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Research Article

An EPQ Model with Unit Production Cost and Set-Up Cost as Functions of Production Rate

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Extensive research has been devoted to economic production quantity (EPQ) problem. However, no attention has been paid to problems where unit production and set-up costs must be considered as functions of production rate. In this paper, we address the problem of determining the optimal production quantity and rate of production in which unit production and set-up costs are assumed to be continuous functions of production rate. Based on the traditional economic production quantity (EPQ) formula, the cost function associated with this model is proved to be nonconvex and a procedure is proposed to solve this problem. Finally, utility of the model is presented using some numerical examples and the results are analyzed.

1. Introduction

The economic production quantity (EPQ) model has been widely used in practice because of its simplicity. However, there are some drawbacks in the assumption of the original EPQ model and many researchers have tried to improve it with different viewpoints. Recently, the classical EPQ model has been generalized in many directions. Some authors extended the EPQ model by incorporating the effect of learning in setups and process quality. Also, set-up time reduction on production run length and varying parameters have received significant attention. The relationship between set-up cost and production run length is also influenced by the learning and forgetting effects. The effect of learning and forgetting in setups and in product quality is investigated by Jaber and Bonney [1]. Porteus studied the effect of process deterioration on the optimal production cycle time [2].

Darwish generalized the EPQ model by considering a relationship between the set-up cost and the production run length [3]. Jaber investigated the lot sizing problem for reduction in setups with reworks and interruptions to restore the process to an "in-control" state [4]. Unlike the model presented by Khouja [5], he considered that the set-up cost and defect rate decrease as the number of restoration activities increases. Afshar-Nadjafi and Abbasi considered an EPQ model with depreciation cost and process quality cost as continuous functions of time [6]. Freimer et al. studied the effect of imperfect yield on EPQ decisions. They considered set-up cost reductions and process quality improvements as types of investments in the production processes [7].

Furthermore, the classical EPQ model has been investigated in many other ways; for example, the effect of varying production rate on the EPQ model was investigated by Khouja [8]. Huang introduced the EPQ model under conditions of permissible delay in payments [9]. Salameh and Jaber developed the EPQ model for items of imperfect quality [10]. Jaber et al. applied first and second laws of thermodynamics on inventory management problem. They showed that their approach yields higher profit than that of the classical EPQ model [11].

Recently, Hou considered an EPQ model with imperfect production processes, in which the set-up cost and process quality are functions of capital expenditure [12]. Tsou presented a modified inventory model which accounts for imperfect items and Taguchi's cost of poor quality [13].

The assumption of the fixed unit production and set-up costs is one of the classical EOQ shortcomings. To the author's knowledge, none of the above EPQ models considered the unit production and set-up costs as continuous functions of the production rate.

In this paper, the classical EPQ model is extended by considering unit production cost and set-up cost as continuous functions of production rate. We use a simple method to solve the extended economic production quantity model of minimizing the total annual cost. Also, numerical examples are used to show the utility of the proposed models. The paper is organized as follows. The model assumptions and notation are presented in Section 2. Then, the model is developed in Section 3. In Section 4 we explain a brief summary of the results given in this paper based on a numerical example. Finally, Section 5 contains the conclusions.

2. Assumptions and Notation

In this section, we derive a mathematical statement for the EPQ model with unit production cost and set-up cost as continuous functions of production rate. The basic EPQ model is that of determining a production quantity of an item, subject to the following conditions related to the production facility and marketplace [14].

(i) Demand rate, is continuous, known, and constant.

(ii) Production rate is greater than or equal to demand rate.

(iii) All demands must be met.

(iv) Holding costs are determined by the value of the item.

(v) Set-up time is assumed to be zero.

(vi) There are no quantity constraints.

(vii) No shortages are allowed.

Most of the assumptions in our mathematical model are the same as those in the conventional EPQ. Besides, we impose the following additional assumptions.

(i) Unit production cost of product is a decreasingly continuous function of production rate.

(ii) Set-up cost is an increasingly continuous function of production rate.

The classic EPQ model assumes that unit production cost and set-up cost are fixed. However, in real production environment, this assumption does not accurately reflect the reality, because it can often be observed that the unit production cost and set-up cost depend on the production rate. Processes with low production rates require less set-up cost than that of high rates. This is because the effort and automation needed to perform a set-up activity are related to the condition of the production process, that is, for high production rates, the production process is more likely to be subjected to higher level of equipment resulting in a higher set-up cost. In mass production processes by increasing the investment in high-tech equipment, the production rate increases but also the set-up cost will increase accordingly. For example, the set-up cost of a CNC is much more than in simple turning machines. Also, processes with high production rates result in low unit production cost. This is because the human resource and raw materials are used efficiently with comparison to processes with low production rates.

These assumptions add complexity of the model where a closed form solution was not possible and the convexity of the cost function was not validated.

In order to state the problem mathematically, let

Q be production quantity (real positive decision variable),

D annual demand rate of product,

P annual production rate of product (real positive decision variable),

C unit production cost of product,

i annually inventory holding cost rate,

h annual unit holding cost ($h = ic$),

A set-up cost of production system,

T cycle length ($T = Q/D$),

T_p production period length in a cycle ($T_p = Q/P$),

T_d only-demand period length in a cycle ($T_p = Q/D - Q/P$),

$I_{\max}$ maximum on-hand inventory level ($I_{\max} = Q(1 - D/P)$),

ATC annual total cost (objective function).

3. Model Development

In this section we develop EPQ model with unit production cost and set-up cost as continuous functions of production rate. The set-up cost in the proposed model is assumed to be an increasingly continuous function of production rate as follows (modified from that of Jaber and Bonney [1]):

$$A(P) = A_0 P^{\psi}; \qquad D \le P \le P_{\max}, \tag{1}$$

where the factor ψ is the shape factor of the set-up cost, the parameter A_0 is a positive constant that can be interpreted as set-up cost associated with the classical EPQ model ($\psi = 0$), and $P_{\max}$ is maximum production rate which serves as an upper limit on the set-up cost. Also, $A_{\max} = A_0 P_{\max}^{\psi}$ is defined as maximum set-up cost (related to maximum production rate, $P_{\max}$). The factor ψ can be estimated by the curve-fitting approach using historical data on set-up cost of the process.

The unit production cost in the proposed models is a decreasingly continuous function of production rate as follows:

$$C(P) = C_0 P^{-\varepsilon}; \qquad D \le P \le P_{\max}, \tag{2}$$

where the factor ε is the shape factor of unit production cost, the parameter C_0 is a positive constant that can be interpreted as unit production cost associated with the classical EPQ model ($\varepsilon = 0$), and $P_{\max}$ is maximum production rate which serves as an upper limit on the set-up cost. Also, $C_{\min} = C_0 P_{\max}^{-\varepsilon}$ is defined as minimum unit production cost (related to maximum production rate, $P_{\max}$). The factor ε can be

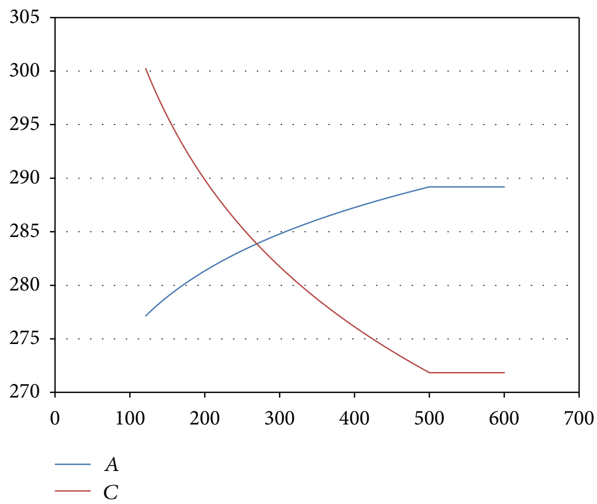

Figure 1: Set-up cost (A) and unit production cost (C) against production rate (P) for $D = 120$, $P_{max} = 500$, $A_0 = 240$, $C_0 = 420$, $\varepsilon = 0.07$, and $\psi = 0.03$.

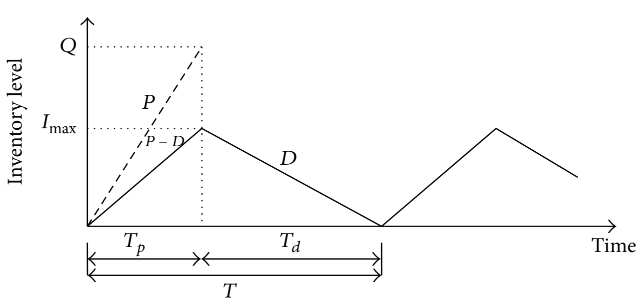

Figure 2: The relation between inventory level and time.

estimated by the curve-fitting approach using historical data on unit production cost of the process.

The behavior of $A(P)$ and $C(P)$ is shown in Figure 1. This figure shows that the set-up cost increases with the production rate. However, after the production rate reaches maximum rate, P_{max}, the process set-up cost is fixed at A_{max}. Inversely, unit production cost decreases with the production rate. However, after the production rate reaches maximum rate, P_{max}, the unit production cost fixed at C_{min}. Although parameters ψ and ε can be any real numbers in general, logically $0 \le \psi \le 1$ and $0 \le \varepsilon \le 1$ are acceptable.

The model presented in this section assumes that all demands are satisfied from inventory; that is, no stock-out situation occurs. The objective is to find economic production quantity Q^* and economic production rate P^*, in order to minimize the annual total cost ATC. The behavior of inventory level in EPQ model is illustrated in Figure 2. It shows that when the inventory level vanishes, production is started at a rate P. Since P exceeds D, the inventory position will increase at a rate $P - D$, satisfying demand and building inventory, until Q units are produced. At that point in the cycle, the inventory level will be a maximum.

ATC is computed as follows:

$$\text{ATC} = C_0 P^{-\varepsilon} D + \frac{D}{Q} A_0 P^{\psi} + \frac{i}{2} Q \left(1 - \frac{D}{P}\right) C_0 P^{-\varepsilon}, \quad (3)$$

where the first term is the production cost, the second term is the set-up cost, and the third term is the holding cost.

The elements of Hessian matrix are

$$\begin{aligned}
\frac{\partial^2 \text{ATC}}{\partial Q^2} &= \frac{2DA_0P^{\psi}}{Q^3}, \\
\frac{\partial^2 \text{ATC}}{\partial Q \partial P} &= \frac{\partial^2 \text{ATC}}{\partial P \partial Q} \\
&= \frac{iC_0}{2}\left(-\varepsilon P^{-\varepsilon-1} + D(\varepsilon+1)P^{-\varepsilon-2}\right) \\
&\quad - \frac{\psi D A_0 P^{\psi-1}}{Q^2}, \\
\frac{\partial^2 \text{ATC}}{\partial P^2} &= \frac{iC_0Q}{2}\left(\varepsilon(\varepsilon+1)P^{-\varepsilon-2} - D(\varepsilon+1)(\varepsilon+2)P^{-\varepsilon-3}\right) \\
&\quad + \frac{\psi(\psi-1)DA_0P^{\psi-2}}{Q} + \varepsilon(\varepsilon+1)C_0DP^{-\varepsilon-2}.
\end{aligned} \quad (4)$$

Evaluation of α_1 and α_2 shows that

$$\alpha_1 = \frac{d^2\text{ATC}}{dQ^2} \ge 0, \quad \forall Q, P, \quad (5)$$

$$\begin{aligned}
\alpha_2 &= \det \nabla^2 \text{ATC}(Q, P) \\
&= \det \begin{bmatrix} \dfrac{d^2\text{ATC}}{dQ^2} & \dfrac{\partial^2\text{ATC}}{\partial Q \partial P} \\ \dfrac{\partial^2\text{ATC}}{\partial P \partial Q} & \dfrac{d^2\text{ATC}}{dP^2} \end{bmatrix} \\
&= \frac{d^2\text{ATC}}{dQ^2} * \frac{d^2\text{ATC}}{dP^2} - \frac{\partial^2\text{ATC}}{\partial Q \partial P} * \frac{\partial^2\text{ATC}}{\partial P \partial Q} \ \text{free in sign.}
\end{aligned} \quad (6)$$

Then ATC is a nonconvex function. Therefore, taking the partial derivatives of ATC with respect to the Q and P and solving equations $(\partial/\partial Q)\text{ATC} = 0$ and $(\partial/\partial P)\text{ATC} = 0$ do not guarantee the necessary conditions for Q and P to be optimal.

For a given amount of production rate P, (3) gives an expression of ATC as a function of only Q which is called *reduced ATC*. Although ATC is a nonconvex function, it is obvious from expression (5) that *reduced ATC* is a convex function. Therefore, taking the first derivative of *reduced ATC* with respect to Q and setting (d/dQ) *reduced ATC* $= 0$ yield the only positive solution at

$$Q^* = \sqrt{\frac{2DA_0P^{(\psi+\varepsilon)}}{iC_0(1 - D/P)}}. \quad (7)$$

This problem poses a difficult computational task due to the nonconvexity involved. To obtain the economic production

Initialize the control parameters $(D, i, \psi, \varepsilon, A_0, C_0, P_{\max}, \lambda)$
set initial solution $(P_0 = D + \lambda)$,
compute Q_0 with relation (7) and $\mathrm{ATC}(P_0, Q_0)$ with relation (3);
set $(P, Q) = (P^*, Q^*) = (P_0, Q_0)$;
set $\mathrm{ATC}(P, Q) = \mathrm{ATC}(P^*, Q^*) = \mathrm{ATC}(P_0, Q_0)$;
while $P + \lambda \le P_{\max}$ *do*
    set $P = P + \lambda$, compute Q with relation (7) and $\mathrm{ATC}(P, Q)$ with relation (3);
    if $\mathrm{ATC}(P, Q) \le \mathrm{ATC}(P^*, Q^*)$
        set $(P^*, Q^*) = (P, Q)$;
    end
end
print (P^*, Q^*);

Algorithm 1

quantity (Q^*) and economic production rate (P^*) of the above mentioned model, we are to minimize ATC in following procedure. In this regard, the steps of Algorithm 1 are briefly presented below where the following notations are used:

(P_0, Q_0): initial solution,

(P, Q): current solution,

(P^*, Q^*): best solution,

$\mathrm{ATC}(P, Q)$: value of the objective function at solution (P, Q),

λ: neighborhood step-length parameter.

Algorithm 1 starts with an initial solution (production rate and production quantity) for the problem and by initializing the control parameters. This first solution is considered as the current and best solution. In the inner cycle of the procedure, repeated, while $P + \lambda \le P_{\max}$, a neighborhood solution of the current solution is obtained by adding to the current solution a fixed amount, λ, where $\lambda > 0$ is the step-length parameter. The generated solution replaces the current one. This procedure continues until P is equal to maximum production rate, $P_{\max}$. Finally, the last best solution is reported as the optimal solution.

4. Numerical Example and Discussion

To illustrate the usefulness of the model developed in Section 3, let us consider the inventory situation where a stock is replenished with Q units. The parameters needed for analyzing the above inventory situation are given as follows.

maximum production rate: 500 unit/year,

demand rate, $D = 220$ units/year,

holding cost rate, $i = 0.2$ annually,

maximum production rate, $P_{\max} = 500$ units/year,

$C_0 = 75$\$/unit,

$A_0 = 100$\$/cycle,

$\varepsilon = 0.09$,

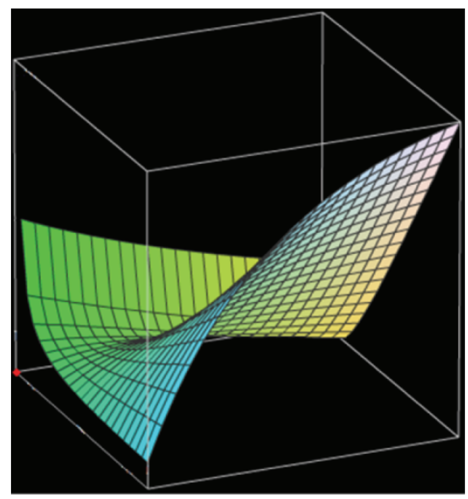

Figure 3: ATC against production quantity (Q) and production rate (P).

$\psi = 0.1$,

$\lambda = 1$.

Using the above mentioned procedure gives $Q^* = 130.614$, $P^* = 500$, and $\mathrm{ATC}^* = 10058.55$, whereas with classical EPQ model (with production rate 500), we have $Q^* = 72.375$ and $\mathrm{ATC}^* = 17107.95$. It is clear that significant losses are realized when the classical EPQ model is used instead of the proposed model. Annual total cost function against production rate and production quantity is sketched in Figure 3.

Now, we demonstrate the utility of the model and study the effect of the shape parameter of the set-up cost and unit production cost on the optimal solution. In order to assess the loss of using the classical EPQ model, its performance is compared with that of the proposed model. Let $\mathrm{ATC}^*_{\mathrm{EPQ}}$ denote the optimal total expected cost using the classical EPQ model ($\varepsilon = \psi = 0$). Now define the percent loss due to using the classical EPQ model instead of the proposed model as in [3] as follows:

$$\%\ \mathrm{LOSS} = \frac{\mathrm{ATC}^*_{\mathrm{EPQ}} - \mathrm{ATC}^*}{\mathrm{ATC}^*_{\mathrm{EPQ}}} \times 100. \quad (8)$$

Table 1: Optimal solutions for different values of ε.

ε	ψ	Q^*	P^*	ATC*	% LOSS
0	0.10	1054.62	221	16571.58	−0.01023
0.02	0.10	1113.12	221	14879.22	10.1206
0.04	0.10	1174.86	221	13359.85	19.2984
0.06	0.10	1240.02	221	11995.82	27.5380
0.08	0.10	126.62	500	10683.06	37.5550
0.10	0.10	134.74	500	9471.08	44.6393
0.12	0.10	143.38	500	8398.54	50.9086
0.14	0.10	152.58	500	7449.28	56.4572
0.16	0.10	162.35	500	6609.02	61.3687
0.18	0.10	172.76	500	5865.14	65.7169
0.20	0.10	183.84	500	5206.48	69.5669
0.30	0.10	250.83	500	2883.93	83.1427
0.50	0.10	466.96	500	913.32	94.6614
0.70	0.10	869.31	500	307.14	98.2047
0.90	0.10	1618.35	500	112.05	99.3451

Table 2: Optimal solutions for different values of ψ.

ε	ψ	Q^*	P^*	ATC*	% LOSS
0.09	0	95.73	500	9891.05	42.1845
0.09	0.02	101.87	500	9920.52	42.0122
0.09	0.04	108.40	500	9951.88	41.8289
0.09	0.06	115.35	500	9985.25	41.6339
0.09	0.08	122.74	500	10020.76	41.4263
0.09	0.10	130.61	500	10058.55	41.2054
0.09	0.12	138.99	500	10098.76	40.9704
0.09	0.14	147.90	500	10141.54	40.7203
0.09	0.16	157.38	500	10187.08	40.4541
0.09	0.18	1668.67	221	10220.20	38.2639
0.09	0.20	1761.22	221	10224.07	38.2405
0.09	0.30	2306.92	221	10246.85	38.1029
0.09	0.50	3957.97	221	10315.79	37.6864
0.09	0.70	6790.66	221	10434.07	36.9720
0.09	0.90	11650.67	221	10637	35.7462

Table 1 gives the optimal solutions for selected values of ε ranging from 0 to 0.9 and ψ fixed at 0.1. It is apparent from Table 1 that the effect of the ε on economic production quantity is not monotonously increasing or decreasing. On the contrary, lot size has a convex behavior as ε increases. Another important observation in this example is that the economic production rate is located in extreme points ($D + 1 = 221$ or $P_{\max} = 500$). The results also indicate that the loss due to using the classical EPQ model increases with ε. This is because of the fact that for high values ε, the unit production cost in the classical EPQ deviates significantly from the actual situation.

In Table 2, the optimal solutions for selected values of ψ ranging from 0 to 0.9 and ε fixed at 0.09 are examined. The results show that the economic production quantity increases as ψ increases; lot size is inflated when ψ approaches unity from below. The results also indicate that the loss due to using the classical EPQ model decreases slightly with ψ.

Table 3 shows the simultaneous effect of values ε and ψ on optimal solutions. The results show that the economic production quantity has a convex behavior as ε and ψ increase; lot size is inflated when ε and ψ approach unity from below. Inversely, the economic production rate has a concave behavior as ε and ψ increase, and also the economic production rate is located in extreme points. The results also indicate that the loss due to using the classical EPQ model increases with ε and ψ. This is because of the fact that for high values ε and ψ, the unit production cost and set-up cost in the classical EPQ deviate significantly from the actual situation.

Table 3: Optimal solutions for different values of ε and ψ.

ε	ψ	Q^*	P^*	ATC*	% LOSS
0	0	850.15	221	16554.65	0.0000
0.02	0.02	896.94	221	14866.05	10.2002
0.04	0.04	999.20	221	13350.26	19.3564
0.06	0.06	105.08	500	11972.33	30.0189
0.08	0.08	118.99	500	10644.08	37.7828
0.10	0.10	134.74	500	9471.08	44.6393
0.12	0.12	152.57	500	8435.17	50.6945
0.14	0.14	172.76	500	7520.34	56.0419
0.16	0.16	195.62	500	6712.43	60.7643
0.18	0.18	221.51	500	5998.95	64.9347
0.20	0.20	250.83	500	5368.86	68.6178
0.30	0.30	466.96	500	3165.31	81.4980
0.50	0.50	11969.42	221	1164.56	92.9654
0.70	0.70	35233.15	221	431.71	97.3922
0.90	0.90	103712.20	221	182.74	98.8961

5. Conclusions

In this paper, economic production quantity (EPQ) model has been developed considering varying both the unit production cost and set-up cost. We have considered the unit production cost as a continuous decreasing function of production rate and the set-up cost as a continuous increasing function of production rate. The problem is described with a mathematical model, and then a simple procedure is proposed to solve it. From the numerical results, we could clearly see that loss due to using the classical EPQ model is significant. Also, the results showed that the shape parameters, ε and ψ, which are a property of the system, have marginal and simultaneous effect on optimal policies, significantly. For example, lot size is inflated when ε and ψ approach unity from below. The results of this study can help managers make optimal decisions on equipment selection (with optimal production rate). Also, the optimal production quantity based on the optimal value of production rate can be determined.

References

[1] M. Y. Jaber and M. Bonney, "Lot sizing with learning and forgetting in set-ups and in product quality," *International Journal of Production Economics*, vol. 83, no. 1, pp. 95–111, 2003.

[2] E. L. Porteus, "Optimal lot sizing process quality improvement and setup cost reduction," *Operations Research*, vol. 34, no. 1, pp. 137–144, 1986.

[3] M. A. Darwish, "EPQ models with varying setup cost," *International Journal of Production Economics*, vol. 113, no. 1, pp. 297–306, 2008.

[4] M. Y. Jaber, "Lot sizing for an imperfect production process with quality corrective interruptions and improvements, and reduction in setups," *Computers and Industrial Engineering*, vol. 51, no. 4, pp. 781–790, 2006.

[5] M. Khouja, "The use of minor setups within production cycles to improve product quality and yield," *International Transactions in Operational Research*, vol. 12, no. 4, pp. 403–416, 2005.

[6] B. Afshar-Nadjafi and B. Abbasi, "EPQ model with depreciation cost and process quality cost as continuous functions of time," *Journal of Industrial Engineering International*, vol. 5, no. 8, pp. 77–89, 2009.

[7] M. Freimer, D. Thomas, and J. Tyworth, "The value of setup cost reduction and process improvement for the economic production quantity model with defects," *European Journal of Operational Research*, vol. 173, no. 1, pp. 241–251, 2006.

[8] M. Khouja, "The economic production lot size model under volume flexibility," *Computers and Operations Research*, vol. 22, no. 5, pp. 515–523, 1995.

[9] Y.-F. Huang, "Optimal retailer's replenishment policy for the EPQ model under the supplier's trade credit policy," *Production Planning and Control*, vol. 15, no. 1, pp. 27–33, 2004.

[10] M. K. Salameh and M. Y. Jaber, "Economic production quantity model for items with imperfect quality," *International Journal of Production Economics*, vol. 64, no. 1, pp. 59–64, 2000.

[11] M. Y. Jaber, R. Y. Nuwayhid, and M. A. Rosen, "Price-driven economic order systems from a thermodynamic point of view," *International Journal of Production Research*, vol. 42, no. 24, pp. 5167–5184, 2004.

[12] K.-L. Hou, "An EPQ model with setup cost and process quality as functions of capital expenditure," *Applied Mathematical Modelling*, vol. 31, no. 1, pp. 10–17, 2007.

[13] J.-C. Tsou, "Economic order quantity model and Taguchi's cost of poor quality," *Applied Mathematical Modelling*, vol. 31, no. 2, pp. 283–291, 2007.

[14] J. R. Tersine, *Principles of Inventory and Materials Management*, North-Holland, New York, NY, USA, 3rd edition, 1988.