Research Article

A Perishable Inventory System with Postponed Demands and Multiple Server Vacations

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A mathematical modelling of a continuous review stochastic inventory system with a single server is carried out in this work. We assume that demand time points form a Poisson process. The life time of each item is assumed to have exponential distribution. We assume (s, S) ordering policy to replenish stock with random lead time. The server goes for a vacation of an exponentially distributed duration at the time of stock depletion and may take subsequent vacation depending on the stock position. The customer who arrives during the stock-out period or during the server vacation is offered a choice of joining a pool which is of finite capacity or leaving the system. The demands in the pool are selected one by one by the server only when the inventory level is above s , with interval time between any two successive selections distributed as exponential with parameter depending on the number of customers in the pool. The joint probability distribution of the inventory level and the number of customers in the pool is obtained in the steady-state case. Various system performance measures in the steady state are derived, and the long-run total expected cost rate is calculated.

1. Introduction

In most of the inventory models considered in the literature, the demanded items are directly delivered from the stock (if available). The demands occurring during the stock-out period are either lost (lost sales case) or satisfied only after the arrival of ordered items (backlogging). In the latter case, it is assumed that either all (full backlogging) or any prefixed number of demands (partial backlogging) that occurred during the stock-out period are satisfied. The often quoted review articles Nahmias [1] and Raafat [2] and the recent review articles N. H. Shah and Y. K. Shah [3] and Goyal and Giri [4] provide excellent summaries of many of these modelling efforts. For some recent references, see Chakravarthy and Daniel [5], Yadavalli et al. [6], and Kalpakam and Shanthi [7, 8].

In the case of continuous review perishable inventory models with random life times for the items, most of the models assume instantaneous supply of order (Lian and Liu [9], Liu and Shi [10], and Kalpakam and Arivarignan [11], and Gürler and Özkaya [12]). When a server is assumed to process the demands in continuous review inventory

systems, the concept of server vacation was also taken up in some works. Daniel and Ramanarayanan [13] introduced the concept of server vacation in an inventory model with two servers. In [14], they studied an (s, S) inventory system in which the server takes a rest when the level of the inventory is zero. The demands that occurred during stock-out period were lost. The interoccurrence times between two successive demands, the lead times, and the rest times are assumed to follow general distributions, which are mutually independent. Using renewal and convolution techniques, they obtained the steady-state transition probabilities. However, server vacations in Queueing systems have been widely studied in different contexts in the literature. We refer the readers to Doshi [15, 16], Takagi [17, 18], and Tian and Zhang [19] for details on various vacations systems and for additional references.

In the context of inventory system, any arriving demands during the server vacation either enter a pool or leave the system. These pooled customers are selected at a later time, preferably after the stock is replenished. This type of inventory problems is called inventory with postponed demands.

TABLE 1: Optimal values corresponding to various demand rates and the replenishment rates ($\gamma = 1.4$, $\beta = 1.3$, $\alpha = 4$, $p = 0.85$).

λ	12				13				14				15				16			
μ																				
4.0	9	56	6	10	60	7	11	63	7	12	66	8	13	69	8					
		37.484085			39.299182			41.092056			42.865498			44.620871						
4.5	9	55	8	9	58	8	10	61	9	11	64	10	12	66	10					
		36.545772			38.252507			39.930717			41.591928			43.235987						
5.0	8	54	9	9	56	9	10	59	10	10	62	11	11	64	11					
		35.729504			37.353218			38.95615			40.53873			42.08412						
5.5	8	52	9	8	55	10	9	57	11	10	60	12	11	62	12					
		35.077762			36.606215			38.119508			39.621105			41.11143						
6.0	7	51	10	8	54	11	9	56	12	9	58	12	10	61	13					
		34.470629			35.957242			37.433969			38.854201			40.262687						

TABLE 2: Optimal values corresponding to various demand rates and perishable rate ($\beta = 1.3$, $\alpha = 4$, $\mu = 4$, $p = 0.85$).

λ	10				11				12				13				14			
γ																				
1.325	7	50	6	8	54	7	9	57	8	10	60	8	11	60	7					
		32.654327			34.455224			36.227262			37.973244			39.725222						
1.350	7	50	6	8	53	6	9	57	7	10	60	8	11	60	7					
		33.0285			34.852712			36.647164			38.416969			40.190166						
1.375	7	50	5	8	53	6	9	57	7	10	60	7	11	60	6					
		33.401547			35.24819			37.066484			38.858742			40.654683						
1.400	7	50	5	8	53	6	9	56	6	10	60	7	11	60	6					
		33.771515			35.643133			37.484085			39.299182			41.115929						
1.425	7	50	5	8	53	5	9	56	6	10	59	6	11	60	6					
		34.140776			36.035305			37.900166			39.738532			41.576681						

The concept of postponed demands in the inventory model has been introduced by Berman et al. [20]. They assumed that both demand and service rates are deterministic. Krishnamoorthy and Islam [21] considered an inventory system with Poisson demand, exponential lead time and the pooled customers are selected according to an exponentially distributed time lag. Sivakumar and Arivarignan [22] considered an inventory model in which the demand occurs according to a Markovian arrival process, lead time has a phase type distribution, life time for the items in the stock has exponential distribution, and the pooled customers are selected exponentially. Manuel et al. [23] dealt with an inventory system in which the positive and negative demands occur according to an independent Markovian arrival processes, the lead time of the reorder, the life time of the items, the interselection time of customers from the pool and the renegeing time points of the customers in the pool are independent, and each has the exponential distribution. In all the above models, the authors assumed that the pool size is finite. Sivakumar and Arivarignan [24] considered an inventory system with independent Markovian arrival processes for positive and negative demands, exponential lead time for the reorders, exponential life times for each item in the stock, and the pool size is infinite.

In earlier works listed above, the server is assumed to be available even during the stock-out periods. In this work, we

have introduced the concept of server vacation. It may be noted that this model also includes situations in which the server may not be availing vacation, but the server may attend some other work during the stock-out period and may return at random times to the inventory system. If the stock is still not replenished, the server goes for another vacation again or to attend the other work. Employees who are assigned extra work on additional remuneration basis can also be thought of such servers.

In this paper, we consider a continuous review perishable inventory system. The unit demands occur according to a Poisson process. The (s, S) ordering policy is implemented to replenish stock. When the inventory is completely depleted, the server is assumed to go on a vacation of random duration. At the end of vacation, if the stock is not replenished, the server again starts his/her vacation. The demands that occurred during stock-out period are offered postponement of their demand after the stock is replenished. Those demands that accept the postponement joins a pool of finite capacity. We assume that fraction p of demands join the pool and the rest leave the system. The pooled demands are selected once the stock is replenished, and the server resumes service. But the selection from the pool will be stopped if the inventory falls below the reorder level, s .

The rest of the paper is organized as follows. In Section 2, we describe the mathematical model of the problem under

TABLE 3: Optimal values corresponding to various demand rates and the vacation time rates ($\gamma = 1.4$, $\alpha = 4$, $\mu = 4$, $p = 0.85$).

λ		12			13			14			15			16		
β																
1.30	9	57	6	10	60	7	11	63	7	12	66	8	13	69	8	
		37.484085			39.299182			41.092056			42.865498			44.620871		
1.35	9	57	8	10	60	9	11	63	9	12	66	10	13	69	10	
		37.167995			38.963248			40.734627			42.487454			44.221881		
1.40	9	57	9	10	60	10	11	63	11	12	66	11	13	69	12	
		36.861364			38.635679			40.388064			42.120116			43.834419		
1.45	9	56	10	9	59	11	10	62	12	11	66	13	12	69	14	
		36.566275			38.316645			40.041077			41.748416			43.440092		
1.50	8	56	11	9	59	12	10	62	13	11	65	14	12	68	15	
		36.260058			37.985767			39.692538			41.381935			43.055515		

TABLE 4: Effect of primary and postponed demand rates on the optimal values ($\gamma = 1.4$, $\beta = 1.3$, $\mu = 4$, $p = 0.85$).

λ		10			11			12			13			14		
α																
4.0	7	50	5	8	53	6	9	56	6	10	60	7	11	63	7	
		33.771515			35.643133			37.484085			39.299182			41.092056		
4.5	7	50	6	8	54	7	9	57	8	10	60	8	11	63	9	
		33.690772			35.556042			37.392308			39.202171			40.990676		
5.0	7	51	7	8	54	8	9	57	8	10	60	9	11	63	10	
		33.618558			35.478749			37.310908			39.116144			40.900342		
5.5	7	51	8	8	54	8	9	57	9	10	60	10	11	63	10	
		33.555137			35.41211			37.2389			39.041211			40.821941		
6.0	7	51	8	8	54	9	9	57	10	10	60	10	11	63	11	
		33.499545			35.352615			37.177226			38.976372			40.752532		

study. The steady-state analysis of the model is presented in Section 3, and some key system performance measures are derived in Section 4. In Section 5, we calculate the total expected cost rate, and, in the final section, we perform sensitivity analyses on the total expected cost rate in terms of numerical illustrations.

2. Problem Formulation

We consider an inventory system with a maximum stock of S unit items. The items are assumed to have identical and independent exponential life-time distributions with parameter $\gamma(> 0)$. When an item perishes, it is removed from the stock. Another process which removes items from the stock is the demand process. The demand is for single item, and the demand occurrences form a Poisson process with rate $\lambda(> 0)$. A server processes the unit demands. The stock is replenished by the server by implementing (s, S) ordering policy, where the lead time is assumed to have independent exponential distribution with parameter $\mu(> 0)$. If the stock is depleted before the supply of reorder, the server goes on a vacation duration which has exponential distribution with parameter $\beta(> 0)$. At the end of a vacation period, if the stock is still empty, the server takes another vacation. The demands that occur during stock-out period are offered postponement at a later time. It is assumed that the demands accept this

FIGURE 1: A three-dimensional plot of the cost function. $\lambda = 14$, $\mu = 4$, $\gamma = 1.2$, $\beta = 1.3$, $\alpha_i = i * \alpha$, $i = 1, 2, \dots, N$, $\alpha = 4$, $p = 0.85$, $c_h = 0.3$, $c_s = 15$, $c_{pr} = 0.3$, $c_{sh} = 5$, $c_p = 3$.

postponement or leave the system according to a Bernoulli distribution with probability p . The demand which accepts an offer of postponement may join a pool of finite capacity fixed as N .

The demands from the pool are selected if the server is on service and the inventory level is above s such that the interval time between successive selections is exponentially

TABLE 5: Optimal values corresponding to various perishable rates and the replenishment rates ($\lambda = 14$, $\beta = 1.3$, $\alpha = 4$, $p = 0.85$).

μ		4.0				4.5				5.0				5.5				6.0			
γ																					
1.30	11	63	9	10	61	11	10	59	12	9	57	13	9	56	13						
		39.231174			38.117777			37.192369			36.39762			35.754118							
1.35	11	63	8	10	61	10	10	59	11	9	57	12	9	56	12						
		40.164239			39.025748			38.075018			37.259064			36.594523							
1.40	11	63	7	10	61	9	10	59	10	9	57	11	9	56	12						
		41.092056			39.930717			38.95615			38.119508			37.433969							
1.45	11	63	6	10	61	8	10	59	9	9	57	10	9	56	11						
		42.013602			40.832153			39.835433			38.978705			38.272741							
1.50	11	62	5	10	61	7	10	59	9	9	56	10	9	56	10						
		42.926105			41.7295			40.711703			39.83621			39.110758							

TABLE 6: Influence of the postponed demand rate and replenishment rate on the optimal values ($\lambda = 14$, $\gamma = 1.4$, $\beta = 1.3$, $p = 0.85$).

μ		4.0				4.5				5.0				5.5				6.0			
α																					
3.50	11	62	4	10	60	7	10	59	9	9	57	9	9	56	10						
		41.194355			40.049416			39.065602			38.230701			37.531787							
3.75	11	62	6	10	61	8	10	59	9	9	57	10	9	56	11						
		41.145131			39.988997			39.009046			38.17254			37.480279							
4.00	11	63	7	10	61	9	10	59	10	9	57	11	9	56	12						
		41.092056			39.930717			38.95615			38.119508			37.433969							
4.25	11	63	8	10	61	10	10	59	11	9	57	11	9	56	12						
		41.039824			39.876714			38.908033			38.071643			37.391719							
4.50	11	63	9	10	61	10	10	59	11	9	57	12	8	56	13						
		40.990676			39.826149			38.863594			38.027026			37.349602

FIGURE 2: Fraction of time server is on vacation— λ versus $\alpha^{(n)}$ ($S = 90$, $s = 15$, $N = 25$, $\beta = 1.4$, $\gamma = 4$, $\mu = 0.8$, $p = 0.75$.) FIGURE 3: Fraction of time server is on vacation— β versus $\alpha^{(n)}$ ($S = 90$, $s = 15$, $N = 25$, $\lambda = 8$, $\gamma = 4$, $\mu = 0.8$, $p = 0.75$.)

TABLE 7: Influence of the vacation time rate and the replenishment rate on the optimal values ($\lambda = 14$, $\gamma = 1.4$, $\alpha = 4$, $p = 0.85$).

μ	4.00				4.25				4.50				4.75				5.00			
β																				
1.300	11	63	7	11	62	8	10	61	9	10	60	10	10	59	10					
		41.092056			40.484338			39.930717			39.41322			38.95615						
1.325	11	63	8	11	62	9	10	61	10	10	60	10	10	59	11					
		40.912412			40.314336			39.754127			39.247492			38.800941						
1.350	11	63	9	10	62	10	10	61	11	10	60	11	9	59	12					
		40.734627			40.145055			39.581237			39.084991			38.635694						
1.375	11	63	10	10	62	11	10	60	11	10	60	12	9	59	12					
		40.559624			39.964654			39.412036			38.926868			38.469604						
1.400	11	63	11	10	61	11	10	60	12	10	59	12	9	58	13					
		40.388064			39.787865			39.246274			38.773279			38.307351						

TABLE 8: Optimal values corresponding to various perishable rates and vacation time rates ($\lambda = 14$, $\alpha = 4$, $\mu = 4$, $p = 0.85$).

β	1.30				1.35				1.40				1.45				1.50			
γ																				
1.30	11	63	9	11	63	11	11	63	12	10	62	14	10	62	15					
		39.231174			38.886899			38.556819			38.226251			37.897017						
1.35	11	63	8	11	63	10	11	63	11	10	62	13	10	62	14					
		40.164239			39.812693			39.474237			39.134974			38.795693						
1.40	11	63	7	11	63	9	11	63	11	10	62	12	10	62	13					
		41.092056			40.734627			40.388064			40.041077			39.692538						
1.45	11	63	6	11	63	8	11	63	10	10	62	11	10	62	12					
		42.013602			41.652036			41.298322			40.944106			40.587172						
1.50	11	62	5	11	63	7	11	63	9	10	63	11	10	62	12					
		42.926105			42.564207			42.204506			41.84354			41.478112

FIGURE 4: Fraction of time server is on vacation— μ versus $\alpha^{(n)}$ ($S = 90$, $s = 15$, $N = 25$, $\beta = 1.4$, $\gamma = 4$, $\lambda = 8$, $p = 0.75$.) FIGURE 5: Fraction of time server is on vacation— γ versus $\alpha^{(n)}$ ($S = 90$, $s = 15$, $N = 25$, $\beta = 1.4$, $\lambda = 8$, $\mu = 0.8$, $p = 0.75$.)

TABLE 9: Influence of the postponed demand rate and vacation time rate on the optimal values ($\lambda = 14$, $\gamma = 1.4$, $\mu = 4$, $p = 0.85$).

β	1.30				1.35				1.40				1.45				1.50			
α																				
4.00	11	63	7	11	63	9	11	63	11	10	62	12	10	62	13					
		41.092056			40.734627			40.388064			40.041077			39.692538						
4.25	11	63	8	11	63	10	11	63	11	10	62	13	10	62	14					
		41.039824			40.674551			40.321923			39.963846			39.611386						
4.50	11	63	9	11	63	10	11	63	12	10	62	13	10	62	14					
		40.990676			40.618727			40.260928			39.893023			39.538037						
4.75	11	63	9	11	63	11	10	63	12	10	63	14	10	62	15					
		40.943838			40.566448			40.204905			39.828332			39.470586						
5.00	11	63	10	11	63	11	10	63	13	10	63	14	10	62	15					
		40.900342			40.518577			40.147674			39.769092			39.409429						

TABLE 10: Optimal values corresponding to various perishable rates and postponed demand rates ($\lambda = 14$, $\beta = 1.3$, $\mu = 4$, $p = 0.85$).

α	4.00				4.25				4.50				4.75				5.00			
γ																				
1.0	11	62	15	11	62	16	11	62	16	11	62	17	11	62	17					
		33.566483			33.512828			33.46372			33.419215			33.378523						
1.1	11	62	13	11	62	13	11	63	14	11	63	15	11	63	15					
		35.463386			35.407926			35.357122			35.310631			35.268167						
1.2	11	63	11	11	63	12	11	63	12	11	63	13	11	63	13					
		37.352776			37.296972			37.245085			37.197543			37.153723						
1.3	11	63	9	11	63	10	11	63	10	11	63	11	11	63	11					
		39.231174			39.176278			39.125134			39.077093			39.033214						
1.4	11	63	7	11	63	8	11	63	9	11	63	9	11	63	10					
		41.092056			41.039824			40.990676			40.943838			40.900342

FIGURE 6: The total expected cost rate— p versus $\alpha^{(n)}$ ($S = 90$, $s = 15$, $N = 25$, $\lambda = 8$, $\beta = 1.4$, $\gamma = 0.7$, $\mu = 8$.) FIGURE 7: Expected Inventory Level— p versus $\alpha^{(n)}$ ($S = 90$, $s = 15$, $N = 25$, $\lambda = 8$, $\beta = 1.4$, $\gamma = 0.7$, $\mu = 0.8$.)

TABLE 11: Variation in optimal values for different values of c_h and c_p ($c_s = 15$, $c_{sh} = 5$, $c_{pr} = 0.3$).

c_h	0.20				0.25				0.30				0.35				0.40			
c_p																				
2.2	11	69	24	10	66	22	10	63	21	10	61	21	10	59	20					
		37.185505			38.452295			39.665423			40.837851			41.974311						
2.4	11	69	19	11	66	18	10	64	18	10	61	17	10	59	16					
		37.599174			38.884988			40.125458			41.301575			42.440661						
2.6	11	69	16	11	66	15	11	64	14	11	61	13	10	59	13					
		37.972831			39.260922			40.503633			41.705424			42.851175						
2.8	11	69	12	11	66	12	11	63	11	11	61	10	11	59	9					
		38.303582			39.590386			40.831047			42.028107			43.188119						
3.0	12	69	9	11	66	8	11	63	7	11	60	6	11	58	5					
		38.563014			39.86128			41.092056			42.278538			43.424151						

TABLE 12: Optimal values corresponding to various holding and perishable costs ($c_s = 15$, $c_{sh} = 5$, $c_p = 3$).

c_h	0.20				0.25				0.30				0.35				0.40			
c_{pr}																				
0.250	12	74	10	12	70	9	12	67	8	11	64	8	11	61	6					
		36.642829			38.025913			39.353537			40.605434			41.809696						
0.275	12	71	10	12	68	9	11	65	8	11	62	7	11	60	6					
		37.617513			38.960668			40.235119			41.452256			42.627321						
0.300	12	69	9	11	66	8	11	63	7	11	60	6	11	58	5					
		38.563014			39.86128			41.092056			42.278538			43.424151						
0.325	11	66	8	11	63	7	11	61	6	11	58	5	11	56	4					
		39.482691			40.72827			41.927648			43.085434			44.202381						
0.350	11	65	8	11	62	7	11	59	6	11	57	5	10	54	3					
		40.359663			41.571843			42.741939			43.871344			44.952053

FIGURE 8: Expected reorder rate— p versus $\alpha^{(n)}$ ($S = 90$, $s = 15$, $N = 25$, $\lambda = 8$, $\beta = 1.4$, $\gamma = 0.7$, $\mu = 0.8$.) FIGURE 9: Expected number of customers in the pool— p versus $\alpha^{(n)}$ ($S = 90$, $s = 15$, $N = 25$, $\lambda = 8$, $\beta = 1.4$, $\gamma = 0.7$, $\mu = 0.8$.)

TABLE 13: Optimal values corresponding to various setup and holding costs ($c_{sh} = 5$, $c_p = 3$, $c_{pr} = 0.3$).

c_h		0.20		0.25		0.30		0.35		0.40					
c_s															
13	12	66	8	12	63	7	12	61	7	11	58	6	11	55	4
		36.775469			38.037617			39.253444			40.42214			41.527663	
14	12	68	9	12	65	8	11	62	7	11	59	6	11	57	5
		37.674734			38.959441			40.189174			41.355166			42.481256	
15	12	69	9	11	66	8	11	63	7	11	60	6	11	58	5
		38.563014			39.86128			41.092056			42.278538			43.424151	
16	11	70	9	11	67	8	11	64	7	11	61	6	10	58	5
		39.433976			40.734783			41.986075			43.19263			44.356396	
17	11	72	9	11	68	9	11	65	8	10	62	7	10	60	6
		40.282373			41.600143			42.871236			44.094213			45.249004	

TABLE 14: Variation in optimal values for different values of c_h and c_{sh} ($c_s = 15$, $c_p = 3$, $c_{pr} = 0.3$).

c_h		0.20			0.25			0.30			0.35			0.40		
c_{sh}																
5.00	12	69	9	11	66	8	11	63	7	11	60	6	11	58	5	
		38.563014			39.86128			41.092056			42.278538			43.424151		
5.25	12	70	12	12	67	11	11	64	10	11	62	10	11	60	9	
		38.811315			40.138832			41.406634			42.618679			43.790972		
5.50	12	71	14	12	68	13	12	66	13	11	63	12	11	61	11	
		39.026709			40.371016			41.668858			42.905412			44.097789		
5.75	12	72	16	12	69	16	12	66	15	12	64	14	11	62	14	
		39.219457			40.577343			41.889389			43.15853			44.366545		
6.00	12	73	19	12	70	18	12	67	17	12	65	16	11	63	16	
		39.394802			40.764441			42.08824			43.370604			44.60868

FIGURE 10: Expected perishable rate— p versus $\alpha^{(n)}$ ($S = 90$, $s = 15$, $N = 25$, $\lambda = 8$, $\beta = 1.4$, $\gamma = 0.7$, $\mu = 0.8$.) FIGURE 11: Mean loss of demand rate— p versus $\alpha^{(n)}$ ($S = 90$, $s = 15$, $N = 25$, $\lambda = 8$, $\beta = 1.4$, $\gamma = 0.7$, $\mu = 0.8$.)

TABLE 15: Effect of c_s and c_p on the optimal cost rate ($c_h = 0.3$, $c_{sh} = 5$, $c_{pr} = 0.3$).

c_s	14.0				14.5				15.0				15.5				16.0			
c_p																				
2.2	10	63	21	10	63	21	10	64	21	10	64	22	10	65	22					
		38.786969			39.226893			39.665423			40.100898			40.535658						
2.4	11	63	17	11	63	17	10	64	18	10	64	18	10	65	18					
		39.22218			39.674866			40.125458			40.559503			40.991876						
2.6	11	63	14	11	63	14	11	64	14	11	64	14	11	65	14					
		39.601437			40.053153			40.503633			40.951362			41.397236						
2.8	11	62	11	11	63	11	11	63	11	11	64	11	11	65	11					
		39.929298			40.380592			40.831047			41.277607			41.723459						
3.0	11	62	7	11	62	7	11	63	7	11	63	7	11	64	7					
		40.189174			40.641335			41.092056			41.540189			41.986075						

TABLE 16: Effect of c_s and c_{pr} on the optimal cost rate ($c_h = 0.3$, $c_{sh} = 5$, $c_p = 3$).

c_s	13.0				13.5				14.0				14.5				15.0			
c_{pr}																				
0.30	12	61	7	12	61	7	11	62	7	11	62	7	11	63	7					
		39.253444			39.726359			40.189174			40.641335			41.092056						
0.32	12	59	6	11	59	6	11	60	6	11	61	7	11	61	6					
		39.914108			40.386834			40.847306			41.306302			41.762515						
0.34	11	57	5	11	58	6	11	59	6	11	59	6	11	60	6					
		40.557466			41.026401			41.492639			41.956904			42.418051						
0.36	11	56	5	11	57	5	11	57	5	11	58	5	11	59	6					
		41.178068			41.652787			42.125454			42.594898			43.062709						
0.38	11	55	4	11	55	4	11	56	4	11	57	5	11	57	5					
		41.786343			42.267918			42.745509			43.220775			43.693443						

distributed with parameter $\alpha_n (> 0)$ where n is the number of customers in the pool.

Notation.

e: a column vector of appropriate dimension containing all ones.

0: a zero matrix of appropriate dimension.

$\mathbf{0}_{(i \times j)}$: a zero matrix with i rows and j columns.

3. Analysis

Let $X(t)$ and $L(t)$, respectively, denote the number of demands in the pool and the on-hand inventory level at time t . We define a binary variable $Y(t)$ (which takes 0 and 1) to denote the status of the server, respectively, “on vacation” and “on service.” Similarly, the binary variable $Z(t)$ will indicate the status of the reorder to replenish the stock: $Z(t) = 0$ or 1, respectively, if “no order is pending” or “an order is pending.”

FIGURE 12: Fraction of time server is on vacation— p versus $\alpha^{(n)}$ ($S = 90$, $s = 15$, $N = 25$, $\lambda = 8$, $\beta = 1.4$, $\gamma = 0.7$, $\mu = 0.8$.)

TABLE 17: Sensitivity of optimal values to variation in c_s and c_{sh} ($c_h = 0.3$, $c_p = 3$, $c_{pr} = 0.3$).

c_s	11				12				13				14				15			
c_{sh}																				
5.00	12	58	6	12	59	6	12	61	7	11	62	7	11	63	7					
		37.330038			38.297398			39.253444			40.189174			41.092056						
5.25	13	60	9	12	61	9	12	62	10	12	63	10	11	64	10					
		37.656022			38.617201			39.561709			40.495801			41.406634						
5.50	13	61	11	12	62	12	12	63	12	12	64	12	12	66	13					
		37.918543			38.88736			39.824379			40.751743			41.668858						
5.75	13	62	14	13	63	14	12	64	14	12	65	15	12	66	15					
		38.149135			39.121004			40.056997			40.978042			41.889389						
6.00	13	63	16	13	64	16	12	65	16	12	66	17	12	67	17					
		38.357356			39.322828			40.267002			41.182026			42.08824						

TABLE 18: Variation in optimal values for different values of c_p and c_{sh} ($c_s = 15$, $c_h = 0.3$, $c_{pr} = 0.3$).

c_{sh}	4.8				5.0				5.2				5.4				5.6			
c_p																				
2.5	10	63	14	11	64	16	11	64	18	11	65	20	11	66	22					
		40.130057			40.320642			40.495101			40.655041			40.804415						
2.6	11	63	12	11	64	14	11	64	16	11	65	18	11	66	20					
		40.299295			40.503633			40.689262			40.859223			41.017575						
2.7	11	63	10	11	64	13	11	64	14	11	65	16	11	66	18					
		40.453574			40.674434			40.873015			41.053664			41.221074						
2.8	11	62	8	11	63	11	11	64	13	11	65	15	11	66	17					
		40.588856			40.831047			41.044544			41.237658			41.415324						
2.9	11	62	6	11	63	9	11	64	11	11	65	13	12	66	15					
		40.701943			40.97071			41.203531			41.410894			41.598786						

The values taken by these random variables are listed in the following.

i_1	i_2	i_3	i_4
0		0	0
			1
			2
			$\vdots$
1	0		Q
			0
2		0	$s+1$
			$s+2$
			$\vdots$
			S
$\vdots$	1		1
			2
			$\vdots$
			s
N			

(1)

We consider the stochastic process $\{X(t), Y(t), Z(t), L(t), t \geq 0\}$. Since the components carry sufficient information on the sample path of the process and from the assumptions made on the input and output processes, this process can easily be seen to be a Markov process.

The state space of the stochastic process $\{X(t), Y(t), Z(t), L(t), t \geq 0\}$ is the collection of all quadruples $\langle i_1 \rangle = (i_1, i_2, i_3, i_4)$ where each entry is selected from each column as we move from left to right; we may cross vertical lines but not horizontal ones. These quadruples can be ordered in the lexicographic order in each box separated by the horizontal lines.

We shall write the transition rate matrix in a block-partitioned form where i_1 th row (block) corresponds to the collection of quadruples $\langle i_1 \rangle$ (those with leftmost entry i_1), $i_1 = 0, 1, 2, \dots, N$. It may be recalled that the pool size (i_1) may increase or decrease by one unit (except the end states) when a demand joins or leaves the pool. Hence, the transition rate matrix is a tridiagonal one.

We also note that when the number of demands in the pool is i ($i = 0, 1, 2, \dots, N-1$), a new demand joining the pool will shift the current set of quadruple $\langle i \rangle$ to $\langle i+1 \rangle$ and the transition rate does not depend upon i . Therefore, all the entries in the upper diagonal of the rate matrix are equal to a constant block matrix, which we shall denote by B_0 . The lower diagonal entries correspond to the transition from

TABLE 19: Effect of c_{sh} and c_{pr} on the optimal cost rate ($c_s = 15$, $c_h = 0.3$, $c_p = 3$).

c_{sh}	5.0				5.2				5.4				5.6				5.8			
c_{pr}																				
0.20	12	72	10	12	73	12	12	73	14	12	74	16	12	75	18					
		37.479871			37.668234			37.836697			37.989434			38.130738						
0.22	12	70	9	12	71	12	12	72	14	12	72	15	12	73	17					
		38.242077			38.439857			38.615717			38.774912			38.92153						
0.24	12	68	9	12	69	11	12	70	13	12	71	15	12	71	17					
		38.987028			39.193609			39.377128			39.543759			39.696724						
0.26	11	66	8	12	67	10	12	68	13	12	69	14	12	70	16					
		39.709844			39.932901			40.12401			40.296785			40.455714						
0.28	11	64	8	11	65	10	12	67	12	12	67	14	12	68	16					
		40.40882			40.653872			40.855615			41.03516			41.199655						

TABLE 20: Variation in Optimal values for different values of c_p and c_{pr} ($c_s = 15$, $c_h = 0.3$, $c_{sh} = 5$).

c_p	2.6				2.7				2.8				2.9				3.0			
c_{pr}																				
0.32	11	62	14	11	62	12	11	62	10	11	62	9	11	61	6					
		41.181731			41.350929			41.506426			41.644627			41.762515						
0.34	11	61	13	11	61	12	11	61	10	11	60	8	11	60	6					
		41.846597			42.015719			42.169423			42.30555			42.418051						
0.36	10	60	13	11	60	11	11	60	10	11	59	8	11	59	6					
		42.4908			42.66911			42.821582			42.953905			43.062709						
0.38	10	59	13	10	58	11	11	58	9	11	58	7	11	57	5					
		43.120144			43.301145			43.460842			43.590387			43.693443						
0.40	10	57	12	10	57	10	10	57	9	11	57	7	11	56	4					
		43.737882			43.918076			44.077826			44.215885			44.3119						

states of $\langle i \rangle$, $i = 1, 2, \dots, N$, to states $\langle i-1 \rangle$. This is due to the selection of a demand from the pool, and the rate of selection of any one of the demands is α_i . Hence, these transition rates depend on row indices. Therefore, the entries of the lower diagonal depend on the row index. We shall denote them by C_i . In the case of diagonal entries, again, the transition rate may depend upon the pool size (row index), if it involves selection of a demand from the pool. Hence, the entries here also depend upon the respective row index. We shall denote them by A_i .

Hence, the infinitesimal generator matrix can be written as

$$\begin{array}{c}
 \begin{array}{cccccccc}
 & 0 & 1 & 2 & 3 & \cdots & N-2 & N-1 & N \\
 0 & \left(\begin{array}{cccccccc}
 A_0 & B_0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\
 C_1 & A_1 & B_0 & 0 & \cdots & \cdots & 0 & 0 \\
 0 & C_2 & A_2 & B_0 & \cdots & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 N-2 & \cdots & \cdots & \cdots & \cdots & \cdots & A_{N-2} & B_0 & \cdots \\
 N-1 & \cdots & \cdots & \cdots & \cdots & \cdots & C_{N-1} & A_{N-1} & B_0 \\
 N & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & C_N & A_N
 \end{array} \right)
 \end{array}
 \end{array} \quad (2)$$

We shall determine the submatrix B_0 , the elements of which correspond to the transitions from $\langle i \rangle$ to $\langle i+1 \rangle$, $i = 0, 1, 2, \dots, N-1$. This corresponds to the case that one demand has joined the pool. In a Markov process, at most one event can occur in a small interval. Hence, for the transitions from (i_1, i_2, i_3, i_4) to $(i_1 + 1, i'_2, i'_3, i'_4)$, we have 0 when $i'_2 \neq i_2$, $i'_3 \neq i_3$, $i'_4 \neq i_4$ and $p\lambda$ when they are equal and $i_2 = 0$. The matrix B_0 is given by

$$\begin{array}{c}
 \begin{array}{cc}
 i'_2 = & 0 & 1 \\
 i'_3 = & 0 & 1
 \end{array}
 \left| \begin{array}{cc}
 & 1 \\
 0 & 1
 \end{array} \right.
 \begin{array}{c}
 i_2 \\
 i_3
 \end{array}
 \begin{array}{c}
 0 \\
 1
 \end{array}
 \left(\begin{array}{cc|cc}
 p\lambda I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 \mathbf{0} & p\lambda & \mathbf{0} & \mathbf{0} \\
 \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
 \end{array} \right)
 \end{array} \quad (3)$$

Block Size = $(Q+1) \quad 1 \quad (Q) \quad (s)$

Next, we consider the submatrix C_i , the elements of which give the transitions from $\langle i \rangle$ to $\langle i-1 \rangle$. This corresponds to the event that a demand from the pool is selected, and this indicates that the server is "on service" and that "no order is pending." Therefore, possible value is $i'_2 = i_2 = 1$. As a result of a selection from the pool, an item from the inventory is used to meet this demand. Hence, $L(t)$ moves from i_4 to $i_4 - 1$, if $i_4 = s+2, s+3, \dots, S$ (with $i'_3 = 0$)

and from $s + 1$ to s (with $i'_3 = 1$). Hence, the submatrix C_i , $i = 1, 2, \dots, N$, is given by

$$\begin{array}{c} i_2 \\ i_3 \\ i_2' \\ i_3' \end{array} \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \left| \begin{array}{cc} 1 & 1 \end{array} \right. \\ \begin{array}{c} i_2 \\ 0 \\ 1 \end{array} \begin{array}{c} i_3 \\ 0 \\ 1 \end{array} \left(\begin{array}{cc|cc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_0^{(i)} & C_0^{(i)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right), \quad (4)$$

Block Size = $(Q + 1) \quad 1 \quad (Q) \quad (s)$

where

$$C_0^{(i)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \alpha_i & 0 & \cdots & 0 & 0 \\ 0 & \alpha_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_i & 0 \end{pmatrix}, \quad (5)$$

$$C_1^{(i)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha_i \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Let us consider the submatrix A_i , $i = 1, 2, \dots, N$. In this case, there is no addition to nor selection from the pool. Firstly, we note the transitions

- (i) from $(i_1, 0, 1, i_4)$ to $(i_1, 1, 0, i_4)$ or $(i_1, 1, 1, i_4)$,
- (ii) from $(i_1, 1, 0, i_4)$ to $(i_1, 0, 0, i_4)$ or $(i_1, 0, 1, i_4)$,
- (iii) from $(i_1, 1, 1, i_4)$ to $(i_1, 0, 0, i_4)$,

as these involve more than one occurrence.

Secondly, we observe that, only in the case of diagonal entries $a((i_1, 1, 0, i_4), (i_1, 1, 0, i_4))$ (i.e., the intensity of passage), the rate α_i (the rate of selection from the pool) will appear.

Hence, the block-partitioned submatrix A_i will have the following structure:

$$\begin{array}{c} i_2 \\ i_3 \\ i_2' \\ i_3' \end{array} \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \left| \begin{array}{cc} 1 & 1 \end{array} \right. \\ \begin{array}{c} i_2 \\ 0 \\ 1 \end{array} \begin{array}{c} i_3 \\ 0 \\ 1 \end{array} \left(\begin{array}{cc|cc} A_{00} & a_{01} & F_{00} & F_{01} \\ a_{10} & a_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & H_{00}^{(i)} & H_{01} \\ \mathbf{0} & G_{11} & H_{10} & H_{11} \end{array} \right) \quad (6)$$

Block Size = $(Q + 1) \quad 1 \quad (Q) \quad (s)$

The following cases are pertaining to the event that the server is on vacation before and after the transition.

- (i) In the case of column vector a_{01} , the reorder status changes from “no order is pending” to “an order is pending.” Thus, $i_4 = 0$ and, after transition, $i'_4 = 0$. No transition is possible from other states $i_4 = 1, 2, \dots, Q$. Thus, we obtain

$$a_{01} = (\beta, 0, \dots, 0)'. \quad (7)$$

- (ii) The row vector a_{10} considers the change of reorder status (from “order is pending” to “no order is pending”). Thus, a supply of Q items is received. This changes $i_2 = 0$, $i_3 = 0$, $i_4 = 0$ to $i'_2 = 0$, $i'_3 = 1$, $i'_4 = Q$. Hence,

$$a_{10} = (0, 0, \dots, 0, \mu). \quad (8)$$

- (iii) The constant a_{11} is a diagonal entry, and, considering all transitions from $(i_1, 0, 1, 0)$ to other states, we get

$$a_{11} = \begin{cases} -(p\lambda + \mu) & \text{if } i_1 = 1, 2, \dots, N - 1, \\ -\mu & \text{if } i_1 = N, \end{cases} \quad (9)$$

or simply $a_{11} = -(p\lambda\bar{\delta}_{i_1N} + \mu)$, where $\bar{\delta}_{i_1N} = 1 - \delta_{i_1N}$.

- (iv) The nondiagonal entry of A_{00} , namely, $a((i_1, 0, 0, i_4), (i_1, 0, 0, i'_4))$ is equal to $i_4\gamma$ for $i'_4 = i_4 - 1$ and zero for other nondiagonal elements. For the diagonal entry (considering the transitions to other states), we get

$$[A_{00}]_{i_4, i_4} = \begin{cases} -(p\lambda + \beta + i_4\gamma) & \text{if } i_4 = 1, 2, \dots, Q, \quad i_1 \neq N, \\ -(\beta + i_4\gamma) & \text{if } i_4 = 1, 2, \dots, Q, \quad i_1 = N, \end{cases} \quad (10)$$

or, simply, $-(p\lambda\bar{\delta}_{i_1N} + \beta + i_4\gamma)$ if $i_4 = 1, 2, \dots, Q$, for all i_1 .

Using arguments similar to the above, we can express the matrix A_i , $i = 1, 2, \dots, N$ as follows:

$$\begin{array}{c} i_2 \\ i_3 \\ i_2' \\ i_3' \end{array} \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \left| \begin{array}{cc} 1 & 1 \end{array} \right. \\ \begin{array}{c} i_2 \\ 0 \\ 1 \end{array} \begin{array}{c} i_3 \\ 0 \\ 1 \end{array} \left(\begin{array}{cc|cc} D_{00}^{(i)} & \beta u_1 & F_{00} & F_{01} \\ \mu u'_{Q+1} & -(\mu + p\lambda\bar{\delta}_{i_1N}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & H_{00}^{(i)} & H_{01} \\ \mathbf{0} & (\lambda + \gamma)u_1 & H_{10} & H_{11} \end{array} \right).$$

Block Size = $(Q + 1) \quad 1 \quad (Q) \quad (s)$ (11)

where

- (i) $u'_k = (0, 0, \dots, 1, \dots, 0, 0)$ a vector having 1 at k th place and 0 at other places,

(ii)

$$D_{00}^{(i)} = \begin{matrix} & \begin{matrix} 0 & 1 & \cdots & Q-1 & Q \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ Q-1 \\ Q \end{matrix} & \begin{pmatrix} -(p\lambda + \beta) & 0 & \cdots & 0 & 0 \\ \gamma & -(p\lambda + \gamma + \beta) & \cdots & 0 & 0 \\ 0 & 2\gamma & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -(p\lambda + (Q-1)\gamma + \beta) & 0 \\ 0 & 0 & \cdots & Q\gamma & -(p\lambda + Q\gamma + \beta) \end{pmatrix} \end{matrix}, \quad (12)$$

(iii)

$$F_{00} = \begin{matrix} & \begin{matrix} s+1 & s+2 & \cdots & Q & Q+1 & \cdots & S \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ \vdots \\ s+1 \\ s+2 \\ \vdots \\ Q \end{matrix} & \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \beta & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \beta & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta & 0 & \cdots & 0 \end{pmatrix} \end{matrix}, \quad (13)$$

(iv)

$$F_{01} = \begin{matrix} & \begin{matrix} 1 & 2 & \cdots & s-1 & s \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ \vdots \\ s-1 \\ s \\ \vdots \\ Q \end{matrix} & \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \beta & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta & 0 \\ 0 & 0 & \cdots & 0 & \beta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \end{matrix}, \quad (14)$$

(v)

$$H_{00}^{(i)} = \begin{matrix} & \begin{matrix} s+1 & s+2 & \cdots & S-1 & S \end{matrix} \\ \begin{matrix} s+1 \\ s+2 \\ \vdots \\ \vdots \\ S \end{matrix} & \begin{pmatrix} -(\lambda + (s+1)\gamma + \alpha_i) & 0 & \cdots & 0 & 0 \\ \lambda + (s+2)\gamma & -(\lambda + (s+2)\gamma + \alpha_i) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda + S\gamma & -(\lambda + S\gamma + \alpha_i) \end{pmatrix} \end{matrix}, \quad (15)$$

(vi)

$$H_{01} = \begin{matrix} & 1 & 2 & \cdots & s-1 & s \\ s+1 & \begin{pmatrix} 0 & 0 & \cdots & 0 & \lambda + (s+1)\gamma \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ S & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \end{matrix}, \quad (16)$$

(vii)

$$H_{10} = \begin{matrix} & s+1 & \cdots & Q & Q+1 & \cdots & S \\ 1 & \begin{pmatrix} 0 & \cdots & 0 & \mu & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ s & 0 & \cdots & 0 & 0 & \cdots & \mu \end{pmatrix} \end{matrix}, \quad (17)$$

(viii)

$$H_{11} = \begin{matrix} & 1 & 2 & \cdots & s-1 & s \\ 1 & \begin{pmatrix} -(\lambda + \gamma + \mu) & 0 & \cdots & 0 & 0 \\ \lambda + 2\gamma & -(\lambda + 2\gamma + \mu) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s & 0 & 0 & \cdots & \lambda + s\gamma & -(\lambda + s\gamma + \mu) \end{pmatrix} \end{matrix}. \quad (18)$$

4. Steady-State Analysis

It can be seen from the structure of the infinitesimal generator A that the time-homogeneous Markov process $\{(X(t), Y(t), Z(t), L(t)); t \geq 0\}$ on the finite state space E is irreducible. Hence, the limiting distribution

$$\begin{aligned} \phi^{(j_1, j_2, j_3, j_4)} &= \lim_{t \rightarrow \infty} \Pr[X(t) = j_1, Y(t) = j_2, \\ &Z(t) = j_3, L(t) = j_4 \mid X(0), Y(0), Z(0), \\ &L(0)] \end{aligned} \quad (19)$$

exists. Let

$$\begin{aligned} \Phi &= (\phi^{(i_1, 0, 0, 0)}, \phi^{(i_1, 0, 0, 1)}, \dots, \phi^{(i_1, 0, 0, Q)}, \\ &\phi^{(i_1, 0, 1, 0)}, \phi^{(i_1, 1, 0, s+1)}, \phi^{(i_1, 1, 0, s+2)}, \dots, \phi^{(i_1, 1, 0, S)}, \\ &\phi^{(i_1, 1, 0, s+1)}, \phi^{(i_1, 1, 0, s+2)}, \dots, \phi^{(i_1, 1, 0, S)}, \\ &\phi^{(i_1, 1, 1, 1)}, \phi^{(i_1, 1, 1, 2)}, \dots, \phi^{(i_1, 1, 1, s)}, \\ &\phi^{(0)}, \phi^{(1)}, \dots, \phi^{(N)}). \end{aligned} \quad (20)$$

Then, the vector of limiting probabilities Φ satisfies

$$\Phi A = \mathbf{0}, \quad \Phi \mathbf{e} = 1. \quad (21)$$

From the structure of A , it is seen that the Markov process under study falls into the class of birth and death process in a Markovian environment as discussed by Gaver et al. [25]. Hence, using the same arguments, we can calculate the limiting probability vectors. For the sake of completeness, we provide the algorithm here.

Algorithm. (1) Determine recursively the matrices

$$\begin{aligned} F_0 &= A_0, \\ F_{i_1} &= A_{i_1} + C_{i_1}(-F_{i_1-1}^{-1})B_0, \quad i_1 = 1, 2, \dots, N. \end{aligned} \quad (22)$$

(2) Compute recursively the vectors $\phi_{(i_1)}$ using

$$\phi^{(i_1)} = \phi^{(i_1+1)} C_{i_1+1}(-F_{i_1}^{-1}), \quad i_1 = N-1, N-2, \dots, 0. \quad (23)$$

(3) Solve the system of equations

$$\phi^{(N)} F_N = \mathbf{0}, \quad (24)$$

$$\sum_{i_1=0}^N \phi^{(i_1)} \mathbf{e} = 1. \quad (25)$$

From the system of (24), vector $\phi^{(N)}$ could be determined uniquely, up to a multiplicative constant. This constant is obtained by using (23) and (25).

5. System Performance Measures

In this section, we give some system performance measures, which are useful in the qualitative interpretation of the model under study.

5.1. Expected Inventory Level. Let η_I denote the mean inventory level in the steady-state. Since $\phi^{(i, k, p, m)}$ denotes the steady state probability when the number of demands in the

pool is i , the server status is k , the status of reorder is p , and the inventory level is m , the mean inventory level is given by

$$\eta_I = \sum_{i=0}^N \left(\sum_{m=1}^Q m \phi^{(i,0,0,m)} + \sum_{m=s+1}^S m \phi^{(i,1,0,m)} + \sum_{m=1}^s m \phi^{(i,1,1,m)} \right). \quad (26)$$

5.2. Expected Reorder Rate. Let η_R denote the expected reorder rate in the steady state. We note that a reorder is triggered

- (i) when the inventory level drops from $s + 1$ to s if the server is in service or
- (ii) when the server returns from the vacation, he/she finds the inventory level less than or equal to s and there is no order pending.

This leads to

$$\eta_R = \sum_{i=0}^N \phi^{(i,1,0,s+1)} (\lambda + (s+1)\gamma + \bar{\delta}_{i0}\alpha_i) + \sum_{i=0}^N \sum_{m=0}^s \beta \phi^{(i,0,0,m)}, \quad (27)$$

where δ_{ij} is the Kronecker delta function.

5.3. Expected Perishable Rate. Let η_{PR} denote the expected reorder Perishable in the steady state. This is given by

$$\eta_{PR} = \sum_{i=0}^N \left(\sum_{m=1}^Q m \gamma \phi^{(i,0,0,m)} + \sum_{m=s+1}^S m \gamma \phi^{(i,1,0,m)} + \sum_{m=1}^s m \gamma \phi^{(i,1,1,m)} \right). \quad (28)$$

5.4. Mean Shortage Rate. Let η_{SR} denote the mean shortage rate in the steady state. It may be noted that a shortage occurs only during the server vacation. The customers who arrive during server vacation prefer to leave the system either by their own choice or the pool has reached its maximum capacity:

$$\eta_{SR} = \sum_{i=0}^{N-1} q \lambda \left[\sum_{m=0}^Q \phi^{(i,0,0,m)} + \phi^{(i,0,1,0)} \right] + \lambda \left[\sum_{m=0}^Q \phi^{(N,0,0,m)} + \phi^{(N,0,1,0)} \right]. \quad (29)$$

5.5. Expected Number of Customers in the Pool. Let η_P denote the expected number of customers in the pool in the steady state. Since $\phi^{(i)}$ is the steady state probability vector for i customers in the pool with each component specifying the probability for the particular combination of the status of

the server, status of the reorder, and the inventory level, the expected number of customers in the pool is given by

$$\eta_P = \sum_{i=1}^N i \phi^{(i)} \mathbf{e}. \quad (30)$$

5.6. Fraction of Time When the Server is on Vacation. Let η_F denote the fraction of time when the server is on vacation. This is given by

$$\eta_F = \sum_{i=0}^N \sum_{m=0}^Q \phi^{(i,0,0,m)} + \sum_{i=0}^N \phi^{(i,0,1,0)}. \quad (31)$$

6. Cost Analysis

The expected total cost per unit time (expected total cost rate) in the steady state for this model is defined to be

$$TC(s, S, N) = c_h \eta_I + c_P \eta_P + c_{pr} \eta_{PR} + c_s \eta_R + c_{sh} \eta_{SR}, \quad (32)$$

where

c_s : setup cost per order,

c_h : the inventory carrying cost per unit item per unit time,

c_P : cost of a waiting customer in the pool per unit time,

c_{pr} : perishable cost per unit item per unit time,

c_{sh} : cost per customer lost per unit time.

Hence, we get

$$\begin{aligned} TC(s, S, N) = & c_h \left\{ \sum_{i=0}^N \left(\sum_{m=1}^Q m \phi^{(i,0,0,m)} + \sum_{m=s+1}^S m \phi^{(i,1,0,m)} + \sum_{m=1}^s m \phi^{(i,1,1,m)} \right) \right\} + c_P \left\{ \sum_{i=1}^N i \phi^{(i)} \mathbf{e} \right\} \\ & + c_s \left\{ \sum_{i=0}^N \phi^{(i,1,0,s+1)} (\lambda + (s+1)\gamma + \bar{\delta}_{i0}\alpha_i) + \sum_{i=0}^N \sum_{m=0}^s \beta \phi^{(i,0,0,m)} \right\} \\ & + c_{pr} \left\{ \sum_{i=0}^N \left(\sum_{m=0}^Q m \gamma \phi^{(i,0,0,m)} + \sum_{m=s+1}^S m \gamma \phi^{(i,1,0,m)} + \sum_{m=1}^s m \gamma \phi^{(i,1,1,m)} \right) \right\} \\ & + c_{sh} \left\{ \sum_{i=0}^{N-1} q \lambda \left[\sum_{m=0}^Q \phi^{(i,0,0,m)} + \phi^{(i,0,1,0)} \right] + \lambda \left[\sum_{m=0}^Q \phi^{(N,0,0,m)} + \phi^{(N,0,1,0)} \right] \right\}. \end{aligned} \quad (33)$$

Due to the complex form of the limiting distribution, it is difficult to discuss the properties of the cost function $TC(s, S, N)$ analytically. Hence, a detailed computational study of the expected cost function is carried out in the next section.

7. Numerical Illustrations

To study the behaviour of the model developed in this work, several examples were performed and a set of representative results is shown here. Although we have not shown the convexity of $TC(s, S, N)$ analytically, our experience with considerable numerical examples indicates the function $TC(s, S, N)$ to be convex. In some cases, it turned out to be an increasing function of s , and we use simple numerical search procedure to get the optimal values of TC, s, S and N (say TC^*, s^*, S^* , and N^*). A typical three-dimensional plot of the expected cost function is given in Figure 1. The optimal cost value $TC^* = 37.352776$ is obtained at $(s^*, S^*, N^*) = (11, 63, 11)$.

We have studied the effect of varying the system parameters and costs on the optimal values, and the results agreed with what one would expect. Some of our results are presented in Tables 1 through 20 where the upper entries in each cell give the s^*, S^* , and N^* , respectively, and the lower entry gives the corresponding TC^* .

Example 1. In the first example, we study the impact of the demand rate λ , the lead time rate μ , the perishable rate γ , the selection rate α_i , and the vacation time parameter β on the optimal values s^*, S^*, N^* , and TC^* . Towards this end, we first fix $\alpha_i = i\alpha, i = 1, 2, \dots, N$, and the cost values as $c_h = 0.3, c_s = 15, c_{pr} = 0.3, c_{sh} = 5$, and $c_p = 3$. We observe the following from Table 1 to 10.

- (1) The optimal values s^*, S^* , and TC^* increase monotonically with increase in λ . This is to be expected since, when the arrivals occurs closer and closer, the optimal reorder level and the maximum inventory level increase to prevent more waiting time of the demands in the pool. We also note that, as λ increases, N^* increases except for the case in which γ increases.
- (2) As is to be expected, the optimal cost rate TC^* decreases monotonically, when μ, α , and β increase and monotonically increases when γ increases.
- (3) When μ increases, s^* and S^* decrease and N^* increases. This can be explained intuitively as follows. If the replenishments occur further and further apart, the inventory level becomes zero frequently. During the stock-out period, a part of arriving demands enter into the pool. To prevent the loss of demand, we have necessity to increase the pool size. For the same reason as β increases, s^* and S^* decrease and N^* increases.
- (4) We note that, as γ increases, s^* and N^* decrease monotonically. We also note that except for the combination of increasing γ, S^* decreases when α increases.

- (5) As α increases, N^* increases and s^* decreases. We also note that, except for the combination of increasing μ, S^* increases when α increases.

Example 2. In this example, we study the impact of the setup cost c_s , holding cost c_h , perishable cost c_{pr} , shortage cost c_{sh} , and the waiting cost c_p on the optimal values s^*, S^*, N^* , and TC^* . Towards this end, we first fix the parameter values as $\lambda = 14, \gamma = 1.4, \beta = 1.3, \alpha_i = i\alpha, i = 1, 2, \dots, N, \alpha = 4, \mu = 4$, and $p = 0.85$. We observe the following from Table 11 to 20:

- (1) The optimal cost increases, when c_h, c_s, c_{pr}, c_{sh} , and c_p increase. The optimal cost is more sensitive to c_h than to c_s, c_{pr}, c_{sh} , and c_p .
- (2) As c_h increases, as is to be expected, the optimal values s^*, S^* , and N^* decrease monotonically. This is to be expected since the holding cost increases, we resort to maintain low stock in the inventory.
- (3) When c_{sh} increases, as is to be expected, the optimal values s^*, S^* , and N^* increase monotonically. This is to be expected since for the high shortage cost to reduce the number of customer to be lost, we have to increase the pool size, maintain high inventory, and place order in the higher level.
- (4) If the waiting cost c_p increases, the optimal values of s^* increase and N^* decreases monotonically. We also note that, except for the combination of increasing the holding cost c_h, S^* decreases when c_p increases.
- (5) If the setup cost c_s increases, it is a common decision that we have to maintain more stock to avoid frequent ordering. This fact is also observed in our model.
- (6) We note that, when the perishable cost c_{pr} increases s^*, S^* , and N^* decrease monotonically.

Example 3. In this example, we look to the impact of the demand rate λ , the reorder rate μ , the selection rate α , and the mean vacation time $1/\beta$ on the fraction, η_F , of time the server is on vacation for the combination of various values of postponed demand rate $\alpha^{(n)} (n = 1, 2, 3)$, $\alpha^{(n)}$ is the increasing, constant, and decreasing functions, where $\alpha^{(1)} = 5 + i * 0.1, \alpha^{(2)} = 5$, and $\alpha^{(3)} = 5 + i * (-0.1), i = 1, \dots, N$. From Figures 2 to 5, we observe the following.

- (i) As is to be expected as λ increases, η_F increases. This is because, if the demands rate increases, the stockout occurs frequently, and hence the server takes vacation frequently. Hence, η_F increases. Similarly the same thing is observed in the case γ .
- (ii) As μ increases, η_F decreases. This happens because, if μ increases, the ordered items are received at higher rate. Hence, fewer stock-out periods occur. Hence the server returns to the system more often. Hence, the intervacation time increases. Therefore η_F decreases. Similarly, the same thing is observed in the case β .
- (iii) In these cases, we observe $\alpha^{(1)} > \alpha^{(2)} > \alpha^{(3)}$.

Example 4. In this example, we now study the impact of p on the performance measures, expected Inventory level η_I , expected reorder rate η_R , expected perishable rate η_{PR} , mean shortage rate η_{SR} , expected number of customers in the pool η_P , fraction of time the server is on vacation η_F and the total expected cost $TC(s, S, N)$, for the combination of various values of postponed demand rate $\alpha^{(n)}$ ($n = 1, 2, 3$), $\alpha^{(n)}$ is the increasing, constant, and decreasing functions, where $\alpha^{(1)} = 5 + i * 0.1$, $\alpha^{(2)} = 5$ and $\alpha^{(3)} = 5 + i * (-0.1)$, $i = 1, \dots, N$. From Figure 6 to 12, we observe the following.

- (i) As $TC(s, S, N)$, η_{SR} and η_P are monotonically increasing, when p increases. In these cases, we observe $\alpha^{(1)} < \alpha^{(2)} < \alpha^{(3)}$.
- (ii) As η_R and η_F are monotonically increasing, when p increases. In these cases, we observe $\alpha^{(1)} > \alpha^{(2)} > \alpha^{(3)}$.
- (iii) As η_I and η_{PR} are monotonically decreasing, when p increases. In these cases, we observe $\alpha^{(1)} < \alpha^{(2)} < \alpha^{(3)}$.

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