Hindawi
Modelling and Simulation in Engineering
Volume 2019, Article ID 2562468, 5 pages
https://doi.org/10.1155/2019/2562468

Research Article

Modeling and Mathematical Analysis of Labor Force Evolution

Sanaa ElFadily [1] **and Abdelilah Kaddar** [2]

[1] *Mohammadia School of Engineering, Mohammed V University, Rabat, Morocco*
[2] *Finance, Entrepreneuriat et Développement, Faculté des Sciences Juridiques, Economiques et Sociales de Salé, Université Mohammed V de Rabat, Sala Al Jadida, Salé, Morocco*

Correspondence should be addressed to Abdelilah Kaddar; a.kaddar@yahoo.fr

Received 19 August 2018; Accepted 4 December 2018; Published 2 January 2019

Academic Editor: Michele Calì

In this paper, we propose a delayed differential system to model labor force (occupied labor force and unemployed) evolution. The mathematical analysis of our model focuses on the local behavior of the labor force around a positive equilibrium position; the existence of a branch of periodic solutions bifurcated from the positive equilibrium is then analyzed according to the Hopf bifurcation theorem. Finally, we performed numerical simulations to illustrate our theoretical results.

1. Introduction

Mathematical modeling has gained considerable importance in recent decades in all fields, particularly in the macroeconomic field. This branch of applied mathematics is interested in the representation, with enough loyalty, of the real phenomena by abstract mathematical objects (variables, parameters, and equations) accessible to analysis and calculation. Its main objective is the exploitation of the conditions of yesterday and/or today to predict those of tomorrow (for example, we can use ordinary differential equations or delayed differential equations).

This work is an opportunity to take stock of the evolution of the labor force over time to explain the reasons for the fluctuations. To do this, we propose delayed differential equations to model labor force evolution (knowing the past and the present, foresee the future).

The study of the labor force evolution (occupied labor force and unemployed) finds all its relevance in the economic and mathematical researches for two reasons, the first relating to the role of labor force in the national economy and the second linking up to the fluctuations in labor force, that occur at different times, with different durations and amplitudes [1–5]. These fluctuations are natural responses of population to the exogenous and/or endogenous causes: technological changes, school failure, vocational or technical training failure, or noncompatibility of the training with the labor market.

The article is organized as follows. In Section 2, we formulate our model of the labor force evolution. In Section 3, local stability of the positive equilibrium and existence of Hopf bifurcation are given. Numerical simulations are performed, in Section 4, to illustrate our theoretical results. Finally, a summary of the results and some suggestions for future research are presented in Section 5.

2. Modeling with Differential Equations

For the estimation of labor force evolution, one needs to know the employment rate γ, the rate of job loss σ, the mortality rate μ, the maximum population growth rate ρ, the carrying capacity N_c, and the time lag needed to contribute in the reproductive process of a new individual looking for a job, τ. Therefore, our labor force evolution model is schematized in Figure 1. In this scheme, the active population is divided into unemployed individuals (compartment U) and occupied labor force individuals (compartment L). The arrows specify the proportion of individuals passing from one compartment to another, and the arrows passing through the small circle indicate the process of reproduction that gives the number of new people looking for a job.

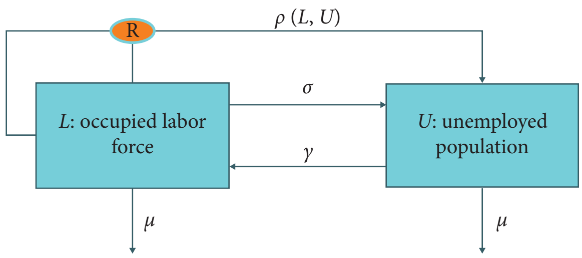

FIGURE 1: Scheme of the labor force evolution.

The temporal evolution of this scheme is governed by the following differential equations:

$$\begin{aligned} \frac{dL}{dt} &= \gamma U - (\sigma+\mu)L, \\ \frac{dU}{dt} &= \rho\left(1-\frac{L_\tau+U_\tau}{N_c}\right)L_\tau + \sigma L - (\mu+\gamma)U, \end{aligned} \tag{1}$$

where the variable L and U can be interpreted as the number of occupied labor force and the number of unemployed, respectively.

The initial conditions for system (1) are

$$\begin{aligned} &L(0) > 0, \\ &U(0) > 0, \\ &(L(\theta), U(\theta)) = (\varphi_1(\theta), \varphi_2(\theta)), \quad \forall\theta \in [-\tau, 0], \end{aligned} \tag{2}$$

where $\varphi_i \in C([-\tau, 0], \mathbb{R}^+)$, $i = 1, 2$.

The first equation models the evolution of occupied labor force population. In this equation, γ is the employment rate. Thus, the term γU can be interpreted as the number of new employees among people looking for a job. The term σL corresponds to the number of employees who have lost their job, and the term μL counts the number of deaths in the occupied labor force compartment. The second equation models the evolution of unemployed population. In this equation, the function $\rho(1-(L+U/N_c))L$ indicates the number of new people looking for a job. It reflects both maximum growth of job seekers when the unemployment rate is very low and very low labor force growth when the unemployment rate is very high. In other words, population growth is limited by the sharp increase in the number of unemployed.

3. Equilibria, Local Stability, and Existence of Hopf Bifurcation

3.1. Equilibria. In the following proposition, we prove the existence of the two equilibria.

Proposition 1. *System (1) has the following equilibria.*

(i) *If $(\gamma\rho/\mu(\gamma+\sigma+\mu)) \le 1$, then system (1) has a unique equilibrium $E_0 = (0, 0)$.*

(ii) *If $(\gamma\rho/\mu(\gamma+\sigma+\mu)) > 1$, then system (1) has two equilibria: the trivial equilibrium $E_0 = (0, 0)$ and the positive equilibrium $E_* = (L_*, U_*)$, where $U_* = ((\sigma+\mu)L_*/\gamma)$ and $L_* = ((\gamma/\gamma+\sigma+\mu) - (\mu/\rho))N_c$.*

Proof. To find equilibria, we consider the following system:

$$\begin{aligned} &\gamma U - (\sigma+\mu)L = 0, \\ &\rho\left(1-\frac{L+U}{N_c}\right)L + \sigma L - (\gamma+\mu)U = 0. \end{aligned} \tag{3}$$

It is easy to verify that $(0, 0)$ is a solution of the system (3), thus $E_0 = (0, 0)$ is a trivial equilibrium of the system (1). In the following, we prove that E_* is the unique positive equilibrium of the system (1).

(L, U) is a positive solution of the system (3) if and only if

$$\begin{aligned} U &= \frac{(\sigma+\mu)L}{\gamma}, \\ \rho\left(1-\frac{\gamma+\sigma+\mu}{\gamma}\frac{L}{N_c}\right) &- \frac{(\gamma+\sigma+\mu)\mu}{\gamma} = 0. \end{aligned} \tag{4}$$

It is clear that if $(\gamma\rho/\mu(\gamma+\sigma+\mu)) > 1$, then the second equation of (4) has a unique positive solution $L_* = ((\gamma/\gamma+\sigma+\mu) - (\mu/\rho))N_c$. This concludes the proof. □

3.2. Stability and Hopf Bifurcation. By analyzing the characteristic equation associated to (1), we prove the existence of Hopf bifurcation around the equilibrium E_*.

Consider the change of variables $x = L - L_*$ and $y = U - U_*$. Then by linearizing system (1) around (L_*, U_*), we have

$$\begin{aligned} \frac{dx}{dt} &= \gamma y - (\sigma+\mu)x, \\ \frac{dy}{dt} &= \sigma x + \left(\frac{\mu(\gamma+\sigma+\mu)}{\gamma} - \frac{\rho L_*}{N_c}\right)x_\tau - (\gamma+\mu)y - \frac{\rho L_*}{N_c}y_\tau. \end{aligned} \tag{5}$$

The characteristic equation associated to system (5) takes the following form:

$$\lambda^2 + a\lambda + b\lambda e^{-\lambda\tau} + c + de^{-\lambda\tau} = 0, \tag{6}$$

where

$$\begin{aligned} a &= \gamma+\sigma+2\mu, \\ b &= \frac{\rho L_*}{N_c}, \\ c &= \mu(\gamma+\sigma+\mu), \\ d &= \rho(\gamma+\sigma+\mu)\frac{L_*}{N_c} - \mu(\gamma+\sigma+\mu). \end{aligned} \tag{7}$$

To give our results, we recall the Routh–Hurwitz criterion and the result of Kuang [6] for the characteristic equations of the second degree.

Lemma 1. *For a second-degree polynomial, $s^2 + a_1 s + a_0$, all the roots are in the left half plane if and only if both coefficients satisfy $a_0 > 0$ and $a_1 > 0$.*

Lemma 2. *Assume $c + d \neq 0$, and $a^2 + b^2 + d^2 \neq 0$.*

(1) *If there is no such imaginary solution of the characteristic equation (6), then the stability of the equilibrium does not change for any $\tau \geq 0$.*

(2) *If there is only one imaginary solution with positive imaginary part of the characteristic equation (6), an unstable equilibrium never becomes stable for any $\tau \geq 0$. If the equilibrium is asymptotically stable for $\tau = 0$, then there exists a critical value, τ_0 of time delay, such that the equilibrium is uniformly asymptotically stable for $\tau \leq \tau_0$, and it becomes unstable for $\tau > \tau_0$.*

(3) *If there are two imaginary roots with positive imaginary part, $i\omega_+$ and $i\omega_-$, such that $\omega_+ > \omega_- > 0$, then the stability of the equilibrium can change a finite number of time at most as τ is increased, and eventually it becomes unstable.*

According to Lemma 1 (Routh–Hurwitz criterion for a second-degree polynomial) and Lemma 2, we have the following results.

Proposition 2. *For $\tau = 0$, the equilibrium (L_*, U_*) is locally asymptotically stable.*

Proof. For $\tau = 0$, the characteristic equation (6) reads as

$$\lambda^2 + \left(\gamma + \sigma + 2\mu + \frac{\rho L_*}{N_c}\right)\lambda + \rho(\gamma + \sigma + \mu)\frac{L_*}{N_c} = 0. \quad (8)$$

Since $\gamma + \sigma + 2\mu + (\rho L_*/N_c) > 0$ and $\rho(\gamma + \sigma + \mu)(L_*/N_c) > 0$, then, according to Lemma 1, all the roots of the characteristic equation (6) are in the left half plane, and hence, the equilibrium (L_*, U_*) is locally asymptotically stable. □

Proposition 3. *If $(\gamma\rho/\mu(\gamma + \sigma + \mu)) > 1$ and μ is close enough to zero, then there exists $\tau_0 > 0$ such that*

(i) *for $\tau \in [0, \tau_0)$, the steady state (L_*, U_*) is locally asymptotically stable;*

(ii) *for $\tau > \tau_0$, (L_*, U_*) is unstable;*

(iii) *for $\tau = \tau_0$, equation (6) has a pair of purely imaginary roots $\pm i\omega_0$, $\omega_0 > 0$,*

where

$$\tau_0 = \frac{1}{\omega_0}\arccos\left(\frac{-ab\omega_0^2 - d(c - \omega_0^2)}{b\omega_0^2 + d^2}\right),$$

$$\omega_0 = \frac{1}{2}\left\{\left(b^2 - 2c - a^2\right) + \sqrt{(b^2 - 2c - a^2)^2 - 4(c^2 - d^2)}\right\}. \quad (9)$$

Proof. Assume that $(\gamma\rho/\mu(\gamma + \sigma + \mu)) > 1$. We calculate

$$c^2 - d^2 = [\rho\gamma - \mu(\gamma + \sigma + \mu)][3\mu(\gamma + \sigma + \mu) - \rho\gamma]. \quad (10)$$

Moreover, we have for $\mu = 0$,

$$c^2 - d^2 = -\rho\gamma. \quad (11)$$

Then by continuity, there exists $\mu_0 > 0$ such that for all $\mu < \mu_0$, then $c^2 - d^2 < 0$. Hence the characteristic equation (6) has only one imaginary solution $i\omega_0 = (1/2)\{(b^2 - 2c - a^2) + \sqrt{(b^2 - 2c - a^2)^2 - 4(c^2 - d^2)}\}i$, with positive imaginary part. Applying Lemma 3.3, we conclude that the statement of our proposition is true. □

Proposition 4. *If $(\gamma\rho/\mu(\gamma + \sigma + \mu)) \geq 3$, then there exists $\tau_0 > 0$ such that*

(i) *for $\tau \in [0, \tau_0)$, the steady state (L_*, U_*) is locally asymptotically stable;*

(ii) *for $\tau > \tau_0$, (L_*, U_*) is unstable;*

(iii) *for $\tau = \tau_0$, equation (6) has a pair of purely imaginary roots $\pm i\omega_0$, $\omega_0 > 0$,*

where

$$\tau_0 = \frac{1}{\omega_0}\arccos\left(\frac{-ab\omega_0^2 - d(c - \omega_0^2)}{b\omega_0^2 + d^2}\right),$$

$$\omega_0 = \frac{1}{2}\left\{\left(b^2 - 2c - a^2\right) + \sqrt{(b^2 - 2c - a^2)^2 - 4(c^2 - d^2)}\right\}. \quad (12)$$

Proof. If $(\gamma\rho/\mu(\gamma + \sigma + \mu)) \geq 3$, then

$$c^2 - d^2 = \mu^2(\gamma + \sigma + \mu)^2\left[\frac{\gamma\rho}{\mu(\gamma + \sigma + \mu)} - 1\right] \cdot \left[3 - \frac{\gamma\rho}{\mu(\gamma + \sigma + \mu)}\right] < 0, \quad (13)$$

hence, the characteristic equation (6) has only one imaginary solution $i\omega_0 = (1/2)\{(b^2 - 2c - a^2) + \sqrt{(b^2 - 2c - a^2)^2 - 4(c^2 - d^2)}\}i$, with positive imaginary part. Applying Lemma 3.3, we conclude that the statement of our proposition is true.

Next, we establish sufficient conditions for the local existence of Hopf bifurcation at the positive equilibrium. □

Theorem 1. *Suppose one of the following situations:*

(H_1): $\gamma\rho/\mu(\gamma + \sigma + \mu) > 1$ and μ is close enough to zero;

(H_2): $\gamma\rho/\mu(\gamma + \sigma + \mu) \geq 3$.

Then, a Hopf bifurcation of periodic solutions of system (1) occurs at (L_*, U_*) when $\tau = \tau_0$.

Proof. For the proof of this theorem, we verify the conditions of Hopf bifurcation theorem (see, for example [7]). From Proposition 4, the characteristic equation (6) has a pair of imaginary roots $\pm i\omega_0$ at $\tau = \tau_0$. It is easy to show that this root is simple.

For the transversality condition, we have

$$\operatorname{sign} \frac{d\operatorname{Re}(\lambda)}{d\tau}\bigg|\tau_0 = \operatorname{sign}\left\{b^2 - 2c - a^2 - 4\left(c^2 - d^2\right)\right\}. \quad (14)$$

After some calculations, we get

$$\begin{aligned} b^2 - 2c - a^2 - 4\left(c^2 - d^2\right) &= \frac{\gamma^2\rho^2\left(1-\mu^2\right) + \left[\gamma\rho\mu - (\gamma+\sigma+\mu)^2\right]^2}{(\gamma+\sigma+\mu)^2} \\ &\quad - 4\mu^2(\gamma+\sigma+\mu)^2\left[\frac{\gamma\rho}{\mu(\gamma+\sigma+\mu)} - 1\right] \\ &\quad \cdot \left[3 - \frac{\gamma\rho}{\mu(\gamma+\sigma+\mu)}\right]. \end{aligned} \quad (15)$$

If one of the two hypotheses (H_1) or (H_2) is verified, then $4\mu^2(\gamma+\sigma+\mu)^2[(\gamma\rho/\mu(\gamma+\sigma+\mu)) - 1][3 - (\gamma\rho/\mu(\gamma+\sigma+\mu))] < 0$. Moreover, we have $0 < \mu < 1$. Consequently,

$$\frac{d\operatorname{Re}(\lambda)}{d\tau}(\tau_0) > 0. \quad (16)$$

This concludes the proof.

From Theorem 1, the following form of Hopf bifurcation theorem can now be stated (see Theorem 2.7., page 291 in [8]). □

Theorem 2. *Under the assumptions of Theorem 1, there is a constant $\varepsilon_0 > 0$ such that, for each $0 \le \varepsilon < \varepsilon_0$, system (1) has a family of periodic solutions $p(\varepsilon)$ with period $T = T(\varepsilon)$, for the parameter values $\tau = \tau(\varepsilon)$ such that $p(0) = 0$, $T(0) = (2\pi/\omega_0)$, and $\tau(0) = \tau_0$, where τ_0 and ω_0 are stated in Proposition 4.*

4. Numerical Simulations

The following numerical simulations are given for system (1), for time series at the top and at the bottom in Figure 2, $\sigma = 0.1$, $\rho = 0.01054$, $\tau = 25$ years, $\gamma = 0.7$, $\mu = 0.000481$, $N_c = 11000000$ and $\sigma = 0.1$, $\rho = 0.01054$, $\tau = 25$ years, $\gamma = 0.2$, $\mu = 0.00481$, $N_c = 51000000$, respectively.

In view of Figure 2, we observe an oscillatory regime on the labor market: the number of occupied labor force and the number of unemployed oscillate and converge towards equilibrium. This shows that time delay, in the reproduction process, is a key parameter for the existence of a branch of bifurcated periodic solutions from the positive equilibrium and therefore of the existence of oscillations when the delay crosses certain critical values.

5. Conclusion

In this paper, we presented a study on the dynamics of a model of the evolution of the employed labor force and the unemployed population. We have shown the existence of a positive equilibrium E_*, and we have studied the analytic behavior around this point. Specially, we have shown the existence of a branch of periodic solutions bifurcated from the positive equilibrium E_*.

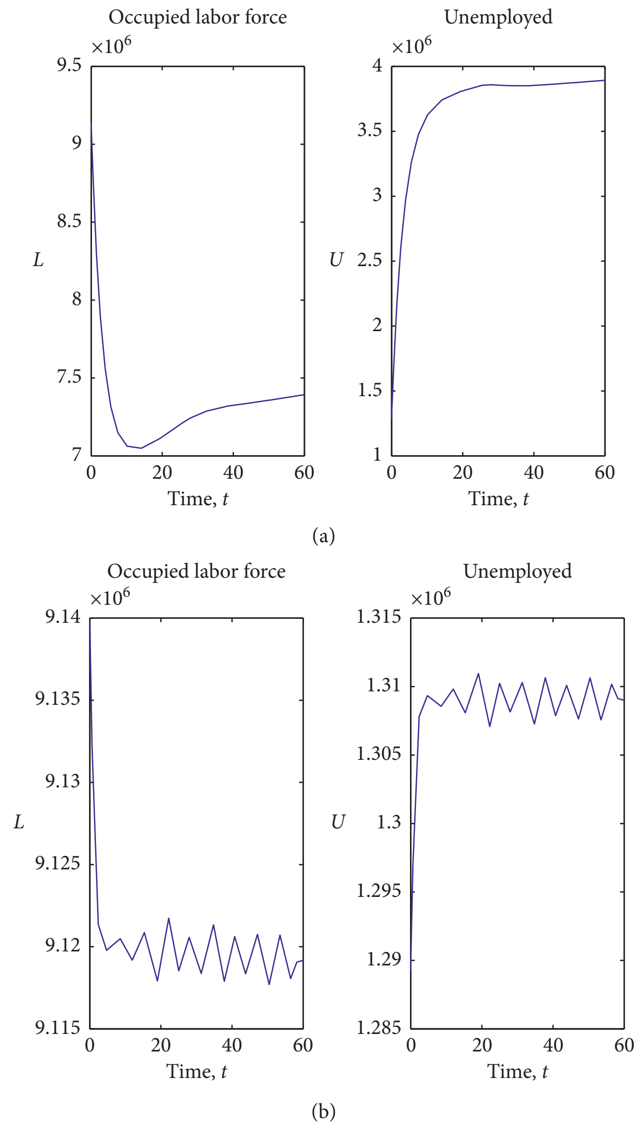

Figure 2: Examples of times series obtained by numerical integration of labor force evolution model (1).

The existence of the limit cycles, for certain positive values of the parameters of the model, supports the phenomenon of coexistence of the two populations: active occupied and unemployed. This coexistence can be explained by the fact that job seekers occur relatively quickly because of population growth.

As a result of this paper, we plan to study the direction of the bifurcating periodic solutions and resonant codimension-two bifurcation (see, for example [9]). Also, we plan to develop this work by introducing the economic growth effect. The resulting model determines the conditions of interaction between economic growth and labor market evolution.

Data Availability

The parameter values can be found at https://en.wikipedia.org/wiki/Demographics_of_Morocco.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] E. Schuss, "Between life cycle model, labor market integration and discrimination: An econometric analysis of the determinants of return migration," in *SOEPpapers on Multidisciplinary Panel Data Research, No. 881*, Deutsches Institut für Wirtschaftsforschung (DIW), Berlin, Germany, 2016.

[2] L. Stanila, M. Ecaterina Andreica, and A. Cristescu, "Econometric analysis of the employment rate for the E.U. Countries," *Procedia-Social and Behavioral Sciences*, vol. 109, pp. 178–182, 2014.

[3] G. Cristina Dimian, B. Ileanu, J. Jablonský, and J. Fábry, "Analysis of european labour market in the crisis context," *Prague Economic Papers*, vol. 22, no. 1, pp. 50–71, 2013.

[4] V. Lima and R. D. Paredes, "The dynamics of the labor markets in Chile," *Estudios de Economía [online]*, vol. 34, no. 2, pp. 163–183, 2007.

[5] A. Isserman, C. Taylor, S. Gerking, and U. Schubert, "Chapter 13 regional labor market analysis,"*Handbook of Regional and Urban Economics*, Vol. 1, pp. 543–580, Elsevier, Amsterdam, Netherlands, 1987.

[6] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Boston, MA, USA, 1993.

[7] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, NY, USA, 1993.

[8] O. Diekmann, S. Van Giles, S. Verduyn Lunel, and H. Walter, *Delay Equations*, Springer-Verlag, New-York, NY, USA, 1995.

[9] Q. Liu, X. Liao, Y. Liu, S. Zhou, and S. Guo, "Dynamics of an inertial two-neuron system with time delay," *Nonlinear Dynamics*, vol. 58, no. 3, pp. 573–609, 2009.