. Sahin, "Warped product pointwise semi-slant submanifolds of Kaehler manifolds," *Portugaliae Mathematica*, vol. 70, pp. 252–268, 2013.

[12] M. A. Khan, "Warped product pointwise semi-slant submanifolds of the complex space forms," *Rendiconti del Circolo Matematico di Palermo Series 2*, vol. 69, no. 1, pp. 195–207, 2020.

[13] I. Mihai and S. Uddin, *Warped Product Pointwise Semi-slant Submanifolds of Sasakian Manifolds*, https://arxiv.org/abs/1706.04305, 2018.

[14] B. Y. Chen and F. Dillen, "Optimal inequalities for multiply warped product submanifolds," *International Electronic Journal of Geometry*, vol. 1, no. 1, pp. 1–11, 2008.

[15] M. A. Khan and K. Khan, "Biwarped product submanifolds of complex space forms," *International Journal of Geometric Methods in Modern Physics*, vol. 16, no. 5, pp. 1–21, 2019.

[16] H. M. Tastan, "Biwarped product submanifolds of a Kaehler manifold," *Filomat*, vol. 32, no. 7, Article ID 23492365, 2018.

[17] A. H. Alkhaldi and A. Ali, "Geometry of bi-warped product submanifolds of nearly trans-sasakian manifolds," *Mathematics*, vol. 9, no. 8, p. 847, 2021.

[18] D. E. Blair, *Contact manifolds in Riemannian Geometry,* Springer-Verlag, Berlin, 1976.

[19] A. Lotta, “Slant submanifolds in contact geometry,” *Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie*, vol. 39, no. 87, pp. 1–4, 1996.
[20] J. L. Cabrerizo, A. Carriazo, L. M. Fernández, and M. Fernandez, “Slant submanifolds in sasakian manifolds,” *Glasgow Mathematical Journal*, vol. 42, no. 1, pp. 125–138, 2000.
[21] F. Etayo, “On quasi-slant submanifolds of an almost Hermitian manifold,” *Publicationes Mathematicae Debrecen*, vol. 53, pp. 217–233, 1998.
[22] B. Y. Chen and O. Garay, “Pointwise slant submanifolds in almost Hermitian manifolds,” *Turkish Journal of Mathematics*, vol. 36, pp. 630–640, 2012.
[23] K. S. Park, “Pointwise slant and pointwise semi-slant submanifolds in almost contact metric manifolds,” *Mathematics*, vol. 8, no. 6, p. 985, 2020.
[24] S. Uddin and A. H. Alkhaldi, “Pointwise slant submanifolds and their warped products in Sasakian manifolds,” *Filomat*, vol. 32, no. 12, pp. 1–12, 2018.
[25] A. Olteanu, “Multiply warped product submanifolds in Kenmotsu space forms,” *Bulletin of the Institute of Mathematics, Academia Sinica*, vol. 5, no. 2, pp. 201–214, 2010.
[26] J. P. Baker, *Twice Warped Products*, M. Sc. Thesis, University of Missouri-Columbia, Columbia, MO, USA, 1997.
[27] B. Y. Chen, *Pseudo-Riemannian Geometry, $\delta-$ Invariants and Applications*, World Scientific Publishing Company, Singapore, 2011.