Research Article

Floquet Theory for Discontinuously Supported Waveguides

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We apply Floquet theory of periodic coefficient second-order ODEs to an elastic waveguide. The waveguide is modeled as a uniform elastic string periodically supported by a discontinuous Winkler elastic foundation and, as a result, a Hill equation is found. The fundamental solutions, the stability regions, and the dispersion curves are determined and then plotted. An asymptotic approximation to the dispersion curve is also given. It is further shown that the end points of the band gap structure correspond to periodic and semiperiodic solutions of the Hill equation.

1. Introduction

Periodic structures often appear in several mechanical systems, ranging from strings and beams [1, 2] to phononic crystals [3–5] to name just a few. Such systems exhibit a typical pass/block band structure when wave propagation is considered. Indeed, periodic structures are especially relevant when employed as waveguides [6, 7] or energy scavenging devices [8]. The analysis of the transmission property of waveguides is best carried out through Floquet theory of periodic coefficient ODEs, although this fact is seldom neglected in favor of a more direct approach by means of the Floquet-Bloch boundary conditions. In this paper, an analysis of the mechanical problem of a uniform string periodically supported on a Winkler foundation is presented from the standpoint of the stability theory of Hill's equation [9]. Besides, a high-frequency asymptotic homogenization procedure is presented, following [10, 11]. The discontinuous character of the support may be due to crack propagation [12–15] or debonding in composite materials [16, 17]. It could also be due to the tensionless character of the substrate [18]. This study follows upon a very vast body of literature on elastic periodic structures [2, 19–24]. The situation of wave propagation through a thin coating layer [25–27] could also be considered. Applications in the realm of civil engineering are also possible [28–32]. The paper is structured as follows: Section 2 sets up the mechanical model and the governing

equations. Section 3 discusses stability of the solution of Hill's equation. The dispersion relation and its asymptotic approximation are presented in Section 4. Finally, conclusions are drawn in Section 5.

2. The Mechanical Model

Let us consider a homogeneous elastic string in uniform tension T , periodically supported on a Winkler elastic foundation (Figure 1). The governing equation for the transverse displacement $w(x, t)$ in the absence of loading is periodic with period L :

$$-T\partial_{xx}w + \rho A\partial_{tt}w + (1-H)\beta w = 0, \quad x \in (-L_1, L - L_1), \quad (1)$$

where ρA is the mass linear density of the string, assumed constant, β is the Winkler subgrade modulus (with physical dimension of stress), and H is Heaviside step function; that is,

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \quad (2)$$

This equation may be rewritten in dimensionless form

$$-\kappa^2 w'' + \ddot{w} + (1-H)w = 0, \quad x_1 \in (-\alpha, 1 - \alpha), \quad (3)$$

FIGURE 1: A homogeneous string periodically supported by an elastic foundation.

having introduced the dimensionless positive ratios:

$$\kappa = \sqrt{\frac{T}{\beta L^2}}, \quad (4)$$

$$\alpha = \frac{L_1}{L} \leq 1,$$

together with $x_1 = x/L \in [0, 1]$, the dimensionless axial coordinate, and $\tau = t/\sqrt{\rho A/\beta}$, the dimensionless time. Here, prime denotes differentiation with respect to x_1 and dot differentiation with respect to τ and $w(x_1, \tau)$ and $H(x_1)$ are assumed. We shall look for the harmonic behavior of w ; that is, $w(x_1, \tau) = u(x_1) \exp(i\Omega\tau)$, whence (3) becomes the constant coefficient ODE for $u(x_1)$:

$$-\kappa^2 u'' - \Omega^2 u + (1 - H)u = 0, \quad x_1 \in (-\alpha, 1 - \alpha). \quad (5)$$

We shift the unit period to range in the interval $(-\alpha/2, 1 - \alpha/2)$ in order to consider an even/odd problem; namely,

$$-\kappa^2 u'' - \Omega^2 u + \left[1 - H\left(x_1 - \frac{\alpha}{2}\right)\right]u = 0, \quad (6)$$

$$x_1 \in \left(-\frac{\alpha}{2}, 1 - \frac{\alpha}{2}\right).$$

The general harmonic solution of (3) in the supported region $x_1 \in (-\alpha/2, \alpha/2)$ is given by

$$u_s(x_1) = A_1 \exp\left(\frac{\sqrt{1 - \Omega^2}}{\kappa} x_1\right) + A_2 \exp\left(-\frac{\sqrt{1 - \Omega^2}}{\kappa} x_1\right), \quad (7)$$

while the solution in the free region $x_1 \in (\alpha/2, 1 - \alpha/2)$ is

$$u_f(x_1) = B_1 \sin\left(\frac{\Omega}{\kappa} x_1\right) + B_2 \cos\left(\frac{\Omega}{\kappa} x_1\right), \quad (8)$$

where A_1, A_2 and B_1, B_2 are integration constants to be determined through the boundary conditions (BCs). The BCs for the system require continuity at the supported/unsupported transition:

$$u_s(0) = u_f(0), \quad (9)$$

$$u'_s(0) = u'_f(0),$$

where prime stands for x_1 differentiation. Furthermore, consideration of the Floquet-Bloch waves lends the periodicity conditions

$$u_s\left(-\frac{\alpha}{2}\right) = u_f\left(1 - \frac{\alpha}{2}\right)\rho, \quad (10)$$

$$u'_s\left(-\frac{\alpha}{2}\right) = u'_f\left(1 - \frac{\alpha}{2}\right)\rho,$$

where $\rho \neq 0$ is the *characteristic multiplier*. For a second-order ODE, there are two nonnecessarily distinct characteristic multipliers, which are denoted by ρ_1 and ρ_2 . Besides, let $\rho_k = \exp(m_k)$ with $k \in \{1, 2\}$; then $m_k = iq$ is the *characteristic exponent*.

3. Boundedness and Periodicity of the Solution

Equation (5) is known as Hill's equation, after Hill who studied it in 1886 in the context of lunar dynamics [33]. For the sake of clarity, we shall write $(0, a)$ for the periodicity interval $(-\alpha/2, 1 - \alpha/2)$. It is easy to find a fundamental system of solutions $\phi_1(x_1), \phi_2(x_1)$ such that

$$\begin{aligned} \phi_1(0) &= 1, \\ \phi'_1(0) &= 0, \\ \phi_2(0) &= 0, \\ \phi'_2(0) &= 1. \end{aligned} \quad (11)$$

This is a set of two particular linearly independent solutions of the ODE (6); see [9] for further details. It is well known that, for any second-order ODE,

$$W(\phi_1(x_1), \phi_2(x_1)) = \det \begin{bmatrix} \phi_1(x_1) & \phi_2(x_1) \\ \phi'_1(x_1) & \phi'_2(x_1) \end{bmatrix} = \phi_1(x_1)\phi_2(x_1)' - \phi_2(x_1)\phi'_1(x_1) \equiv 1, \quad (12)$$

where W is the Wronskian of the fundamental solutions [34]. In light of the problem's symmetry, ϕ_1 is an *even* and ϕ_2 an *odd* function. Let us introduce the *discriminant*

$$D = \phi_1(a) + \phi'_2(a) = \rho_1 + \rho_2$$

$$= 2f(\Omega) \sin\left[\frac{(1 - \alpha)\Omega}{\kappa}\right] \sin\left(\frac{\alpha\sqrt{\Omega^2 - 1}}{\kappa}\right) + 2 \cos\left[\frac{(1 - \alpha)\Omega}{\kappa}\right] \cos\left(\frac{\alpha\sqrt{\Omega^2 - 1}}{\kappa}\right), \quad (13)$$

being

$$f(\Omega) = \frac{1 - 2\Omega^2}{2\Omega\sqrt{\Omega^2 - 1}} = -\frac{1 - 1/2\Omega^2}{\sqrt{1 - 1/\Omega^2}}. \quad (14)$$

Hill's equation is said to be *unstable* if all nontrivial solutions are unbounded in $\mathbb{R}$ and *conditionally stable* if there is a

FIGURE 2: Stability regions (shaded in gray) for Hill's equation (5) at $\kappa = 0.5$. Solid and dashed curves represent the level lines $D = \pm 2$, respectively.

nontrivial solution bounded in $\mathbb{R}$ and *stable* if all solutions are bounded in $\mathbb{R}$.

By Theorem 1.3.1 of [9], the solution is stable if $|D| < 2$ and unstable when $|D| > 2$ and special consideration is required for the case $|D| = 2$. Indeed, when $|D| < 2$, the characteristic multipliers ρ_1 and ρ_2 are complex conjugated and have unit modulus, whereupon the characteristic exponents are opposite; that is, $q_1 = -q_2$. Figure 2 plots the bounding curves $D = 2$ (dashed) and $D = -2$ (solid) as a function of Ω and α for $\kappa = 0.5$. Stable regions are shaded in gray. It is seen that Band Gaps (BGs) are bounded by dashed and solid curves in succession. When $D = \pm 2$ it is $\rho_1 = \rho_2 = \pm 1$, respectively.

It is also well known [9, Section 1.2] that the bounding curve $D = 2$ corresponds to solutions in the form

$$\begin{aligned} \psi_1(x_1) &= p_1(x_1), \\ \psi_2(x_1) &= p_2(x_1), \end{aligned} \quad (15)$$

where $p_1(x_1)$ and $p_2(x_1)$ are periodic functions with period 1, provided that the following condition which grants the availability of two eigenvectors for the repeated eigenvalue $\rho_{1,2} = 1$ holds

$$\phi_2(a) = \phi_1'(a) = 0. \quad (16)$$

Under the same condition, the bounding curve $D = -2$ corresponds to solutions in the form

$$\begin{aligned} \psi_1(x_1) &= P_1(x_1), \\ \psi_2(x_1) &= P_2(x_1), \end{aligned} \quad (17)$$

FIGURE 3: Periodic and semiperiodic solutions for Hill's equation (5) at $\kappa = 0.5$. The red and the blue dotted curves represent the solution of the first and the second of the conditions (16), respectively.

where $P_1(x_1)$ and $P_2(x_1)$ are *semiperiodic* function with period 1; that is,

$$P_i(x_1 + 1) = -P_i(x_1), \quad i = 1, 2. \quad (18)$$

In Figure 3, the solution curves for the first and the second of the conditions (16) are plotted in dotted line style and they partly overlies the conditionally stable curves $D = \pm 2$. However, it is observed that such conditions are satisfied only at some very special points. Indeed, we have

$$\begin{aligned} (1 - \Omega^2) \phi_2(a) - \kappa^2 \phi_1'(a) \\ = \frac{\kappa}{\Omega} \sin\left(\frac{(1 - \alpha)\Omega}{\kappa}\right) \cosh\left(\frac{\alpha\sqrt{1 - \Omega^2}}{\kappa}\right), \end{aligned} \quad (19)$$

whence crossing is possible either when

$$\Omega = n \frac{\kappa\pi}{1 - \alpha}, \quad n \in \{0, 1, 2, 3, \dots\} \quad (20)$$

or when $\alpha = 1$. The case $\alpha = 1$ relates to a fully supported string for which $\phi_1'(a)$ and $\phi_2(a)$ are both proportional to

$$\sqrt{\Omega^2 - 1} \sinh \frac{\sqrt{\Omega^2 - 1}}{\kappa}, \quad (21)$$

so that they vanish at the isolated points $\Omega = \sqrt{1 + n^2 \kappa^2 \pi^2}$, $n \in \{0, 1, 2, \dots\}$. The first point, $\Omega = 1$, corresponds to the pivotal frequency for the 0th BG. Conversely, the case $\alpha = 0$ corresponds to a free string, which is a nondispersive situation. Indeed, BGs collapse into single points which, according to (20), are located at $\Omega = n\kappa\pi$. Other (Ω, α, κ) combinations exist which satisfy $|D| = 2$ and (16), such as $(3.22313, 0.512649, 0.5)$. In all such points two periodic

(or semiperiodic) solutions exist and conditional stability is reverted to stability. However, in the general case, (16) never hold together and thus conditional stability remains. Indeed, the second of the conditions (15) is replaced by

$$\psi_2(x_1) = x_1 p_1(x_1) + p_2(x_1), \quad (22)$$

which is obviously unstable. By the same token, the second of the conditions (17) becomes

$$\psi_2(x_1) = x_1 P_1(x_1) + P_2(x_1). \quad (23)$$

Thus, the *coexistence problem* for period 1 is generally answered in the negative; that is, a single periodic (or semiperiodic) solution exists which is accompanied by a nonperiodic one. Besides, from a mechanical standpoint, it is observed that, in the general case, a periodic solution $p(x)$ is not acceptable on physical grounds, as it conveys a jump discontinuity at the boundary, unless

$$p(0) = p(a). \quad (24)$$

Likewise, a semiperiodic solution $P(x)$ is not acceptable unless

$$P(0) = -P(a). \quad (25)$$

We now prove that such conditions can always be met. To this aim we now take advantage of the problem's symmetry and recall that a nontrivial even solution exists if and only if

$$\begin{aligned} \phi_1'\left(\frac{a}{2}\right) &= 0 \\ \text{or } \phi_1\left(\frac{a}{2}\right) &= 0, \end{aligned} \quad (26)$$

respectively, are periodic and semiperiodic, whereas an odd solution stands if and only if

$$\begin{aligned} \phi_2'\left(\frac{a}{2}\right) &= 0 \\ \text{or } \phi_2\left(\frac{a}{2}\right) &= 0, \end{aligned} \quad (27)$$

again for periodic and semiperiodic [9, Theorem 1.3.4]. Now, when $D = 2$ we have the nontrivial periodic solution $p_1(x_1)$ and let us define

$$p_{1+}(x_1) = \frac{p_1(x_1) + p_1(a - x_1)}{p_1(0) + p_1(a)}. \quad (28)$$

By the periodicity of p_1 , clearly p_{1+} is proportional to the even part of p_1 and it is periodic. Besides,

$$\begin{aligned} p_{1+}(0) &= p_{1+}(a), \\ p_{1+}'\left(\frac{a}{2}\right) &= 0, \end{aligned} \quad (29)$$

which are, respectively, (24) and the first of the conditions (26). With the choice for the denominator in (28), it is also $p_{1+}(0) = 1$ by which we conclude that $p_{1+} \equiv \phi_1$ extended in

FIGURE 4: Stability regions (shaded in gray) for Hill's equation (5) at $\kappa = 0.25$.

periodic fashion (it is also easy to show that no slope jump discontinuity is admitted by Hill's equation. Then, one can prove that $\phi_1'(0) = \phi_1'(a) = 0$ and $\phi_2'(0) = \pm\phi_2'(a)$, where the sign is given in the periodic and semiperiodic situation, resp.). Likewise, when $D = -2$, we have the nontrivial semiperiodic solution $P_1(x_1)$ and let us define

$$P_{1-}(x_1) = \frac{P_1(x_1) - P_1(a - x_1)}{P_1(0) - P_1(a)}, \quad (30)$$

which, in light of P_1 being semiperiodic, is again proportional to the *even* part of P_1 . It is the sum of two semiperiodic functions; $P_{1-}(x)$ is semiperiodic; besides

$$\begin{aligned} P_{1-}(0) &= -P_{1-}(a), \\ P_{1-}\left(\frac{a}{2}\right) &= 0, \end{aligned} \quad (31)$$

which are, respectively, (25) and the second of the conditions (27). With the choice for the denominator in (30), it is $P_{1-}(0) = 1$ and we conclude that $P_{1-} \equiv \phi_1$ extended in semiperiodic fashion. Through the analogous definitions of p_{1-} and p_{1+} , one gets the odd function ϕ_2 extended in periodic and semiperiodic fashion, respectively.

The role of κ is illustrated comparing Figure 2 with Figure 4. It is seen that the frequency axis roughly scales with κ . Besides, reducing κ brings wider BGs which tilt and tend to cluster together. It is noticed that no symmetry exists about $\alpha = 0.5$.

4. Dispersion Relation

Imposing the BCs ((9), (10)) gives the dispersion relation:

$$\frac{1}{2}D - \cos q = 0. \quad (32)$$

FIGURE 5: Dispersion relation for a string on a periodic support ($\kappa = 0.5$, $\alpha = 0.25$).

This relation can also be written as (see also [35])

$$\begin{aligned} & \frac{1}{2} [1 + f(\Omega)] \cos\left(\frac{(1-\alpha)\Omega - \alpha\sqrt{\Omega^2 - 1}}{\kappa}\right) \\ & + \frac{1}{2} (1 - f(\Omega)) \cos\left(\frac{(1-\alpha)\Omega + \alpha\sqrt{\Omega^2 - 1}}{\kappa}\right) \\ & - \cos q = 0 \end{aligned} \quad (33)$$

and it is plotted in Figure 5 for $\kappa = 0.5$ and $\alpha = 0.25$. Equation (32) conforms to the form of (4) in [19].

It is easy to prove that $f(\Omega)$ quickly asymptotes -1 from below for $\Omega > 1$, whence we can give a simple expression for the dispersion relation:

$$\frac{(1-\alpha)\Omega + \alpha\sqrt{\Omega^2 - 1}}{\kappa} = q + k\pi, \quad k \in \{0, 1, 2, \dots\}. \quad (34)$$

Such approximation is superposed onto the dispersion curves in Figure 8 whereupon it is seen that it is very effective already at Ω close to 1, although it is unable to capture the BGs.

The relevant bounding values for each band gap (BG) are obtained considering the periodic and the semiperiodic eigenvalue problems, for which two sets of BCs need to be considered, respectively:

- (1) the periodicity conditions $u(0) = u(a)$ and $u'(0) = u'(a)$;
- (2) the semiperiodicity conditions $u(0) = -u(a)$ and $u'(0) = -u'(a)$.

The relevant eigenfrequencies are denoted by $\lambda_0, \lambda_1, \dots$ and $\mu_0, \mu_1, \dots$, respectively, for the periodic and the semiperiodic eigenproblems. The eigenmode for the first periodic eigenfrequency λ_0 is shown in Figure 6 at $\kappa = 0.5$, $\alpha = 0.25$. Likewise,

FIGURE 6: The first eigenmode, relative to the eigenfrequency $\lambda_0 = 0.488445$ at $\kappa = 0.5$, $\alpha = 0.25$, and $q = 0$, is a periodic function.

FIGURE 7: The eigenmode at the lower end of the first BG, relative to the eigenfrequency $\mu_0 = 1.57858$ at $\kappa = 0.5$, $\alpha = 0.25$, and $q = \pi$, is a semiperiodic function.

the eigenmode for the first semiperiodic eigenfrequency μ_0 is shown in Figure 7. Such eigenfrequency is the lower boundary of the system's first BG.

The dispersion curve intersection with the axis $q = 0$ is given by λ_m while the intersection with the axis $q = \pi$ is given by μ_m , where $m \in \mathbb{N}$. BGs' size is obtained by $\mu_{2m} - \mu_{2m+1}$ and $\lambda_{2m+1} - \lambda_{2m+2}$.

5. Conclusions

In this paper, the Floquet theory of periodic coefficient ODEs is applied to describe the behavior of a mechanical waveguide. A homogeneous elastic string periodically supported by an elastic Winkler foundation is considered and it is found that the governing equation is given by a second-order Hill equation. Floquet-Bloch periodic boundary conditions are enforced. The stability regions together with the dispersion relation are found in terms of the Floquet theory through the discriminant. An asymptotic approximation to the lowest cutoff frequency is given. The fundamental eigensolutions of the periodic and of the semiperiodic problem are also determined. It is remarked that the present methodology can be extended to tackle functionally graded beams [36–39] or plates [40, 41]. Applications in the realm of civil engineering are also possible [28–32, 42]. A nice extension of the present theory could be related to temperature [43, 44] or viscoelastic effects in the fiber [45–48].

FIGURE 8: Dispersion relation (solid) superposed onto its first approximation (34) (dashed).

Competing Interests

The author declares that there are no competing interests.

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