Research Article

Thermal Boundary Layer in Flow due to an Exponentially Stretching Surface with an Exponentially Moving Free Stream

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A numerical investigation is made to study the thermal boundary layer for flow of incompressible Newtonian fluid over an exponentially stretching sheet with an exponentially moving free stream. The governing partial differential equations are transformed into self-similar ordinary differential equations using similarity transformations in exponential forms. Then those are solved numerically by shooting technique using Runge-Kutta method. The study reveals that the momentum boundary layer thickness for this flow is considerably smaller than the linear stagnation point flow past a linearly stretching sheet. The momentum and thermal boundary layer thicknesses reduce when the velocity ratio parameter increases. For the temperature distribution, in addition to the heat transfer from the sheet, the heat absorption at the sheet also occurs in certain situations and both heat transfer and absorption increase with the velocity ratio parameter and the Prandtl number. The temperature inside the boundary layer significantly decreases with higher values of velocity ratio parameter and the Prandtl number.

1. Introduction

The viscous fluid flow due to a stretching sheet is very significant problem in fluid dynamics due to its huge applications in many manufacturing processes, for example, the cooling of an infinite metallic plate in a cooling bath, the boundary layer along material handling conveyers, the aerodynamic extrusion of plastic sheets, the boundary layer along a liquid film in the condensation processes, hot rolling, paper production, metal spinning, glass-fiber production, and drawing of plastic films. The heat transfer from a stretching surface is of interest in polymer extrusion processes where the object, after passing through a die, enters the fluid for cooling below a certain temperature. During the processes, mechanical properties are greatly dependent upon the rate of cooling and the rate at which such objects are cooled has an important bearing on the properties of the final product. So, the quality of the final product depends on the rate of heat transfer from the stretching surface. The suitable choice of cooling liquid is very much crucial as it has a direct impact on rate of cooling of the surface.

Crane [1] first investigated the steady boundary layer flow of an incompressible viscous fluid over a linearly stretching plate and gave an exact similarity solution in closed analytical form. Crane’s work was extended by many researchers such as P. S. Gupta and A. S. Gupta [2], Chen and Char [3], Pavlov [4], and Ali [5] considering the effects of heat and mass transfer and magnetic field under various physical conditions. Later, Ali [6] discussed the thermal boundary layer on a surface stretching with power law velocity. The mixed convection for the flow due to linearly stretching sheet with mass suction/injection was studied by Ali and Al-Yousef [7]. Ali [8] also investigated the buoyancy effects on the boundary layers induced by a rapidly stretching surface. Tsai et al. [9] discussed the effect of nonuniform heat source/sink on the flow and heat transfer from an unsteady stretching sheet through a quiescent fluid medium extending to infinity. In two important papers, Bhattacharyya and Layek [10, 11] explained the behaviour of chemically reactive solute distribution in MHD flow over a stretching sheet and in slip flow towards a vertical stretching sheet. Recently, Bhattacharyya [12, 13] presented effect of heat source/sink...
on the MHD boundary layer flow over steady and unsteady shrinking/stretching sheet and Bhattacharyya et al. [14] found the analytic solution of MHD flow of non-Newtonian Casson fluid flow induced due to stretching/shrinking sheet with wall mass transfer.

On the other hand, Hiemenz [15] first studied the steady flow in the neighbourhood of a stagnation point. Chiam [16] investigated a problem which is a combination of the works of Hiemenz [15] and Crane [1], that is, the linear stagnation point flow towards a linear stretching sheet when the stretching rate of the plate is equal to the strain rate of the stagnation point flow and he found no boundary layer structure near the plate. After few years, Mahapatra and Gupta [17] reinvestigated the same stagnation point flow towards a stretching sheet with different stretching and straining velocities and they found two kinds of boundary layers near the sheet depending on the ratio of the stretching and straining velocity rates. Wang [18] demonstrated the stagnation point flow towards a shrinking sheet. Furthermore, the behaviour of stagnation point flow over stretching/shrinking sheet under different physical aspects was discussed by many researchers [19–35].

In last few decades in almost all investigations of the flow over a stretching sheet, the flow occurs because of linear variation of stretching velocity of the flat sheet with the distance from the origin. So, the boundary layer flow induced by an exponentially stretching sheet is not studied much though it is very important and practical flow frequently appeared in many engineering processes. In 1999, Magyari and Keller [36] first consider the boundary layer flow due to an exponentially stretching sheet and he also studied the heat transfer in the flow taking exponentially varied wall temperature. After that, Elbashbeshy [37] numerically examined the flow and heat transfer over an exponentially stretching surface considering wall mass suction. Khan and Sanjayanand [38] investigated the flow of viscoelastic fluid and heat transfer over an exponentially stretching sheet with viscous dissipation effects. Partha et al. [39] presented a similarity solution for mixed convection flow past an exponentially stretching sheet was demonstrated by Bhattacharyya and Layek [48].

Inspired by the above investigations, in the present paper, the thermal boundary layer in incompressible flow over an exponentially stretching sheet in an exponentially moving free stream is studied. The wall temperature distribution is taken variable in exponential form. Using similarity transformations, the governing partial differential equations are transformed into self-similar ordinary differential equations. Then the transformed equations are solved by standard shooting technique using fourth order Runge-Kutta method. The numerical computations are presented through some figures and the various characteristics of flow and heat transfer are thoroughly described. This type of flow is possible in reality in some special situations. The motive of the investigation is to study new type of flow dynamics and the heat transfer in the flow field.

2. Analysis of Motion and Heat Transfer

Consider the steady two-dimensional laminar boundary layer flow of a viscous incompressible fluid and heat transfer over an exponentially stretching sheet (of velocity $U_\infty$) with an exponentially moving free stream (of velocity $U_\infty$). The sheet coincides with the plane $y = 0$ and the flow confined to $y > 0$. The governing equations for the flow and the temperature are written in usual notation as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{U_\infty}{\rho_c} \frac{dU_\infty}{dx} + v \frac{\partial^2 u}{\partial y^2},$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2},$$

where $u$ and $v$ are the velocity components in $x$ and $y$ directions, respectively, $U_\infty$ is the fluid velocity in free stream, $\nu = \mu/\rho$ is the kinematic fluid viscosity, $\rho$ is the fluid density, $\mu$ is the coefficient of fluid viscosity, $T$ is the temperature, $k$ is the fluid thermal conductivity, and $c_p$ is the specific heat. The diagram of the physical problem is given in Figure 1.

The boundary conditions corresponding to the velocity components and the temperature are given by

$$u = U_\infty (x) \quad \text{at} \quad y = 0;$$

$$u \rightarrow U_\infty (x) \quad \text{as} \quad y \rightarrow \infty,$$

$$T = T_\infty + T_0 \exp \left( \frac{\lambda x}{2L} \right) \quad \text{at} \quad y = 0;$$

$$T \rightarrow T_\infty \quad \text{as} \quad y \rightarrow \infty,$$

where $U_\infty$ is the variable temperature of the sheet, $T_\infty$ is the free stream temperature assumed to be constant, $T_0$ is a constant which measures the rate of temperature increase
along the sheet, \(L\) denotes the reference length, and \(\lambda\) is a parameter which is physically very important in controlling the exponential increment of temperature along the sheet and it may have both positive and negative values. The stretching and free stream velocities \(U_w\) and \(U_\infty\) are, respectively, given by

\[
U_w(x) = c \exp\left(\frac{x}{L}\right), \quad U_\infty(x) = a \exp\left(\frac{x}{L}\right),
\]

where \(c\) and \(a\) are constants with \(c > 0\) and \(a \geq 0\) (for without any free stream \(a = 0\)).

Now, the stream function \(\psi(x, y)\) is given by

\[
u = \frac{\partial \psi}{\partial y}, \quad \nu = -\frac{\partial \psi}{\partial x}.
\]

For relations in (7), the continuity equation (1) is identically satisfied and the momentum equation (2) and the temperature equation (3) are reduced to the following forms:

\[
\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = U_\infty \frac{dU_\infty}{dx} + \nu \frac{\partial^2 \psi}{\partial x^2},
\]

\[
\frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} = \frac{\kappa}{\rho \nu} \frac{\partial^2 T}{\partial y^2}.
\]

The boundary conditions in (4) for the flow become

\[
\frac{\partial \psi}{\partial y} = U_w(x) \quad \text{at} \ y = 0; \quad \frac{\partial \psi}{\partial y} \rightarrow U_\infty(x) \quad \text{as} \ y \rightarrow \infty.
\]

Next, the dimensionless variables for \(\psi\) and \(T\) are introduced as

\[
\psi = \sqrt{\frac{2\nu L c}{f''}} f(\eta) \exp\left(\frac{x}{2L}\right),
\]

\[
T = T_\infty + (T_w - T_\infty) \theta(\eta),
\]

where \(\eta\) is the similarity variable and is defined as \(\eta = y \sqrt{c/2\nu L} \exp(x/2L)\).

Using (10), finally the following nonlinear self-similar equations are obtained:

\[
f''' + ff'' - 2f'^2 + 2\left(\frac{a}{c}\right)^2 = 0, \quad (11)
\]

\[
\theta'' + Pr \left(\theta f' - \lambda f \theta'\right) = 0, \quad (12)
\]

where \(a/c\) is the velocity ratio parameter and \(Pr = \mu c_p / k\) is the Prandtl number.

The boundary conditions in (9) and (5) reduce to the following forms:

\[
f(\eta) = 0, \quad f'(\eta) = 1 \quad \text{at} \ \eta = 0; \quad f'(\eta) \rightarrow \frac{a}{c} \quad \text{as} \ \eta \rightarrow \infty, \quad (13)
\]

\[
\theta(\eta) = 1 \quad \text{at} \ \eta = 0; \quad \theta(\eta) \rightarrow 0 \quad \text{as} \ \eta \rightarrow \infty.
\]

When \(c = a\), (11) gives closed form analytical solution \(f(\eta) = \eta\) which is the same as the linear stagnation-point flow over a stretching sheet with velocity varied linearly to the distance from the origin.

The physical quantities of interest in this problem are the local skin-friction coefficient and the local Nusselt number. The dimensionless local skin-friction coefficient is expressed as

\[
C_{f_x} = \frac{\tau_w}{\rho U_w^2}, \quad (14)
\]

where \(\tau_w\) is the shear stress at the wall and is given by \(\tau_w = \mu [\partial u/\partial y]_{y=0}\). So,

\[
\sqrt{2 \text{Re} C_{f_x}} = f''(0), \quad (15)
\]

where \(\text{Re} = U_w L / \nu\) is the Reynolds number.

The local Nusselt number can be written as

\[
\text{Nu}_x = -\frac{x}{T_w - T_\infty} \frac{\partial T}{\partial y} \bigg|_{y=0}. \quad (16)
\]

So,

\[
\sqrt{\text{Re}_x} \text{Nu}_x = -\frac{x}{2L} \theta'(0), \quad (17)
\]

where \(\text{Re}_x = U_w x / \nu\) is the local Reynolds number.

3. Numerical Method for Solution

The nonlinear coupled differential equations (11) and (12) along with the boundary conditions (13) form a two point boundary value problem (BVP) and is solved using shooting method, by converting it into an initial value problem (IVP). In this method, it is necessary to choose a suitable finite value of \(\eta \rightarrow \infty\), say \(\eta_{\infty}\). The following first-order system is set:

\[
f' = p, \quad p' = q, \quad q' = 2p^2 - fq - 2\left(\frac{a}{c}\right)^2, \quad (18)
\]

\[
\theta' = z, \quad z' = -Pr (fz - \lambda \rho \theta).
\]
with the boundary conditions

\[ f(0) = 0, \quad p(0) = 1, \quad \theta(0) = 1. \]  

To solve (18) with (19) as an IVP, the values for \( q(0) \), that is, \( f''(0) \), and \( z(0) \), that is, \( \theta'(0) \), are needed but no such values are given. The initial guess values for \( f''(0) \) and \( \theta'(0) \) are chosen and the fourth-order Runge-Kutta method is applied to obtain the solution. The calculated values of \( f'(\eta) \) and \( \theta(\eta) \) at \( \eta_{\infty}(=25) \) are compared with the given boundary conditions \( f'(\eta_{\infty}) = a/c \) and \( \theta(\eta_{\infty}) = 0 \) and adjust values of \( f''(0) \) and \( \theta'(0) \) using “secant method” to give better approximation for the solution. The step size is taken as \( \Delta \eta = 0.01 \). The process is repeated until we get the results correct up to the desired accuracy of \( 10^{-6} \) level.

4. Results and Discussion

The numerical computations have been carried out for various values of the physical parameters involved in the equations, namely, the velocity ratio parameter \( a/c \), the Prandtl number \( Pr \), and the parameter \( \lambda \). For illustration of the results, computed values are plotted through graphs and the physical explanations are rendered for all cases.

To ensure the accuracy of the numerical scheme, we compare the values of \( f''(0) \) and \( f(\infty) \) with the published data by Magyari and Keller \[36\] in Table 1 without any exponential free stream, \( U_{\infty} = 0 \) that is \( a/c = 0 \) (\( a = 0 \)) and those are found in excellent agreement. Thus, we are feeling confident that the presented results are accurate.

As similar to that of stagnation point flow with linear straining velocity towards a linearly stretching sheet, in the boundary layer flow with exponentially varied free stream velocity over exponentially stretching sheet two different kinds of boundary layer structures have formed near the sheet depending upon the ratio of two constants relating to the stretching and free stream velocities, that is, on the velocity ratio parameter \( a/c \), for \( a/c > 1 \) and \( a/c < 1 \). Also, it is important to note that for \( a/c = 1 \) no velocity boundary layer is formed near the sheet. The velocity profiles for various values of \( a/c \) are depicted in Figure 2. Also, the variations of \( f(\eta) \) and the dimensionless temperature for different values of \( a/c \) are plotted in Figures 3 and 4. It is observed that \( f(\eta) \) increases with increasing values of \( a/c \) and the temperature at a point decreases with \( a/c \).

The momentum and thermal boundary layer thicknesses are denoted by \( \delta \) and \( \delta_T \), respectively, and are described by the equations \( \delta = \eta_R \sqrt{2V_0L/c} \exp(-x/2L) \) and \( \delta_T = \eta_{RT} \sqrt{2V_0L/c} \exp(-x/2L) \). The dimensionless boundary layer thicknesses \( \eta_R \) and \( \eta_{RT} \) are defined as the values of \( \eta \) (nondimensional distance from the surface) at which the difference of dimensionless velocity \( f'(\eta) \) and the parameter \( a/c \) has been reduced to 0.001 and the dimensionless temperature \( \theta(\eta) \) has been decayed to 0.001, respectively. Both boundary layer thicknesses are very vital in practical and theoretical standpoints. The values \( \eta_R \) and \( \eta_{RT} \) are given in Table 2 for several values of \( a/c \). From the table, it is seen that the momentum boundary layer thickness decreases when \( a/c \) increases (both for \( a/c > 1 \) and \( a/c < 1 \)). But, it is worth noting that the momentum boundary layer thickness is considerably thinner than that of the boundary layer of linear

![Figure 2: Velocity profiles for various values of \( a/c \).](image)

![Figure 3: The variations of \( f(\eta) \) for various values of \( a/c \).](image)

![Figure 4: Temperature profiles \( \theta(\eta) \) for various values of \( a/c \).](image)

**Table 1:** The values of \( f''(0) \) and \( f(\infty) \) for \( a/c = 0 \).

<table>
<thead>
<tr>
<th>( a/c )</th>
<th>Magyari and Keller [36]</th>
<th>Present study</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0.1 )</td>
<td>1.281808</td>
<td>1.2818084</td>
</tr>
<tr>
<td>0.2</td>
<td>0.905639</td>
<td>0.9056433</td>
</tr>
</tbody>
</table>

```
Table 2: Values of \( \eta_\delta \) and \( \eta_{\delta T} \) for several values of \( a/c \) with \( Pr = 0.5 \) and \( \lambda = 1 \).

<table>
<thead>
<tr>
<th>( a/c )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.5</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mahapatra and Gupta [17]</td>
<td>( \eta_\delta )</td>
<td>6.96</td>
<td>5.91</td>
<td>4.36</td>
<td>—</td>
<td>2.62</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>( \eta_{\delta T} )</td>
<td>5.04</td>
<td>4.17</td>
<td>2.91</td>
<td>1.87</td>
<td>1.79</td>
<td>1.68</td>
</tr>
</tbody>
</table>

The effect of the parameter \( \lambda \) on the temperature distribution is very significant because the increase of wall temperature along the sheet depends upon \( \lambda \). For various values of \( \lambda \), the dimensionless temperature profiles \( \theta(\eta) \) are presented in Figures 5 and 6 for \( a/c = 2 \) and \( a/c = 0.1 \), respectively. In both the cases, the figures exhibit the overshoot of temperature profiles for some negative values of \( \lambda \). Hence, when the wall temperature in inverse exponential forms, the thermal overshoot occurs. Also, it is noticed that the temperatures at particular point and the thermal boundary layer thickness decrease with increase of \( \lambda \). Furthermore, it is very important to note that the overshoot pick is of higher length when free stream velocity is large compared to the stretching velocity (\( a/c = 2 \)). The dimensionless temperature gradient profiles \( \theta'(\eta) \) for different \( \lambda \) are depicted in Figures 7 and 8 for \( a/c = 2 \) and \( a/c = 0.1 \), respectively. From these two figures, it is clearly seen that the temperature gradient at the sheet is positive for some values of \( \lambda \) and negative for some other values of \( \lambda \). So, heat transfer from the sheet and heat absorption at the sheet occur depending on the values of \( \lambda \) for both values of \( a/c \). Because of inverse exponential distribution of surface temperature \( (\lambda > 0) \), the heat absorption is found in the thermal boundary layer. Thus the parameter \( \lambda \) related to wall temperature plays a vital role in controlling the cooling.
process of the surface and consequently it has a great impact on the quality of the manufactured products.

The temperature profiles for various values of Prandtl number Pr are demonstrated in Figures 9–12 with \( \frac{a}{c} = 2 \), \( \frac{a}{c} = 0.1 \), and \( \lambda = 1 \). In Figures 9 and 10, for \( \lambda = 1 \), it is observed that the temperature at a fixed point decreases with Pr for both values of \( \frac{a}{c} \). Accordingly, the thermal boundary layer thickness decreases with increasing values of Pr and the heat transfers from the sheet to the adjacent fluid. But, when \( \lambda = -1.5 \) (Figures 11 and 12), the temperature overshoot is observed; that is, the sheet absorbs the heat and interestingly for both values of \( \frac{a}{c} \) the height of the overshoot pick increases with Pr, so the heat absorption at the sheet also increases. However, similar to the linear case, in this situation the thermal boundary layer thickness decreases with increasing values of Pr. For increase of Prandtl number, the fluid thermal conductivity reduces and consequently the thermal boundary layer thickness becomes thinner.

The values of \( f''(0) \) related to local skin-friction coefficient and \( -\theta'(0) \) related to local Nusselt number are plotted in Figures 13–15 for various values of \( \frac{a}{c} \), \( \lambda \) and \( \text{Pr} \). The value of \( f''(0) \) increases with velocity ratio parameter \( \frac{a}{c} \). Also, the value of \( -\theta'(0) \) increases with \( \lambda \). Moreover, for the larger negative values of \( \lambda \), the heat absorption \( [-\theta'(0) < 0] \) at the surface is found. With the increase of \( \frac{a}{c} \) and Pr, the heat transfer enhances and also heat absorption becomes larger (Figures 14 and 15).

5. Concluding Remarks

The momentum and heat transfer characteristics of the laminar boundary layer flow induced by an exponentially stretching sheet in an exponentially moving free stream are investigated. The transformed governing equations are solved by shooting technique using Runge-Kutta method. The findings of this study can be summarized as follows.

(a) The momentum boundary layer thickness for this flow is significantly smaller than that of the linear stagnation point flow past a linearly stretching sheet.
(b) The momentum and thermal boundary layer thicknesses decrease with increase of velocity ratio parameter \( \frac{a}{c} \).
(c) For positive values of \( \lambda \) and smaller negative values of \( \lambda \), heat transfers from surface to the ambient fluid and importantly for greater negative value of \( \lambda \), heat
transfers from the ambient fluid to the surface, the heat absorption occurs.

(d) Temperature and thermal boundary layer thicknesses decrease with \( \lambda \), a parameter associated with wall temperature.

(e) The heat transfer from the sheet and the heat absorption at the sheet are enhanced with the increase in both velocity ratio parameter and Prandtl number.

**Nomenclature**

\( a \): Constant related to free stream velocity  
\( c \): Constant related to stretching velocity  
\( a/c \): Velocity ratio parameter  
\( c_p \): Specific heat  
\( f \): Dimensionless stream function  
\( L \): Reference length  
\( Pr \): Prandtl number  
\( T \): Temperature  
\( T_w \): Variable temperature of the sheet  
\( T_\infty \): Free stream temperature  
\( u \): Velocity component in \( x \) direction  
\( v \): Velocity component in \( y \) direction  
\( U_w \): Stretching velocity  
\( U_\infty \): Free stream velocity  
\( \eta \): Similarity variable  
\( \kappa \): Fluid thermal conductivity  
\( \lambda \): Parameter  
\( \mu \): Coefficient of fluid viscosity  
\( \nu \): Kinematic viscosity of fluid  
\( \rho \): Density of fluid  
\( \psi \): Stream function  
\( \theta \): Dimensionless temperature.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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