Research Article

Global Optimization for Solving Linear Multiplicative Programming Based on a New Linearization Method

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This paper presents a new global optimization algorithm for solving a class of linear multiplicative programming (LMP) problem. First, a new linear relaxation technique is proposed. Then, to improve the convergence speed of our algorithm, two pruning techniques are presented. Finally, a branch and bound algorithm is developed for solving the LMP problem. The convergence of this algorithm is proved, and some experiments are reported to illustrate the feasibility and efficiency of this algorithm.

1. Introduction

Consider the following linear multiplicative programming (LMP) problem:

LMP:

$$\begin{aligned} v = \min \quad & \phi(x) = \sum_{i=1}^p (c_i^T x + d_i) (e_i^T x + f_i) \\ \text{s.t.} \quad & x \in D = \{x \in R^n \mid Ax \leq b\}, \end{aligned} \quad (1)$$

where $p \geq 2$, $c_i^T = (c_{i1}, c_{i2}, \dots, c_{in})$, $e_i^T = (e_{i1}, e_{i2}, \dots, e_{in}) \in R^n$, $d_i, f_i \in R$, $i = 1, \dots, p$, $A = (a_{ij})_{m \times n} \in R^{m \times n}$ is a matrix, $b = (b_i)_{m \times 1} \in R^m$ is a vector, and $D \subseteq R^n$ is nonempty and bounded.

As a special case of nonconvex programming problem, the problem LMP has been paid more attention since the 1990s. There are two reasons. The first one is that, from a practical point of view, LMP problem appears in a wide variety of practical applications, such as financial optimization [1], data mining/pattern recognition [2], plant layout design [3], VLSI chip design [4], and robust optimization [5]. The second one is that, from a research point of view, LMP is N-hard; that is, it usually possesses multiple local optimal solutions that are not globally optimal. So, it is hard to find its global optimal solution, and it is necessary to put forward good methods.

In the past few decades, for all $x \in D$, under the assumption that $c_i^T + d_i > 0$, $e_i^T x + f_i > 0$, a number of practical algorithms have been proposed for globally solving problem LMP. These methods can be classified into parameterization based methods [6, 7], branch-and-bound methods [8–10], decomposition method [11], cutting plane method [12], and so on.

The purpose of this paper is to present an effective method for globally solving problem LMP. Compared with other algorithms, the main features of this algorithm are (1) by using the special structure of LMP, a new linear relaxation technique is presented, which can be used to construct the linear relaxation programming (LRP) problem; (2) two pruning techniques are presented, which can be used to improve the convergence speed of the proposed algorithm; (3) the problem investigated in this paper has a more general form than those in [6–12]; it does not require $c_i^T x + d_i > 0$ and $e_i^T x + f_i > 0$; (4) numerical results and comparison with methods [8, 13–22] show that our algorithm works as well as or better than those methods.

This paper is organized as follows. In Section 2, the new linear relaxation programming (LRP) problem for LMP problem is proposed, which provides a lower bound for the optimal value of LMP. In order to improve the convergence speed of our algorithm, two pruning techniques are presented in Section 3. In Section 4, the global optimization algorithm

is given, and the convergence of the algorithm is proved. Numerical experiments are carried out to show the feasibility and efficiency of our algorithm in Section 5.

2. Linear Relaxation Programming (LRP)

To solve problem LMP, the principal task is the construction of lower bound for this problem and its partitioned subproblems. A lower bound of LMP problem and its partitioned subproblems can be obtained by solving a linear relaxation programming problem. For generating this linear relaxation, the strategy proposed by this paper is to underestimate the objective function $\phi(x)$ with a linear function. All the details of this procedure will be given in the following.

First, we solve $2n$ linear programming problems: $l_j^0 = \min_{x \in D} x_j$, $u_j^0 = \max_{x \in D} x_j$, $j = 1, \dots, n$, and construct a rectangle $H^0 = \{x \in R^n \mid l_j^0 \leq x_j \leq u_j^0, j = 1, \dots, n\}$. Then, the LMP problem can be rewritten as the following form:

LMP:

$$\begin{aligned} v = \min \quad & \phi(x) = \sum_{i=1}^p (c_i^T x + d_i) (e_i^T x + f_i) \\ \text{s.t.} \quad & x \in D = \{x \in R^n \mid Ax \leq b\}, \\ & H^0 = \{x \mid l_j^0 \leq x_j \leq u_j^0, j = 1, \dots, n\}. \end{aligned} \quad (2)$$

Let $H = [l, u]$ be the initial rectangle H^0 or some subrectangle of H^0 that is generated by the proposed algorithm. Next, we will show how to construct the linear relaxation programming problem LRP for LMP.

Towards this end, for $i = 1, \dots, p$, compute

$$\begin{aligned} \xi_i &= \sum_{j=1}^n \min \{c_{ij} l_j, c_{ij} u_j\} + d_i, \\ \bar{\xi}_i &= \sum_{j=1}^n \max \{c_{ij} l_j, c_{ij} u_j\} + d_i, \\ \underline{\eta}_i &= \sum_{j=1}^n \min \{e_{ij} l_j, e_{ij} u_j\} + f_i, \\ \bar{\eta}_i &= \sum_{j=1}^n \max \{e_{ij} l_j, e_{ij} u_j\} + f_i, \end{aligned} \quad (3)$$

and consider the product term $(c_i^T x + d_i)(e_i^T x + f_i)$ in $\phi(x)$.

Since $c_i^T x + d_i - \xi_i \geq 0$, $e_i^T x + f_i - \underline{\eta}_i \geq 0$, we have

$$(c_i^T x + d_i - \xi_i)(e_i^T x + f_i - \underline{\eta}_i) \geq 0; \quad (4)$$

that is,

$$\begin{aligned} (c_i^T x + d_i)(e_i^T x + f_i) - \underline{\eta}_i (c_i^T x + d_i) - \xi_i (e_i^T x + f_i) \\ + \xi_i \underline{\eta}_i \geq 0. \end{aligned} \quad (5)$$

Furthermore, we have

$$\begin{aligned} (c_i^T x + d_i)(e_i^T x + f_i) \geq \underline{\eta}_i (c_i^T x + d_i) + \xi_i (e_i^T x + f_i) \\ - \xi_i \underline{\eta}_i. \end{aligned} \quad (6)$$

In addition, since $c_i^T x + d_i - \xi_i \geq 0$, $e_i^T x + f_i - \bar{\eta}_i \leq 0$, we have

$$(c_i^T x + d_i - \xi_i)(e_i^T x + f_i - \bar{\eta}_i) \leq 0. \quad (7)$$

Furthermore, we can obtain

$$\begin{aligned} (c_i^T x + d_i)(e_i^T x + f_i) \leq \bar{\eta}_i (c_i^T x + d_i) + \xi_i (e_i^T x + f_i) \\ - \xi_i \bar{\eta}_i. \end{aligned} \quad (8)$$

From (6) and (8), we have the following relations:

$$\begin{aligned} \phi(x) &= \sum_{i=1}^p (c_i^T x + d_i)(e_i^T x + f_i) \\ &\geq \sum_{i=1}^p [\underline{\eta}_i (c_i^T x + d_i) + \xi_i (e_i^T x + f_i) - \xi_i \underline{\eta}_i] \\ &= \phi^l(x), \\ \phi(x) &= \sum_{i=1}^p (c_i^T x + d_i)(e_i^T x + f_i) \\ &\leq \sum_{i=1}^p [\bar{\eta}_i (c_i^T x + d_i) + \xi_i (e_i^T x + f_i) - \xi_i \bar{\eta}_i] \\ &= \phi^u(x). \end{aligned} \quad (9)$$

Based on the above discussion, the linear relaxation programming (LRP) problem can be established as follows, which provides a lower bound for the optimal value of LMP problem over H :

LRP:

$$\begin{aligned} \min \quad & \phi^l(x) \\ \text{s.t.} \quad & Ax \leq b, \\ & x \in H. \end{aligned} \quad (10)$$

Theorem 1. For all $x \in H$, let $\Delta x = u - l$ and consider the functions $\phi^l(x)$, $\phi(x)$, and $\phi^u(x)$. Then, one has $\lim_{\Delta x \rightarrow 0} (\phi(x) - \phi^l(x)) = \lim_{\Delta x \rightarrow 0} (\phi^u(x) - \phi(x)) = 0$.

Proof. We first prove $\lim_{\Delta x \rightarrow 0} (\phi(x) - \phi^l(x)) = 0$. By the definitions $\phi(x)$ and $\phi^l(x)$, we have

$$\begin{aligned} |\phi(x) - \phi^l(x)| &= \left| \sum_{i=1}^p (c_i^T x + d_i)(e_i^T x + f_i) \right. \\ &\quad \left. - \sum_{i=1}^p [\underline{\eta}_i (c_i^T x + d_i) + \xi_i (e_i^T x + f_i) - \xi_i \underline{\eta}_i] \right| \end{aligned}$$

$$\begin{aligned}
& \leq \left| \sum_{i=1}^p \left[(c_i^T x + d_i) (e_i^T x + f_i) - \underline{\eta}_i (c_i^T x + d_i) \right] \right| \\
& + \left| \sum_{i=1}^p \left[\xi_i (e_i^T x + f_i) - \underline{\xi}_i \underline{\eta}_i \right] \right| \\
& \leq \sum_{i=1}^p |c_i^T x + d_i| \left| (e_i^T x + f_i) - \underline{\eta}_i \right| \\
& + \sum_{i=1}^p |\xi_i| \left| e_i^T x + f_i - \underline{\eta}_i \right|. \tag{11}
\end{aligned}$$

Since D is nonempty and bounded, there exists M_i such that $M_i = \max_{x \in D} |c_i^T x + d_i|$. From the above inequality, we have

$$|\phi(x) - \phi^l(x)| \leq \sum_{i=1}^p M_i |\bar{\eta}_i - \underline{\eta}_i| + \sum_{i=1}^p |\xi_i| |\bar{\eta}_i - \underline{\eta}_i|. \tag{12}$$

By the definitions of $\underline{\eta}_i$ and $\bar{\eta}_i$, we know that $\Delta s = \bar{\eta}_i - \underline{\eta}_i \rightarrow 0$ as $\Delta x \rightarrow 0$. Combining (12), we have $\lim_{\Delta x \rightarrow 0} (\phi(x) - \phi^l(x)) = 0$.

Similarly, we can prove $\lim_{\Delta x \rightarrow 0} (\phi^u(x) - \phi(x)) = 0$, and the proof is complete. $\square$

Theorem 1 implies that $\phi^l(x)$ and $\phi^u(x)$ will approximate the function $\phi(x)$ as $\Delta x \rightarrow 0$.

3. Pruning Technique

To improve the convergence speed of this algorithm, we present two pruning techniques, which can be used to eliminate the region in which the global optimal solution of LMP problem does not exist.

Assume that UB and LB are the current known upper bound and lower bound of the optimal value v of the problem LMP. Let

$$\alpha_j = \sum_{i=1}^p (\underline{\eta}_i c_{ij} + \xi_i e_{ij}), \quad j = 1, \dots, n,$$

$$\Lambda_1 = \sum_{i=1}^p (\underline{\eta}_i d_i + \xi_i f_i - \underline{\eta}_i \underline{\xi}_i),$$

$$\gamma_k = \text{UB} - \sum_{j=1, j \neq k}^n \min \{ \alpha_j l_j, \alpha_j u_j \} - \Lambda_1,$$

$$k = 1, \dots, n,$$

$$\beta_j = \sum_{i=1}^p (\bar{\eta}_i c_{ij} + \xi_i e_{ij}), \quad j = 1, \dots, n,$$

$$\Lambda_2 = \sum_{i=1}^p (\bar{\eta}_i d_i + \xi_i f_i - \bar{\eta}_i \bar{\xi}_i),$$

$$\rho_k = \text{LB} - \sum_{j=1, j \neq k}^n \max \{ \beta_j l_j, \beta_j u_j \} - \Lambda_2,$$

$$k = 1, \dots, n. \tag{13}$$

The pruning techniques are derived as in the following theorems.

Theorem 2. For any subrectangle $H \subseteq H^0$ with $H_j = [l_j, u_j]$, if there exists some index $k \in \{1, 2, \dots, n\}$ such that $\alpha_k > 0$ and $\gamma_k < \alpha_k u_k$, then there is no globally optimal solution of LMP problem over H^1 ; if $\alpha_k < 0$ and $\gamma_k < \alpha_k l_k$, for some k , then there is no globally optimal solution of LMP problem over H^2 , where

$$H^1 = (H_j^1)_{n \times 1} \subseteq H,$$

$$\text{with } H_j^1 = \begin{cases} H_j, & j \neq k, \\ \left(\frac{\gamma_k}{\alpha_k}, u_k \right] \cap H_k, & j = k, \end{cases} \tag{14}$$

$$H^2 = (H_j^2)_{n \times 1} \subseteq H,$$

$$\text{with } H_j^2 = \begin{cases} H_j, & j \neq k, \\ \left[l_k, \frac{\gamma_k}{\alpha_k} \right) \cap H_k, & j = k. \end{cases}$$

Proof. For all $x \in H^1$, we first show that $\phi(x) > \text{UB}$. Consider the k th component x_k of x . Since $x_k \in (\gamma_k/\alpha_k, u_k]$, we can obtain that

$$\frac{\gamma_k}{\alpha_k} < x_k \leq u_k. \tag{15}$$

From $\alpha_k > 0$, we have $\gamma_k < \alpha_k x_k$. For all $x \in H^1$, by the above inequality and the definition of γ_k , it implies that

$$\text{UB} - \sum_{j=1, j \neq k}^n \min \{ \alpha_j l_j, \alpha_j u_j \} - \Lambda_1 < \alpha_k x_k; \tag{16}$$

that is,

$$\begin{aligned}
\text{UB} & < \sum_{j=1, j \neq k}^n \min \{ \alpha_j l_j, \alpha_j u_j \} + \alpha_k x_k + \Lambda_1 \\
& \leq \sum_{j=1}^n \alpha_j x_j + \Lambda_1 = \phi^l(x).
\end{aligned} \tag{17}$$

Thus, for all $x \in H^1$, we have $\phi(x) \geq \phi^l(x) > \text{UB} \geq v$; that is, for all $x \in H^1$, $\phi(x)$ is always greater than the optimal value v of the problem LMP. Therefore, there cannot exist globally optimal solution of LMP problem over H^1 .

Similarly, for all $x \in H^2$, if there exists some k such that $\alpha_k < 0$ and $\gamma_k < \alpha_k l_k$, it can be derived that there is no globally optimal solution of LMP problem over H^2 . $\square$

Theorem 3. For any subrectangle $H \subseteq H^0$ with $H_j = [l_j, u_j]$, if there exists some index $k \in \{1, 2, \dots, n\}$ such that $\beta_k > 0$ and $\rho_k > \beta_k l_k$, then there is no globally optimal solution of LMP problem over H^3 ; if $\beta_k < 0$ and $\rho_k > \beta_k u_k$, for some k , then there is no globally optimal solution of LMP problem over H^4 , where

$$H^3 = (H_j^3)_{n \times 1} \subseteq H,$$

$$\text{with } H_j^3 = \begin{cases} H_j, & j \neq k, \\ \left[l_k, \frac{\rho_k}{\beta_k} \right) \cap H_k, & j = k, \end{cases} \quad (18)$$

$$H^4 = (H_j^4)_{n \times 1} \subseteq H,$$

$$\text{with } H_j^4 = \begin{cases} H_j, & j \neq k, \\ \left(\frac{\rho_k}{\beta_k}, u_k \right] \cap H_k, & j = k. \end{cases}$$

Proof. First, we show that, for all $x \in H^3$, $\phi(x) < \text{LB}$. Consider the k th component x_k of x . By the assumption and the definitions of β_k and ρ_k , we have

$$l_k \leq x_k < \frac{\rho_k}{\beta_k}. \quad (19)$$

Note that since $\beta_k > 0$, we have $\rho_k > \beta_k x_k$. For all $x \in H^3$, by the above inequality and the definition of ρ_k , it implies that

$$\begin{aligned} \text{LB} &> \sum_{j=1, j \neq k}^n \max\{\beta_j l_j, \beta_j u_j\} + \beta_k x_k + \Lambda_2 \\ &\geq \sum_{j=1}^n \beta_j x_j + \Lambda_2 = \Phi^u(x) \geq \phi(x). \end{aligned} \quad (20)$$

Thus, for all $x \in H^3$, we have $v \geq \text{LB} > \phi(x)$. Therefore, there cannot exist globally optimal solution of LMP problem over H^3 .

For all $x \in H^4$, if there exists some k such that $\beta_k < 0$ and $\rho_k > \beta_k u_k$, from arguments similar to the above, it can be derived that there is no globally optimal solution of LMP problem over H^4 . $\square$

4. Algorithm and Its Convergence

Based on the previous results, this section presents the branch and bound algorithm and gives its convergence.

4.1. Branching Rule. In branch and bound algorithm, branch rule is a critical element in guaranteeing convergence. This paper chooses a simple and standard bisection rule, which is sufficient to ensure convergence since it drives the intervals shrinking to a singleton for all the variables along any infinite branch of the branch and bound tree.

Consider any node subproblem identified by rectangle $H = \{x \in R^n \mid l_j \leq x_j \leq u_j, j = 1, \dots, n\} \subseteq H^0$. The branching rule is as follows:

- (i) let $k = \text{argmax}\{u_j - l_j \mid j = 1, \dots, n\}$;
- (ii) let $\tau = (l_k + u_k)/2$;
- (iii) let

$$\begin{aligned} H^1 &= \{x \in R^n \mid l_j \leq x_j \leq u_j, j \neq k, l_k \leq x_k \leq \tau\}, \\ H^2 &= \{x \in R^n \mid l_j \leq x_j \leq u_j, j \neq k, \tau \leq x_k \leq u_k\}. \end{aligned} \quad (21)$$

Through using this branching rule, the rectangle H is partitioned into two subrectangles H^1 and H^2 .

4.2. Branch and Bound Algorithm. From the above discussion, the branch and bound algorithm for globally solving LMP problem is summarized as follows.

Let $\text{LB}(H^k)$ be the optimal function value of LRP over the subrectangle $H = H^k$ and $x^k = x(H^k)$ be an element of the corresponding argmin.

Algorithm Statement

Step 1. Choose $\epsilon \geq 0$. Find an optimal solution $x^0 = x(H^0)$ and the optimal value $\text{LB}(H^0)$ for problem LRP with $H = H^0$. Set $\text{LB}_0 = \text{LB}(H^0)$, and $\text{UB}_0 = \phi(x^0)$. If $\text{UB}_0 - \text{LB}_0 \leq \epsilon$, then stop: x^0 is an ϵ -optimal solution of problem LMP. Otherwise, set $Q_0 = \{H^0\}$, $k = 1$, and go to Step 2.

Step 2. Set $\text{UB}_k = \text{UB}_{k-1}$. Subdivide H^{k-1} into two subrectangles via the branching rule, and denote the set of new partition rectangles as $\bar{H}^k$.

Step 3. For each new rectangle $H \in \bar{H}^k$, utilizing the pruning techniques of Theorems 2 and 3 to prune rectangle H . For $i = 1, \dots, m$, if there exists some i such that $\sum_{j=1}^n \min\{a_{ij} l_j, a_{ij} u_j\} \geq b_i$ over rectangle H , then remove the rectangle H from $\bar{H}^k$; that is, $\bar{H}^k = \bar{H}^k \setminus H$.

Step 4. If $\bar{H}^k \neq \emptyset$, solve LRP to obtain $\text{LB}(H)$ and $x(H)$ for each $H \in \bar{H}^k$. If $\text{LB}(H) > \text{UB}_k$, set $\bar{H}^k = \bar{H}^k \setminus H$. Otherwise, let $\text{UB}_k = \min\{\text{UB}_k, \phi(x(H))\}$. If $\text{UB}_k = \phi(x(H))$, set $x^k = x(H)$.

Step 5. Set

$$Q_k = \{Q_{k-1} \setminus H^{k-1}\} \cup \bar{H}^k. \quad (22)$$

Step 6. Set $\text{LB}_k = \min\{\text{LB}(H) \mid H \in Q_k\}$. Let H^k be the subrectangle which satisfies that $\text{LB}_k = \text{LB}(H^k)$. If $\text{UB}_k - \text{LB}_k \leq \epsilon$, then stop: x^k is a global ϵ -optimal solution of problem LMP. Otherwise, set $k = k + 1$, and go to Step 2.

4.3. Convergence Analysis. In this subsection, the convergence properties of the algorithm are given.

Theorem 4. The algorithm either terminates finitely with globally ϵ -optimal solution or generates an infinite sequence

TABLE 1: Comparison results of Examples 1–10.

Example (p, m, n)	Methods	Time (s)	Iter	Optimal solution	Optimal value
1 (min) (1, 8, 2)	[13]	5.0780	48	(2.0, 8.0)	10
	[14]	0.3	53	(2.0, 8.0)	10
	[15]	10.83	27	(2.0003, 7.9999)	10.0042
	Ours	0.062	1	(2.0, 8.0)	10
2 (min) (2, 6, 2)	[13]	0.2030	2	(0.0, 4.0)	3
	[16]	—	3	(0.0, 4.0)	3
	Ours	0.086	1	(0.0, 4.0)	3
3 (min) (1, 12, 4)	[13]	0.1880	1	(1.3148, 0.1396, 0.0, 0.4233)	0.8902
	[17]	—	6	(1.3148, 0.1396, 0.0, 0.4233)	0.8902
	Ours	0.093	1	(1.3148, 0.1396, 0.0, 0.4233)	0.8902
4 (min) (2, 5, 2)	[8]	—	—	(0.0, 4.0)	3
	[18]	—	—	(0.0, 4.0)	3
	Ours	0.0842	1	(0.0, 4.0)	3
5 (min) (2, 4, 2)	[19]	—	—	(1.0, 3.0)	−13
	Ours	0.0150	1	(1.0, 3.0)	−13
6 (min) (2, 4, 2)	[19]	—	—	(1.0, 4.0)	−22
	Ours	0.0160	1	(1.0, 4.0)	−22
7 (min) (3, 7, 3)	[20]	—	57	(5.5556, 1.7778, 2.6667)	−112.754
	Ours	1.5930	34	(5.5556, 1.7778, 2.6667)	−112.7531
8 (max) (2, 4, 2)	[21]	—	13	(0.74984, 0.74984)	−38.87628
	Ours	0.5	16	(0.75, 0.75)	−38.8750
9 (min) (2, 6, 2)	[22]	—	7	(1.547, 2.421)	−16.2837
	Ours	0.659	23	(1.5480, 2.4152)	−16.2893
10 (min) (2, 6, 2)	[22]	—	29	(1.5549, 0.7561)	10.6756
	Ours	0.7662	27	(1.5568, 0.7545)	10.6753

$\{x^k\}$, where any accumulation point is a globally optimal solution of LMP.

Proof. If the algorithm terminates finitely, without loss of generality, assume that the algorithm terminates at the k th step; by the algorithm, we have

$$UB_k - LB_k \leq \varepsilon. \quad (23)$$

So, x^k is a global optimal solution of the problem LMP.

If the algorithm is infinite, then an infinite sequence $\{x^k\}$ will be generated. Since the feasible region of LMP is bounded, the sequence $\{x^k\}$ must have a convergence subsequence. Without loss of generality, set $\lim_{k \rightarrow \infty} x^k = x^*$. By the algorithm, we have

$$\lim_{k \rightarrow \infty} LB_k \leq \nu. \quad (24)$$

Since x^* is a feasible solution of problem LMP, $\nu \leq \Phi(x^*)$. Taken together, the following relation holds:

$$\lim_{k \rightarrow \infty} LB_k \leq \nu \leq \Phi(x^*). \quad (25)$$

On the other hand, by the algorithm and the continuity of $\phi^l(x)$, we have

$$\lim_{k \rightarrow \infty} LB_k = \lim_{k \rightarrow \infty} \phi^l(x^k) = \phi^l(x^*). \quad (26)$$

By Theorem 1, we can obtain

$$\phi(x^*) = \phi^l(x^*). \quad (27)$$

Therefore, $\nu = \phi(x^*)$; that is x^* is a global optimal solution of problem LMP. $\square$

5. Numerical Experiments

To verify the performance of the proposed algorithm, some numerical experiments are carried out and compared with some other methods [8, 13–22]. The algorithm is compiled with Matlab 7.1 on a Pentium IV (3.06 GHZ) microcomputer. The simplex method is applied to solve the linear relaxation programming problems. In our experiments, for Examples 1–10, the convergence tolerance ϵ is 10^{-6} ; for Example 11, the convergence tolerance ϵ is 10^{-2} .

The results of problems 1–10 are summarized in Table 1, where the following notations have been used in row headers: Iter is the number of algorithm iterations; Time (s) is

TABLE 2: Comparison results of Algorithms 1 and 2 for Examples 1–10.

Example (p, m, n)	Methods	Time (s)	Iter	Optimal solution	Optimal value
1 (min)	Algorithm 1	0.062	1	(2.0, 8.0)	10
(1, 8, 2)	Algorithm 2	0.062	1	(2.0, 8.0)	10
2 (min)	Algorithm 1	0.086	1	(0.0, 4.0)	3
(2, 6, 2)	Algorithm 2	0.086	1	(0.0, 4.0)	3
3 (min)	Algorithm 1	0.093	1	(1.3148, 0.1396, 0.0, 0.4233)	0.8902
(1, 12, 4)	Algorithm 2	0.093	1	(1.3148, 0.1396, 0.0, 0.4233)	0.8902
4 (min)	Algorithm 1	0.0842	1	(0.0, 4.0)	3
(2, 5, 2)	Algorithm 2	0.0842	1	(0.0, 4.0)	3
5 (min)	Algorithm 1	0.0150	1	(1.0, 3.0)	−13
(2, 4, 2)	Algorithm 2	0.0150	1	(1.0, 3.0)	−13
6 (min)	Algorithm 1	0.0160	1	(1.0, 4.0)	−22
(2, 4, 2)	Algorithm 2	0.0160	1	(1.0, 4.0)	−22
7 (min)	Algorithm 1	1.5930	34	(5.5556, 1.7778, 2.6667)	−112.7531
(3, 7, 3)	Algorithm 2	1.9091	58	(5.5556, 1.7778, 2.6667)	−112.7531
8 (max)	Algorithm 1	0.5	16	(0.75, 0.75)	−38.8750
(2, 4, 2)	Algorithm 2	1.6425	64	(0.75, 0.75)	−38.8750
9 (min)	Algorithm 1	0.659	23	(1.5480, 2.4152)	−16.2893
(2, 6, 2)	Algorithm 2	1.7235	39	(1.5480, 2.4152)	−16.2893
10 (min)	Algorithm 1	0.7662	27	(1.5568, 0.7545)	10.6753
(2, 6, 2)	Algorithm 2	1.4862	68	(1.5480, 2.4152)	10.6753

execution time in seconds. Except for the results of our algorithm, the results of the other eleven algorithms are taken directly from the corresponding references. In Table 1, “—” denotes the corresponding value is not available.

For problems 1–10, the efficiency of the algorithm proposed by this paper (named Algorithm 1) and the algorithm proposed by this paper but without using the pruning techniques (named Algorithm 2) is compared. The comparison results are given in Table 2.

Example 1 (see [13–15]).

$$\begin{aligned}
 \min \quad & (x_1 + x_2)(x_1 - x_2 + 7) \\
 \text{s.t.} \quad & 2x_1 + x_2 \leq 14, \\
 & x_1 + x_2 \leq 10, \\
 & -4x_1 + x_2 \leq 0, \\
 & -2x_1 - x_2 \leq -6,
 \end{aligned}$$

$$-x_1 - 2x_2 \leq -6,$$

$$x_1 - x_2 \leq 3,$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

(28)

Example 2 (see [13, 16]).

$$\begin{aligned}
 \min \quad & x_1 + (2x_1 - 3x_2 + 13)(x_1 + x_2 - 1) \\
 \text{s.t.} \quad & -x_1 + 2x_2 \leq 8, \\
 & -x_2 \leq -3, \\
 & x_1 + 2x_2 \leq 12, \\
 & x_1 - 2x_2 \leq -5, \\
 & x_1 \geq 0, \quad x_2 \geq 0.
 \end{aligned}$$

(29)

Example 3 (see [13, 17]).

$$\begin{aligned}
 \min \quad & (0.813396x_1 + 0.67440x_2 + 0.305038x_3 + 0.129742x_4 + 0.217796) \\
 & \times (0.224508x_1 + 0.063458x_2 + 0.932230x_3 + 0.528736x_4 + 0.091947) \\
 \text{s.t.} \quad & 0.488509x_1 + 0.063565x_2 + 0.945686x_3 + 0.210704x_4 \leq 3.562809, \\
 & -0.324014x_1 - 0.501754x_2 - 0.719204x_3 + 0.099562x_4 \leq -0.052215, \\
 & 0.445225x_1 - 0.346896x_2 + 0.637939x_3 - 0.257623x_4 \leq 0.427920, \\
 & -0.202821x_1 + 0.647361x_2 + 0.920135x_3 - 0.983091x_4 \leq 0.840950,
 \end{aligned}$$

$$\begin{aligned}
& -0.886420x_1 - 0.802444x_2 - 0.305441x_3 - 0.180123x_4 \leq -1.353686, \\
& -0.515399x_1 - 0.424820x_2 + 0.897498x_3 + 0.187268x_4 \leq 2.137251, \\
& -0.591515x_1 + 0.060581x_2 - 0.427365x_3 + 0.579388x_4 \leq -0.290987, \\
& 0.423524x_1 + 0.940496x_2 - 0.437944x_3 - 0.742941x_4 \leq 0.373620, \\
& x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ x_4 \geq 0.
\end{aligned} \tag{30}$$

Example 4 (see [8, 18]).

$$\begin{aligned}
\min \quad & x_1 + (x_1 - x_2 + 5)(x_1 + x_2 - 1) \\
\text{s.t.} \quad & -2x_1 - 3x_2 \leq -9, \\
& 3x_1 - x_2 \leq 8, \\
& -x_1 + 2x_2 \leq 8, \\
& x_1 + 2x_2 \leq 12, \\
& x_1 \geq 0.
\end{aligned} \tag{31}$$

Example 5 (see [19]).

$$\begin{aligned}
\min \quad & (x_1 + x_2)(x_1 - x_2) \\
& + (x_1 + x_2 + 1)(x_1 - x_2 + 1) \\
\text{s.t.} \quad & x_1 + 2x_2 \leq 10, \\
& x_1 - 3x_2 \leq 20, \\
& 1 \leq x_1 \leq 3, \\
& 1 \leq x_2 \leq 3.
\end{aligned} \tag{32}$$

Example 6 (see [19]).

$$\begin{aligned}
\min \quad & (x_1 + x_2)(x_1 - x_2) \\
& + (x_1 + x_2 + 2)(x_1 - x_2 + 2) \\
\text{s.t.} \quad & x_1 + 2x_2 \leq 20, \\
& x_1 - 3x_2 \leq 20, \\
& 1 \leq x_1 \leq 4, \\
& 1 \leq x_2 \leq 4.
\end{aligned} \tag{33}$$

Example 7 (see [20]).

$$\begin{aligned}
\min \quad & (2x_1 - 2x_2 + x_3 + 2)(-2x_1 + 3x_2 + x_3 - 4) \\
& + (-2x_1 + x_2 + x_3 + 2)(x_1 + x_2 - 3x_3 + 5) \\
& + (-2x_1 - x_2 + 2x_3 + 7)(4x_1 - x_2 - 2x_3 - 5) \\
\text{s.t.} \quad & x_1 + x_2 + x_3 \leq 10, \\
& x_1 - 2x_2 + 3x_3 \leq 10, \\
& -2x_1 + 2x_2 + 3x_3 \leq 10,
\end{aligned}$$

$$\begin{aligned}
& -x_1 + 2x_2 + 3x_3 \geq 6, \\
& x_1 \geq 1, \ x_2 \geq 1, \ x_3 \geq 1.
\end{aligned} \tag{34}$$

Example 8 (see [21]).

$$\begin{aligned}
\max \quad & (-2x_1 + 3x_2 - 6)(3x_1 - 5x_2 + 3) \\
& + (4x_1 - 5x_2 - 7)(-3x_1 + 3x_2 + 4) \\
\text{s.t.} \quad & x_1 + x_2 \leq 1.5, \\
& x_1 - x_2 \leq 0, \\
& x_1 \geq 0, \ x_2 \geq 0.
\end{aligned} \tag{35}$$

Example 9 (see [22]).

$$\begin{aligned}
\min \quad & (x_1 + 2x_2 - 2)(-2x_1 - x_2 + 3) \\
& + (3x_1 - 2x_2 + 3)(x_1 - x_2 - 1) \\
\text{s.t.} \quad & -2x_1 + 3x_2 \leq 6, \\
& 4x_1 - 5x_2 \leq 8, \\
& 5x_1 + 3x_2 \leq 15, \\
& -4x_1 - 3x_2 \leq -12, \\
& x_1 \geq 0, \ x_2 \geq 0.
\end{aligned} \tag{36}$$

Example 10 (see [22]).

$$\begin{aligned}
\min \quad & (-x_1 + 2x_2 - 0.5)(-2x_1 + x_2 + 6) \\
& + (3x_1 - 2x_2 + 6.5)(x_1 + x_2 - 1) \\
\text{s.t.} \quad & -5x_1 + 8x_2 \leq 24, \\
& 5x_1 + 8x_2 \leq 44, \\
& 6x_1 - 3x_2 \leq 15, \\
& -4x_1 - 5x_2 \leq -10, \\
& x_1 \geq 0, \ x_2 \geq 0.
\end{aligned} \tag{37}$$

To further verify the effectiveness of Algorithm 1, a random problem with variable scale is constructed, which is defined as follows.

Example 11.

$$\begin{aligned} \min \quad & f(x) = \sum_{i=1}^p (c_i^T x + d_i) (e_i^T x + f_i) \\ \text{s.t.} \quad & x \in D = \{x \mid x \in R^n, Ax \leq b\}, \end{aligned} \quad (38)$$

where the real elements of c_i, d_i, e_i, f_i are pseudorandomly generated in the range $[-0.5, 0.5]$; the real elements of A, b are pseudorandomly generated in the range $[0.01, 1]$. For Example 11, Algorithms 1 and 2 are used to solve 10 different random instances for each size and present statistics of the results. The computational results are summarized in Table 3, where the following notations have been used in row headers: Avg.Iter is the average number of iterations; Avg.Time is the average execution time in seconds; m is the number of constraints; n is the number of variables.

From Table 1, it can be seen that our algorithm can determine the global optimal solution more effectively than that of [8, 13–22] in most cases. For Examples 8 and 9, although the number of iterations of our algorithm is more than that of the literatures [21, 22], the optimal values and optimal solutions obtained by our algorithm are better than them.

From Table 2, it can be seen that, for Examples 1–6, Algorithms 1 and 2 all only need one iteration to find the optimal solution; the advantage of Algorithm 1 is not reflected. However, for Examples 7–10, the performance of Algorithm 1 is better than that of Algorithm 2.

From Table 3, we can see that, for small scale problems, the advantage of Algorithm 1 is not much better than the Algorithm 2, but with the increase of the scale of the problem, the advantage of Algorithm 1 is more and more powerful. For example, when $(p, m, n) = (10, 5, 5)$, the average running time of Algorithms 1 and 2 is 17.0344 and 29.5778, respectively; the average iterations of Algorithms 1 and 2 are 17.6 and 30.4. However, when $(p, m, n) = (10, 100, 100)$, the average running time of Algorithms 1 and 2 is 42.8972 and 402.0530, respectively; the average iterations of Algorithms 1 and 2 are 56.8 and 215.3. It is clear that the efficiency of Algorithm 1 is much better than that of Algorithm 2 for large scale problems. In addition, from Table 3, we also can see that, compared with m and n , the impact of p on our algorithm is even greater; the Avg.Time and Avg.Iter of Algorithm 1 are not increased significantly with the increase of the problem size.

The comparison results of Tables 2 and 3 show that the pruning techniques are very good at improving the convergence speed of our algorithm.

The test results show that our algorithm is both feasible and efficient.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

TABLE 3: Comparison results of Algorithms 1 and 2 for Example 11.

(p, m, n)	Methods	Avg.Time	Avg.Iter
(3, 5, 5)	Algorithm 1	0.8520	3.4
	Algorithm 2	3.8780	13.6
(3, 10, 10)	Algorithm 1	1.3142	4.2
	Algorithm 2	16.1132	46.8
(3, 20, 20)	Algorithm 1	23.2650	49.6
	Algorithm 2	32.5486	66.8
(3, 100, 100)	Algorithm 1	42.8972	62.5
	Algorithm 2	75.0706	32.2
(6, 5, 5)	Algorithm 1	3.7030	5.2
	Algorithm 2	8.6610	16.6
(6, 10, 10)	Algorithm 1	5.1134	7.8
	Algorithm 2	40.2840	69.8
(6, 20, 20)	Algorithm 1	12.7132	13.4
	Algorithm 2	100.4470	128.8
(6, 100, 100)	Algorithm 1	31.9760	25.3
	Algorithm 2	254.5428	183.4
(10, 5, 5)	Algorithm 1	17.0344	17.6
	Algorithm 2	29.5778	30.4
(10, 10, 10)	Algorithm 1	26.1132	46.8
	Algorithm 2	195.9464	136.8
(10, 20, 20)	Algorithm 1	32.5126	27.8
	Algorithm 2	249.4000	169.2
(10, 100, 100)	Algorithm 1	42.8972	56.8
	Algorithm 2	402.0530	215.3

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