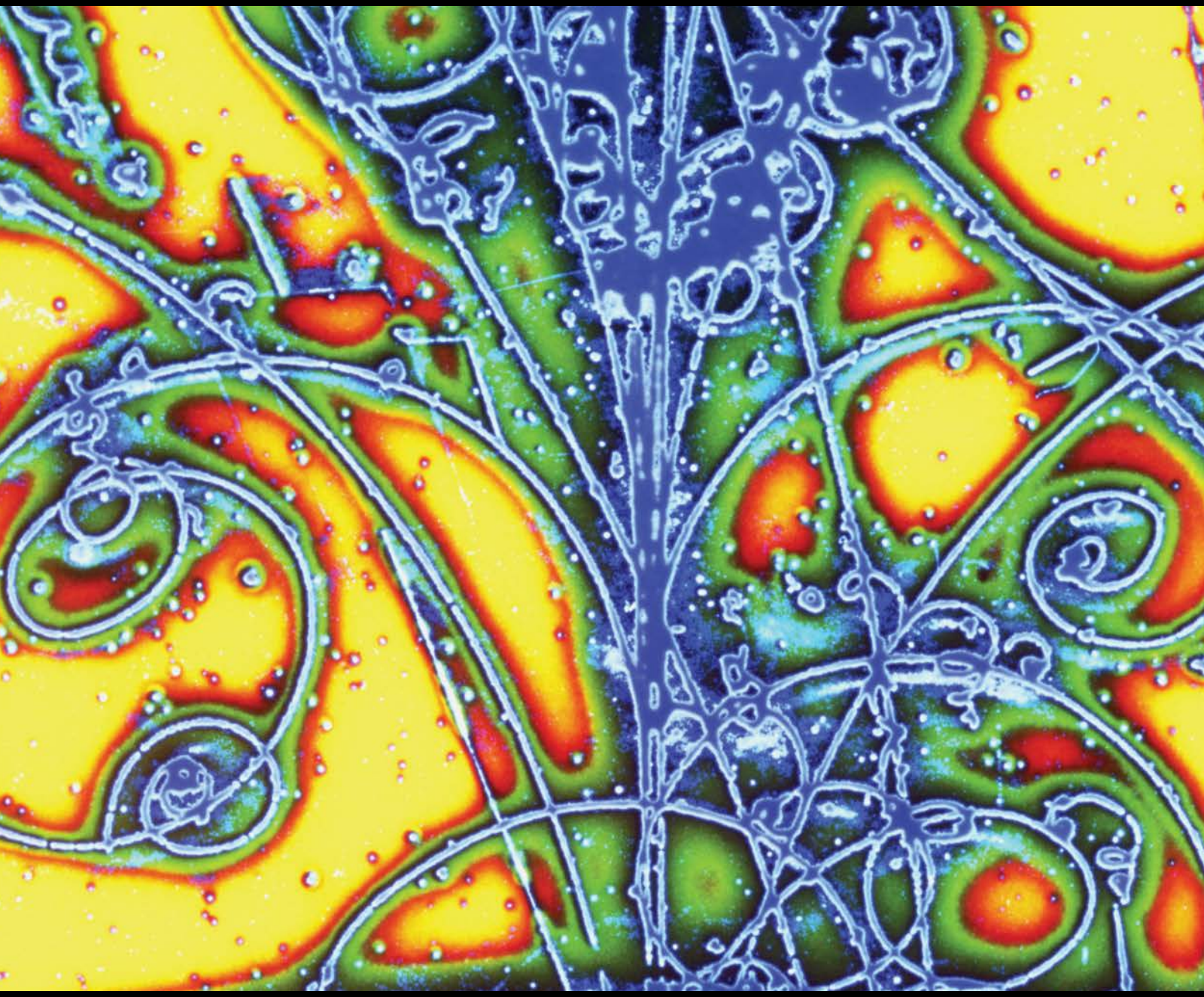
Advances in High Energy Physics

Phenomenological Aspects of Quantum Gravity and Modified Theories of Gravity

Guest Editors: Ahmed Farag Ali, Giulia Gubitosi, Mir Faizal, and Barun Majumder



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Editorial Board

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Editorial

Phenomenological Aspects of Quantum Gravity and Modified Theories of Gravity

Ahmed Farag Ali,[1,2] Giulia Gubitosi,[3] Mir Faizal,[4] and Barun Majumder[5]

[1] *Benha University, Benha, Egypt*
[2] *Netherlands Institute for Advanced Study, Korte Spinhuissteeg 3, 1012 CG Amsterdam, Netherlands*
[3] *Imperial College London, London, UK*
[4] *British Columbia University, Vancouver, BC, Canada*
[5] *Indian Institute of Technology-Gandhinagar, Gujarat, India*

Correspondence should be addressed to Ahmed Farag Ali; ahmed.ali@fsc.bu.edu.eg

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The phenomenological aspects of various approaches to quantum gravity and modified theories of gravity are getting a lot of attention and cover many different topics such as theories with minimal length, noncommutative geometry, theories with generalized uncertainty principle, violations of Lorentz symmetries, and loop quantum cosmology. The recent developments in these approaches with their phenomenological implications in particle physics, cosmology, and astrophysics are the main theme of this special issue. In this special issue on quantum gravity phenomenology, we have invited papers that address such issues.

In the paper by S. Dutta et al., the authors present a study on emergent universe scenario with modified Chaplygin gas and its implications with the initial big bang singularity. In the paper by P. Pedram, the author uses the modified commutation relation, which has quadratic order in the momentum as deformation, to study the Beckner, Bialynicki-Birula, and Mycielski (BBM) inequality. With a particular choice of self-adjoint representation, the author has shown that the BBM inequality remains valid for various choices of the deformation parameter. In the paper by H. Moradpour and R. Dehghani, the authors tackled the unified first law of thermodynamics and its genuine connection with the apparent horizon of FRW universe in which they found that whenever there is no energy exchange between the various parts of cosmos, one could get an expression for the apparent horizon entropy in quasi-topological gravity. In the paper by S.-Z. Yang et al., the authors studied the Hawking radiation in the context of Lorentz invariance violation which has been predicted in different approaches to quantum gravity. In the paper by M. J. Soleimani et al, the authors investigate tunneling of charged massive particles in charged TeV-scale black hole with the help of a generalized uncertainty relation that has a minimal length and maximal momentum. In the paper by M. Ronco, the author tackled Planck-scale dynamical dimensional reduction in which he observed that the number of UV dimensions can be used to constrain the ambiguities in the choice of these loop quantum gravity-based modifications of the Dirac space-time algebra. In the paper by T. Moon and P. Oh, the authors studied spontaneous symmetry breaking in 5D conformally invariant gravity and found that the dimensional reduction via ADM decomposition gives rise to the 4D Minkowski vacuum. In the paper by Z.-W. Feng et al., the authors studied the implications of minimal length theories on the entropic force realization of gravity and they derived, in this context, the modified Einstein field equation and modified Friedman equations which may have various phenomenological implications. In the paper by A. E. Bernardini and R. Rocha, the authors have studied a family of asymmetric thick brane configurations which are formed by defects. A left-right asymmetric chiral localization of spin 1/2 particles is produced in the localization of fermion fields of

the thick branes. In the paper by S.-Z. Yang et al., the authors investigated the origin of Hawking radiation in the context of fermions tunneling from higher-dimensional Reissner-Nordström black hole and calculated the entropy of the black hole. In the paper by F. Ghobakhloo and H. Hassanabadi, the authors investigated the free particle propagator in the context of minimal length theories and discussed possible implications of this modified propagator.

Ahmed Farag Ali
Giulia Gubitosi
Mir Faizal
Barun Majumder

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Research Article

Investigation of Free Particle Propagator with Generalized Uncertainty Problem

F. Ghobakhloo and H. Hassanabadi

Physics Department, Shahrood University of Technology, P.O. Box 316, Shahrood 3619995161, Iran

Correspondence should be addressed to F. Ghobakhloo; fatemeh.ghobakhloo@gmail.com

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We consider the Schrödinger equation with a generalized uncertainty principle for a free particle. We then transform the problem into a second-order ordinary differential equation and thereby obtain the corresponding propagator. The result of ordinary quantum mechanics is recovered for vanishing minimal length parameter.

1. Introduction

The generalization of classical action principle into quantum theory appears in path integral formulation. Instead of a single classical path, the quantum version considers a sum, or better, say, integral, of infinite possible paths [1, 2]. Although the main idea of path integral approach was released by N. Wiener, in an attempt to solve diffusion and Brownian problems, it was introduced in Lagrangian formulation of quantum mechanics of P. A. M. Dirac [1, 2]. Nevertheless, the present comprehensive formulation is named after Feynman and extracted from his Ph.D. thesis supervised by J. A. Wheeler [1, 2]. Feynman's formulation is now an essential ingredient in many fundamental theories of theoretical physics including quantum field theory, quantum gravity, and high energy physics [1–3].

On the other hand, we are now almost sure from fundamental theories such as string theory and quantum gravity that the ordinary quantum mechanics ought to be reformulated. In more precise words, a generalization of Heisenberg uncertainty principle, called generalized uncertainty principle (GUP), should be considered at energies of order Planck scale [4–7]. This generalization corresponds to a generalization of wave equation of quantum mechanics. Till now, various wave equations of quantum mechanics, different interactions, and other related mathematical aspects and physical concepts have been considered in this framework [8–18].

In our paper, we are going to combine these two subjects. Namely, we study the free particle propagator in Schrödinger framework in minimal length formulation. In Section 2, we review the essential concepts of GUP and write the generalized Hamiltonian for free particle. In Section 3, we obtain the propagator for this system in which the details of calculations are brought.

2. GUP-Corrected Hamiltonian

An immediate consequence of the ML is the GUP

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha l_p^2 \frac{\Delta p}{\hbar}, \tag{1}$$

where the GUP parameter α is determined from a fundamental theory. At low energies, that is, energies much smaller than the Planck mass, the second term on the right hand side of (1) vanishes and we recover the well-known Heisenberg uncertainty principle. The GUP of (1) corresponds to the generalized commutation relation

$$\left[x_{op}, p_{op}\right] = i\hbar\left(1 + \beta p^2\right) \quad 0 \leq \beta \leq 1, \tag{2}$$

where $x_{op} = x$, $p_{op} = p[1 + \beta(p)^2]$ and $0 \leq \beta \leq 1$. The limits $\beta \to 0$ and $\beta \to 1$ correspond to the standard quantum mechanics and extreme quantum gravity, respectively.

Equation (2) gives the minimal length in this case as $(\Delta x)_{\min} = 2l_p\sqrt{\alpha}$. It should be noted that, in the deformed Schrödinger equation, the Hamiltonian does not have any explicit time dependence [19]:

$$\left(\frac{p_{op}^2}{2m} + V(x)\right)\psi_n(x) = E_n\psi_n(x). \tag{3}$$

This deformed momentum operator modifies the original Hamiltonian as

$$H = \frac{p^2}{2m} + V_{ff}(x), \tag{4}$$

where

$$V_{ff}(x) = \beta\frac{p^4}{m} + V(x). \tag{5}$$

The problem becomes much simpler if we consider [16]

$$\begin{aligned} p^2 &= 2m\left(E_n^{(0)} - V(x)\right), \\ p^4 &= 4m^2\left(E_n^{(0)} - V(x)\right)^2. \end{aligned} \tag{6}$$

In the free particle case, we therefore have

$$H = \frac{p^2}{2m} + 4\beta m\left(E_n^{(0)}\right)^2. \tag{7}$$

We now calculate the single free particle propagator corresponding to this deformed Hamiltonian in Section 3.

3. Perspicuous Form of Propagator

If the wave function $\psi(x,t')$ is known at a time t', we can explicitly write the wave function $\psi(x,t'')$ at a later time t'' using the propagation relation as [20]

$$\psi\left(x,t''\right) = \exp\left(-\frac{iH\left(t''-t'\right)}{\hbar}\right)\psi\left(x,t'\right). \tag{8}$$

For a small time interval $t''-t' = \Delta t$, we have

$$\begin{aligned} &\left\langle x\left|\exp\left(-\frac{iH\Delta t}{\hbar}\right)\right|x\right\rangle = \frac{1}{2\pi\hbar}\int dp \\ &\cdot\left(\left\langle x\left|\exp-\frac{i}{\hbar}\left(\frac{p^2}{2m}\Delta t\right)\right|p\right\rangle\right. \\ &\left.\cdot\left\langle p\left|\exp-\frac{i}{\hbar}\left(4\beta m\left(E_n^{(0)}\right)^2\Delta t\right)\right|x\right\rangle\right) = \int\frac{dp}{2\pi\hbar} \\ &\cdot\exp-\frac{i}{\hbar}\left(\frac{p^2}{2m} + 4\beta m\left(E_n^{(0)}\right)^2\Delta t\right). \end{aligned} \tag{9}$$

Therefore, the quantum mechanical propagator for small time interval $\Delta t = t - t'$, corresponding to this nonlocal Hamiltonian, can be written as

$$K\left(x'',t'';x',t'\right) = \int\exp\left(\frac{i}{\hbar}\int_{t'}^{t'+\Delta t}L(t)\,dt\right)\frac{dp}{2\pi\hbar}, \tag{10}$$

in which the Lagrangian is given by [20]

$$L(t) = p\cdot\frac{\left(x''-x'\right)}{\left(t''-t'\right)} - \frac{p^2}{2m} - 4\beta m\left(E_n^{(0)}\right)^2. \tag{11}$$

Therefore, the propagator appears as

$$\begin{aligned} K\left(x'',t'';x',t'\right) &= \frac{1}{2\pi\hbar} \\ &\cdot\exp\frac{i}{\hbar}\left(\frac{\left(x''-x'\right)^2 m}{2\Delta t} - 4\beta m\left(E_n^{(0)}\right)^2\Delta t\right) \\ &\cdot\int dp\cdot\exp\left(-\frac{i\Delta t}{2m\hbar}\left(p - \frac{\left(x''-x'\right)m}{\Delta t}\right)^2\right) \end{aligned} \tag{12}$$

or

$$\begin{aligned} K\left(x'',t'';x',t'\right) &= \frac{1}{2\pi\hbar} \\ &\cdot\exp\frac{i}{\hbar}\left(\frac{\left(x''-x'\right)^2 m}{2\Delta t} - 4\beta m\left(E_n^{(0)}\right)^2\Delta t\right)\int dU \\ &\cdot\exp\left(-\frac{i\Delta t}{2m\hbar}U^2\right) \end{aligned} \tag{13}$$

with

$$U = \left(p - \frac{\left(x''-x'\right)m}{\Delta t}\right). \tag{14}$$

In a more explicit form, the propagator for free particle under minimal length is

$$\begin{aligned} K\left(x'',t'';x',t'\right) &= \left(\sqrt{\frac{m}{2i\pi\hbar\Delta t}}\right) \\ &\cdot\exp\frac{i}{\hbar}\left(\frac{\left(x''-x'\right)^2 m}{2\left(t''-t'\right)} - 4\beta m\left(E_n^{(0)}\right)^2\left(t''-t'\right)\right). \end{aligned} \tag{15}$$

Now, if we assume $\beta = 0$, then the one-dimensional free particle propagator is given by

$$\begin{aligned} &K\left(x'',t'';x',t'\right) \\ &\quad= \left(\sqrt{\frac{m}{2i\pi\hbar\Delta t}}\right)\exp\frac{i}{\hbar}\left(\frac{\left(x''-x'\right)^2 m}{2\left(t''-t'\right)}\right). \end{aligned} \tag{16}$$

In order to obtain free particle propagator for a finite time interval $(t''-t')$ we divide the interval into N subintervals of

equal length Δt such that $(t''-t') = N\Delta t$. Now, the propagator of a finite time interval is written as

$$K\left(x'',t'';x',t'\right) = \left(\sqrt{\frac{m}{2i\pi\hbar\Delta t}}\right)^N \cdot \int dx_1 dx_2 dx_3 \cdots dx_{N-1} \exp\frac{im}{2\hbar\Delta t} \cdot \left(\left(x_1-x_0\right)^2 + \left(x_2-x_1\right)^2 + \cdots + \left(x_N-x_{N-1}\right)^2\right) \cdot \exp\left(\frac{-i4\beta m}{\hbar}\left(E_n^{(0)}\right)^2\left(t''-t'\right)\right). \tag{17}$$

The integral in (17) can be calculated as [20]

$$\int dx_1 dx_2 dx_3 \cdots dx_{N-1} \exp i\lambda \cdot \left(\left(x_1-x_0\right)^2 + \left(x_2-x_1\right)^2 + \cdots + \left(x_N-x_{N-1}\right)^2\right) = \frac{1}{\sqrt{N}}\left(\frac{i\pi}{\lambda}\right)^{(N-1)/2} \exp\left(\frac{i\lambda\left(x_N-x_0\right)^2}{N}\right). \tag{18}$$

Substituting (18) into (17), the propagator is obtained as

$$K\left(x'',t'';x',t'\right) = \left(\sqrt{\frac{m}{2i\pi\hbar\Delta t}}\right)^N \frac{1}{\sqrt{N}}\left(\frac{i\pi}{\lambda}\right)^{(N-1)/2} \cdot \exp\left(\frac{i\lambda\left(x_N-x_0\right)^2}{N}\right) \cdot \exp\left(\frac{-i4\beta m}{\hbar}\left(E_n^{(0)}\right)^2\Delta t\right), \tag{19}$$

where $\lambda = m/2\hbar\Delta t$. Replacing x_N and x_0 by x'' and x', respectively, and using $(t''-t') = N\Delta t$, we obtain the final expression as

$$K\left(x'',t'';x',t'\right) = \left(\sqrt{\frac{m}{2i\pi\hbar\Delta t}}\right)^N \frac{1}{\sqrt{N}}\left(\frac{\lambda}{i\pi}\right)^{(N-1)/2} \cdot \exp\left(\frac{i\lambda\left(x_N-x_0\right)^2}{N}\right) \cdot \exp\left(\frac{-i4\beta m}{\hbar}\left(E_n^{(0)}\right)^2\Delta t\right). \tag{20}$$

Now, if we calculate the probability of detecting the particle at a finite region Δx, enclosing final point x'', from (20), we get

$$K\left(x'',t'';x',t'\right) = \left(\sqrt{\frac{m}{2i\pi N\hbar\Delta t}}\right) \cdot \exp\left(\frac{im\left(x_N-x_0\right)^2}{2\hbar N\Delta t}\right) \cdot \exp\left(\frac{i4\beta m}{\hbar}\left(E_n^{(0)}\right)^2\Delta t\right). \tag{21}$$

In the limit $\beta \to 0$, the final form of propagator is given by

$$K\left(x'',t'';x',t'\right) = \left(\sqrt{\frac{m}{2i\pi\hbar\left(t''-t'\right)}}\right)\exp\left(\frac{im\left(x''-x'\right)^2}{2\hbar\left(t''-t'\right)}\right) \tag{22}$$

which is the result in ordinary quantum mechanics.

4. Conclusion

We considered the nonrelativistic free particle propagation problem in an analytical manner in minimal length formalism. We first transformed arising differential equation into a second-order differential equation which included a modified effective potential. We next calculated the propagator. Apart from the application of the study, the work is of pedagogical interest in graduate physics.

Competing Interests

The authors declare that they have no competing interests.

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Research Article

Fermions Tunneling from Higher-Dimensional Reissner-Nordström Black Hole: Semiclassical and Beyond Semiclassical Approximation

ShuZheng Yang,[1] Dan Wen,[1,2,3] and Kai Lin[1,4]

[1] *Department of Astronomy, China West Normal University, Nanchong, Sichuan 637002, China*
[2] *College of Physics and Information Science, Hunan Normal University, Changsha, Hunan 410081, China*
[3] *Faculdade de Engenharia de Guaratinguetá, Universidade Estadual Paulista, 12516-410 Guaratinguetá, SP, Brazil*
[4] *Instituto de Física e Química, Universidade Federal de Itajubá, 37500-903 Itajubá, MG, Brazil*

Correspondence should be addressed to Kai Lin; lk314159@hotmail.com

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Based on semiclassical tunneling method, we focus on charged fermions tunneling from higher-dimensional Reissner-Nordström black hole. We first simplify the Dirac equation by semiclassical approximation, and then a semiclassical Hamilton-Jacobi equation is obtained. Using the Hamilton-Jacobi equation, we study the Hawking temperature and fermions tunneling rate at the event horizon of the higher-dimensional Reissner-Nordström black hole space-time. Finally, the correct entropy is calculation by the method beyond semiclassical approximation.

Hawking radiation is an important prediction in modern gravitation theory [1–5]. Recently, Kraus et al. proposed quantum tunneling theory to explain and study Hawking radiation [6–31], and then semiclassical Hamilton-Jacobi method is put forward to research the properties of scalar particles' tunnels [32–36]. In 2007, Kerner and Mann investigated the 1/2 spin fermion tunneling from static black holes [37]. In their work, the spin up and spin down cases are researched, respectively, and the radial equations are obtained, so that they can finally determine the Hawking temperature and tunneling rate at the event horizon. Subsequently, Kerr and Kerr-Newman black holes cases, the charged dilatonic black hole case, the de Sitter horizon case, the BTZ black hole case, 5-dimensional space-time cases, and several nonstationary black hole cases were all researched, respectively [38–48], and we used Hamilton-Jacobi method to study the fermion tunneling from higher-dimensional uncharged black holes [49, 50]. However, up to now, no one has studied higher-dimensional charged black holes cases, so we set out to research that case. In our work, we developed the Kerner and Mann method and proved that the semiclassical Hamilton-Jacobi equation can be obtained not only with the Klein-Gordon equation of curved space-time, but also with the Dirac equation in curved space-time. Applying the Hamilton-Jacobi equation, we can then obtain semiclassical Hawking temperature and tunneling rate at the event horizon of higher-dimensional Reissner-Nordström black hole.

In modern physics theory, the concept of an extra dimension can help to solve some theoretical issues, so several higher-dimensional metrics of curved space-time were investigated. The metric of static charged $(n+2)$-dimensional Reissner-Nordström black hole is given by [10, 51–55]

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2 d\Omega_n^2, \tag{1}$$

where $d\Omega_n^2$ is the metric of n-dimensional sphere

$$\begin{aligned} d\Omega_n^2 &= \sum_{i=1}^{n} h^{ii} d\theta_i^2 \\ &= d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2 d\theta_3^2 + \cdots \\ &\quad + \prod_{i=1}^{n-1} \sin^2\theta_i d\theta_n^2, \end{aligned}$$

$$f(r) = 1 - \frac{\omega_n M}{r^{n-1}} + \frac{\omega_n Q^2}{2(n-1)V_n r^{2n-2}}, \quad \omega_n = \frac{16\pi}{nV_n}, \tag{2}$$

where M and Q are mass and electric charge of black hole, and the electromagnetic potential is

$$A_\mu = \left(\frac{Q}{(n-1)V_n r^{n-1}}, 0, 0, 0, \ldots\right), \tag{3}$$

where V_n is volume of unit n-sphere (we can adopt the units $G = c = \hbar = 1$). The outer/inner horizon located at

$$r_\pm^{n-2} = \frac{\omega_n}{2}\left[M \pm \sqrt{M^2 - \frac{nQ^2}{8\pi(n-1)}}\right]. \tag{4}$$

Obviously, at the horizons, the equation $f(r_\pm) = 0$ should be satisfied. However, the physical property near the inner horizon cannot be researched, so we just study the fermion tunneling at the outer event horizon of this black hole. The charged Dirac equation in curved space-time is

$$\gamma^\mu D_\mu \Psi + \frac{m}{\hbar}\Psi = 0, \quad \mu = t, r, \theta_1, \ldots, \theta_n, \tag{5}$$

where

$$\begin{aligned} D_\mu &= \partial_\mu + \Gamma_\mu + \frac{iqA_\mu}{\hbar} \\ \Gamma_\mu &= \frac{1}{8}\left[\tilde{\gamma}^a, \tilde{\gamma}^b\right] e_a{}^\nu e_{b\nu;\mu}, \end{aligned} \tag{6}$$

where m and q are mass and electric charge of the particles, and $e_{b\nu;\mu} = \partial_\mu e_{b\nu} - \Gamma^\alpha_{\mu\nu} e_{ab}$ is the covariant derivative of tetrad $e_{b\nu}$. The gamma matrices in curved space-time need to be satisfied

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I. \tag{7}$$

After the gamma matrices are defined, we choose the gamma matrices in $(n+2)$-dimensional flat space-time as

$$\begin{aligned} \tilde{\gamma}^1_{m\times m} &= \begin{pmatrix} I_{m/2\times m/2} & 0 \\ 0 & -I_{m/2\times m/2} \end{pmatrix} \\ \tilde{\gamma}^2_{m\times m} &= \begin{pmatrix} 0 & I_{m/2\times m/2} \\ I_{m/2\times m/2} & 0 \end{pmatrix} \\ \tilde{\gamma}^\eta_{m\times m} &= \begin{pmatrix} 0 & i\tilde{\gamma}^{\eta-2}_{m/2\times m/2} \\ -i\tilde{\gamma}^{\eta-2}_{m/2\times m/2} & 0 \end{pmatrix}, \\ &\qquad \eta = 3, 4, 5, \ldots, n+2, \end{aligned} \tag{8}$$

where $I_{m/2\times m/2}$ and $\tilde{\gamma}^\eta_{m/2\times m/2}$ are unit matrices and flat gamma matrices with $m/2 \times m/2$ order and $m = 2^{(n+2)/2}$ is the order of the matrices in even (odd) dimensional space-time. Corresponding to the flat case, the gamma matrices can be chosen as

$$\begin{aligned} \gamma^t_{m\times m} &= \frac{i}{\sqrt{f}}\tilde{\gamma}^1_{m\times m} \\ \gamma^r_{m\times m} &= \sqrt{f}\tilde{\gamma}^2_{m\times m} \\ \gamma^\eta_{m\times m} &= r^{-1}\sqrt{h^{\eta\eta}}\tilde{\gamma}^\eta_{m\times m}, \quad \eta = 3, 4, 5, \ldots, n+2. \end{aligned} \tag{9}$$

Now, let us simplify the Dirac equation via semiclassical approximation, and rewrite the spinor function as

$$\Psi = \begin{pmatrix} A_{m/2\times 1}(t, r, \ldots, x^\eta, \ldots) \\ B_{m/2\times 1}(t, r, \ldots, x^\eta, \ldots) \end{pmatrix} e^{(i/\hbar)S(t,r,\ldots,x^\eta,\ldots)}, \tag{10}$$

where $A_{m/2\times 1}(t, r, \ldots, x^\eta, \ldots)$ and $B_{m/2\times 1}(t, r, \ldots, x^\eta, \ldots)$ are column matrices with $m/2 \times 1$ order and S is classical action. Via semiclassical approximation method, Substituting (10) into (5) and dividing the exponential term and multiplying by $\hbar$, we can get

$$\begin{pmatrix} C & D \\ E & F \end{pmatrix}\begin{pmatrix} A_{m/2\times 1} \\ B_{m/2\times 1} \end{pmatrix} = 0 \tag{11}$$

$$\begin{aligned} C = &-\frac{1}{\sqrt{f}}\left(\frac{\partial S}{\partial t} + qA_t\right) I_{m/2\times m/2} \\ &+ mI_{m/2\times m/2} \end{aligned} \tag{12}$$

$$\begin{aligned} D = &\, i\sqrt{f}\frac{\partial S}{\partial r} I_{m/2\times m/2} \\ &- \sum_\eta r^{-1}\sqrt{h^{\eta\eta}}\frac{\partial S}{\partial x^\eta}\tilde{\gamma}^{\eta-2}_{m/2\times m/2} \end{aligned} \tag{13}$$

$$\begin{aligned} E = &\, i\sqrt{f}\frac{\partial S}{\partial r} I_{m/2\times m/2} \\ &+ \sum_\eta r^{-1}\sqrt{h^{\eta\eta}}\frac{\partial S}{\partial x^\eta}\tilde{\gamma}^{\eta-2}_{m/2\times m/2} \end{aligned} \tag{14}$$

$$\begin{aligned} F = &\frac{1}{\sqrt{f}}\left(\frac{\partial S}{\partial t} + qA_t\right) I_{m/2\times m/2} \\ &+ mI_{m/2\times m/2}. \end{aligned} \tag{15}$$

Solving (11), we have

$$\begin{aligned} \left(E - FD^{-1}C\right) A_{m/2\times 1} &= 0 \\ \left(F - EC^{-1}D\right) B_{m/2\times 1} &= 0. \end{aligned} \tag{16}$$

It is evident that the coefficient matrices of (16) must vanish, when $A_{m/2\times 1}$ and $B_{m/2\times 1}$ have nontrivial solutions. Due to the fact that $CD = DC$, we can write the condition that determinant of coefficient vanish as

$$\det(ED - FC) = 0. \tag{17}$$

From the relation of flat gamma matrices $\{\tilde{\gamma}^{\mu}, \tilde{\gamma}^{\nu}\} = 2\delta_{\mu\nu}$, we can obtain the semiclassical Hamilton-Jacobi equation in $(n+2)$-dimensional Reissner-Nordström space-time

$$-\frac{1}{f}\left(\frac{\partial S}{\partial t}+qA_t\right)^2+f\left(\frac{\partial S}{\partial r}\right)^2+\cdots+g^{\eta\eta}\left(\frac{\partial S}{\partial x^{\eta}}\right)+\cdots+m^2=0. \tag{18}$$

Using the Hamilton-Jacobi equation, in charged static space-time, we can separate the variables for the action as

$$S=-\omega t+R(r)+Y(\ldots,x^{\eta},\ldots)+K, \quad (K \text{ is a constant}) \tag{19}$$

and the Hamilton-Jacobi equation is broken up as

$$-\frac{1}{f}(\omega-qA_t)^2+f\left(\frac{dR}{dr}\right)^2+m^2=\frac{\lambda}{r^2} \tag{20}$$

$$\sum_{\eta}h^{\eta\eta}\left(\frac{\partial Y}{\partial x^{\eta}}\right)^2+\lambda=0, \tag{21}$$

where (20) and (21) are radial and nonradial equations, respectively, and λ is a constant. However, we only research on the radial equation, because the tunnel at the event horizon is radial. From (20), we can get

$$\frac{dR(r)}{dr}=\pm\frac{\sqrt{(\omega-qA_t)^2r^2+f(\lambda-m^2r^2)}}{fr}. \tag{22}$$

Near the event horizon, the radial action is given by

$$\operatorname{Im}R_{\pm}=\pm\frac{\pi(\omega-\omega_0)}{f'(r_+)}+\operatorname{Im}C, \tag{23}$$

where R_+ is part of the outgoing solution, while R_- is the part of incoming solution, and

$$\omega_0=q\frac{Q}{(n-1)V_nr_+^{n-1}}. \tag{24}$$

So the tunneling rate is

$$\begin{aligned}\Gamma&=\frac{\text{Prob [out]}}{\text{Prob [in]}}=\frac{\exp(-2\operatorname{Im}S_+)}{\exp(-2\operatorname{Im}S_-)}\\&=\frac{\exp(-2\operatorname{Im}R_++\operatorname{Im}K)}{\exp(-2\operatorname{Im}R_-+\operatorname{Im}K)}\\&=\exp\left(\frac{-4\pi(\omega-\omega_0)}{f'(r_+)}\right),\end{aligned} \tag{25}$$

where Im represents the imaginary part of the function, and the Hawking temperature is

$$T_H=\frac{f'(r_+)}{4\pi}. \tag{26}$$

However, above calculation is worked on semiclassical approximation, because we ignored all higher order terms of $\mathcal{O}(\hbar)$. Recently, Banerjee and Majhi proposed a new method beyond semiclassical approximation to research the quantum tunneling, and their works show that the conclusion should be corrected [56–65], and this correct entropy may be applied in quantum gravity theory. Now let us generalize this work in higher-dimensional Reissner-Nordström black hole space-time.

Because the tetrad $e_{\mu}{}^{a}$ in the space-time are given by

$$e_{\mu}{}^{a}=\operatorname{diag}\left(\sqrt{f},\frac{1}{\sqrt{f}},r,r\sin\theta_1,\ldots,r\prod_{i=1}^{n-1}\sin\theta_i\right), \tag{27}$$

so that Γ_{μ} is

$$\begin{aligned}\tilde{\gamma}^{a}e_{a}{}^{\mu}\Gamma_{\mu}&=\tilde{\gamma}^{1}\sqrt{f}\left(\frac{n}{2r}+\frac{f'}{4f}\right)\\&\quad+\frac{1}{2r}\sum_{k=1}^{n-1}\tilde{\gamma}^{k+1}\frac{(n-k)\cot\theta_k}{\prod_{i=1}^{k-1}\sin\theta_i}.\end{aligned} \tag{28}$$

It means the Dirac equation becomes

$$\begin{aligned}&i\frac{\tilde{\gamma}^0}{\sqrt{f}}\left(\frac{\partial}{\partial t}+\frac{iqA_t}{\hbar}\right)\Psi+\tilde{\gamma}^1\sqrt{f}\left(\frac{\partial}{\partial r}+\frac{n}{2r}+\frac{f'}{4f}\right)\Psi\\&\quad+\sum_{k=1}^{n-1}\frac{\tilde{\gamma}^{k+1}}{r\prod_{i=1}^{k-1}\sin\theta_i}\left(\frac{\partial}{\partial\theta_k}+\frac{(n-k)\cot\theta_k}{2}\right)\Psi\\&\quad+\frac{\tilde{\gamma}^{n+1}}{r\prod_{i=1}^{n-1}\sin\theta_i}\frac{\partial\Psi}{\partial\theta_n}+\frac{m}{\hbar}\Psi=0,\end{aligned} \tag{29}$$

and this equation can be simplified at event horizon

$$i\tilde{\gamma}^0\left(\frac{\partial}{\partial t}+\frac{iqA_0}{\hbar}\right)\Psi+\tilde{\gamma}^1\left(\frac{\partial}{\partial r_*}+\frac{f'}{4}\right)\Psi=0, \tag{30}$$

because $f\to 0$ at event horizon, and $dr_*=dr/f$ is tortoise coordinate. On the other hand, the space-time background is static, and Ψ can be rewritten as

$$\Psi=\begin{bmatrix}A(r)\\B(r)\end{bmatrix}e^{-(i/\hbar)\omega t}, \tag{31}$$

where $A(r)$ and $B(r)$ are matrices with $m/2\times 1$ and ω is frequency or energy of Dirac particle. Finally, applying the definitions of $\tilde{\gamma}^0$ and $\tilde{\gamma}^1$, we get

$$\begin{pmatrix}\omega-\omega_0 & \hbar f\left(\frac{\partial}{\partial r}-\frac{f'}{4f}\right)\\ \hbar f\left(\frac{\partial}{\partial r}-\frac{f'}{4f}\right) & -(\omega-\omega_0)\end{pmatrix}\begin{pmatrix}A_q\\B_q\end{pmatrix}=0; \tag{32}$$

here A_q and B_q are qth elements of matrices $A(r)$ and $B(r)$, respectively. Above equation becomes

$$\frac{\partial B_q/\partial r}{\partial A_q/\partial r}=\frac{(\omega-\omega_0)A_q-\hbar\left(f'B_q/4\right)}{-(\omega-\omega_0)B_q-\hbar\left(f'A_q/4\right)}, \tag{33}$$

so

$$\frac{\omega-\omega_0}{2}\frac{\partial}{\partial r}\left(A_q^2+B_q^2\right)-\hbar\frac{f'}{4}\left(B_q\frac{\partial A_q}{\partial r}-A_q\frac{\partial B_q}{\partial r}\right)=0. \tag{34}$$

At event horizon $f'(r_0)\neq 0$ and depends on the position r_0, so above equation implies

$$\frac{\partial}{\partial r}\left(A_q^2+B_q^2\right)=0,\qquad B_q\frac{\partial A_q}{\partial r}-A_q\frac{\partial B_q}{\partial r}=0, \tag{35}$$

and the solution is

$$A_q^2+B_q^2=0. \tag{36}$$

Above relation means A_q and B_q can be rewritten as

$$A_q=C_q e^{(i/\hbar)R_q(r)},\qquad B_q=F_q e^{(i/\hbar)R_q(r)}, \tag{37}$$

and $C_q=\pm iF_q$ are constants.

Next, let us use the method beyond semiclassical approximation to expand $R_q(r)$ and $K=\omega-\omega_0$ as

$$R_q=R_{q0}+\sum_{i=1}^{\infty}\hbar^i R_{qi}(r),\qquad K=\omega-\omega_0=K_0+\sum_{i=1}^{\infty}\hbar^i K_i, \tag{38}$$

so we get

$$\begin{aligned}
\hbar^0&:\begin{pmatrix}-i\frac{K_0}{f} & \frac{\partial R_{q0}}{\partial r}\\ \frac{\partial R_{q0}}{\partial r} & -i\frac{K_0}{f}\end{pmatrix}\begin{pmatrix}C_q\\ F_q\end{pmatrix}=0,\\
\hbar^1&:\begin{pmatrix}-i\frac{K_1}{f} & \frac{\partial R_{q1}}{\partial r}+\frac{if'}{4f}\\ \frac{\partial R_{q1}}{\partial r}+\frac{if'}{4f} & -i\frac{K_1}{f}\end{pmatrix}\begin{pmatrix}C_q\\ F_q\end{pmatrix}=0,\\
\hbar^k&:\begin{pmatrix}-i\frac{K_k}{f} & \frac{\partial R_{qk}}{\partial r}\\ \frac{\partial R_{qk}}{\partial r} & -i\frac{K_k}{f}\end{pmatrix}\begin{pmatrix}C_q\\ F_q\end{pmatrix}=0,\qquad k\geq 2.
\end{aligned} \tag{39}$$

Therefore, the determinants of matrices vanish:

$$\begin{aligned}
\hbar^0&:\ R_{q0\pm}=\pm\int\frac{K_0}{f}\,dr,\\
\hbar^1&:\ R_{q1\pm}=\pm\int\frac{K_1-if'(r_+)/4}{f}\,dr,\\
\hbar^k&:\ R_{qk\pm}=\pm\int\frac{K_k}{f}\,dr,\quad k\geq 2,\\
\operatorname{Im}R_{qi}&=\operatorname{Im}R_{qi+}-\operatorname{Im}R_{qi-}=\frac{2\pi K_i}{f'(r_+)}.
\end{aligned} \tag{40}$$

In order to calculate the tunneling rate and Hawking temperature, we rewrite $\operatorname{Im}R_{qi}=(\beta_i/A_h^i)\operatorname{Im}R_{q0}$ ($i\geq 0$, A_h is area of black hole, and β_i are dimensionless constant parameters) since the forms of R_{qj} are the same, so the total radial action is

$$\begin{aligned}
\operatorname{Im}R_q&=\operatorname{Im}R_{q0}(r)+\sum_{i=1}^{\infty}\hbar^i R_{qi}(r)\\
&=\left(1+\sum_{i=1}^{\infty}\beta_i\frac{\hbar^i}{A_h^i}\right)R_{q0}.
\end{aligned} \tag{41}$$

The tunneling rate of Dirac particle at event horizon is given by

$$\begin{aligned}
\overline{\Gamma}_h&=\exp\left[-\frac{2}{\hbar}\left(1+\sum_{i=1}^{\infty}\beta_i\frac{\hbar^i}{A_h^i}\right)R_{q0}\right]\\
&=\exp\left[-\frac{4\pi K_0}{\hbar f'(r_+)}\left(1+\sum_{i=1}^{\infty}\beta_i\frac{\hbar^i}{A_h^i}\right)\right].
\end{aligned} \tag{42}$$

From the relation between tunneling rate and Hawking radiation, we get the temperature of black holes

$$T_h=\left(1+\sum_{i=1}^{\infty}\beta_i\frac{\hbar^i}{A_h^i}\right)T_H. \tag{43}$$

Finally, the laws of black hole thermodynamics request

$$\begin{aligned}
S_h&=\int dS_h=\int\left.\frac{dM-AdQ}{T_h}\right|_{r=r_+}\\
&=\frac{A_h}{4\pi}+\pi\beta_1\ln(A_h)+\cdots\\
&=S_H+\pi\beta_1\ln(S_H)+\cdots
\end{aligned} \tag{44}$$

and $S_H=A_h/4\pi$ is entropy of semiclassical approximation, and this result shows the correction of entropy is logarithmic correction.

In this paper, we studied fermions tunneling from higher-dimensional Reissner-Nordström black holes and obtained the Hamilton-Jacobi equation from charged Dirac equation. This work shows that semiclassical Hamilton-Jacobi equation can describe the property of both 0 spin scalar particles

and 1/2 spin fermions. In this work, we did not emphasize dimensions of space-time larger than (3 + 1) dimensions, so the method also can be used in the research of (3 + 1)-dimensions and lower cases.

As we all know, the information loss is an open problem in black hole physics, and the information of particles maybe vanishes at the singularity. In order to solve this difficulty, Horowitz and Maldacena proposed a boundary condition, which is called the black hole final state, at singularity of black hole to perfectly entangle the incoming Hawking radiation particles and the collapsing matter [66, 67]. Due to the boundary condition, any particle which is falling into the black holes completely annihilates. It is a new and interesting idea to investigate the black hole physics and quantum gravity, so we also will work on this area in the future.

Competing Interests

The authors declare that they have no competing interests.

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Research Article

Matter Localization on Brane-Worlds Generated by Deformed Defects

Alex E. Bernardini[1] and Roldão da Rocha[2]

[1] *Departamento de Física, Universidade Federal de São Carlos, P.O. Box 676, 13565-905 São Carlos, SP, Brazil*
[2] *Centro de Matemática, Computação e Cognição, Universidade Federal do ABC (UFABC), 09210-580 Santo André, SP, Brazil*

Correspondence should be addressed to Roldão da Rocha; roldao.rocha@ufabc.edu.br

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Localization and mass spectrum of bosonic and fermionic matter fields of some novel families of asymmetric thick brane configurations generated by deformed defects are investigated. The localization profiles of spin 0, spin 1/2, and spin 1 bulk fields are identified for novel matter field potentials supported by thick branes with internal structures. The condition for localization is constrained by the brane thickness of each model such that thickest branes strongly induce matter localization. The bulk mass terms for both fermion and boson fields are included in the global action as to produce some imprints on mass-independent potentials of the Kaluza-Klein modes associated with the corresponding Schrödinger equations. In particular, for spin 1/2 fermions, a complete analytical profile of localization is obtained for the four classes of superpotentials here discussed. Regarding the localization of fermion fields, our overall conclusion indicates that thick branes produce a *left-right asymmetric chiral* localization of spin 1/2 particles.

1. Introduction

The brane-world model is a prominent paradigm that has been addressed to solve several questions in physics. Within this framework, brane-worlds are required to render a consistent $4D$ physics of our Universe, at least up to certain sensible limits [1]. In the brane-world scenario all kinds of matter fields should be localized on the brane. In the RS brane-world model [2], the brane is generated by a scalar field coupled to gravity [3, 4], in a particular scenario which may be interpreted as the thin brane limit of thick brane scenarios. Generically, a prominent test that thick brane-world models must pass, to be physically consistent, regards their stability, with respect to tensor, vector, and scalar fluctuations of the background fields that generate the field configurations, namely, the thick brane itself. At least the zero modes of Standard Model matter fields were shown to be localized on several brane-world models [5–10], suggesting that such kind of models is physically viable in high energy physics. Several alternative scenarios, including Gauss-Bonnet terms, $f(R)$ gravity, tachyonic potentials, cyclic defects, and Bloch branes, have been further studied [11–15], and analogous scenarios in an expanding Universe have been approached [16, 17]. The curvature nature of the brane-world, namely, to be a de Sitter, Minkowski, or anti-de Sitter one, is in general obtained *a posteriori*, by solving the $5D$ Einstein field equations. In fact, the bulk and the brane cosmological constants depend upon the brane and the bulk gravitational field content, governed by curvature, and must obey the intrinsic fine-tuning, in the Randall-Sundrum-like models limit.

The analytical study of stability can be uncontrollably intricate, due to the involved structure of the scalar field coupled to gravity. To circumvent the complicated and not analytical approaches, linearized formulations have been commonly worked out. In this context, supported by the stability of deformed defect generated brane-world models, scalar, vector, and tensor perturbations are investigated throughout this work.

Localization aspects of various matter fields with spin 0, spin 1/2, and spin 1 on analytical thick brane-world models are indeed a main concern in deriving brane-world models, since they must describe our physical $4D$ world. The localization

of the spin 1/2 fermions deserves a special attention, since there is no scalar field to couple with in this model, in contrast to thick branes generated by deforming defect mechanisms [18]. Otherwise, Kalb-Ramond fields, although already investigated [19], will not be the main aim here. The spin 1/2 issue has been previously studied in some other contexts [20], including further coupling of more scalar fields in the action [21] and asymmetric brane-worlds generated by a plenty of scalar field potentials [8, 22–25]. In particular, asymmetric Bloch branes in the context of the hierarchy problem have been addressed in [13].

Our aim is to investigate the localization of bulk matter and gauge fields on the brane, in the context where the mass-independent potentials of the corresponding Schrödinger-like equations, regarding the $1D$ quantum mechanical analogue problem, can be suitably acquired from a warped metric. In particular, for a bulk mass proportional to the fermion mass term enclosed by the global action, the possibility of trapping spin 1/2 fermions on asymmetric branes is discussed and quantified.

To accomplish this aim, this paper is organized as follows. In Section 2, a brief review of brane-world scenarios supported by an effective action driven by a (dark sector) scalar field is presented. Warp factors and the corresponding internal brane structure are described for four different analytical models. In Section 3, the *left-right*-chiral asymmetric aspects of matter localization for spin 1/2 fermion fields on thick branes are investigated. Extensions to scalar boson and vector boson fields are obtained in Sections 4 and 5, respectively. Final conclusions are drawn in Section 6.

2. Brane-World Preliminaries and Some Analytical Models

Let one start considering a $5D$ space-time warped into $4D$. The most general $5D$ metric compatible with a brane-world spatially flat cosmological background has the form given by

$$ds^2 = g_{MN}dx^M dx^N = e^{2A(y)} g_{\mu\nu}(x^\alpha)\, dx^\mu dx^\nu + dy^2, \quad (1)$$

where $e^{2A(y)}$ denotes the warp factor, and the signature $(-++++)$ is employed, with $M, N = 0, 1, 2, 3, 5$. $g_{\mu\nu}$ stands for the components of the $4D$ metric tensor ($\mu, \nu = 0, 1, 2, 3$). One can identify $y \equiv x_4$ as the infinite extra dimension coordinate (which runs from $-\infty$ to ∞) and notice that the normal to surfaces of constant y is orthogonal to the brane, into the bulk (brane tension terms have been suppressed/absorbed by the metric (c.f. Equations (24) and (25) from [10] for real scalar field Lagrangians in the context of thick brane solutions)).

The brane-world scenario examined here is set up by an effective action, driven by a (dark sector) scalar field, ζ, coupled to $5D$ gravity, given by

$$S_{\text{eff}} = -\int dx^5 \sqrt{\det g_{MN}} \left[\frac{1}{4}\left(\kappa_5^{-2} R - 2\Lambda_5\right) + \frac{1}{2} g_{MN}\partial^M\zeta\partial^N\zeta - V(\zeta)\right], \quad (2)$$

where R is the $5D$ scalar curvature, $R_{NQ} = g^{BM} R_{BNQM}$ is the Ricci tensor, and $\kappa_5 = (8\pi G_5)^{1/2}$ denotes the $5D$ gravitational coupling constant, hereon set to be equal to unity, where G_5 is the $5D$ Newton constant. The Einstein equations read

$$R_{MN} - \frac{1}{2} R g_{MN} = -\Lambda_5 g_{MN} + \kappa_5^2 T^\zeta_{MN}, \quad (3)$$

where T^ζ_{MN} denotes the energy-momentum tensor corresponding to the matter Lagrangian, regarding the matter field ζ. After solving the $5D$ Einstein field equations, the bulk cosmological constant turns out, in general, to be positive or negative, thus realising a de Sitter or anti-de Sitter brane-world, respectively, generated by curvature. It realises and emulates the interplay involving the $4D$ and $5D$ cosmological constants. Some further possibilities are devised, for example, in [14, 26]; however it is worth mentioning that an additional scalar field can be still added in the action, whose isotropisation will precisely define the nature of the brane-world. This latter case is however beyond the scope of our analysis. Obviously, whatever the possibility to be considered, the thin brane limit must obey the fine-tuning relation [27] $\Lambda_4 = (\kappa_5^2/2)((1/6)\kappa_5^2\sigma^2 + \Lambda_5)$, among the effective $4D$ and $5D$ cosmological constants and the brane tension σ as well.

Considering the real scalar field action, (2), one can compute the stress-energy tensor

$$T^\zeta_{MN} = \partial_M\zeta\partial_N\zeta + g_{MN}V(\zeta) - \frac{1}{2} g_{MN} g^{AB}\partial_A\zeta\partial_B\zeta, \quad (4)$$

which, supposing that both the scalar field and the warp factor dynamics depend only upon the extra coordinate, y, leads to an explicit dependence of the energy density in terms of the field, ζ, and of its first derivative, $d\zeta/dy$, as

$$T^\zeta_{00}(y) = \left[\frac{1}{2}\left(\frac{d\zeta}{dy}\right)^2 + V(\zeta)\right] e^{2A(y)}. \quad (5)$$

With the same constraints on ζ about the dependence on y, the equations of motion currently known from [3, 4], which arise from the above action, are

$$\frac{d^2\zeta}{dy^2} + 4\frac{dA}{dy}\frac{d\zeta}{dy} - \frac{d}{d\zeta}V(\zeta) = 0, \quad (6)$$

through a variational principle relative to the scalar field, ζ, and

$$\frac{3}{2}\frac{d^2A}{dy^2} = -\left(\frac{d\zeta}{dy}\right)^2, \quad (7)$$

through a variational principle relative to the metric, or equivalently to A, manipulated to result into

$$3\left(\frac{dA}{dy}\right)^2 = \frac{1}{2}\left(\frac{d\zeta}{dy}\right)^2 - V(\zeta), \quad (8)$$

after an integration over y.

For the scalar field potential written in terms of a *superpotential*, w, as

$$V(\zeta) = \frac{1}{8}\left(\frac{dw}{d\zeta}\right)^2 - \frac{1}{3}w^2, \quad (9)$$

the above equations are mapped into first-order equations [3, 4] as

$$\frac{d\zeta}{dy} = \frac{1}{2}\frac{dw}{d\zeta}, \tag{10}$$

$$\frac{dA}{dy} = -\frac{1}{3}w, \tag{11}$$

for which the solutions can be found straightforwardly through immediate integrations [3] (see also [10] and references therein). The energy density follows from (9) as

$$T_{00}^{\zeta}(y) = \left[\frac{1}{4}\left(\frac{dw}{d\zeta}\right)^2 - \frac{1}{3}w^2\right]e^{2A(y)}. \tag{12}$$

The analysis of localization aspects of brane-world scenarios will be constrained by some known examples, I, II, III, and IV, for which the warp factor, $A(y)$, and the energy density, $T_{00}(y)$, can be analytically computed. Model I is supported by a sine-Gordon-like superpotential given by

$$w^{\mathrm{I}}(\zeta) = \frac{2}{\sqrt{2}a}\sin\left(\sqrt{\frac{2}{3}}\zeta\right), \tag{13}$$

which reproduces the results from [4]. Model II corresponds to a deformed $\lambda\zeta^4$ theory with the superpotential given by

$$w^{\mathrm{II}}(\zeta) = \frac{3\sqrt{3}}{a}\left(1 - \frac{\zeta^2}{9}\right)^{3/2}. \tag{14}$$

Models III and IV are deformed topological solutions from [28] supported by superpotentials like

$$\begin{aligned} w^{\mathrm{III}}(\zeta) &= \frac{2}{a}\arctan\left[\sinh(\zeta)\right], \\ w^{\mathrm{IV}}(\zeta) &= \frac{1}{4a}\left[\zeta\left(5 - 2\zeta^2\right)\sqrt{1-\zeta^2} + 3\arctan\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right], \end{aligned} \tag{15}$$

where the parameter a fixes the thickness of the brane described by the warp factor, $e^{2A(y)}$. Besides exhibiting analytically manipulable profiles, the above superpotentials have already been discussed in the context of thick brane localization [4, 5, 10]. Models I and II are, respectively, motivated by sine-Gordon and $\lambda\zeta^4$ theories, and models III and IV are obtained (also analytically) from deformed versions of the $\lambda\zeta^4$ model [12]. In particular, models III and IV can also be mapped onto tachyonic Lagrangian versions of scalar field brane models [6, 10, 28].

From the above superpotentials, the respective solutions for $\zeta(y)$ are set as

$$\begin{aligned} \zeta^{\mathrm{I}}(y) &= \sqrt{6}\arctan\left[\tanh\left(\frac{y}{2\sqrt{2}a}\right)\right], \\ \zeta^{\mathrm{II}}(y) &= 3\,\mathrm{sech}\left(\frac{\sqrt{3}y}{2a}\right), \\ \zeta^{\mathrm{III}}(y) &= \mathrm{arcsinh}\left(\frac{y}{a}\right), \\ \zeta^{\mathrm{IV}}(y) &= \frac{y}{\sqrt{a^2+y^2}}, \end{aligned} \tag{16}$$

where one has suppressed any additional (irrelevant) constant of integration for convenience, and one has just considered the positive solutions (in (16) there could be explicit constant of integration that amounts to letting $y \mapsto y + C$, corresponding to the position of the brane in the extra dimension, for which one has set $C = 0$).

The obtained expressions for the warp factor as resulting from (11) are, respectively, given by

$$\begin{aligned} A^{\mathrm{I}}(y) &= -\ln\left[\cosh\left(\frac{y}{\sqrt{2}a}\right)\right], \\ A^{\mathrm{II}}(y) &= \tanh\left(\frac{\sqrt{3}y}{2a}\right)^2 - 2\ln\left[\cosh\left(\frac{\sqrt{3}y}{2a}\right)\right], \\ A^{\mathrm{III}}(y) &= \frac{1}{3}\left[\ln\left(1 + \frac{y^2}{a^2}\right) - 2\frac{y}{a}\arctan\left(\frac{y}{a}\right)\right], \\ A^{\mathrm{IV}}(y) &= -\frac{1}{12}\left[\frac{y^2}{a^2+y^2} + 3\frac{y}{a}\arctan\left(\frac{y}{a}\right)\right], \end{aligned} \tag{17}$$

where integration constants are introduced as to set a normalization criterion for which $A(0) = 0$.

The solutions for A^{I} and A^{II} are depicted in Figure 1. The corresponding localized energy densities computed through (12) are, respectively, given by

$$\begin{aligned} T_{00}^{\mathrm{I}}(y) &= \frac{3}{4a^2}\,\mathrm{sech}\left(\frac{y}{\sqrt{2}a}\right)^2\left[\mathrm{sech}\left(\frac{y}{\sqrt{2}a}\right)^2 - 2\tanh\left(\frac{y}{\sqrt{2}a}\right)^2\right], \\ T_{00}^{\mathrm{II}}(y) &= \frac{9}{8a^2}\,\mathrm{sech}\left(\frac{\sqrt{3}y}{2a}\right)^8\tanh\left(\frac{\sqrt{3}y}{2a}\right)^2 \cdot \left[7\cosh\left(\frac{\sqrt{3}y}{a}\right) - \cosh\left(\frac{2\sqrt{3}y}{a}\right)\right] \cdot e^{2\tanh(\sqrt{3}y/2a)^2}, \end{aligned}$$

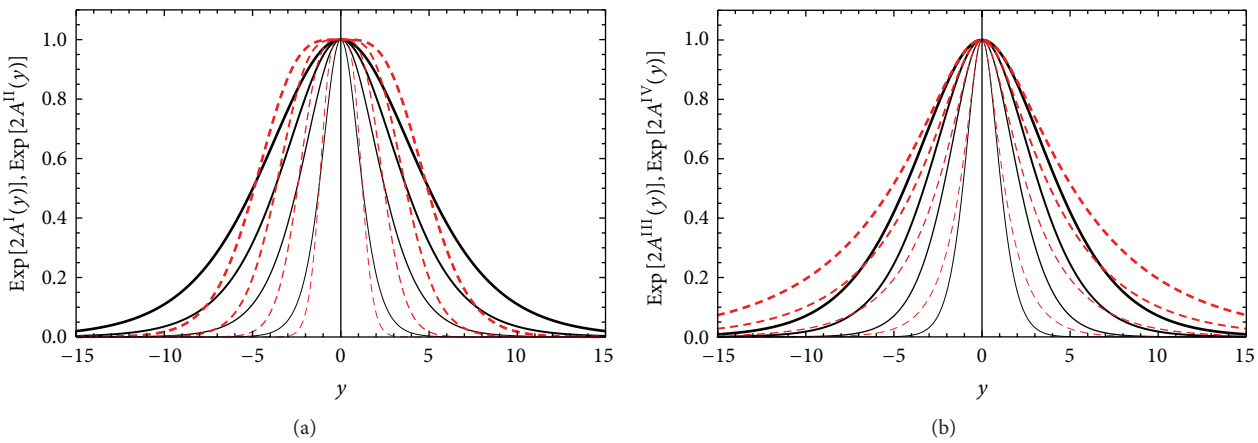

FIGURE 1: (a) Warp factors, $e^{2A(y)}$ for models I (solid (black) lines) and II (dashed (red) lines). (b) Warp factors, $e^{2A(y)}$ for models III (solid (black) lines) and IV (dashed (red) lines). One has considered integer values of the brane width parameter a, running from 1 (thinnest line) to 4 (thickest line), corresponding to an increasing thickness.

$$T_{00}^{\mathrm{III}}(y) = \frac{1}{a^2}\left(1+\frac{y^2}{a^2}\right)^{-1/3}\left(1 - \frac{4}{3}\left(1+\frac{a^2}{y^2}\right)\arctan\left(\frac{y}{a}\right)^2\right)e^{(4y/3a)\arctan(y/a)},$$

$$T_{00}^{\mathrm{IV}}(y) = \left(\frac{a^4}{(a^2+y^2)^3} - \frac{5a^3y+3ay^3+3\left(a^2+y^2\right)^2\arctan\left(y/a\right)^2}{48a^2\left(a^2+y^2\right)^4}\right) \cdot e^{-(y/2a)((ay/3(a^2+y^2))+\arctan(y/a))}. \tag{18}$$

The brane scenarios for models from I to IV are depicted in Figure 1 for the warp factors and in Figure 2 for the energy densities, from which one can observe that models from I to IV give rise to thick branes, most of them with no internal structures. In fact, only the potential that controls the scalar field from model II allows the emergence of thick branes that host internal structures in the form of a layer of a novel phase enclosed by two separate interfaces, inside which the energy density of the matter field gets more concentrated. It is related to the extension/localization of the warp factor; namely, when the profiles depicted in Figure 1 approach a *plateau* form in the region inside the brane, the corresponding internal structure is observed through its energy profile.

The appearance of negative energy densities in the plots for T_{00} may be related to a predominance of the scalar field potential over the kinetic-like term related to the coordinate y. Speculatively, it indicates that the vacuum minimal energy can be adjusted by the inclusion of some additional term, eventually related to the cosmological constant.

The localization of bulk matter fields on thick branes generated by each one of these models will be identified in the following sections. Spin 0, spin 1/2, and spin 1 fields will evolve coupled to gravity and, as usual, the bulk matter field contribution to the bulk energy will be neglected. It means that the obtained solutions hold in the presence of the bulk matter, without disturbing the bulk geometry.

3. Asymmetric *Left-Right* Matter Localization for Spin 1/2 Fermion Fields

To investigate the localization of bulk matter on the brane, one first considers that fermion localization on brane-worlds is usually accomplished when the $5D$ Dirac algebra is realised by the objects $\Gamma^M = e^M_{\overline{M}}\Gamma^{\overline{M}}$, where $e^M_{\overline{M}}$ denotes the *fünfbein*, Γ^M satisfy the Clifford relation $\{\Gamma^M, \Gamma^N\} = 2g^{MN}$, and $\Gamma^{\overline{M}}$ are the gamma matrices in the $5D$ flat space-time. Hereupon $\overline{M}, \overline{N}, \ldots = 0, 1, 2, 3, 5$ and $\overline{\mu}, \overline{\nu}, \ldots = 0, 1, 2, 3$ denote the $5D$ and $4D$ local Lorentz indexes, respectively. The *fünfbein* is provided by $e^{\overline{M}}_M = \{e^A\tilde{e}^{\overline{\nu}}_\mu, e^A\}$, where $\Gamma^M = e^{-A}(\gamma^\mu, \gamma^5)$, and $\gamma^\mu = \tilde{e}^\mu_{\overline{\nu}}\gamma^{\overline{\nu}}$ and γ^5 are, respectively, the $4D$ gamma matrices and the $4D$ volume element, respectively. The Dirac action for a spin 1/2 fermion with a mass term can be expressed as [20, 25]

$$S_{1/2} = \int d^5x\sqrt{-g}\left[\overline{\Psi}\Gamma^M\left(\partial_M + \omega_M\right)\Psi - MF(z)\overline{\Psi}\Psi\right]. \tag{19}$$

Here $\omega_R = (1/4)\omega_R^{\overline{RS}}\Gamma_{\overline{R}}\Gamma_{\overline{S}}$ is the spin connection, where

$$\omega_R^{\overline{RS}} = -\frac{1}{2}e^{T\overline{R}}e^{Q\overline{S}}\partial_{[T}e_{Q]\overline{T}}e^{\overline{T}}_R + \frac{1}{2}e^{S[\overline{R}}\partial_{[R}e_{S]}{}^{\overline{S}]}, \tag{20}$$

and $F(z)$ is some general scalar function, providing a mass term with a kink-like profile, which from this point is written in terms of a conformal variable z such that $dz = e^{-A(y)}dy$ regards a transformation to conformal coordinates. This kind of mass term is introduced in the action, for it has played

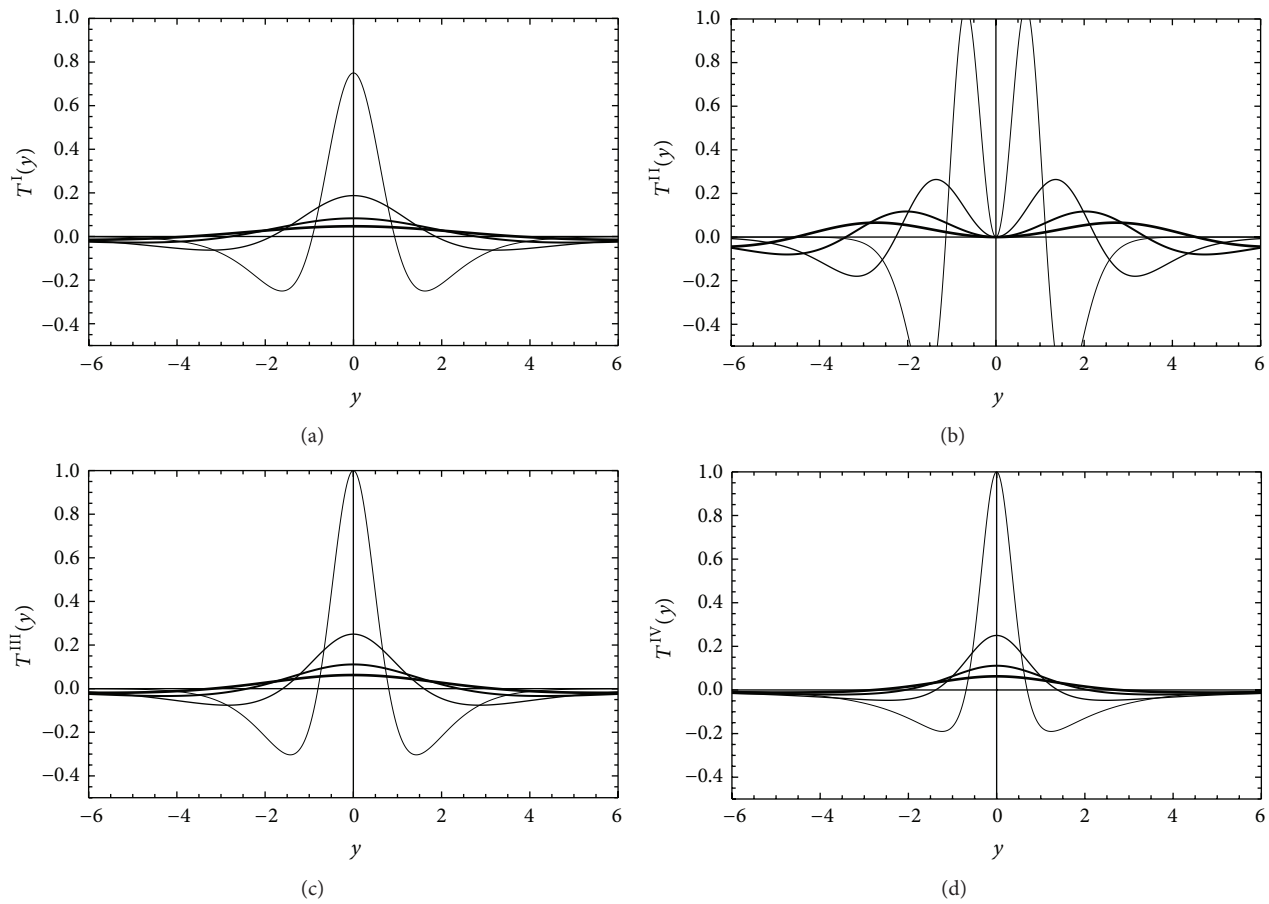

Figure 2: Energy density, $T_{00}(y)$, for models from I (a) to IV (d), where integer values of the brane width parameter a, running from 1 (thinnest line) to 4 (thickest line), corresponding to an increasing thickness.

a critical role on the localization of fermionic fields on a Minkowski brane. The components of the spin connection ω_M with respect to (1) are $\omega_\mu = (1/2)(\partial_z A)\gamma_\mu\gamma_5 + \widehat{\omega}_\mu$, where $\widehat{\omega}_\alpha = (1/4)\overline{\omega}_\alpha^{\overline{\mu}\overline{\nu}}\Gamma_{\overline{\mu}}\Gamma_{\overline{\nu}}$ is the spin connection derived from the metric $g_{\mu\nu} = \widetilde{e}_\mu^{\overline{\mu}}\widetilde{e}_\nu^{\overline{\nu}}\eta_{\overline{\mu}\overline{\nu}}$. Thus, the equation of motion corresponding to the action (19) reads

$$\left[\gamma^\mu\left(\partial_\mu + \widehat{\omega}_\mu\right) + \gamma^5\left(\partial_z + 2\partial_z A\right) - e^A MF(z)\right]\Psi = 0. \quad (21)$$

The $5D$ Dirac equation can be hence studied by taking spinors with respect to $4D$ effective fields. In this way the chiral splitting yields

$$\Psi = e^{-2A(z)}\left(\sum_n \psi_{Ln}(x^\mu) L_n(z) + \psi_{Rn}(x^\mu) R_n(z)\right), \quad (22)$$

where $L_n(z)$ and $R_n(z)$ are the well-known KK modes, and $\psi_{Rn}(x^\mu) = +\gamma^5\psi_{Rn}(x^\mu)$ $[\psi_{Ln}(x^\mu) = \gamma^5\psi_{Ln}(x^\mu)]$ is the right-chiral [left-chiral] component of a $4D$ Dirac field, respectively. In addition, the sum over n can be both continuous and discrete. Assuming that $\gamma^\mu(\partial_\mu + \widehat{\omega}_\mu)\psi_{(R,L)n} = m_n\psi_{(L,R)n}$, the $L_n(z)$ and $R_n(z)$ functions should then satisfy the subsequent coupled equations,

$$\left[\partial_z - e^A MF(z)\right] R_n(z) = -m_n L_n(z), \quad (23a)$$

$$\left[\partial_z + e^A MF(z)\right] L_n(z) = m_n R_n(z). \quad (23b)$$

The associated Schrödinger-like equations can be thus acquired for the left- and right-chiral KK modes of fermions, respectively, as

$$\begin{aligned}\left(-\partial_z^2 + V_L(z)\right) L_n &= m_{L_n}^2 L_n,\\ \left(-\partial_z^2 + V_R(z)\right) R_n &= m_{R_n}^2 R_n,\end{aligned} \quad (24)$$

where the mass-independent potentials are given by

$$\begin{aligned}V_L(z) = {} & e^{2A}M^2F^2(z) - e^A A' MF(z)\\ & - e^A M\partial_z F(z),\end{aligned} \quad (25a)$$

$$\begin{aligned}V_R(z) = {} & e^{2A}M^2F^2(z) + e^A A' MF(z)\\ & + e^A M\partial_z F(z).\end{aligned} \quad (25b)$$

Note that the Schrödinger-like equations (24) can be transformed into $U^{\dagger}UL_n = m_n^2 L_n$ and $UU^{\dagger}R_n = m_n^2 R_n$, where $U \equiv \partial_z + e^A MF(z)$. This observation is based upon supersymmetric quantum mechanics, implying that the mass squared is nonnegative.

In order to lead these results to the standard $4D$ action for a massless fermion and a series of massive chiral fermions, the action $S = \sum_n \int d^4x\sqrt{-g}\overline{\psi}_n[\gamma^\mu(\partial_\mu + \widehat{\omega}_\mu) - m_n]\psi_n$ is employed, for orthonormalization conditions

$$\int_{-\infty}^{+\infty} L_m L_n dz = \delta_{mn} = \int_{-\infty}^{+\infty} R_m R_n dz, \qquad \int_{-\infty}^{+\infty} L_m R_n dz = 0. \tag{26}$$

In formulae (23a) and (23b), by setting $m_n = 0$, thus it yields

$$L_0 \propto e^{-M\int e^A F dz}, \qquad R_0 \propto e^{M\int e^A F dz}. \tag{27}$$

Hence, either the massless left- or right-chiral KK fermion modes can be localized on the brane, being the other one nonnormalizable.

By taking $F(z) = \zeta(z)$, regarding (16), it yields

$$V_L(z(y)) = e^{2A(y)}\left(M^2\zeta^2(y) - \frac{dA(y)}{dy}M\zeta(y) - M\frac{d\zeta(y)}{dy}\right), \tag{28a}$$

$$V_R(z(y)) = e^{2A(y)}\left(M^2\zeta^2(y) + \frac{dA(y)}{dy}M\zeta(y) + M\frac{d\zeta(y)}{dy}\right). \tag{28b}$$

Equations (28a) and (28b) evince that when the mass term in the action (19) regards $M = 0$, the potentials for left- and right-chiral KK modes $V_{L,R}(z)$ vanish. Then both chiral fermions cannot be localized on the thick brane. Moreover, if $V_L(z)$ and $V_R(z)$ are demanded to be $\mathbb{Z}_2$-even with respect to the extra dimension z, then the mass term $MF(z)$ must be an odd function of z [29]. In fact, some useful classes of brane-world models have the extra dimension topology $S^1/\mathbb{Z}_2$. If the background scalar is an odd function of extra warped dimension, the Yukawa coupling, between the fermion and the background scalar field, assures the localization mechanism for fermions [29]. For the majority of brane-world models, the scalar field ζ is, usually, a kink, being an odd function of the extra dimension. Here we do not necessarily impose this condition, in order to not preclude asymmetric solutions, with respect to the extra dimension.

In what follows the profile of the above *left-right* potentials is depicted in Figure 3 for different values of the localization parameter a. In fact, the potentials $V_{L,R}(z)$ have asymptotic behaviors that tend to zero from up, as $y \to \pm\infty$, for all models from I to IV. In model I, at $y = 0$ the potential $V_R(z)$ attains its maximum positive value, a global maximum, for $a = 1$. The potential $V_R(z)$ changes to a volcano-type profile along the interval of $1 < a < 2$, such that for $a = 2, 3, 4, \ldots$ the point $y = 0$ regards a local minimum, which allows for producing unstable resonances, which can be tunneled to the outside of the potential. Nevertheless, the potential $V_L(z)$ has the associated minima at $y = 0$ for all positive integer values of a, $a = 2, 3, 4, \ldots$, and it creates the conditions for producing bound states. A very similar behavior is exhibited by model III, in spite of showing different amplitudes. Model IV is quite similar to these models, with the only qualitative difference concerning the fact that, at $y = 0$, the potential $V_R(z)$ attains its maximum positive value, a global maximum, for $a = 1$ and $a = 2$. For model IV, the stability conditions created by the right- and left-chiral volcano-type potentials are more sensible to the increasing of the brane width ($a \gtrsim 3$), in comparison to models I and II ones ($a \gtrsim 2$), inducing no mass gap to separate the fermion zero mode from the excited KK massive modes. In these cases, there exist continuous spectra for the Kaluza-Klein modes of fermions of both chiralities. These volcano-type potentials imply the existence of resonant or metastable states of fermions which can tunnel from the brane to the bulk [9]. The left-chiral KK mode has a continuous gapless spectrum for models I, III, and IV, according to Figure 3. Since the potential for left-chiral fermions presents a negative value at the brane location for these models, the zero modes of right- and left-chiral fermions, $R_0(y)$ and $L_0(y)$, are the only necessary ingredient to be tested to be localized on the brane. For model II, both potentials for the left- and right-chiral fermions have positive values of the potential, irrespective of y. However for both cases $V_{R,L}(y)$, when $a \lesssim 1$, an asymmetric behavior emerges and produces a totally odd symmetric well-barrier profile in the limit of $a \to 0$. Except for $0 < a \lesssim 1$, the zero mode of left- and right-chiral fermions can not be trapped. All potentials for model II are asymmetric (except for $a = 0$, which is nonsense in the brane context), have maxima at $y = 0$, and tend to zero at $y \to \pm\infty$, and there is no bound state for right-chiral fermions. In particular, for $V_{R,L}(y)$ when $a = 1$, the minima occur at $y \sim \pm 0.87$.

4. Matter Localization for Spin 0 Scalar Fields

The localization of scalar fields on thick branes generated by deformed defects can also be considered from this point. In particular, an interesting approach on domain walls can be also found in [30]. In fact, a massive scalar field coupled to gravity can be described by the following action:

$$S_0 = -\frac{1}{2}\int d^5x\sqrt{-g}\left(g_{MN}\partial^M\Phi\partial^N\Phi + m_0^2\Phi^2\right), \tag{29}$$

where m_0 denotes the effective mass of a bulk scalar field, Φ, and from where one can check whether spin 0 matter fields can be trapped on the thick brane. By employing the metric

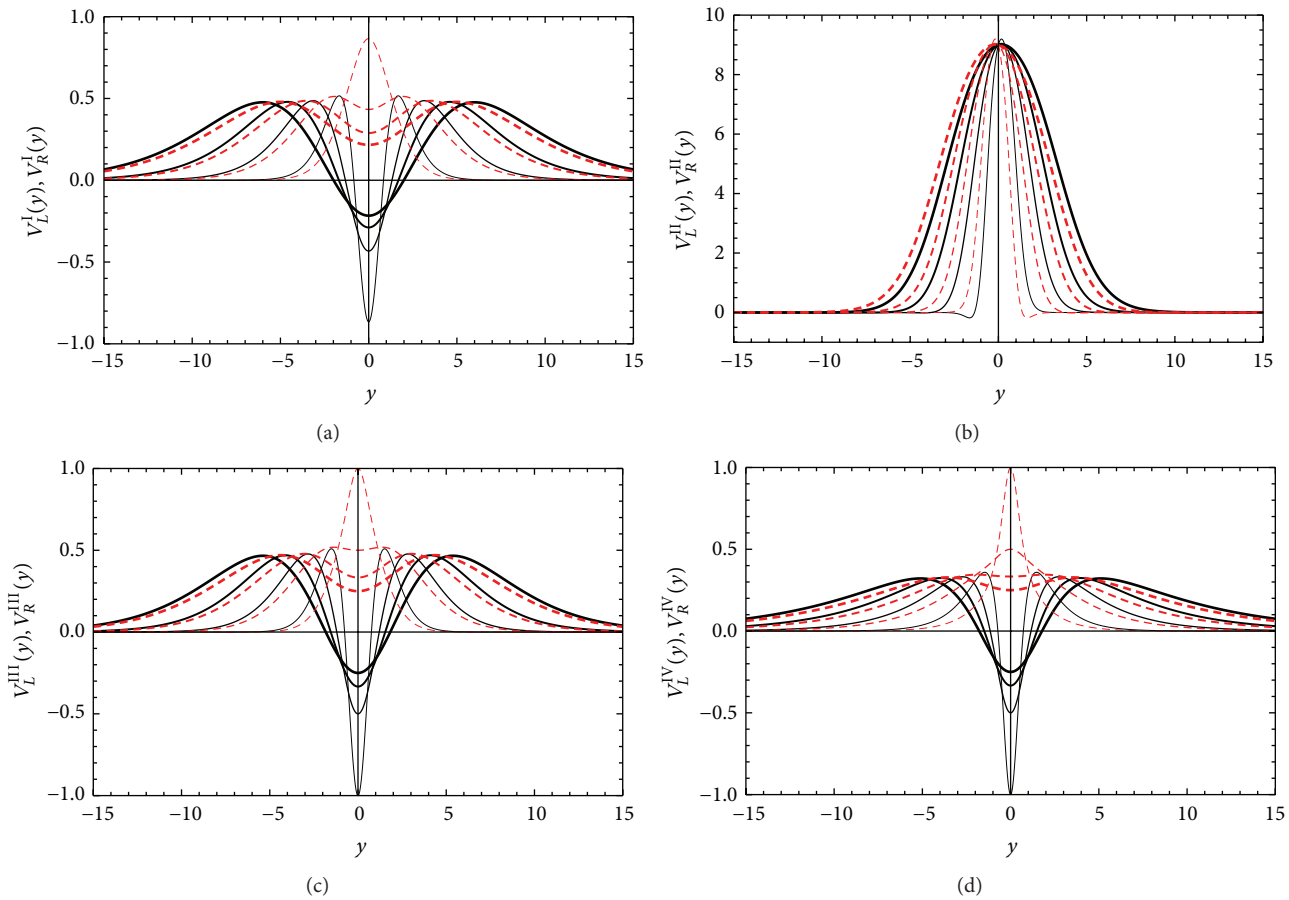

Figure 3: Associated Schrödinger-like quantum mechanical potentials, $V_L(y)$ and $V_R(y)$, respectively, for left-chiral (solid (black) lines) and right-chiral (dashed (red) lines) KK modes of fermions, for models from I (a) to IV (d). Again, one has considered integer values of the brane width parameter a, running from 1 (thinnest line) to 4 (thickest line), corresponding to an increasing thickness. The fermion mass parameter has been assumed to be equal to unit, $M = 1$.

(1), the associated equation of motion from the action in (29) reads

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) + e^{-3A}\partial_z\left(e^{3A}\partial_z\Phi\right) - e^{2A}m_0^2\Phi = 0. \tag{30}$$

Hence, by the KK decomposition $\Phi(x^\mu, z) = \sum_n \chi_n(x^\mu)\xi_n(z)e^{-3A/2}$, where ξ_n is assumed to satisfy the $4D$ Klein-Gordon equation $[\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)/\sqrt{-g} - m_n^2]\xi_n(x^\mu) = 0$, m_n being the $4D$ mass of the KK excitation of the scalar field. Then the scalar KK mode $\xi_n(z)$ is ruled by the following equation:

$$\left[-\partial_z^2 + V_0(z)\right]\xi_n(z) = m_n^2\xi_n(z). \tag{31}$$

This equation is a Schrödinger one, with effective potential given by

$$\begin{aligned} V_0(z(y)) &= \frac{3}{2}A''(z) + \frac{9}{4}A'^2(z) + e^{2A(z)}m_0^2 \\ &= e^{2A(z)}\left(\frac{3}{2}\frac{d^2A(y)}{dy^2} + \frac{15}{4}\left(\frac{dA(y)}{dy}\right)^2 + m_0^2\right). \end{aligned} \tag{32}$$

The profile of the above scalar boson potential is depicted in Figure 4 (solid (black) lines) for different values of the localization parameter a. For $m_0 = 1$, only brane scenarios with $a \lesssim 2$ provide conditions to have a localized scalar field. Even in this case, such localized states behave much more as resonances than as bound states, given that they can be tunneled out of the potential. Bound states appear only for noninteger values of the brane width such that $a < 1$, which will correspond to typical volcano-type potentials.

5. Matter Localization for Spin 1 Vector Fields

One now turns to spin 1 vector fields and begins with the $5D$ action of a vector field:

$$S_1 = -\frac{1}{4}\int d^5x\sqrt{-g}g^{MN}g^{RS}F_{MR}F_{NS}, \tag{33}$$

where $F_{MN} = \partial[_M A_N]$ denotes the field strength tensor. A $5D$ spin 1 field can be now studied via the KK decomposition $A_M(x^\rho, z) = \sum_n a_M^{(n)}(x^\rho)\tau_n(z)$. The action of the $5D$ massless

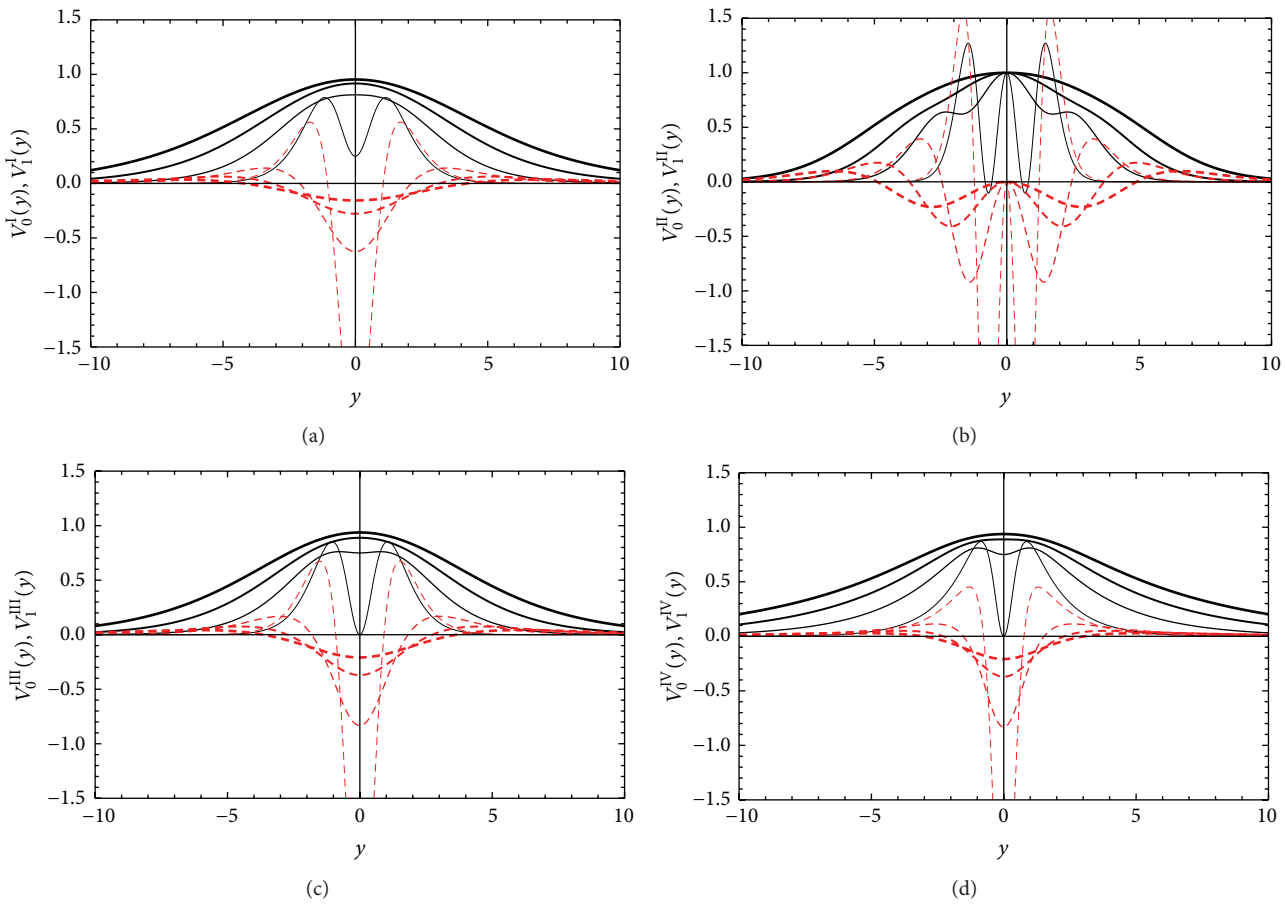

FIGURE 4: Associated Schrödinger-like quantum mechanical potentials, $V_0(y)$ and $V_0(y)$, respectively, for spin 0 (solid (black) lines) and spin 1 (dashed (red) lines) bosons for models from I (a) to IV (d). Again, one has considered integer values of the brane width parameter a running from 1 (thinnest line) to 4 (thickest line), corresponding to an increasing thickness. The spin 0 boson mass parameter has been assumed to be equal to unit.

vector field (33) is invariant under the following gauge transformation:

$$\begin{aligned} A_M(x^\rho, z) &\longmapsto \widetilde{A}_M(x^\rho, z) \\ &= A_M(x^\rho, z) + \partial_M F(x^\rho, z), \end{aligned} \tag{34}$$

where $F(x^\rho, z)$ denotes any arbitrary regular scalar function, for $M = \mu, 5$. The field component $A_5(x^\rho, z)$ equals zero [25], by this gauge. In fact, (34) yields

$$\widetilde{A}_5(x^\rho, z) = \sum_n a_5^{(n)}(x^\rho)\tau_n(z) + \partial_z F(x^\rho, z). \tag{35}$$

By choosing $F(x^\rho, z) = -\sum_n a_5^{(n)}(x^\rho)\int \tau_n(z)dz$, [25] then $\widetilde{A}_5 = 0$, and hence the action (33) leads to the space-time action

$$S_1 = -\frac{1}{4}\int d^5x\sqrt{-g}\left(F^{\mu\nu}F_{\mu\nu} + 2e^{-A}g^{\mu\nu}A'_\mu A'_\nu\right), \tag{36}$$

where $(\,)' = \partial_z$. Given a set of orthonormal functions $\tau_n(z)$, playing the role of spin 1 Kaluza-Klein modes, and the decomposition of the vector field $A_\mu(x^\rho, z) = \sum_n a_\mu^{(n)}(x^\mu)\tau_n(z)e^{-A/2}$, the action (36) reads

$$S_1 = -\frac{1}{2}\sum_n \int d^4x\sqrt{-g}\left(\frac{1}{2}f^{(n)\mu\nu}f_{\mu\nu}^{(n)} + m_n^2 a^{(n)\mu}a_\mu^{(n)}\right), \tag{37}$$

where $f_{\mu\nu}^{(n)} = \partial_{[\mu}a_{\nu]}^{(n)}$ stands for the $4D$ field strength tensor. The KK modes $\tau_n(z)$ satisfy the Schrödinger equation

$$\left(-\partial_z^2 + V_1(z)\right)\tau_n(z) = m_n^2\tau_n(z), \tag{38}$$

where the mass-independent potential reads [31]

$$\begin{aligned} V_1(z) &= \frac{1}{4}A'^2(z) + \frac{1}{2}A''(z) \\ &= e^{2A(z)}\left[\frac{1}{2}\frac{d^2A(y)}{dy^2} + \frac{3}{4}\left(\frac{dA(y)}{dy}\right)^2\right]. \end{aligned} \tag{39}$$

The profile of the above vector boson potential is depicted in Figure 4 (dashed (red) lines) for different values of

the localization parameter a. All the thick brane scenarios with $a \gtrsim 1$ provide localization conditions to have vector field bound states. Increasing values of a lead to more stable bound states.

6. Conclusions and Discussion

Thick branes driven by superpotentials supported by deformed defects (c.f. (13)–(15)) for various bulk matter fields of spin 0, spin 1/2, and spin 1 have been investigated. For spin 1 gauge fields, the profile of the associated vector boson potential showed that, in the thick brane models for $a \gtrsim 1$, localization conditions hold as to guarantee the existence of vector field bound states. Quantitatively, increasing values of the brane thickness parameter, a, lead to more stable bound states.

Concerning spin 0 (scalar) fields, the profile of the potential evinces that only thick brane scenarios with $a \gtrsim 4$ provide localization conditions compatible to scalar field bound states.

The most intricate result is related to spin 1/2 fields. In fact, for fermionic fields, left-right potentials were deeply studied and for the four models considered here, the issue of localization has been scrutinized. It is worth pointing out that models I, III, and IV admit volcano-type potentials, inducing no mass gap to separate the fermion zero mode from the excited KK massive modes. Hence continuous spectra for the Kaluza-Klein modes of fermions of both chiralities are allowed. A refined analysis of the values of the a parameter in these models, influencing the localization of fermionic fields, was provided. Model II, induced by the superpotential (14), reveals a peculiar behavior. In this model, right-chiral fermions have positive values of the potential irrespective of the extra dimension when $a = 1$. Hence, except for this value, the zero mode of left- and right-chiral fermions can not be trapped. All potentials for model II are asymmetric and have maxima at $y = 0$ and minima at $y \to \pm\infty$, and there is no bound state for right-chiral fermions, but, again, for $V_{R,L}(y)$ when $a = 1$.

It is worth mentioning that, for the localization of a fermion zero mode, the mass term $MF(z)\overline{\Psi}\Psi$ was considered in the $5D$ action. An interesting approach concerning such mass term in (19) has been studied, corresponding to the so-called singular dark spinors [32, 33]. Such massive mass dimension one quantum fields are prime candidates for the dark matter problem, also presenting possible signatures at LHC [34]. It generates a slightly different action responsible for spin 1/2 matter fields localization [35, 36]. Such approach can be also extended in the context of deformed defects here presented.

Competing Interests

The authors declare that they have no competing interests.

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Research Article

The Effects of Minimal Length, Maximal Momentum, and Minimal Momentum in Entropic Force

Zhong-Wen Feng,[1] Shu-Zheng Yang,[2] Hui-Ling Li,[1,3] and Xiao-Tao Zu[1]

[1] *School of Physical Electronics, University of Electronic Science and Technology of China, Chengdu 610054, China*
[2] *Department of Astronomy, China West Normal University, Nanchong 637009, China*
[3] *College of Physics Science and Technology, Shenyang Normal University, Shenyang 110034, China*

Correspondence should be addressed to Zhong-Wen Feng; zwfengphy@163.com

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The modified entropic force law is studied by using a new kind of generalized uncertainty principle which contains a minimal length, a minimal momentum, and a maximal momentum. Firstly, the quantum corrections to the thermodynamics of a black hole are investigated. Then, according to Verlinde's theory, the generalized uncertainty principle (GUP) corrected entropic force is obtained. The result shows that the GUP corrected entropic force is related not only to the properties of the black holes but also to the Planck length and the dimensionless constants α_0 and β_0. Moreover, based on the GUP corrected entropic force, we also derive the modified Einstein's field equation (EFE) and the modified Friedmann equation.

1. Introduction

The existence of thermodynamics of black holes is a great discovery for the foundations of physics [1–11]. This idea was proposed by Bekenstein who proved that the entropy of a black hole is $S = Ak_Bc^3/4\hbar G$, where A is the horizon area, k_B is the Boltzmann constant, $\hbar$ is the Planck constant, and G is Newton's gravitational constant, respectively [2]. Later, based on the entropy of black hole, Hawking showed that the Schwarzschild (SC) black hole emits thermal radiation and the temperature of SC black hole is proportional to the surface gravity on the event horizon κ; namely, $T = \kappa/2\pi$ [5, 6]. Furthermore, in [9, 10], the authors proved that the entropy and the temperature of black hole satisfy the laws of thermodynamics. These discoveries indicate that the thermodynamics of black hole have profound connection with the gravity.

In order to investigate the deeper-seated relation between thermodynamics and the gravity, Jacobson assumed the spacetime as a kind of gas, its entropy is proportional to the area, then using the fundamental Clausius relation and the equivalence principle, he demonstrated that the Einstein field equation is nothing but a state equation of this kind of gas. Following this agreement, the Einstein field equation can be derived from the first law of thermodynamics together with the relation between the entropy and the horizon area of a black hole [12]. Subsequently, Padmanabhan pointed out that, in the static spherically symmetric spacetimes, the gravitational field equations on the horizon can be rewritten as a form of the ordinary first law of thermodynamics [13]. Inspired by Padmanabhan's idea, people found that Einstein's equation is a thermodynamic identity. Those works explain why the field equations should encode information of horizon thermodynamics [14–19].

In 2011, Verlinde proposed a remarkable new perspective on the relation between the gravity and the thermodynamics. Based on Sakharov's idea [20] and the holographic principle, Verlinde pointed out that the gravity is no longer a fundamental force; instead, it can be explained as an entropic force which arises from the change of information when material bodies move away from the screens of holographic systems [21]. In his paper, Verlinde showed various interesting results. For example, the the second law of Newton can be obtained by incorporating the entropic force with the Unruh temperature. Using the entropic force together with the holographic principle and the equipartition law of energy, one can yield Newton's law of gravitation. Moreover,

an astounding discovery should be mentioned that Einstein's equation can be derived from the theory of entropic force. This new proposal of the gravity has received wide attention causing people to do many relevant works to discuss the entropic force [22–25].

On the other hand, a lot of works showed that the original thermodynamics of black holes would not be held when considering the quantum gravity effects [26–31]. Various theories of quantum gravity suggest the existence of a minimal observable length, which can be identified with the Planck scale. This view is supported by many Gedanken experiments and is applied to different physical systems [32–35]. The generalized uncertainty principle (GUP) is one of the most important theories, which is modified by the minimum measurable length. The applications of GUP has been widely studied [36–45]. In particular, the GUP effect on the micro black holes has been deeply discussed in [46–48]. In [49], combining the GUP with the thermodynamics of black holes, the authors investigated the modified Hawking temperature and the entropy; their results showed that the GUP corrected thermodynamics of black holes are different from those of the original cases. The GUP can stop the Hawking radiation in the final stages of black holes' evolution and lead to the remnants of black holes. Therefore, the GUP is considered as a good tool to solve the information paradox problem of black holes. In [50–57], people have investigated the GUP corrected entropic force by using the GUP corrected thermodynamics. The implications of modified entropic force have been investigated in many contexts such as cosmology [50–53], Hamiltonians of the quantum systems in quasispace [54], quantum walk [55], and quarkonium binding [56, 57]. In fact, the expression of GUP is not unique. However, in most of the papers, the expression of GUP is limited to two forms. One form only has a minimal length $\Delta x\Delta p \geq \hbar[1+\alpha_0^2\ell_p^2(\Delta p)^2]$ [30]. The other form has a minimal length and a maximal momentum $\Delta x\Delta p \geq \hbar[1-\alpha_0\ell_p\Delta p+\alpha_0^2\ell_p^2(\Delta p)^2]$ [58]. According to the previous works, it is interesting to raise the question whether it is possible to derive a general form of GUP which contains a minimal length, a minimal momentum, and a maximal momentum. Actually, in [59], the authors pointed out that the existence of a minimal length $\Delta x_{\min}$ comes from the fact that a string cannot probe its distances smaller than the length, and the maximal momentum $\Delta p_{\max}$ originated in the doubly special relativity (DSR) that predicts there exists an upper bound for the momentum of a particle. For the minimal momentum $\Delta p_{\min}$, it is well known that the notion of a plane wave does not appear in the a general curved spacetime; hence, it indicates that there exists a limit to the precision with which the corresponding momentum can be described. According to this phenomenon, one can express a nonzero minimal uncertainty in momentum measurement, that is, $\Delta p_{\min}$. Taking these facts into account, Nozari and Saghafi introduced the most general form of GUP, which admits a minimal length, a minimal momentum, and a maximal momentum. The GUP is given by

$$\Delta x\Delta p \geq \hbar\left[1-\alpha_0\ell_p\Delta p+\alpha_0^2\ell_p^2(\Delta p)^2+\beta_0^2\ell_p^2(\Delta x)^2\right], \quad (1)$$

where Δx and Δp are the uncertainties for position and momentum, α_0 and β_0 are dimensionless constants of the order of unity that depends on the details of the quantum gravity hypothesis, and $\ell_p=\sqrt{\hbar G/c^3}\approx 10^{-35}$ m is the Planck length, respectively [60]. From (1), one can easily obtain the minimal length $\Delta x_{\min}=\alpha_0\ell_p$, the minimal momentum $\Delta p_{\min}=2\beta_0\ell_p$, and the maximal momentum $\Delta p_{\max}=[1+\sqrt{1-(1+\alpha_0^2\beta_0^2\ell_p^4)}]/\alpha_0\ell_p$. Under the standard limit, that is, $\Delta x \gg \ell_p$, (1) becomes the Heisenberg uncertainty principle (HUP). It is well known that GUP has a great effect on the thermodynamics of black holes and the entropic force that would bring many new results. Therefore, in this paper, using (1), we first investigate the modified thermodynamics of a black hole. Then, following Verlinde's viewpoint, the modified number of bits N is obtained. The modified number of bits N leads to the GUP corrected entropic force. Considering the GUP corrected entropic force, the gravitational force, the gravitational potential, Einstein's field equation, and the Friedmann equation are modified.

This paper is organized as follows. The quantum corrections to the entropy and Hawking temperature are derived in Section 2. According to Verlinde's theory, the GUP corrected entropic force is calculated in Section 3. In Sections 4 and 5, using the GUP corrected entropic force, the modified Einstein's field equation and the modified Friedmann equation is investigated. The last section is devoted to the conclusions.

2. The GUP Impact on the Thermodynamics of a Black Hole

In order to calculate the GUP impact on the thermodynamics of a black hole, one needs to solve (1) as a quadratic equation in Δp. The result is given by

$$\Delta p \geq \frac{\Delta x+\hbar\alpha_0\ell_p}{2\hbar\alpha_0^2\ell_p^2}\left\{1-\sqrt{1-\frac{4\hbar^2\alpha_0^2\ell_p^2\left[1+\beta_0^2(\Delta x)^2\ell_p^2\right]}{\left(\Delta x+\hbar\alpha_0\ell_p\right)^2}}\right\}, \quad (2)$$

where we choose the negative-signed solution since only it can recover the HUP in the classical limit $\ell_p\to 0$. Employing the Taylor expansion, (2) can be rewritten as

$$\Delta p \geq \frac{1}{\Delta x+\alpha_0\ell_p}\left\{1+\left[\beta_0^2(\Delta x)^2+\frac{\alpha_0^2}{\left(\Delta x+\alpha_0\ell_p\right)^2}\right]\ell_p^2+\mathcal{O}\left(\ell_p^4\right)\right\}, \quad (3)$$

where we are setting $\hbar=1$. As [49, 61] have pointed out, the uncertainty momentum Δp can be defined as the energy ω of

emitted photon from black hole. Therefore, based on (3), the low bound for the energy is

$$\omega \geq \frac{1}{\Delta x + \alpha_0 \ell_p} \left\{ 1 + \left[\beta_0^2 (\Delta x)^2 + \frac{\alpha_0^2}{\left(\Delta x + \alpha_0 \ell_p\right)^2} \right] \ell_p^2 + \mathcal{O}\left(\ell_p^4\right) \right\}. \tag{4}$$

Next, assuming an emitted particle with energy ω and size R, for any black hole absorbing or emitting this particle, the minimal change in the horizon area of a black hole can be expressed as

$$\Delta A_{\min} \geq 8\pi\omega R \ell_p^2. \tag{5}$$

In the classical model, people can set $R = 0$. However, according to the arguments of [2], the size of a quantum particle cannot be smaller than Δx. Therefore, it implies the existence of a finite boundary given by

$$\Delta A_{\min} \geq 8\pi\omega \Delta x \ell_p^2. \tag{6}$$

By substituting (4) into inequality (6), then, using the relation of minimal length $\Delta x_{\min} = \alpha_0 \ell_p$, the boundary turns out to be

$$\Delta A_{\min} \geq 4\pi \ell_p^2 \left\{ 1 + \left[\beta_0^2 (\Delta x)^2 + \frac{\alpha_0^2}{(2\Delta x)^2} \right] \ell_p^2 + \mathcal{O}\left(\ell_p^4\right) \right\}. \tag{7}$$

Now, consider the case that a photon is emitted by the SC black hole [49, 61, 62]. Near the event horizon of SC black hole, the position uncertainty of a photon is the order of the radius of the black hole; that is, $\Delta x = 2r_s$, where r_s is the radius of SC black hole. According to the area of the SC black hole $A = 4\pi r_s^2$, the relation between Δx and A can be expressed as $(\Delta x)^2 = 4r_s^2 = A/\pi$. Substituting this relation into (7), the abovementioned equation can be rewritten as the following expression:

$$\Delta A_{\min} \simeq \lambda \ell_p^2 \left[1 + \left(\beta_0^2 \frac{A}{\pi} + \frac{\pi\alpha_0^2}{4A} \right) \ell_p^2 + \mathcal{O}\left(\ell_p^4\right) \right], \tag{8}$$

with λ being an undetermined coefficient that is greater than 4π. In the previous works, people proved that the entropy of black holes depended on the area of horizon. Moreover, the ideas of information theory also showed the minimal increase of entropy is conjectured related to the value of the area A. According to [2, 63], the fundamental unit of entropy as one bit of information can be denoted as $\Delta S_{\min} = b = \ln 2$, so that one can easily obtain

$$\frac{dS}{dA} = \frac{\Delta S_{\min}}{\Delta A_{\min}} = \frac{b}{\lambda \ell_p^2} \left[1 + \left(\beta_0^2 \frac{A}{\pi} + \frac{\pi\alpha_0^2}{4A} \right) \ell_p^2 + \mathcal{O}\left(\ell_p^4\right) \right]^{-1}. \tag{9}$$

In accordance with the idea of entropy-area law, we obtain a constant $b/\lambda = k_B/4$. Putting this constant into the abovementioned equation and expanding it and then integrating the result, the GUP corrected entropy is obtained as follows:

$$S = \frac{Ak_B}{4\ell_p^2} \left\{ 1 - \left[\frac{\alpha_0^2 \pi}{4A} \ln\left(\frac{A}{4\ell_p^2} \right) + \frac{A\beta_0^2}{2\pi} \right] \ell_p^2 + \mathcal{O}\left(\ell_p^4\right) \right\}. \tag{10}$$

It is clear that the GUP corrected entropy is proportional to the area of horizon A, Planck length ℓ_p, and the dimensionless constants α_0 and β_0. When ignoring α_0 and β_0, (10) reduces to the original entropy of the black hole. Moreover, it can be found that the first correction term of (10) is logarithmic in A, ℓ_p, and α_0, which is coincident with previous findings [64–66]. It should be noted that the second correction term is proportional to β_0 and goes like A^2; if $\beta_0 \geq \sqrt{2\pi/A}$, the second correction term is in principle larger than the leading term proportional to A. In order to avoid this paradoxical situation, it requires that $\beta_0 < \sqrt{2\pi/A}$. Meanwhile, one can calculate the GUP corrected Hawking temperature based on (10):

$$T = \frac{\kappa}{8\pi} \frac{dA}{dS} = \frac{\kappa}{2\pi k_B} \ell_p^2 \left[1 + \left(\frac{A\beta_0^2}{\pi} + \frac{\pi\alpha_0^2}{4A} \right) \ell_p^2 + \mathcal{O}\left(\ell_p^4\right) \right], \tag{11}$$

where κ is the surface gravity of black holes. For the SC black hole, one sets $\kappa = 1/4M$, if one sets $\alpha_0 = \beta_0 = 0$, the GUP corrected temperature reduces to the original Hawking temperature.

3. The Modified Newton's Law of Gravitation due to the GUP

In this section, we will investigate the GUP impact on Newton's law of gravitation. For revealing the entropic force, Verlinde used the holographic principle and the first law of thermodynamics. When a test particle approaches a holographic screen, the entropic force of a gravitational system is expressed as

$$F\Delta x = T\Delta S, \tag{12}$$

where F is the entropic force, T and ΔS are the temperature and the change of entropy on holographic screen, and Δx is the displacement of the particle from the holographic screen, respectively [21]. Equation (12) implies a nonzero force is proportional to a nonzero acceleration. Using the argument of Bekenstein, that is, the change of entropy associated with the information on the boundary $\Delta S = 2\pi k_B$, it is found that the change in the entropy near the holographic screen is linear in Δx:

$$\Delta S = \frac{2\pi k_B mc\Delta x}{\hbar}, \tag{13}$$

where $\Delta x = \hbar/mc$ and m and $\hbar$ are the mass of elementary component and the Planck constant, respectively. Equation (13) is reminiscent of the osmosis across a semipermeable membrane. Meanwhile, it should be noted that ΔS is proportional to the mass of the elementary component. In order to understand this idea, one can postulate that a particle near the holographic screen is made up of two or more subparticles and each subparticle leads to the associated change in entropy after displacement. Because the mass of the elementary component and the entropy are additive, it leads to the fact that ΔS is proportional to m. Based on the conclusions in [67], the horizon of black holes can be taken as a storage device for information. If one denotes the amount of information by N bits, the information is proportional to the area $N = Ac^3/G\hbar$. With the help of the entropy-area law $S = Ak_Bc^3/4\hbar G$, the number of bits obeys the following relation:

$$N = \frac{4S}{k_B}. \tag{14}$$

Obviously, the number of bits is proportional to the entropy. Substituting (10) into (14), the number of bits is modified as follows:

$$\begin{aligned} N &= \frac{A}{\ell_p^2}\left\{1 - \left[\frac{\alpha_0^2\pi}{4A}\ln\left(\frac{A}{4\ell_p^2}\right) + \frac{A\beta_0^2}{2\pi}\right]\ell_p^2 + \mathcal{O}\left(\ell_p^4\right)\right\} \\ &= \frac{Ac^3}{G\hbar}\left\{1 - \left[\frac{\alpha_0^2\pi}{4A}\ln\left(\frac{A}{4\ell_p^2}\right) + \frac{A\beta_0^2}{2\pi}\right]\ell_p^2 \right. \\ &\quad \left. + \mathcal{O}\left(\ell_p^4\right)\right\}, \end{aligned} \tag{15}$$

where $\ell_p^2 = G\hbar/c^3$. By setting the total energy of the black hole (or holographic system) as E and noting that the energy is divided evenly over the bits N, it is easy to obtain the notion that each bit carries an energy equal to $k_BT/2$. According to the equipartition rule, the total energy can be expressed as

$$E = \frac{k_BNT}{2}. \tag{16}$$

Using the relation $E = Mc^2$ and then putting (12) and (13) into the abovementioned equation, one gets

$$F = \frac{4\pi c^3Mm}{\hbar N}. \tag{17}$$

Next, substituting N from (15) into (17), the GUP corrected Newton's law of gravitation becomes

$$\begin{aligned} F &= \frac{GMm}{R^2}\left\{1 + \left[\frac{\alpha_0^2}{16R}\ln\left(\frac{\pi R}{\ell_p^2}\right) + 2\beta_0^2R\right]\ell_p^2 \right. \\ &\quad \left. + \mathcal{O}\left(\ell_p^4\right)\right\}. \end{aligned} \tag{18}$$

In the abovementioned equation, one has $A = 4\pi R^2$. It is well known that the Newtonian gravitational force dominates at large scales; however, it becomes weak at small scales (recent experiments show that the Newtonian gravitational force is led down to 0.13 mm~0.16 mm [68]). Meanwhile, it is hard to combine the Newtonian gravitational force with quantum mechanics. In (18), it is clear that the GUP corrected Newton's law is dependent not only on Newton's gravitational constant G, the mass of two bodies M and m, and the distances R but also on the dimensionless constants α_0 and β_0 as well as the Planck length ℓ_p. Therefore, due to the effect of GUP, the result shows that the Newtonian gravitational force is valid at scales which is smaller than the order of a millimeter. When $\alpha_0 = \beta_0 = 0$, (18) reduces to the original Newton's law. Moreover, one can obtain the Newtonian potential from (18):

$$\begin{aligned} V(R) &= -\frac{GMm}{R}\left\{1 \right. \\ &\quad + \left\{\frac{\alpha_0^2}{32R}\left[\frac{1}{32} - \ln\left(\frac{\pi R}{\ell_p^2}\right)\right] - 2R\beta_0^2\ln R\right\}\ell_p^2 \\ &\quad \left. + \mathcal{O}\left(\ell_p^4\right)\right\}. \end{aligned} \tag{19}$$

It is interesting to compare (19) with the predictions that came from higher order corrections to the Newtonian potential in the Randall-Sundrum II (RS II) [69]; the modification in Newton's gravitational potential on brane is [70]

$$V(R) \sim \begin{cases} \dfrac{GMm}{r}\left(1 + \dfrac{4l_\mu}{3\pi r} - \cdots\right) & \text{for } l_\mu \gg r \\ \dfrac{GMm}{r}\left(1 + \dfrac{2l_\mu}{3\pi r^2} - \cdots\right) & \text{for } l_\mu \ll r, \end{cases} \tag{20}$$

where r and l_μ are the radius and the characteristic length scale of the theory, respectively. Our result agrees with the Newtonian potential in RS II when $l_\mu \gg r$. Hence, it suggests that $(\alpha_0, \beta_0) \sim l_\mu$ can help us to set a new upper bound on the dimensionless constants (α_0, β_0). Besides, both (20) and the correction terms in (19) become susceptible at a short distance; they indicate that GUP and brane world may predict the similar phenomena.

4. The Quantum Corrections to Einstein's Field Equation

A lot of works predict that Einstein's field equation can be derived from entropic force. In this section, we will further investigate the laws of gravity and extend them to the relativistic case, so that we can obtain the modified Einstein's field equation via the GUP corrected entropic force. According to the GUP corrected number of bits, the bit density on the screen can be expressed as

$$dN = \frac{1}{\ell_p^2}\left[1 - \left(\frac{\alpha_0^2\pi}{4A} + \frac{A\beta_0^2}{\pi}\right)\ell_p^2 + \mathcal{O}\left(\ell_p^4\right)\right]dA, \tag{21}$$

where A represents the area of the holographic screen, and we use the natural units $c = k_B = 1$ in the equation above.

Here, we assume that the energy associated with the mass M is divided over N. Moreover, due to the equipartition law, it is easy to find that each bit carries $T/2$ mass. Therefore, the total mass is

$$M = \frac{1}{2}\int_{\mathcal{S}} T\, dN, \tag{22}$$

where $\mathcal{S}$ is the holographic screen. The local temperature can be expressed as

$$T = \frac{\hbar e^{\phi} n^{b} \nabla_{b}\varphi}{2\pi}, \tag{23}$$

where e^{ϕ} is the redshift factor as the local temperature is measured by an observer from infinity [71]. Substituting (21) and (23) into (22), one has

$$M = \frac{1}{4\pi G} \cdot \int_{\mathcal{S}} e^{\phi}\nabla\phi \left[1 - \left(\frac{\alpha_0^2\pi}{4A} + \frac{A\beta_0^2}{\pi}\right)\ell_p^2 + \mathcal{O}\left(\ell_p^4\right)\right] dA. \tag{24}$$

It is necessary to mention that the integral on the right side of the equation above represents the modified Komar mass (the original Komar mass contained inside a volume in a static curved space time is defined as $M_K = (1/4\pi G)\int_{\mathcal{S}} e^{\phi}\nabla\phi\, dA$) [21, 71, 72]; hence, (24) is the modified Gauss law in general relativity. Using the Stokes theorem and the Killing equation $\nabla^a\nabla_a\xi^b = -R_a^b\xi^a$, the Komar mass in terms of the Killing vector ξ^a and Ricci tensor R_{ab} can be rewritten as [21, 73]

$$M_K = \frac{1}{4\pi G}\int_{\Sigma} R_{ab} n^a \xi^b dV. \tag{25}$$

Therefore, (24) becomes

$$M = \frac{1}{4\pi G}\int_{\Sigma} R_{ab} n^a \xi^b dV + \frac{1}{4\pi G} e^{\phi} \nabla\phi \int_{\mathcal{S}} \left[-\left(\frac{\alpha_0^2\pi}{4A} + \frac{A\beta_0^2}{\pi}\right)\ell_p^2 + \mathcal{O}\left(\ell_p^4\right)\right] dA, \tag{26}$$

where Σ is the three-dimensional volume bounded by the holographic screen $\mathcal{S}$ and its normal is n^a. In [72], the authors showed that M can be expressed in terms of the stress-energy tensor T_{ab}:

$$M = 2\int_{\Sigma} dV \left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right) n^a \xi^b. \tag{27}$$

As a result, substituting (27) into (26), one yields

$$\int_{\Sigma}\left[R_{ab} - 8\pi G\left(T_{ab} - \frac{1}{2}T g_{ab}\right)\right] n^a\xi^b dV = \ell_p^2 \int_{\mathcal{S}} e^{\phi}\nabla\phi \left[\frac{\alpha_0^2\pi}{4A} + \frac{A\beta_0^2}{\pi} + \mathcal{O}\left(\ell_p^4\right)\right] dA. \tag{28}$$

With the help of $F = -me^{\phi}\nabla\phi$, one can obtain the GUP corrected Einstein's field equation by some manipulations:

$$R_{ab} = 8\pi G\left(T_{ab} - \frac{1}{2}T g_{ab}\right) \cdot \left[1 - \left(\frac{\alpha_0^2\pi}{4A} + \frac{A\beta_0^2}{\pi}\right)\frac{\ell_p^2}{2\pi} + \mathcal{O}\left(\ell_p^4\right)\right], \tag{29}$$

with the area of the holographic screen A. Obviously, this field equation is dependent not only on the geometry of the space time and energy-momentum tensor but also on the GUP terms. For large horizon area or $\alpha_0 = \beta_0 = 0$, the modified Einstein's field equation reduces to the original case.

5. The Quantum Corrections to the Friedmann Equation

In [19, 22, 51, 73–76], people analyzed the Friedmann equation by using the entropic force. Hence, we will study the effect of the GUP arising from (1) on the form of the Friedmann equation. In the homogeneous and isotropic spacetime, the Friedmann-Robertson-Walker (FRW) universe is described by the line element:

$$ds^2 = h_{\mu\nu}dx^{\mu}dx^{\nu} + \tilde{r}^2 d\Omega^2, \tag{30}$$

where $\tilde{r} = ra(t)$, $x^{\mu} = (t, r)$, $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric of two-dimensional unit sphere, $h_{\mu\nu} = \mathrm{diag}[-1, a^2/(1-kr^2)]$ is the two-dimensional metric with $\mu = \nu = 0, 1$, and k is the spatial curvature constant, respectively. Using the relation $h^{\mu\nu}\partial_{\mu}\tilde{r}\partial_{\nu}\tilde{r} = 0$, the dynamical apparent horizon of the FRW universe can be expressed as

$$\tilde{r} = ar = \frac{1}{\sqrt{H^2 + k/a^2}}, \tag{31}$$

where $H = \dot{a}/a$ is the Hubble parameter. Now, supposing that the matter source in the FRW universe is a perfect fluid, stress-energy tensor is

$$T_{\mu\nu} = (\rho + p)\, u_{\mu}u_{\nu} + p g_{\mu\nu}, \tag{32}$$

where u_{μ} is the four velocities of the fluid. The conservation law of energy-momentum leads to the following continuity equation:

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{33}$$

In order to investigate the GUP corrected Friedmann equation, one should consider a compact spatial region $V = (4/3)\pi\tilde{r}^3$ with a compact boundary $\Sigma = 4\pi\tilde{r}^2$. By combining (18) with Newton's second law, one has

$$F = m\left(\frac{\partial^2\tilde{r}}{\partial t^2}\right) = m\ddot{a}r = -G\frac{Mm}{\tilde{r}^2}\chi\left(\ell_p, \alpha_0, \beta_0\right), \tag{34}$$

where $\chi(\ell_p, \alpha_0, \beta_0) = 1 + [2\beta_0^2\tilde{r}^2 + \alpha_0^2\ln(\pi\tilde{r}^2/\ell_p^2)/16\tilde{r}^2]\ell_p^2 + \mathcal{O}(\ell_p^4)$ and m represents the test particle near the holographic

screen; the higher order terms $\mathcal{O}(\ell_p^4)$ can be ignored since ℓ_p is a very small value. The total physical mass M inside the volume $\mathscr{V}$ can be defined as

$$M = \int_{\mathscr{V}} dV \left(T_{\mu\nu} u^\mu u^\nu\right) = \frac{4}{3}\pi \tilde{r}^3 \rho, \tag{35}$$

where $\rho = M/V$ is the energy density of the matter in the spatial region V. Putting (34) into (33), one has the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G \rho \chi\left(\ell_p, \alpha_0, \beta_0\right). \tag{36}$$

For deriving the Friedmann equation, it is necessary to use the active gravitational mass (or Tolman-Komar mass) $\mathscr{M}$ instead of the total mass M because the acceleration in a dynamical background is produced by the active gravitational mass. According to [22], one can express the active gravitational mass in terms of energy-momentum tensor $T_{\mu\nu}$:

$$\mathscr{M} = 2\int_{\mathscr{V}} dV \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) u^\mu u^\nu = \frac{4}{3}\pi \tilde{r}^3 \left(\rho + 3p\right). \tag{37}$$

Replacing the total mass M by the active gravitational mass $\mathscr{M}$, (35) can be rewritten as

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G \left(\rho + 3p\right) \chi\left(\ell_p, \alpha_0, \beta_0\right). \tag{38}$$

The equation above is the GUP corrected acceleration equation for the dynamical evolution of the FRW universe. Using continuity equation (32) and multiplying both sides of (37) with $\dot{a}a$ and then integrating it, the result is [51]

$$\frac{d}{dt}\left(\dot{a}^2\right) = \frac{8\pi G}{3}\left[\frac{d}{dt}\left(\rho a^2\right)\right] \chi\left(\ell_p, \alpha_0, \beta_0\right). \tag{39}$$

Integrating both sides for each term of (38), one has

$$\begin{aligned} \dot{a}^2 + k = {} & \frac{8\pi G}{3}\rho a^2 \left\{1 + \frac{1}{\rho a^2}\right. \\ & \cdot \int \left[2\left(ra\right)^2 \beta_0^2 + \frac{\alpha_0^2 \pi}{16\pi\left(ra\right)^2} \ln\left(\frac{\pi\left(ra\right)^2}{\ell_p^2}\right)\right] \\ & \left. \cdot \ell_p^2 d\left(\rho a^2\right)\right\}; \end{aligned} \tag{40}$$

the above equation can be rewritten as

$$\begin{aligned} H^2 + \frac{k}{a^2} = {} & \frac{8\pi G}{3}\rho \left\{1 + \frac{1}{\rho a^2}\right. \\ & \cdot \int \left[2\left(ra\right)^2 \beta_0^2 + \frac{\alpha_0^2 \pi}{16\pi\left(ra\right)^2} \ln\left(\frac{\pi\left(ra\right)^2}{\ell_p^2}\right)\right] \\ & \left. \cdot \ell_p^2 d\left(\rho a^2\right)\right\}. \end{aligned} \tag{41}$$

It should be noted that k is the spatial curvature which takes the values -1, 0, 1, and the values correspond to a close, flat, and open FRW universe, respectively. For calculating the correction term of (40), we assume an equation of state parameter is $\omega = p/\rho$, where ω is a constant independent of time (or redshift), so, integrating the continuity equation (33), it yields

$$\rho = \rho_0 a^{-3(1+\omega)}, \tag{42}$$

where ρ_0 is an integration constant. Substituting (41) into (40) and integrating, the result is given by

$$\begin{aligned} H^2 + \frac{k}{a^2} = {} & \frac{8\pi G}{3}\rho \left\{1 + \left(1 + 3\omega\right)\left[\frac{\alpha_0^2}{72\tilde{r}^2\left(1+\omega\right)^2}\right.\right. \\ & \left.\left. + \frac{2\beta_0^2 \tilde{r}^2}{3\omega - 1} + \frac{\alpha_0^2}{24\tilde{r}^2\left(1+\omega\right)} \ln\left(\frac{\tilde{r}\sqrt{\pi}}{\ell_p}\right)\right] \ell_p^2\right\}. \end{aligned} \tag{43}$$

With the help of (31), the equation above can be further rewritten as

$$\begin{aligned} & \left(H^2 + \frac{k}{a^2}\right)\left\{1 - \left(1 + 3\omega\right)\right. \\ & \cdot \ell_p^2 \left[\frac{\alpha_0^2}{72\left(1+\omega\right)^2}\left(H^2 + \frac{k}{a^2}\right)\right. \\ & + \frac{2\beta_0^2}{3\omega - 1}\left(H^2 + \frac{k}{a^2}\right)^{-1} \\ & \left.\left. + \frac{\alpha_0^2}{24\left(1+\omega\right)}\left(H^2 + \frac{k}{a^2}\right) \ln\left[\left(H^2 + \frac{k}{a^2}\right)\frac{\ell_p^2}{\pi}\right]\right]\right\} \\ & = \frac{8\pi G}{3}\rho. \end{aligned} \tag{44}$$

The equation above is the GUP corrected Friedmann equation of the FRW universe, which derived from the entropic force. It should be noted that (43) is not only determined by the Hubble parameter H, the expansion scale factor of universe a, the spatial curvature k, and the constant ω but also affected by the dimensionless constants α_0 and β_0 and the Planck length ℓ_p. For the present universe, (43) is nothing but a usual Friedmann equation since the apparent horizon radius is very large. However, the correction terms make sense when the apparent horizon radius is at a short scale. Hence, one can use the GUP corrected Friedmann to investigate the early stage of the universe. Moreover, in [51, 52], the authors concluded that the impact of quantum corrections at the early stage of the universe can affect the inflation, so that people may detect those consequences by astronomical observation.

6. Conclusions

In this paper, we studied the quantum corrections to the entropic force via a new kind of GUP that admits a minimal

length, a minimal momentum, and a maximal momentum. Firstly, we derived the modified entropy-area law of a black hole. Then, using the the modification of entropy, the GUP corrected number of bits N was obtained. Subsequently, based on the GUP corrected number of bits and Verlinde's conjecture about the entropic force, the GUP corrected Newton's law of gravitation and Einstein's field equation as well as the Friedmann equation have been investigated. The results showed that the GUP corrected Newton's law of gravitation, the GUP corrected Einstein's field equation, and the GUP corrected Friedmann equation are dependent on the quantum correction terms α_0 and β_0 and the Planck length ℓ_p. These results agree with the original cases at a larger scale. However, when the length approaches the order of Planck scale, the corrected results depart from the original cases since the GUP effect becomes susceptible at a short scale. Besides, it is found that the GUP corrected Newton's law of gravitation is working at the sub-μm range, and it can predict the similar phenomenon as the Randall-Sundrum II model; this can help us to set a new upper bound on (α_0, β_0). Moreover, we found that the GUP corrected Friedmann equation can help people to study the properties of the early universe, and the impact of GUP may be detected by the astronomical observation.

Competing Interests

The authors declare that they have no competing interests.

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Research Article

Spontaneous Symmetry Breaking in 5D Conformally Invariant Gravity

Taeyoon Moon[1,2] and Phillial Oh[1]

[1] *Department of Physics and Institute of Basic Science, Sungkyunkwan University, Suwon 440-746, Republic of Korea*
[2] *Institute of Basic Science and Department of Computer Simulation, Inje University, Gimhae 621-749, Republic of Korea*

Correspondence should be addressed to Phillial Oh; ploh@skku.edu

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We explore the possibility of the spontaneous symmetry breaking in 5D conformally invariant gravity, whose action consists of a scalar field nonminimally coupled to the curvature with its potential. Performing dimensional reduction via ADM decomposition, we find that the model allows an exact solution giving rise to the 4D Minkowski vacuum. Exploiting the conformal invariance with Gaussian warp factor, we show that it also admits a solution which implements the spontaneous breaking of conformal symmetry. We investigate its stability by performing the tensor perturbation and find the resulting system is described by the conformal quantum mechanics. Possible applications to the spontaneous symmetry breaking of time-translational symmetry along the dynamical fifth direction and the brane-world scenario are discussed.

1. Introduction

Conformal symmetry is an important idea which has appeared in diverse area of physics, and its application to gravity has started with the idea that conformally invariant gravity in four dimensions (4D) [1–4] might result in a unified description of gravity and electromagnetism. The Einstein-Hilbert action of general relativity is not conformally invariant. In realizing the conformal invariance of this, a conformal scalar field is necessary [5, 6] in order to compensate the conformal transformation of the metric, and a quartic potential for the scalar field can be allowed. Its higher dimensional extensions are straightforward. In five dimensions (5D), conformal symmetry can be preserved with a fractional power potential [7, 8] for the scalar field. So far, it seems that little attention has been paid to the 5D conformal gravity with its fractional power potential. Such a potential renders a perturbative approach inaccessible, but nonperturbative treatment may reveal novel aspects. One can also construct a conformally invariant gravity with the Weyl tensor via R-squared gravity, but we focus on the Einstein gravity with a conformal scalar.

If the scalar field spontaneously breaks the conformal invariance with a Planckian VEV, the theory reduces to the 4D Einstein gravity with a cosmological constant [9, 10]. On the other hand, the spontaneous symmetry breaking of Lorentz symmetry [11] or gauge symmetry [12] in 5D brane-world scenario [13–16] was studied, but little is known in the context of 5D conformal gravity. In this paper, we explore 5D conformal gravity with a conformal scalar and investigate possible consequences in view of the spontaneous symmetry breaking.

Let us consider 5D conformal scalar action of the form

$$S = \int d^5x \sqrt{-g} \left[\frac{1}{2}\xi\phi^2 R - \frac{\omega}{2} g^{AB} D_A\phi D_B\phi - V(\phi) \right] + S_m, \tag{1}$$

where R is the five-dimensional curvature scalar, S_m is the action for some matter[1], and A, B run over 0, 1, 2, 3, 4. Here, ξ is a dimensionless parameter describing the nonminimal coupling of the scalar field to the spacetime curvature. Also a parameter $\omega = \pm 1$ with $+(-)$ corresponds to canonical (ghost)

scalar. For $\omega = \pm 1$, the conformal invariance of action (1) without matter term forces ξ to be

$$V(\phi) = V_0 |\phi|^{10/3}, \qquad \xi = \mp \frac{3}{16}, \tag{2}$$

where V_0 is a constant and the corresponding conformal transformation is given by

$$g_{AB} \longrightarrow e^{2\sigma(x)} g_{AB}, \qquad \phi \longrightarrow e^{-(3/2)\sigma(x)} \phi(x). \tag{3}$$

When $\omega = -1$, the conformal scalar has a negative kinetic energy term, but we regard it as a gauge artifact [17] which can be eliminated from the beginning through field redefinition. Even with no scalar field remaining after gauging away for both cases ($\omega = \pm 1$), the physical mass scale can be set since the corresponding vacuum solution requires introduction of a scale which characterizes the conformal symmetry breaking. In 4D conformal gravity, it is known that the conformal symmetry can be spontaneously broken at electroweak [10, 18] or Planck [9, 10] scale. In all cases, the action (1) becomes 5D Einstein action with a cosmological constant by redefinition of the metric, $\widehat{g}_{AB} = \phi^{4/3} g_{AB}$, but we stick to the above conformal form (1) of the action to argue with the spontaneous breakdown of the conformal symmetry.

The paper is organized as follows. In Section 2, we perform the dimensional reduction from five to four dimensions by using the ADM decomposition. In Section 3, we present exact solutions with four-dimensional Minkowski vacuum ($R_{(4)} = 0$) and check if they can give a spontaneous breaking of the conformal symmetry. In Section 4, the gravitational perturbation and their stability for the solutions are considered. In Section 5, we include the summary and discussions.

2. Dimensional Reduction (5D to 4D)

In order to derive the 4D action from the 5D conformal gravity (1), we make use of the following ADM decomposition where the metric in 5D can be written as[2]

$$\begin{aligned} dS^2 &= g_{AB} dx^A dx^B \\ &= g_{\mu\nu}(x, y)(dx^\mu + N^\mu dy)(dx^\nu + N^\nu dy) \\ &\quad + \epsilon N^2(x, y) dy^2. \end{aligned} \tag{4}$$

To describe the background solution, we go to the "comoving" gauge and choose $N^\mu = 0$. In this case, we can recover our 4D spacetime by going onto a hypersurface $\Sigma_y : y = y_0 =$ constant, which is orthogonal to the 5D unit vector:

$$\widehat{n}^A = \frac{\delta_4^A}{N}, \qquad n_A n^A = \epsilon, \tag{5}$$

along the extra dimension, and $g_{\mu\nu}$ can be interpreted as the metric of the 4D spacetime. Using the metric ansatz (4), one obtains

$$\begin{aligned} R^{(5)} = R^{(4)} - 2\frac{\nabla^2 N}{N} + \frac{\epsilon}{N^2}\Bigg(&\frac{\overset{*}{N}\, g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}}{N} - g^{\alpha\beta}\overset{**}{g}_{\alpha\beta} \\ &- \frac{3\overset{*}{g}{}^{\alpha\beta}\overset{*}{g}_{\alpha\beta}}{4} - \frac{\left(g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right)^2}{4}\Bigg), \end{aligned} \tag{6}$$

where the asterisk * denotes the differentiation with respect to y and $\nabla^2 = \nabla_\mu \nabla^\mu$ is the four-dimensional Laplacian. Using this, we find that action (1) becomes

$$\begin{aligned} S = \int d^5x \sqrt{-g^{(4)}} N \Bigg[&\frac{1}{2}\xi\phi^2 R^{(4)} \\ &+ \epsilon \frac{\xi\phi^2}{8N^2}\left(\overset{*}{g}{}^{\alpha\beta}\overset{*}{g}_{\alpha\beta} + \left(g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right)^2 + \frac{8\overset{*}{\phi}}{\phi} g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right) \\ &- \xi\nabla^2\phi^2 - \frac{\omega}{2} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \epsilon\frac{\omega}{2N^2}\overset{*}{\phi}\overset{*}{\phi} - V(\phi)\Bigg] \\ &+ S_m. \end{aligned} \tag{7}$$

One can check that the above action (7) is invariant with respect to four-dimensional diffeomorphism $x^\mu \to x'^\mu \equiv x'^\mu(x)$ with $N'(x', y) = N(x, y)$. It is also invariant under $y \to y' \equiv y'(y)$ and $N \to N' \equiv (dy'/dy)^{-1} N$ apart from the matter action.

Before going further, we would like to comment on the homogeneous solution to the equations of motion given in 5D conformal gravity (1) without matter term. To this end, we first consider the Einstein equation for action (1), whose form is given by

$$R_{AB} - \frac{1}{2} g_{AB} R = T_{AB}, \tag{8}$$

$$\begin{aligned} T_{AB} &= \frac{1}{\xi\phi^2}\left(\partial_A\phi\partial_B\phi - \frac{1}{2} g_{AB}\partial^C\phi\partial_C\phi - V g_{AB}\right) \\ &\quad + \frac{1}{\phi^2}\left(D_A D_B \phi^2 - g_{AB}\Box^{(5)}\phi^2\right) \end{aligned} \tag{9}$$

and the scalar equation can be written as

$$0 = \omega\Box^{(5)}\phi + \xi\phi R - V'(\phi), \tag{10}$$

where D_A is 5D covariant derivative, $\Box^{(5)} \equiv D_A D^A$, and the prime $'$ denotes the differentiation with respect to ϕ. One can

easily check that the homogeneous solution to (8) and (10) is given by

$$R_{AB} = \Lambda g_{AB},\quad \phi = \phi_0,\quad \Lambda = \frac{2V_0}{3\xi}\left(\phi_0\right)^{4/3}. \tag{11}$$

We note that this solution can be approached in diverse ways. Firstly, field redefinition $\tilde{g}_{MN} = (\phi/\phi_0)^{4/3} g_{MN}$ necessitates introduction of scale, which leads to five-dimensional Planck mass $M_5 = \xi\phi_0^{2/3}$. Secondly, solution (11) corresponds to a gauge fixed case ($\phi = \phi_0$) through the conformal transformation (3). Lastly, it can be interpreted as a vacuum solution obtained when considering an effective potential $V_{\text{eff}} = -\xi R\phi^2/2 + V_0|\phi|^{10/3}$ (we will study the effective potential V_{eff} for details at the end of the next section). In all cases, conformal symmetry is spontaneously broken with a symmetry breaking scale $\sim \phi_0 \neq 0$. In addition, they yield the physically equivalent results: de Sitter $V_0/\xi > 0$ or anti-de Sitter $V_0/\xi < 0$. It is to be noticed that both cases are classically stable. Also there is a huge degeneracy of vacuum solutions due to conformal invariance such that if $(g_{AB}^{(0)}(x,y), \phi^{(0)}(x,y))$ is a solution, then $(\tilde{g}_{AB} = e^{2\sigma(x,y)} g_{AB}^{(0)}, \tilde{\phi} = e^{-(3/2)\sigma(x,y)}\phi^{(0)})$ is also a solution for an arbitrary function $\sigma(x,y)$. In the next section, we will investigate the explicit solution form, starting from the reduced action (7) without matter term.

3. Exact Solutions

From action (7) without matter term, we find the equation of motion for N as

$$\begin{aligned} 0 = {} & \frac{\xi\phi^2}{2}R^{(4)} \\ & - \frac{\epsilon\xi\phi^2}{8N^2}\left(\overset{*}{g}{}^{\alpha\beta}\overset{*}{g}_{\alpha\beta} + \left(g^{\mu\nu}\overset{*}{g}_{\mu\nu}\right)^2 + \frac{8\overset{*}{\phi}}{\phi}g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right) \\ & - \xi\nabla^2\phi^2 - \frac{\omega}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \frac{\epsilon\omega}{2N^2}\overset{*}{\phi}\overset{*}{\phi} - V(\phi) \end{aligned} \tag{12}$$

and the equation for four-dimensional metric $g^{\mu\nu}$ is given by

$$\begin{aligned} & \frac{1}{2}\xi\phi^2\left(R_{\mu\nu}^{(4)} - \frac{1}{2}g_{\mu\nu}R^{(4)}\right) = T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)} + T_{\mu\nu}^{(3)} + T_{\mu\nu}^{(4)} \\ & \quad + T_{\mu\nu}^{(5)}, \\ & T_{\mu\nu}^{(1)} = \frac{\xi}{2N}\left[\nabla_\mu\nabla_\nu\left(N\phi^2\right) - g_{\mu\nu}\nabla^2\left(N\phi^2\right)\right], \\ & T_{\mu\nu}^{(2)} = \frac{\epsilon\xi\phi^2}{8N^2}\left[\frac{1}{2}\overset{*}{g}{}^{\alpha\beta}\overset{*}{g}_{\alpha\beta}g_{\mu\nu} + 2\frac{N}{\phi^2}\left(\frac{\overset{*}{\phi}{}^2}{N}\right)\overset{*}{g}_{\mu\nu} + 2\overset{**}{g}_{\mu\nu}\right. \\ & \quad + g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\overset{*}{g}_{\mu\nu} + \overset{*}{g}{}^{\alpha\beta}g_{\beta\mu}\overset{*}{g}_{\nu\alpha} + g^{\alpha\beta}\overset{*}{g}_{\beta\nu}\overset{*}{g}_{\mu\alpha} \\ & \quad - \frac{1}{2}\left(g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right)^2 g_{\mu\nu} - 2\left(g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right)g_{\mu\nu} \\ & \quad \left. - 2\frac{N}{\phi^2}\left(\frac{\overset{*}{\phi}{}^2}{N}\right)g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}g_{\mu\nu}\right], \\ & T_{\mu\nu}^{(3)} = -\frac{\xi}{N}\left[\nabla_{(\mu}N\nabla_{\nu)}\phi^2 - \frac{1}{2}g_{\mu\nu}\nabla_\rho N\nabla^\rho\phi^2\right], \\ & T_{\mu\nu}^{(4)} = \left[\frac{\omega}{2}\nabla_\mu\phi\nabla_\nu\phi - \frac{\omega}{4}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi - \frac{1}{2}V(\phi)g_{\mu\nu}\right], \\ & T_{\mu\nu}^{(5)} = -\frac{\epsilon}{N}\left[\xi\left(\frac{\overset{*}{\phi}\overset{*}{\phi}}{N}\right)g_{\mu\nu} + \frac{\omega}{4}\frac{\overset{*}{\phi}\overset{*}{\phi}}{N}g_{\mu\nu}\right]. \end{aligned} \tag{14}$$

Also the equation of motion for the scalar field can be written as

$$\begin{aligned} 0 = {} & \xi N\phi R^{(4)} - 2\xi\phi\nabla^2 N + \frac{\epsilon\xi\phi}{4N}\left(\overset{*}{g}{}^{\alpha\beta}\overset{*}{g}_{\alpha\beta} - \left(g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right)^2\right. \\ & \left. - 4\left(g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right) + 4\frac{\overset{*}{N}}{N}g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right) + N\left(\omega\nabla_\mu\nabla^\mu\phi\right. \\ & \left. + \frac{\omega}{N}\nabla_\mu N\nabla^\mu\phi - V'(\phi)\right) + \frac{\epsilon\omega}{2N}\left(g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\overset{*}{\phi} - 2\frac{\overset{*}{N}\overset{*}{\phi}}{N}\right. \\ & \left. + 2\overset{**}{\phi}\right). \end{aligned} \tag{15}$$

Now one can equate (12) and trace of (13) and then obtain

$$\begin{aligned} V(\phi) = {} & -\frac{\xi}{2}\nabla^2\phi^2 - 2\frac{\xi}{N}\nabla_\mu N\nabla^\nu\phi^2 - \frac{3\xi\phi^2}{2N}\nabla^2 N \\ & - \frac{\epsilon}{2N}\left[8\xi\left(\frac{\overset{*}{\phi}\overset{*}{\phi}}{N}\right) + 3\omega\frac{\overset{*}{\phi}\overset{*}{\phi}}{N}\right] + \frac{\epsilon\xi\phi^2}{8N^2}\left[-3\overset{*}{g}{}^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right. \\ & \left. - 6g^{\alpha\beta}\overset{**}{g}_{\alpha\beta} - 4\frac{\overset{*}{\phi}}{\phi}g^{\alpha\beta}\overset{*}{g}_{\alpha\beta} + 6\frac{\overset{*}{N}}{N}g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right]. \end{aligned} \tag{16}$$

Substituting the above equation back into (12), we arrive at

$$\begin{aligned} \frac{\xi\phi^2}{2}R^{(4)} = {} & \frac{\xi}{2}\nabla^2\phi^2 - 2\frac{\xi}{N}\nabla_\mu N\nabla^\nu\phi^2 - \frac{3\xi\phi^2}{2N}\nabla^2 N + \frac{\omega}{2} \\ & \cdot g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{2\epsilon\omega}{N^2}\overset{*}{\phi}\overset{*}{\phi} - \frac{4\epsilon\xi}{N}\left(\frac{\overset{*}{\phi}\overset{*}{\phi}}{N}\right) \\ & + \frac{\epsilon\xi\phi^2}{8N^2}\left(-2\overset{*}{g}{}^{\alpha\beta}\overset{*}{g}_{\alpha\beta} + \left(g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right)^2 - 6g^{\alpha\beta}\overset{**}{g}_{\alpha\beta}\right. \\ & \left. + \frac{4\overset{*}{\phi}}{\phi}g^{\alpha\beta}\overset{*}{g}_{\alpha\beta} + \frac{6\overset{*}{N}}{N}g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right). \end{aligned} \tag{17}$$

Let us focus on the conformally invariant case with $\omega = -16\xi/3$ and $V(\phi) = V_0|\phi|^{10/3}$. In order to find solutions for this case, we consider the following ansatz:

$$\begin{aligned} g_{\mu\nu}(x,y) &= e^{-2\mu^2(y-y_0)^2}\hat{g}_{\mu\nu}(x), \\ N(x,y) &= N_0 e^{-z_1\mu^2y^2}, \\ \phi(x,y) &= \phi_0 e^{(3/2)z_2\mu^2y^2}, \end{aligned} \tag{18}$$

where N_0 and ϕ_0 are constants. For this ansatz, (15)–(17) become

$$\begin{aligned} \frac{10}{3}V_0\phi_0^{7/3} &= e^{2(z_1-z_2)\mu^2y^2}\frac{\phi_0}{N_0^2} \\ &\cdot \epsilon\xi\left[80(1-z_2)(z_1-1)\mu^4y^2\right. \\ &+ 80zy_0y(2-z_1-z_2)\mu^4 + 40(1-z_2)\mu^2 \\ &\left.- 80y_0^2\mu^4\right], \end{aligned} \tag{19}$$

$$\begin{aligned} V_0\phi_0^{10/3} &= e^{2(z_1-z_2)\mu^2y^2}\frac{\phi_0^2}{N_0^2} \\ &\cdot \epsilon\xi\left[24(1-z_2)(z_1-1)\mu^4y^2\right. \\ &+ 24y_0y(2-z_1-z_2)\mu^4 + 12(1-z_2)\mu^2 \\ &\left.- 24y_0^2\mu^4\right], \end{aligned} \tag{20}$$

$$\begin{aligned} R^{(4)} &= e^{2z_1\mu^2y^2}\frac{2\epsilon}{N_0^2}\left[24(1-z_2)(z_1-z_2)\mu^4y^2\right. \\ &\left.- 4y_0y(z_1-z_2)\mu^4 + 12(1-z_2)\mu^2\right]. \end{aligned} \tag{21}$$

The above equation (21) determines $R^{(4)}$ as a function of y; namely, each hypersurface $y = \overline{y}$ has different values of $R^{(4)}$. But, we will restrict our attention to four-dimensional Minkowski space. Then, we notice that (19)–(21) always allow trivial vacuum solution $\phi_0 = 0$, $R^{(4)}$ = arbitrary, independent of z_1 and z_2, in general. The search for nontrivial vacuum with $\phi_0 \neq 0$ is facilitated by the fact that the coefficients in (19) and (20) come out right so that the two equations are identical. Finally, for the 4D Minkowski vacuum ($R^{(4)} = 0$), we can obtain two solutions: (i) the *r.h.s* of (19)–(21) vanishes when $y_0 = 0$ and $z_2 = 1$, which yields $V_0 = 0$, $R^{(4)} = 0$ and in this case, z_1 is arbitrary; (ii) for $V_0 \neq 0$ and $y_0 \neq 0$, they allow the solution of $z_1 = z_2 = 1$. We summarize the 4D Minkowski solutions as follows:

$$\begin{aligned} &\text{(i)}\quad g_{\mu\nu} = e^{-2\mu^2y^2}\eta_{\mu\nu}, \\ &N = N_0 e^{-z_1\mu^2y^2}, \\ &\phi = \phi_0 e^{(3/2)\mu^2y^2}, \\ &V_0 = 0 \\ &\text{(ii)}\quad g_{\mu\nu} = e^{-2\mu^2(y-y_0)^2}\eta_{\mu\nu}, \\ &N = N_0 e^{-\mu^2y^2}, \\ &\phi = \phi_0 e^{(3/2)\mu^2y^2}, \\ &V_0 = -24\epsilon\xi\frac{y_0^2\mu^4}{N_0^2\phi_0^{4/3}}. \end{aligned} \tag{22}$$

Now we turn to an issue related to the spontaneous breaking of the conformal symmetry. We notice that the spontaneous symmetry breaking can be realized for a negative value of the curvature scalar with $R < 0$ and $V_0 > 0$. To see this, we consider an effective potential V_{eff} for the canonical scalar field ($\omega = 1$) and $V_0 > 0$ in action (1) as

$$V_{\text{eff}} = -\frac{1}{2}\xi\phi^2R + V_0|\phi|^{10/3}. \tag{23}$$

As was mentioned in (2), here ξ is fixed as a negative value of $\xi = -3/16$ for the canonical scalar $\omega = 1$, which preserves the conformal symmetry of action (1). Since solution (i) ($V_0 = 0$) with a stable equilibrium $\phi_v = 0$ does not provide a symmetry-broken phase, we focus on the case (ii) with $\epsilon = 1$ giving $V_0 > 0$ (hereafter we fix $\epsilon = 1$).

It turns out that, for (ii) with the positive 5D curvature scalar $R > 0$, we have only one vacuum solution of $\phi_v = 0$, while for $R < 0$, there exist two vacua of $\pm\phi_v$ with nonzero value:

$$\phi_v = e^{(3/2)\mu^2y_*^2}\phi_0, \tag{24}$$

where y_* is given by $y_* = 4y_0/3 \pm \sqrt{3/\mu^2 + 16y_0^2}/3$. In this case, the conformal symmetry is spontaneously broken with the symmetry breaking scale ϕ_v given by (24). This result is summarized in Figure 1 which shows that the effective potential with $R > 0$ has only one minimum $\phi_v = 0$ (a), while for $R < 0$, it has two minima with $\pm\phi_v$ (b) which corresponds to the case of spontaneous breaking of the conformal symmetry.

4. Tensor Perturbation

In this section, we explore the stability of solutions (i), (ii) by performing a tensor fluctuation around the solutions. Note that the impact of conformal invariance shows up in the perturbation theory. One can always go to the unitary gauge and choose $\varphi = 0$ in the scalar perturbation with $\phi = \phi_0 + \varphi$. On the other hand, when $\omega/\xi \neq -16/3$, it is no longer possible to gauge away the fluctuation, and φ is a dynamical field coupled to other fluctuations.

To investigate a tensor fluctuation around solutions (i), (ii), we need to consider the tensor perturbation of the metric as

$$ds^2 = a^2\left(\eta_{\mu\nu} + h_{\mu\nu}\right)dx^\mu dx^\nu + N^2dy^2 \tag{25}$$

$$= a^2\left[\left(\eta_{\mu\nu} + h_{\mu\nu}\right)dx^\mu dx^\nu + dz^2\right], \tag{26}$$

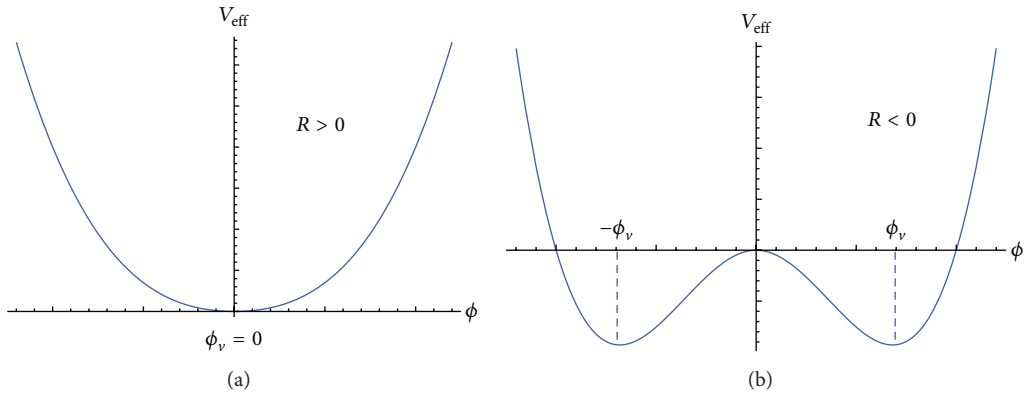

FIGURE 1: The effective potential $V_{\text{eff}} = -\xi R\phi^2/2 + V_0|\phi|^{10/3}$ with $\xi < 0$ and $V_0 > 0$, where (a) [unbroken phase] corresponds to the effective potential with a positive value of the curvature scalar R, while (b) [broken phase] is a case of $R < 0$.

where a new variable z given in a conformally flat metric (26) satisfies $Ndy = adz$. It is found that the equation of motion for the tensor modes $h_{\mu\nu}$ is given by

$$\partial_z^2 h_{\mu\nu} + A(z)\,\partial_z h_{\mu\nu} - \Box^{(4)} h_{\mu\nu} = 0, \tag{27}$$

where $h_{\mu\nu}$ satisfy the transverse-traceless gauge conditions ($\partial^\mu h_{\mu\nu} = 0$, $h = \eta^{\mu\nu} h_{\mu\nu} = 0$) and $A(z)$ is given by

$$A(z) = \frac{3}{a}\frac{\partial a}{\partial z} + \frac{2}{\phi}\frac{\partial \phi}{\partial z}. \tag{28}$$

Consideration of a separation of variables $h_{\mu\nu}(x,z) = f(z)H_{\mu\nu}(x)$ splits (27) into two parts:

$$\left[-\partial_z^2 + V_{\text{QM}}\right] f = m^2 f, \tag{29}$$

$$\Box^{(4)} H_{\mu\nu} = m^2 H_{\mu\nu}. \tag{30}$$

Here m is the mass of four-dimensional Kaluza-Klein modes and the corresponding quantum mechanical potential V_{QM} reads

$$V_{\text{QM}} = \frac{1}{2}\frac{\partial A}{\partial z} + \frac{A^2}{4}. \tag{31}$$

One can check easily that V_{QM} vanishes for solution (i), which yields just plane wave solution with constant zero mode. On the other hand, for solution (ii), it gives an inverse square potential[3] as

$$V_{\text{QM}} = \frac{15}{4z^2}, \tag{32}$$

where the corresponding Hamiltonian can be written as

$$H = \frac{1}{2}\left(p_z^2 + gz^{-2}\right) \quad \text{with } g = \frac{15}{4}. \tag{33}$$

It is known in [19, 20] that, for the Schrödinger equation (29) with the inverse square potential ($V_{\text{QM}} = g/z^2$), the stability of the mode f is determined by the condition[4]

$$g > -\frac{1}{4}, \tag{34}$$

which guarantees that the graviton mode along the fifth dimension with $g = 15/4$ is stable.

Before closing the section, we remark that the graviton mode along the fifth dimension preserves the residual conformal $SO(2,1)$ symmetry. It is well known that the quantum mechanical system of Hamiltonian (33) is conformally invariant [21], being referred to as *conformal quantum mechanics* (CQM). To see this, we first construct the CQM action for H (33) from the Lagrangian formalism:

$$S_{\text{CQM}} = \frac{1}{2}\int dt\left(\dot{z}^2 - \frac{g}{z^2}\right), \tag{35}$$

which is invariant under the nonrelativistic conformal transformations:

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad z' = \frac{z}{\gamma t + \delta} \qquad \text{with } \alpha\delta - \beta\gamma = 1. \tag{36}$$

In this case, it is known that three generators, that is, H (time translation), D (dilatation), and K (special conformal) generators, can act with the transformation rules:

$$H;\ t' = t + \tilde{t}, \qquad D;\ t' = \tilde{d}^2 t, \qquad K;\ t' = \frac{t}{\tilde{k}t + 1}, \tag{37}$$

where $\tilde{t}$, $\tilde{d}$, $\tilde{k}$ are some constants and the generators D and K at $t = 0$ in addition to H (33) are given by

$$D = -\frac{1}{4}\left(zp_z + p_z z\right), \qquad K = \frac{1}{2}z^2. \tag{38}$$

These generators obey $SO(2,1)$ commutations rules given by

$$[H,D] = iH,$$
$$[K,D] = iK, \tag{39}$$
$$[H,K] = 2iD,$$

whose Casimir invariant $\mathscr{C}$ is given by $\mathscr{C} = (HK + KH)/2 - D^2 = 3/4$.

5. Summary and Discussion

In this paper, we considered the conformally invariant gravity in 5D, which consists of a scalar field nonminimally coupled to the curvature with its potential. We found two solutions (i) and (ii) giving 4D Minkowski vacuum. By analyzing the dynamics of the metric perturbations around the solutions, we showed that two solutions are stable, since the former yields a plane wave solution with the constant zero mode, whereas the latter gives an inverse square potential. In particular, it was shown for solution (ii) that one has unbroken phase when $R > 0$, $V_0 > 0$, while for $R < 0$, $V_0 > 0$ the spontaneous breaking of the conformal symmetry can be realized with the scale ϕ_v given by (24).

We point out that solution (ii) may lead to a different mechanism which allows the possibility of a spontaneous breaking of translational invariance along the extra dimension. To this end, we consider a solution explicitly to (29) for $m^2 = 0$ as

$$f(z)_{m^2=0} = c_1 z^{5/2} + c_2 z^{-3/2} \tag{40}$$

with arbitrary constants c_1 and c_2. The first term of the *r.h.s* in (40) is not normalizable since the function $f(z)$ diverges at infinity, while the second term cannot lead to a normalizable solution due to its divergence at the origin. Thus, there is no normalizable zero mode solution. To resolve this problem, we define a new evolution operator $\mathscr{R}$ [21] given in terms of K (38) and H (33):

$$\mathscr{R} \equiv \frac{1}{2}\left(\frac{1}{a}K + aH\right), \tag{41}$$

which yields the eigenvalues of $\mathscr{R}$ as follows:

$$r_n = r_0 + n, \quad r_0 = \frac{3}{2}. \tag{42}$$

Here, a is some constant with the length dimension. It turns out that the new evolution operator $\mathscr{R}$ (41) provides a normalizable ground state f_0:

$$f_0(z) = c_0 z^{5/2} e^{-z^2/2}, \quad z \geq 0, \tag{43}$$

where a constant c_0 is given by $c_0 = 1$, being obtained from the normalization condition $\int_0^\infty |f_0(z)|^2 dz = 1$. Importantly, even if we have the normalizable ground state (43) by introducing the new evolution operator $\mathscr{R}$ (41), it implements the spontaneous breaking of the conformal symmetry in the sense that the fundamental length scale a is not included in the Lagrangian but generated by the particular form of the vacuum. On the other hand, it should be pointed out that since the well-defined ground state described by the Hamiltonian H which generates the time translation is not present, it may lead to a spontaneous symmetry breaking of time-translational invariance along the dynamical fifth direction [22–25].

We conclude with the following remark. We see that both solutions (i) and (ii) can be characterized by Gaussian warp factor [26–28], where the maximum value is located at $y = 0$ and $y = y_0$, respectively. But, the vacuum mode ϕ_v of (24) of the scalar field and the massless mode of gravity for the broken phase ($R < 0$, $V_0 > 0$) are not localized on the Gaussian brane, because a value of y_* in (24) cannot be equivalent to y_0 and zero mode (40) is written as $f(y)_{m^2=0} = \tilde{c}_1 e^{-5\mu^2 y_0 y} + \tilde{c}_2 e^{3\mu^2 y_0 y}$ in y-coordinate. Thus, it seems that it is hard to describe the brane-world scenario with the current approach. One possible alternative to apply our result to the scenario would be to treat the conformal gravity discussed in this study (or its variation) as the conformal matter sector and introduce 5D Einstein-Hilbert action separately. Then, there could be a possibility of addressing some of the related issues, especially the brane stabilization as a consequence of the spontaneous symmetry breaking. The details will be reported elsewhere.

Competing Interests

The authors declare that they have no competing interests.

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Endnotes

1. We assume that the matter is confined on a hypersurface at $y = y_m$, where y is the fifth coordinate. It is known that this matter in the brane has a geometrical origin in space-time-matter (STM) theory or induced-matter theory (IMT) [29–31]. Also, the IMT has been extended to the modified Brans-Dicke theory (MBDT) of type (1), where the induced matter exhibits interesting cosmological consequences [32, 33]. However, in this paper we are interested in the phenomena of spontaneous symmetry breaking and, thus, we will be neglecting the matter sector.
2. x^μ are the coordinates in 4D and y is the noncompact coordinate along the extra dimension (see [32, 33] and references therein). We use spacetime signature $(-,+,+,+,\pm)$, while $\epsilon = \pm 1$ denotes spacelike or timelike extra dimension.
3. The inverse square potential (32) corresponds to a repulsion from the origin. For an attractive potential ($\alpha < -1/4$) case with $V(x) = \alpha x^{-2}$, it was shown in

[34] that the quantum mechanical system has infinite continuous bound states from negative infinity to zero.

4. This condition yields exactly the BF bound [35] given in the d-dimensional AdS spacetime for the one-dimensional Schödinger equation:

$$-\partial_x^2\psi + \frac{m^2 + \left(d^2-1\right)/4}{x^2} = E\psi \longrightarrow m^2 \geq m_{\mathrm{BF}}^2 = -\frac{d^2}{4}, \quad (*)$$

when replacing g with $m^2 + (d^2-1)/4$.

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Research Article

On the UV Dimensions of Loop Quantum Gravity

Michele Ronco[1,2]

[1] *Dipartimento di Fisica, Sapienza-Università di Roma, Piazzale A. Moro 2, 00185 Roma, Italy*
[2] *INFN, Sez. Roma1, Piazzale A. Moro 2, 00185 Roma, Italy*

Correspondence should be addressed to Michele Ronco; michele.ronco@roma1.infn.it

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Planck-scale dynamical dimensional reduction is attracting more and more interest in the quantum-gravity literature since it seems to be a model independent effect. However, different studies base their results on different concepts of space-time dimensionality. Most of them rely on the *spectral* dimension; others refer to the *Hausdorff* dimension; and, very recently, the *thermal* dimension has also been introduced. We here show that all these distinct definitions of dimension give the same outcome in the case of the effective regime of Loop Quantum Gravity (LQG). This is achieved by deriving a modified dispersion relation from the hypersurface-deformation algebra with quantum corrections. Moreover, we also observe that the number of UV dimensions can be used to constrain the ambiguities in the choice of these LQG-based modifications of the Dirac space-time algebra. In this regard, introducing the *polymerization* of connections, that is, $K \to \sin(\delta K)/\delta$, we find that the leading quantum correction gives $d_{\mathrm{UV}} = 2.5$. This result may indicate that the running to the expected value of two dimensions is ongoing, but it has not been completed yet. Finding d_{UV} at ultrashort distances would require going beyond the effective approach we here present.

1. Introduction

There is an increasing interest in the quantum-gravity literature about the effect of dynamical dimensional reduction of space-time. It consists in scale dependence of the dimension d that runs from the standard IR value of four space-time dimensions to the lower value $d \simeq 2$ at Planckian energies. Remarkably, despite the fact that quantum-gravity approaches start from different conceptual premises and adopt different formalism types, this dimensional running has been found in the majority of them, such as Causal Dynamical Triangulation (CDT) [1], Horava-Lifshitz gravity [2], Causal Sets [3], Asymptotic Safety [4], Space-Time Noncommutativity [5], and LQG [6–8], which is here of interest.

However, in quantum gravity, even the concept of space-time dimension is a troublesome issue and it requires some carefulness. In fact, nonperturbative, background independent approaches (e.g., LQG [9, 10] and CDT [11]) generally rely on nongeometric quantities and they have discreteness as their core feature. For this reason, in order to extract phenomenological predictions, a coarse-graining process would be necessary aimed at deriving a more manageable effective description from the fundamental discrete blocks, which characterize the Planckian realm. It is a common expectation that this procedure should leave some traces in a semiclassical regime where the emerging picture would be given in terms of quantum space-time. This reduction has the advantage of allowing us to recover at least some of our more familiar physical observables or, when it would not be possible, analogous ones with potential departures from their classical counterparts. The dimension belongs to this latter set of semiclassical observables because the usual Hausdorff dimension is ill-defined for quantum space-time [12]. In the CDT approach [1, 11], it was recognized for the first time that a proper "quantum analogue" could be the spectral dimension d_S, which is the scaling of the heat-kernel trace, and it reproduces the standard Hausdorff dimension when the classical smooth space-time is recovered. What is more, it was found that in the UV $d_S \simeq 2$ (see however [13] for recent CDT simulations favouring a smaller value of the dimension), which is now a recurring number in the literature [14–17]. In the Asymptotic Safety program, such a value is also intimately connected to the hope of having a fixed point in the UV. In fact, it has been proven that renormalizability is accomplished only if

the dimension runs to two [18]. Furthermore, this prediction finds support in a recently developed approach [19] that has the advantage of relying on a minimal set of assumptions. Provided that quantum gravity will host an effective limit characterized by the presence of a minimum allowed length (identified with the Planck length), then it is shown [19] that the Euclidean volume becomes two-dimensional near the Planck scale (the author is grateful to Thanu Padmanabhan for pointing this out).

However, in a recent paper [20], the physical significance of d_S has been questioned. Such a concern is based on two observations: the computation of d_S requires preliminary Euclideanization of the space-time and also it turns out to be invariant under diffeomorphisms on momentum space. Both of these features are regarded as evidence of the fact that d_S is an unphysical quantity [20]. Given that, it has been proposed to describe the phenomenon of dimensional reduction in terms of the thermal (or thermodynamical) dimension d_T, which can be defined as the exponent of the Stefan-Boltzmann law. Then, the UV flowing of d_T is realized through a modified dispersion relation (MDR) that affects the partition function used to compute the energy density (see [20] or Section 3 for further details). Thus, the value of d_T near the Planck scale depends crucially on the specific form of the MDR. Furthermore, it has been recently noticed (see [21, 22]) that, from the MDR, it is also possible to infer the Hausdorff dimension d_H of energy-momentum space. If the duality between space-time and momentum space is preserved in quantum gravity, this framework should provide another alternative characterization of the UV running. In this way, we are in presence of proliferation of distinct descriptions of the UV dimensionality of quantum space-time. These pictures make use of very different definitions of the dimension and, in principle, there is no reason why they should give the same outcome. On the other hand, they all coincide in the IR-low-energy regime where they reduce to 4 and, thus, we could expect that this should happen also in the UV.

In this paper, we show that this advisable convergence can be achieved in the semiclassical limit of LQG under rather general assumptions. The insight we gain is based on the recently proposed quantum modifications of the hypersurface-deformation algebra (or the algebra of smeared constraints) [23–27], which reduces to a correspondingly deformed Poincaré algebra in the asymptotic region, as shown in [28]. These Planckian deformations of the relativistic symmetries are a key feature of the Deformed Special Relativity scenario [29, 30], as already pointed out in [28], and, what is more, it has been recently shown in [31] that they are consistent with κ-Minkowski noncommutativity of the space-time coordinates [32, 33]. We here exploit them to compute the MDR, thereby linking the LQG-based quantum corrections to the deformation of the dispersion relation. The general form of the MDR we derive allows us to find that the spectral, the thermal, and the Hausdorff (notice that we are always referring to the Hausdorff dimension of momentum space since, as we mentioned, that of quantum space-time cannot be defined) dimensions follow the same UV flowing; that is, $d_S = d_T = d_H$.

Another significant observation we make is that, following our analysis "in reverse," we can get information about the LQG quantum-geometric deformations, which affect the Dirac algebra, from the value of the UV dimension. The importance of this recognition resides in the fact that these modifications are subjected to many sources of possible ambiguities [34–36] coming, for example, from the regularization techniques used to formally quantize the Hamiltonian constraint. These ambiguities are far from being resolved and it is still not clear whether they may affect potentially physical outcomes [37]. Thus, they are usually addressed only on the basis of mere theoretical arguments or being guided by a principle of technical simplification. The main sources of ambiguities are the spin representations of the quantum states of geometry as well as the choice of the space lattice and, in effective models we will here consider, they correspond, respectively, to holonomy and inverse-triad corrections. We here partially fix them with what is believed to be a phenomenological prediction, that is, the number of UV dimensions. Notably, we notice that the leading-order correction provided by the holonomy corrections of homogeneous connections [38, 39], which are often implemented by simply taking the expression $\sin(\delta K)/\delta$ instead of K, for example, in Loop Quantum Cosmology (LQC) [40, 41] (see also [42] for a recent review on symmetry reduced models of LQG), is compatible with $d_{\mathrm{UV}} = 2.5$. Thus, as we would have expected, the number of dimensions is correctly flowing to lower values even if it has not already reached the value of 2, a value which is favoured in the quantum-gravity literature for the aforementioned reasons. On the basis of the steps we sketch out in the analysis, we are here reporting that it should be possible to exclude all the deformation functions $f(K)$ that are not consistent with $d_{\mathrm{UV}} = 2$. Remarkably, those quantum corrections, which are related to LQC as well as to the semiclassical limit of the theory, seem to point toward the right UV flowing. As already stated, the prediction of Planckian dimensional reduction has been also confirmed by previous LQG analyses [6, 7] but more refined computations of [8] have revealed that the "magic" number of 2 can be reproduced only focusing on specific superposition of kinematical spin-networks states (see [8] for the details). Relying on the recently developed effective methods for LQG, we here provide further support to the idea that the effective space-time of LQG may be two-dimensional at ultra-Planckian scales.

2. Modified Dispersion Relation

We start considering the classical hypersurface-deformation algebra (HDA), which was first introduced by Dirac [43]. It is the set of Poisson brackets closed by the smeared constraints of the Arnowitt-Deser-Misner formulation of General Relativity (GR) [44]. The Dirac algebra of constraints is the way in which general covariance is implemented once the space-time manifold has been split into the time direction and the three spatial surfaces, that is, $\mathscr{M} = \mathbb{R} \times \Sigma$.

It is given by

$$\begin{aligned}\left\{D\left[M^k\right], D\left[N^j\right]\right\} &= D\left[\mathscr{L}_{\vec{M}} N^k\right],\\ \left\{D\left[N^k\right], H[M]\right\} &= H\left[\mathscr{L}_{\vec{N}} M\right],\\ \{H[N], H[M]\} &= D\left[h^{jk}\left(N\partial_j M - M\partial_j N\right)\right],\end{aligned} \tag{1}$$

where $H[N]$ is the Hamiltonian (or scalar) constraint, while $D[N^k]$ is the momentum (or vector) constraint. The function N is called the lapse and it is needed to implement time diffeomorphisms, while N^i is the shift vector necessary to move along a given hypersurface and, finally, h^{ij} is the inverse metric induced on Σ. Thus, $H[N]$ and $D[N^k]$ have to be understood as the generators of gauge transformations which, in the case of GR, are space-time diffeomorphisms. For the purposes of our analysis, it is relevant to point out the well established fact (see [45]) that when the spatial metric is flat $h_{ij} = \delta_{ij}$, if we take $N = \Delta t + v_a x^a$ (where v_a is the infinitesimal boost parameter) and $N^k = \Delta x^k + R^k_l x^l$ (where R^k_l is the matrix that generates infinitesimal rotations), we can infer the Poincaré algebra from the Dirac algebra [45]. This classical relation is expected to hold also at the quantum level.

One of the open issues in the LQG research is the search for fully quantized versions of the constraints $H[N]$ and $D[N^k]$ on a Hilbert space. While it is known how to treat spatial diffeomorphisms and also how to solve the momentum constraint [46, 47] thereby obtaining the kinematical Hilbert space of the theory, the finding of a Hamiltonian operator is far from being completed. However, over the last fifteen years, several techniques have been developed, using both effective methods and discrete operator computations. In this way, some candidates for an effective scalar constraint $H^{\mathrm{Q}}[N]$ have been identified [48–50]. For the analysis we are here reporting, the interesting fact is that the semiclassical corrections introduced in the Hamiltonian leave trace in the algebra of constraints. Remarkably, even if these calculations use different formalism types and they are based on different assumptions, the general form of the modified HDA turns out to be the same in all these studies; that is, only the Poisson bracket between two scalar constraints is affected by quantum effects [39, 51]:

$$\left\{H^{\mathrm{Q}}[M], H^{\mathrm{Q}}[N]\right\} = D\left[\beta h^{ij}\left(M\partial_j N - N\partial_j M\right)\right], \tag{2}$$

where the specific form of the deformation function β as well as its dependence on the phase space variables, which are (h_{ij}, π_{ij}) if we use the metric formulation or (A^a_i, E^i_a) if we use Ashtekar's one, varies with the quantum corrections considered to define $H^{\mathrm{Q}}[N]$.

One of the causes of these quantum modifications of the scalar constraint is the fact that LQG cannot be quantized directly in terms of the Ashtekar variables A^a_i, which have to be replaced with their parallel propagators (or holonomies) [9, 10, 38, 39] $h_\alpha(A) = \mathscr{P} e^{-\int_\alpha t^i A^a_i \tau_a}$ (where $\mathscr{P}$ is the path-ordering operator, t^i the tangent vector to the curve α, and $\tau_a = -(i/2)\sigma_a$ the generators of $SU(2)$). If a is the spatial index of a direction along which space-time is homogeneous, then one has to consider just the local point-wise holonomies $h_i(A) = \cos(\delta A/2)\mathbb{I} + \sin(\delta A/2)\sigma_i$ (where $\delta \propto l_P = \sqrt{G} \approx 10^{-35}$ m is connected to the square root of the minimum eigenvalue of the area operator [41]). These are the kinds of quantum effects considered in effective (semiclassical) LQG theories as well as both in spherically reduced models and in cosmological contexts [42]. In particular, for spherically symmetric LQG (see, e.g., [28, 50, 51]), the deformation function depends on the homogeneous angular connection K_ϕ and it is directly related to the second derivative of the square of the holonomy correction $f(K_\phi)$; that is, $\beta = (1/2)(d^2 f^2(K_\phi)/dK_\phi^2)$. Then, the important contribution of [28] has been to establish a link between these LQG-inspired quantum corrections and DSR-like deformations of the relativistic symmetries, thanks to the recognition that the angular connection $K_\phi \propto P_r$ is proportional to the Brown-York radial momentum [52] that generates spatial translations at infinity (see [28, 31] for the details). In fact, it has been shown that, taking the Minkowski limit of (2) as we sketched above for the classical case, the LQG-deformed HDA produces a corresponding Planckian deformation of the Poincaré algebra:

$$\left[B_r, P_0\right] = iP_r\beta\left(l_P P_r\right), \tag{3}$$

where the other commutation relations remain unmodified. Here, B_r is the generator of radial boosts and P_0 is the energy. The explicit form of β is unknown and as we already stressed it is affected by ambiguities. In light of this, for our analysis, we assume a rather general form:

$$\beta\left(\lambda P_r\right) = 1 + \alpha l_P^\gamma P_r^\gamma \tag{4}$$

which is motivated by the above considerations and, obviously, satisfies the necessary requirement: $\lim_{l_P\to 0}\beta(l_P P_r) = 1$; that is, we want to recover the standard Poincaré algebra in the continuum limit. We leave the constants of order one α and γ that parametrize the aforementioned ambiguities unspecified. These parameters should encode at least the leading-order quantum correction to the Poincaré algebra.

Using (3) and (4) and taking into account the fact that $[B_r, P_r] = iP_0$ and $[P_r, P_0] = 0$, a straightforward computation gives us the following MDR:

$$E^2 = p^2 + \frac{2\alpha}{\gamma+2} l_P^\gamma p^{\gamma+2}. \tag{5}$$

This completes the analysis started in [28] and carried on in [31], which aimed at building a bridge between the formal structures of loop quantization to the more manageable DSR scenario with the objective of enhancing the possibilities of linking mathematical constructions to observable quantities. The remarkable fact of having derived (5) from the LQG-deformed algebra of constraints (2) is that it will give us the opportunity to constrain experimentally the formal ambiguities of the LQG approach exploiting the ever-increasing phenomenological implications of MDRs (see [53] and the references therein). We are often in the situation in which quantum-gravity phenomenology misses clear derivation from full-fledged developed approaches to quantum gravity

or, on the contrary, the high complexity of these formalism types does not allow inferring testable effects. Following the work initiated in [31], we are here giving a further contribution to fill this gap. We also find it interesting to notice that our MDR confirms a property of two previously proposed MDRs (see [54, 55]); that is, LQG corrections affect only the momentum sector of the dispersion relation leaving the energy dependence untouched. Therefore, this property, which has rigorous justification in the spherically symmetric framework [28, 50, 51] we are here adopting, seems to be a recurring feature of LQG. Moreover, all the precedent analyses were confined to the kinematical Hilbert space of LQG, while we have here obtained (5) from the flat-space-time limit of the full HDA including also the semiclassical Hamiltonian constraint (2). Thus, even if we are working *off shell* (i.e., we do not solve the constraint equations), the MDR (5) should contain at least part of the dynamical content of LQG. In the next section, we will see that the form of (5) is crucial to prove that the running of dimensions does not depend on the chosen definition of the dimension.

3. Dimensions and Quantum Corrections

Our next task is to use the MDR (5) we derived in Section 2 in order to show that, regardless of the value of the unknown parameters α and γ, the different characterizations of the UV flowing introduced in the literature predict the same number of dimensions if we consider the effective regime of LQG in the sense introduced in [28, 50, 51] and sketched in the previous section. To see this, we start by the computation of the spectral dimension, which is defined as follows:

$$d_S = -2\lim_{s\to 0}\frac{d\log P(s)}{d\log(s)}, \tag{6}$$

where $P(s)$ is the average return probability of a diffusion process in Euclidean space-time with fictitious time s. Following [20, 56–60], we compute d_S from the Euclidean version of our MDR (5) which is a d'Alembertian operator on momentum space:

$$\Delta^E = E^2 + p^2 + \frac{2\alpha}{\gamma+2} l_P^\gamma p^{\gamma+2}. \tag{7}$$

Then, a lengthy but straightforward computation (see [58–60]) leads to the following result:

$$d_S = 1 + \frac{6}{2+\gamma}. \tag{8}$$

Notice that the value of d_S does not depend on α but only on γ, that is, only on the order of Planckian correction to the dispersion relation (see (5)). We will use this fact later on.

Now we want to show that also the thermal dimension d_T is given by (8). To this end, we recall the definition of d_T introduced in [20]. If you have deformed Lorentzian d'Alembertian $\Delta^L_{\gamma_t\gamma_x} = E^2 - p^2 + l_t^{2\gamma_t} E^{2(1+\gamma_t)} - l_x^{2\gamma_x} p^{2(1+\gamma_x)}$, then d_T is the exponent of the temperature T in the modified Stefan-Boltzmann law:

$$\rho_{\gamma_t\gamma_x} \propto T^{1+3((1+\gamma_t)/(1+\gamma_x))} \tag{9}$$

which can be obtained as usual deriving the logarithm of the thermodynamical partition function [20] with respect to the temperature T. Evidently, in our case, we have that $\gamma_t = 0$ and $\gamma_x = \gamma/2$ and, thus, we find $d_T = 1 + 6/(2+\gamma)$; that is, the thermal dimension agrees with the spectral dimension $d_T \equiv d_S$.

Finally, we can calculate also the Hausdorff dimension of momentum space that if the duality with space-time is not broken by quantum effects should agree with both d_S and d_T. As pointed out in [21], a way to compute d_H is to find a set of momenta that "linearize" the MDR. Given (5), a possible choice is given by

$$k = \sqrt{p^2 + \frac{2\alpha}{\gamma+2} l_P^\gamma p^{\gamma+2}}. \tag{10}$$

In terms of these new variables (E, k), the UV measure on momentum space becomes

$$p^2 dp dE \longrightarrow k^{(4-\gamma)/(\gamma+2)} dk dE. \tag{11}$$

From (11), we can read off d_H:

$$d_H = 2 + \frac{4-\gamma}{\gamma+2} = 1 + \frac{6}{2+\gamma} \tag{12}$$

which, evidently, coincides with both d_S and d_T. Thus, no matter which definition of dimensionality is used, in the semiclassical limit (or in symmetry reduced models) of LQG, the UV running is free of ambiguities since we have found that $d_S \equiv d_T \equiv d_H$.

The last consideration we want to make concerns what the number of UV dimensions can teach us about LQG. In the analysis we here reported, the value of γ should be provided by the LQG corrections used to build $H^Q[N]$, which, though, are far from being unique. On the other hand, we mentioned that the number of dimensions runs to two in the UV, a prediction that seems to be model independent. Moreover, support in LQG has been also found by the studies of [6, 8] under certain assumptions. It is evident from (8) and (12) that in order to reproduce such a shared expectation we should take $\gamma = 4$. A recurring form of holonomy corrections both in spherically symmetric LQG [24, 28, 39, 61] and in LQC [40, 41] is represented by the choice $f(K) = \sin(\delta K)/\delta$ that implies $\beta = \cos(2\delta K)$. In light of the above arguments, if we keep restricted to mesoscopic scales where the MDR is well approximated by the first-order correction, it is easy to realize that this implies $\gamma = 2$ (see [31] for the explicit computation). In this way, we would have $d = 2.5$ at scales near but below the Planck scale. We find the fact that we obtain a value which is greater than 2 rather encouraging, since such an outcome may signal that the descent from the classical value of 4 to the UV value is in progress but it has not been completed yet. In fact, at ultra-Planckian energies, the Taylor expansion of the correction function $f(K)$ in series of powers of δ is no more reliable and, as a consequence, it cannot fully capture the flowing of d. Therefore, we are led to conclude that the much-used polymerization of homogeneous connections, which is a direct consequence of evaluating the holonomies in the

fundamental $j = 1/2$ representation of $SU(2)$, realizes at least partially the expected running of dimensions. Polymerizing configuration variables is used not only in LQC and in symmetry reduced contexts but also in the definition of the semiclassical regime of LQG, which is based on the introduction of spin-network states peaked around a single representation, in the majority of cases $j = 1/2$. We have here shown that these quantum modifications can be related to the phenomenon of dimensional reduction. This link we have established gives us the possibility of constraining part of the quantization ambiguities in LQG. In fact, the value of d_{UV} fixes a specific choice of the parameter γ in (4), thereby restricting the form of the allowed deformation functions $K \to f(K)$. In light of our analysis, one should select only those modifications which are compatible with the prediction of two-dimensional space-time at ultrashort distances, that is, those giving $2 \lesssim d < 4$ (or equivalently $0 < \gamma \lesssim 4$) to a first approximation.

4. Outlook

In the recent quantum-gravity literature, basically three different definitions of dimension have been proposed with the aim of generalizing this notion for quantum space-time. It is well known that they all provide a possible characterization of the UV running but, in the majority of cases, they also give different outcomes for the value of the dimension. This clashes with the growing consensus on the fact that the phenomenon of dynamical dimensional reduction is a model independent feature of quantum gravity that gives the unique predictions $d_{\mathrm{UV}} \simeq 2$. Thus, if d_S, d_T, and d_H are really proper definitions of the UV dimension, we would like to have that $d_S \equiv d_T \equiv d_H$.

In this paper, we showed that this striking convergence can be accomplished in the case of the LQG approach. To achieve such a result, we relied on the deformations of the constraint algebra recently proposed in the framework of both effective spherically symmetric LQG and LQC. Remarkably, in the former case, it has been proven that these quantum corrections leave traces in the Minkowski limit in terms of a DSR-like Poincaré algebra where the relevant deformations are functions of the spatial momentum. We here exploited these LQG-motivated deformations of the relativistic symmetries to infer the generic form of the MDR. Interestingly, our MDR is qualitatively of the same type of the two previously proposed MDRs for LQG [54, 55]; namely, the modifications affect only the momentum sector. From the MDR, we derived the spectral, the thermal, and the Hausdorff dimensions proving that they all agree. Thus, in the top-down approach of LQG, the desirable convergence of different characterization of the concept of dimensionality is accomplished. On the other hand, the analysis we here reported may provide a guiding principle for the construction of bottom-up approaches.

Moreover, our analysis led us to give a contribution toward enforcing the fecund bond between theoretical formalism types and phenomenological predictions. In fact, we found that the simple polymerization of connections, which is much used both in midi-superspace models and in LQC where semiclassical states are exploited to compute effective constraints, is able to generate the running of the dimension. In this way, we have provided further evidence that the phenomenon of UV dimensional reduction can be realized also in the LQG approach, thereby confirming the results of previous studies. Remarkably, the value of d_{UV} is sensible to the specific choice of quantum corrections which are considered in the model. Therefore, we pointed out that there is an observable whose value can be used to select a particular form for the quantum correction functions, thereby reducing the LQG quantization ambiguities. In particular, we showed that the evaluation of the Hamiltonian constraint over semiclassical states peaked at $j = 1/2$ (that can be also implemented with the substitution $K \to \sin(\delta K)/\delta$ at an effective level) corresponds to $d_{\mathrm{UV}} = 2.5$. Since we inferred such a value from the parametrization of the first LQG correction (4) (which corresponds to a second-order correction in the Planck length $\sim l_P^2$), then it is reasonable to regard our result as a first approximation that cannot capture the ending outcome of the dimensional running. From this perspective, we observed that obtaining a dimension greater than 2 at energies near but below the Planck scale might be significant, because it can be read as a hint that the dimension is flowing to the "magic" number of 2 that we can expect to be reached in the deep UV. Developing the off-shell constraint algebra in full generality without any symmetrical reduction or semiclassical approximation would be of fundamental importance to extend our observations to the full LQG framework.

Competing Interests

The author declares that he has no competing interests.

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Research Article

Charged Massive Particle's Tunneling from Charged Nonrotating Microblack Hole

M. J. Soleimani,[1] N. Abbasvandi,[2] Shahidan Radiman,[2] and W. A. T. Wan Abdullah[1]

[1] *Department of Physics, University of Malaya, 50603 Kuala Lumpur, Malaysia*
[2] *School of Applied Physics, FST, University Kebangsaan Malaysia, 43600 Bangi, Malaysia*

Correspondence should be addressed to M. J. Soleimani; msoleima@cern.ch

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In the tunneling framework of Hawking radiation, charged massive particle's tunneling in charged nonrotating TeV-scale black hole is investigated. To this end, we consider natural cutoffs as a minimal length, a minimal momentum, and a maximal momentum through a generalized uncertainty principle. We focus on the role played by these natural cutoffs on the luminosity of charged nonrotating microblack hole by taking into account the full implications of energy and charge conservation as well as the backscattered radiation.

1. Introduction

One of the most exciting consequences of models of low scale gravity [1–4] is the possibility of production of small black holes [5–7] at particle colliders such as the Large Hadron Collider (LHC) as well as in Ultrahigh Energy Cosmic Ray Air Showers (UECRAS) [8–10]. Incorporation of gravity in quantum field theory supports the idea that the standard Heisenberg uncertainty principle should be reformulated by the so-called generalized uncertainty principle near the Planck scale [11–13]. In particular, the existence of a minimum observable length is indicated by string theory [14], TeV-scale black hole physics [15], and loop quantum gravity [16]. Moreover, some black hole Gedanken experiments support the idea of existence of a minimal measurable length in a fascinating manner [17, 18] On the other hand, Doubly Special Relativity theories [19–24] suggest that a test particle's momentum cannot be arbitrarily imprecise and there is an upper bound for momentum fluctuation. It means that there is also a maximal particle momentum. It has been shown that incorporation of quantum gravity effects in black hole physics and thermodynamics through a generalized uncertainty principle (GUP) with the mentioned natural cutoffs modifies the result dramatically, specially, the final stage of black hole evaporation. Parikh and Wilczek on their pioneering work [25] constructed a procedure to describe the Hawking radiation emitted from a Schwarzschild black hole as a tunneling through its quantum horizon. The emission rate (tunneling probability) which arising from the reduction of the black hole mass is related to the change of black hole entropies before and after the emission. In this paper, charged particle's tunneling from charged nonrotating microblack hole is investigated. We consider a more general framework of GUP that admits a minimal length, minimal momentum, and maximal momentum to study the effects of natural cutoffs on the tunneling mechanism and luminosity of charged nonrotating TeV-scale black holes with extra dimensions in Arkani-Hamed, Dimopoulos, and Dvali (ADD) model [1] in the context of this GUP. The calculation shows that the emission rate satisfies the first law of black hole thermodynamics. The paper is organized as follows: in Section 2, we introduce a generalized uncertainty principle with minimal length, minimal momentum, and maximal momentum. In Section 3, we obtain an expression for emission rate of charged particle from charged nonrotating microblack hole based on the ADD model and the mentioned GUP. We consider the backscattering of the emitted radiation taking into account energy and charge conservation to evaluate the luminosity of TeV-scale black hole in presence of natural cutoffs. The last part is the discussion and calculation.

2. Generalized Uncertainty Principle

The existence of a minimal measurable length of the order of the Planck length, $l_p \sim 10^{-35}$ m, was indicated by most of quantum gravity approaches [26, 27] which modifies the Heisenberg uncertainty principle (HUP) to the so-called generalized (gravitational) uncertainty principle (GUP). The minimal position uncertainty, Δx_0, could be not made arbitrarily small toward zero [12] in the GUP framework due to its essential restriction on the measurement precision of the particle's position. On the other hand, Doubly Special Relativity (DSR) theories [21–24] have considered that existence of a minimal measurable length would restrict a test particle's momentum to take arbitrary values and therefore there is an upper bound for momentum fluctuation [28, 29]. So there is a maximal particle's momentum due to the fundamental structure of space-time at the Planck scale [30, 31]. Based on the above arguments, the GUP that predicts both a minimal length and a maximal momentum can be written as follows [19, 20]:

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left[1 - \alpha \langle \Delta p \rangle + 2\alpha^2 \langle \Delta p \rangle^2\right]. \tag{1}$$

The relation (1) can lead us to the following commutator relation:

$$[x, p] = i\hbar\left(1 - \alpha p + 2\alpha^2 p^2\right), \tag{2}$$

where α is GUP dimensionless positive constant of both minimal length and maximal momentum that depends on the details of the quantum gravity hypothesis. It has been developed that particle's momentum cannot be zero if the curvature of space-time becomes important and its effects are taken into account [32, 33]. In fact, there appears to be a limit to the precision of which the corresponding momentum can be expressed as a nonzero minimal uncertainty in momentum measurement. Based on this more general framework as a consequence of small correction to the canonical commutation relation, this GUP can be represented as [34]

$$\Delta x \Delta p \geq \hbar\left(1 - \alpha l_p \Delta p + \alpha^2 {l_p}^2 (\Delta p)^2 + \beta^2 {l_p}^2 (\Delta x)^2\right) \tag{3}$$

which in extra dimensions can be written as follows:

$$\begin{aligned} &\Delta x_i \Delta p_i \\ &\quad \geq \hbar\left(1 - \alpha l_p (\Delta p_i) + \alpha^2 {l_p}^2 (\Delta p_i)^2 + \beta^2 {l_p}^2 (\Delta x_i)^2\right). \end{aligned} \tag{4}$$

Here, α and β are dimensionless positive coefficients which are independent of Δx and Δp. In general they may depend on expectation value of x and p. According to the generalized Heisenberg algebra, we suppose that operators of position and momentum obey the following commutation relation:

$$[x, p] = i\hbar\left(1 - \alpha p + \alpha^2 p^2 + \beta^2 x^2\right). \tag{5}$$

In what follows, we use this more general framework of GUP to find the tunneling rate of emitted particles through charged nonrotating TeV-scale black holes.

3. Tunneling Mechanism

The idea of large extra dimensions might allow studying interactions at Trans-Planckian energies in particle colliders and the ADD model used d new large space-like dimensions. So, in order to investigate the Hawking radiation via tunneling from charged nonrotating TeV-scale black holes of higher dimensional, a natural candidate is that of Reissner-Nordstrom d-dimensional modified solution in presence of generalized uncertainty principle. In this case, the line element of d-dimensional Reissner-Nordstrom solution of Einstein field equation is given by [35]

$$\begin{aligned} ds^2 &= f(r) c^2 dt^2 - f^{-1}(r) dr^2 - r^2 d\Omega_{d-2}^2 \\ &= g_{\mu\nu} dx^\mu dx^\nu, \end{aligned} \tag{6}$$

where Ω_{d-2} is the metric of the unit S^{d-2} as $\Omega_{d-2} = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$ and

$$f = f(M, Q, r) = 1 - \frac{\omega_{d-2} M}{r^{d-3}} + \frac{\omega_{d-2} Q^2}{2(d-3)\Omega_{d-2} r^{2(d-3)}}, \tag{7}$$

where $\omega_{d-2} = 16\pi/(d-2)\Omega_{d-2}$. Here M and Q are the mass and electric charge of the black hole, respectively; units $G_d = c = \hbar = 1$ are adopted throughout this paper. The black hole has an outer/inner horizon located at

$$r_\pm^{d-3} = \frac{\omega_{d-2}}{2}\left[M \pm \sqrt{M^2 - \frac{(d-2)Q^2}{8\pi(d-3)}}\right]. \tag{8}$$

Therefore, the event horizon shrinks, and the inner one appears; when the black hole becomes charged the inner radius is related to the amount of charge and the outer one r_+ corresponds to the radius of Schwarzschild black hole. In this case, (8) can be rewritten as follows:

$$r_+ = \left(\frac{\omega_{d-2}}{2}\left[M + \sqrt{M^2 - \frac{(d-2)Q^2}{8\pi(d-3)}}\right]\right)^{1/(d-3)}. \tag{9}$$

In order to apply the semiclassical tunneling analysis, one can find a proper coordinate system for the black hole metric where all the constant lines are flat and the tunneling path is free of singularities. In this manner, Painlevé coordinates are suitable choices. In these coordinates, the d-dimensional Reissner-Nordstrom metric is given by

$$ds^2 = -f dt^2 \pm 2\sqrt{1-f}\, dt dr + dr^2 + r^2 d\Omega_{d-2}^2 \tag{10}$$

which is stationary, nonstatic, and nonsingular at the horizon and plus (minus) sign corresponds to the space-time line element of the outgoing (incoming) particles across the event horizon, respectively. The trajectory of charged massive particles as a sort of de Broglie s-wave can be approximately determined as [36, 37]

$$\dot{r} = \frac{dr}{dt} = -\frac{g_{tt}}{2g_{tr}} = \pm\frac{f}{2\sqrt{1-f}}, \tag{11}$$

where the plus (minus) sign denotes the radial geodesics of the outgoing (incoming) charged particles tunneling across the event horizon, respectively. We incorporate quantum gravity effects in the presence of the minimal length, minimal momentum, and maximal momentum via the GUP which motivates modification of the standard dispersion relation in the presence of extra dimensions based on ADD model. If the GUP is a fundamental outcome of quantum gravity proposal, it should appear that the de Broglie relation is as follows:

$$\lambda_{\pm} = \frac{p_i}{2\beta^2 l_p^2}\left(1 \pm \sqrt{1 - \frac{4\beta^2 l_p^2\left(1 - \alpha l_p p_i + \alpha^2 l_p^2 p_i^2\right)}{p_i^2}}\right). \quad (12)$$

One can find easily that positive sign does not recover ordinary relation in the limits $\alpha \to 0$ and $\beta \to 0$. So we consider the minus sign as

$$\lambda_- = \left(\frac{3\beta^2 l_p^4}{p_i} + p_i l_p^2\right)\alpha^2 - \left(\frac{2\beta^2 {l_p}^3}{p_i^2} + l_p\right)\alpha + \frac{\beta^2 l_p^2}{p_i^3} + \frac{1}{p_i} \quad (13)$$

or equivalently

$$\varepsilon = \left(3E\beta^2 l_p^4 + E^3 l_p^2\right)\alpha^2 - \left(2\beta^2 l_p^3 + E^2 l_p\right)\alpha + E + \frac{\beta^2 l_p^2}{E}. \quad (14)$$

Here, for investigating Hawking radiation of charged massive particles from the event horizon of charged nonrotating microblack hole, we use this more general uncertainty principle and take into consideration the response of background geometry to radiated quantum of energy E with GUP correction, that is, ε. The emitted particle which can be treated as a shell of energy ε and charge q moves on the geodesics of a space-time with central mass $M - \varepsilon$ substituted for M and charge parameter $Q - q$ replaced with Q. We set the total Arnowitt-Deser-Misner (ADM) mass, M, and the ADM charge of the space-time to be fixed but allow the hole mass and charge to fluctuate and replace M by $M-\varepsilon$ and Q by $Q-q$ in both the metric and the geodesic equation. So the outgoing radial geodesics of the charged massive particle tunneling out from the event horizon and the nonzero component of electromagnetic potential are

$$\dot{r} = \frac{f\left(M - \varepsilon, Q - q, r\right)}{2\sqrt{1 - f\left(M - \varepsilon, Q - q, r\right)}}, \qquad A_t = \frac{Q - q}{(d-3)\,\Omega_{d-2} r^{d-3}}. \quad (15)$$

So the Lagrangian for the matter-gravity system is

$$L = L_m + L_e, \quad (16)$$

where $L_e = -(1/4)F_{\mu\nu}F^{\mu\nu}$ is the Lagrangian function of the electromagnetic field corresponding to the generalized coordinates $A_\mu = (A_t, 0, 0, 0)$ [38].

We assume the tunneling mechanism as a semiclassical method producing Hawking radiation. In this case, using WKB approximation, the emission rate of tunneling massive charged particle can be obtained from the imaginary part of the particle action at the stationary phase for the tunneling trajectory; namely, [39, 40]

$$\Gamma \sim \exp\left(-2\,\mathrm{Im}\, I\right). \quad (17)$$

Assuming the generalized coordinate A_t is an ignorable one, to eliminate this degree of freedom completely, we can obtain the action of the matter-gravity system as

$$I = \int_{t_i}^{t_f}\left(L - P_{A_t}\dot{A}_t\right)dt = \int_{r_i}^{r_f}\left[\int_{(0,0)}^{(p_r, p_{A_t})}\left(\dot{r}\,dp'_r - \dot{A}_t\,dp'_{A_t}\right)\right]\frac{dr}{\dot{r}}, \quad (18)$$

where r_i and r_f are the location of the event horizon corresponding to t_i and t_f, respectively, before and after the particle of energy ε and charge q tunnels out, in which p_{A_t} and p_r are the canonical momentum conjugate to the coordinates A_t and r, respectively.

In order to consider the effect of quantum gravity, the commutation relation between the radial coordinate components and conjugate momentums should be modified based on (1) and (2) of the expressed GUP as follows [41]:

$$[r, p_r] = i\left(1 - \alpha l_p p_r + \alpha^2 l_p^2 p_r^2\right). \quad (19)$$

So as it is clear from the more general GUP and based on (5) the commutation relation should be modified as

$$[r, p_r] = i\left(1 - \alpha l_p p_r + \alpha^2 l_p^2 p_r^2 + \beta^2 l_p^2 r^2\right). \quad (20)$$

In the classical limit it is replaced by Poisson bracket as follows:

$$\{r, p_r\} = \left(1 - \alpha l_p p_r + \alpha^2 l_p^2 p_r^2 + \beta^2 l_p^2 r^2\right). \quad (21)$$

Now, we apply the deformed Hamiltonian equation:

$$\dot{r} = \{r, H\} = \{r, p_r\}\left.\frac{dH}{dr}\right|_r, \qquad dH|_{(r, A_t, p_t)} = d\left(M - \varepsilon\right), \quad (22)$$
$$\dot{A}_t = \frac{dH}{dp_{A_t}} = \frac{dE'_Q}{dp_{A_t}}, \qquad dH|_{(A_t, r, p_r)} = A_t d\left(Q - q\right).$$

In (18) as the Hamiltonian is $H = M - \varepsilon'$, one can set $p^2 \simeq \varepsilon'^2$ and $p \simeq \varepsilon'$ and eliminate the momentum in favor of the energy in integral (18) and switching the order of integration yields the imaginary part of the action as follows:

$$\begin{aligned}\operatorname{Im} I &= \operatorname{Im}\int_{r_i}^{r_f}\int_{(M,Q)}^{(M-\varepsilon,Q-q)}\left[\left(1-\alpha l_p\varepsilon'+\alpha^2 l_p^2\varepsilon'^2+\beta l_p^2 r^2\right)d\left(M-\varepsilon'\right)-\frac{Q-q'}{(d-3)\,\Omega_{d-2}r^{d-3}}d\left(Q-q'\right)\right]\frac{dr}{\dot{r}}\\ &= \operatorname{Im}\int_{r_i}^{r_f}\int_{(M,Q)}^{(M-\varepsilon,Q-q)}\frac{2\sqrt{1-f\left(M-\varepsilon',Q-q',r\right)}}{f\left(M-\varepsilon',Q-q',r\right)}\left[\left(1-\alpha l_p\varepsilon'+\alpha^2 l_p^2\varepsilon'^2+\beta l_p^2 r^2\right)d\left(M-\varepsilon'\right)-\frac{Q-q'}{(d-3)\,\Omega_{d-2}r^{d-3}}d\left(Q-q'\right)\right]dr.\end{aligned} \tag{23}$$

r integral can be evaluated by deforming the contour of the single pole at the outer horizon. During r integral first, we find

$$\begin{aligned}\operatorname{Im} I &= \operatorname{Im}\int_0^{\varepsilon}2\left(-\pi i\right)r_+\left(M-\varepsilon',Q-q\right)\\ &\quad\cdot\left(1-\alpha l_p\varepsilon'+\alpha^2 l_p^2\varepsilon'^2+\beta l_p^2\right)d\left(-\varepsilon'\right)\\ &\quad-\int_0^{Q-q}2\left(-\pi i\right)r_+\left(M-\varepsilon,Q-q'\right)\\ &\quad\cdot\frac{\left(Q-q'\right)}{(d-3)\,\pi r^{d-3}}d\left(Q-q'\right).\end{aligned} \tag{24}$$

This allows us to consider the leading order correction to be just proportional to second order of αl_p and also second order of βl_p for simplicity without loss of generality. In this regard, we can finish the integration by applying Taylor series and obtain the imaginary part of the action. Although integral (24) is complicated, one can find such terms as an example for $d = 5$ as follows:

$$\begin{aligned}\operatorname{Im} I_{d=5} &\approx \cdots + 4096\alpha l_p\pi M^2 - 8192\alpha l_p\pi M\left(3E\beta^2 l_p^4\right.\\ &\quad\left.+E^3 l_p^2\right)\alpha^2 + 8192\alpha l_p\pi M\left(2\beta^2 l_p^3+E^2 l_p\right)\alpha\\ &\quad-8192\alpha l_p\pi ME - 8192\alpha l_p\pi M\frac{\beta^2 l_p^2}{E}\\ &\quad+4096\alpha l_p\pi\left(\left(3E\beta^2 l_p^4+E^3 l_p^2\right)\alpha^2\right.\\ &\quad\left.-\left(2\beta^2 l_p^3+E^2 l_p\right)\alpha+E+\frac{\beta^2 l_p^2}{E}\right)^2-192\alpha l_p Q^2\\ &\quad+\cdots.\end{aligned} \tag{25}$$

Substituting (24) into (17), the tunneling probability of charged particles from charged nonrotating TeV-scale black holes is obtained as

$$\begin{aligned}\Gamma = \exp\left(-2\operatorname{Im} I\right) \simeq \exp\Bigg[&\operatorname{Im}\int_0^{\varepsilon}2\left(-\pi i\right)r_+\left(M-\varepsilon',Q-q\right)\left(1-\alpha l_p\varepsilon'+\alpha^2 l_p^2\varepsilon'^2+\beta l_p^2\right)d\left(-\varepsilon'\right)\\ &-\int_0^{Q-q}2\left(-\pi i\right)r_+\left(M-\varepsilon,Q-q'\right)\frac{\left(Q-q'\right)}{(d-3)\,\pi r^{d-3}}d\left(Q-q'\right)\Bigg] = \exp\left(\Delta s\right),\end{aligned} \tag{26}$$

where Δs is the difference in black hole entropies before and after emission [42–47]. It was shown that the emission rates on the high energy scales correspond to differences between the counting of states in the microcanonical and in the canonical ensembles [48, 49]. By performing integration on (24), one can find that the first order of E in the exponential gives a thermal, Boltzmannian spectrum. The existence of extra terms in relation (25) shows that the radiation is not completely thermal. In fact, these extra terms enhance the nonthermal character of the radiation. Also, it is easy to find that the tunneling rate should be greater than the ordinary one in any stage of tunneling process. This tunneling rate compared to the tunneling rate which is calculated in [50] obviously shows that, by considering all natural cutoffs in generalized uncertainty principle relation, many additional terms appeared. The additional terms show strong deviation of microblack holes radiation from ordinary thermal radiation.

4. Backscattering and Luminosity

It has been shown [51] that black holes radiate a thermal spectrum of particles. So microblack holes emit black body radiation at the Hawking temperature. Following a heuristic argument [52], the energy of the Hawking particles is $\Delta E \approx c\Delta p$ and it is deduced for the Hawking temperature of black hole based on LED scenario,

$$T_H \simeq \frac{(d-3)\,\Delta p}{4\pi}, \tag{27}$$

where $(d-3)/4\pi$ is a calibration factor in d-dimensional space-time. By saturating inequality (4), one can find

momentum uncertainty in terms of position uncertainty as follows:

$$\Delta P_i = \left(\frac{\alpha l_p + \Delta x_i}{4\alpha^2 l_p^2}\right) \cdot \left(1 \pm \sqrt{1 - \frac{4\alpha^2 l_p^2 \left(1 + 4\beta^2 l_p^2 (\Delta x_i)^2\right)}{\left(\alpha l_p + \Delta x_i\right)^2}}\right). \tag{28}$$

So the modified black hole Hawking temperature in the presence of natural cutoffs becomes

$$T_H = \frac{(d-3)(2r_+ + \alpha l_p)}{16\pi\alpha^2 l_p^2}\left(1 - \sqrt{1 - \frac{4\alpha^2 l_p^2 \left(1 + 16\beta^2 l_p^2 r_+^2\right)}{\left(2r_+ + \alpha l_p\right)^2}}\right). \tag{29}$$

Based on (29), GUP give rise to the existence of a minimal mass of a charged nonrotating microblack hole given by

$$M_{\min}^{\mathrm{GUP}} = \frac{(d-2)\,\Omega}{16\pi}\left(\left(\frac{\left(1 + 2\sqrt{1 - 12\beta^2\alpha^2 L_{pl}^4}\right) L_{pl}\alpha}{32\beta^2\alpha^2 L_{pl}^4 - 2}\right)^{2d-6} + \frac{8\pi Q^2}{(d-2)(d-3)}\right) \cdot \left(\left(\frac{\left(1 + 2\sqrt{1 - 12\beta^2\alpha^2 L_{pl}^4}\right) L_{pl}\alpha}{32\beta^2\alpha^2 L_{pl}^4 - 2}\right)^{d-3}\right)^{-1}. \tag{30}$$

Therefore, there are some black hole remnants without radiation based on (30). A radiated particle state corresponding to an arbitrary finite number of virtual pairs inside the black hole event horizon is as follows [53]:

$$\left|\psi\right\rangle = N \sum e^{-\pi n\varepsilon/\hbar k} \left|n_{\mathrm{out}}^{(L)}\right\rangle \otimes \left|n_{\mathrm{out}}^{(R)}\right\rangle, \tag{31}$$

where $N^2 = e^{\gamma\varepsilon}/(e^{\gamma\varepsilon} - 1)$ is a normalization constant and k is the surface gravity. This quantum state is transformed with respect to an observer outside the horizon. In order to obtain the average particle number in the energy state ε with respect to an observer, one can trace out the inside degrees of freedom to yield the reduced density matrix of the form

$$\rho_{\mathrm{reduced}} = \left(1 - \exp\left(-\frac{2\pi\varepsilon}{\hbar k}\right)\right) \sum_{n=0}^{\infty} e^{-\gamma n\varepsilon} \left|n_{\mathrm{out}}^{(R)}\right\rangle \otimes \left\langle n_{\mathrm{out}}^{(R)}\right|. \tag{32}$$

In this regard, the number distribution with respect to ε is given by

$$\left\langle n_\varepsilon \right\rangle = \mathrm{trace}\left(n\rho_{\mathrm{reduced}}\right) = \frac{1}{e^{\gamma\varepsilon} - 1}, \tag{33}$$

where $\gamma = 1/T_H$. Whenever a particle is radiated from the microblack hole event horizon, its wave function satisfies a wave equation with an effective potential that depends on outer event horizon. As the potential represents a barrier to the outgoing radiation, one part of the radiation is backscattered.

In this way, it can be shown that the distribution $\langle n_\varepsilon \rangle$ for the Hawking radiation will be modulated by grey body factor [54] which for a charged nonrotating TeV-scale black hole is given as

$$\Lambda = 4\varepsilon^2 r_+^2, \tag{34}$$

which Λ is the standard approximated grey body factor. In this way, one can take energy and charge conservation into account [54] and get the straightforward result by substituting (14) into (34). So we obtain

$$\Lambda_{EC} = 4\left[E\left(1 - \alpha l_p E + \alpha^2 l_p^2 E^2\right) \cdot \left(1 + \frac{\beta^2 l_p^2}{E^2}\left(1 - \alpha l_p E + \alpha^2 l_p^2 E^2\right)\right)\right]^2 r_+^2 (M - \varepsilon, Q - q). \tag{35}$$

On the other hand, if we consider the full consequences of energy and charge conservation, for total flux, including backscattering [55], the luminosity modulated according to the grey body factor has to be written as

$$L^d(M) = \frac{1}{2\pi}\int_0^{M - M_{\min}^{\mathrm{GUP}}} \left\langle n_\varepsilon \right\rangle \Lambda_{EC}\varepsilon d\varepsilon = \frac{1}{2\pi}\int_0^{M - M_{\min}^{\mathrm{GUP}}} \frac{4\left[E\left(1 - \alpha l_p E + \alpha^2 l_p^2 E^2\right)\left(1 + \left(\beta^2 l_p^2/E^2\right)\left(1 - \alpha l_p E + \alpha^2 l_p^2 E^2\right)\right)\right]^2 r_+^2 (M - \varepsilon, Q - q)\, d\varepsilon}{\exp\left(\left(16\pi\alpha^2 l_p^2 \varepsilon/(d-3)\left(2r_+ + \alpha l_p\right)\right)\left[1 - \sqrt{1 - 4\alpha^2 l_p^2\left(1 + l_p^2\beta^2 r_+^2\right)/\left(2r_+ + \alpha l_p\right)^2}\right]^{-1}\right) - 1}. \tag{36}$$

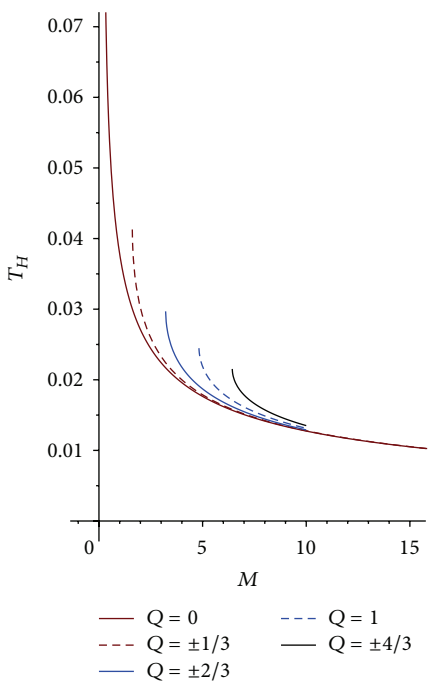

Figure 1: Hawking temperature for different amount of charges in presence of GUP.

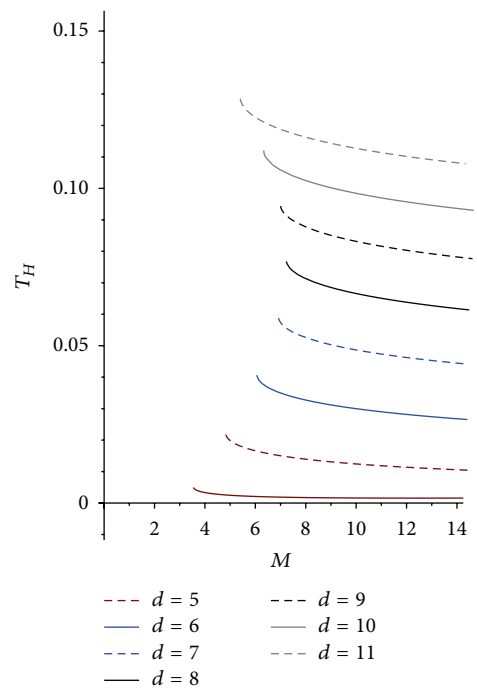

Figure 2: Hawking temperature with respect to the mass in terms of GUP.

It is important to remark that the total luminosities for microblack hole would be ten times bigger if we neglect backscattering effect. We are taking into account in the integration limits that the maximum energy of a radiated particle could be $M - M_{\min}^{\mathrm{GUP}}$. Equation (36) gives larger luminosities for smaller masses. The results show that, in large extra dimension scenario, Hawking temperature of charged black hole increases and leads to faster decay and less classical behaviors for black holes (Figure 2). On the other hand, it has been shown [56, 57] that the allowed particles forming the black hole at the LHC are quarks, antiquarks, and gluons which formed nine possible electric charge states: $\pm 4/3, \pm 1, \pm 2/3, \pm 1/3, 0$. In this case, as far as the electric charge of the black hole increases, the minimum mass and its order of magnitude increase and the temperature peak is displaced to the lower temperature (see Figure 1). As (36) is related to the black hole temperature, based on the above arguments, the luminosity of charged nonrotating TeV-scale black hole has different amount with respect to the charge of black hole and also extra dimensions.

5. Conclusion and Discussion

In this paper, we have investigated Hawking radiation of the charged massive particles as a semiclassical tunneling process from the charged nonrotating microblack hole. In this respect, we considered possible effect of natural cutoffs as a minimal length, a maximal momentum, and a minimal momentum on the tunneling rate. We have shown that, in the presence of generalized uncertainty principle, the tunneling rate of charged massive particle is deviated from thermal emission. In order to study the evolution of the TeV-scale microblack hole as it evaporates respecting energy and charge conservation, we have also modified the grey body factor, which allows considering the effect of the backscattered emitted radiation. We have calculated Hawking temperature based on the GUP which admitted a minimal length, a maximal momentum, and a minimal momentum. The adopted GUP predict a minimal mass remnant with respect to the charge of black hole. So we have been able to derive an expression for the luminosity that takes into account natural cutoffs in presence of large extra dimension based on ADD scenario for different amount of charge of black hole (Figures 2 and 1). The investigation implies that, considering natural cutoffs in the presence of LED, information conservation of charged nonrotating microblack hole is still possible.

Competing Interests

The authors declare that they have no competing interests.

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Research Article

Lorentz Invariance Violation and Modified Hawking Fermions Tunneling Radiation

Shu-Zheng Yang,[1] Kai Lin,[2,3] Jin Li,[4] and Qing-Quan Jiang[1]

[1] *Department of Astronomy, China West Normal University, Nanchong, Sichuan 637002, China*
[2] *Instituto de Física e Química, Universidade Federal de Itajubá, 37500-903 Itajubá, MG, Brazil*
[3] *Instituto de Física, Universidade de São Paulo, CP 66318, 05315-970 São Paulo, SP, Brazil*
[4] *Department of Physics, Chongqing University, Chongqing 400030, China*

Correspondence should be addressed to Kai Lin; lk314159@hotmail.com

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Recently the modified Dirac equation with Lorentz invariance violation has been proposed, which would be helpful to resolve some issues in quantum gravity theory and high energy physics. In this paper, the modified Dirac equation has been generalized in curved spacetime, and then fermion tunneling of black holes is researched under this correctional Dirac field theory. We also use semiclassical approximation method to get correctional Hamilton-Jacobi equation, so that the correctional Hawking temperature and correctional black hole's entropy are derived.

1. Introduction

In 1974, Hawking proved black holes could radiate Hawking radiation once considering the quantum effect near the horizons of black holes [1, 2]. This theory indicates that black hole would be viewed as thermodynamic system, so that black hole physics can be connected closely with gravity, quantum theory, and thermodynamics physics. According to the view point of quantum tunneling theory, the virtual particles inside of black hole could cross the horizon due to the quantum tunneling effect and become real particles and then could be observed by observers as Hawking radiation. Wilczek et al. proposed a semiclassical method to study the quantum tunneling from the horizon of black hole [3–21]. Along with this method, the Hamilton-Jacobi method was applied to calculate the Hawking tunneling radiation. According to the Hamilton-Jacobi method, the wave function of Klein-Gordon equation can be rewritten as $\Phi = C\exp(iS/\hbar)$ (where S is semiclassical action) and Hamilton-Jacobi equation is obtained via semiclassical approximation. Using the Hamilton-Jacobi equation, the tunneling rate could be calculated by the relationship $\Gamma \sim \exp(-2\,\mathrm{Im}\,S)$ (where Γ is the tunneling rate at the horizon of black hole), and then the Hawking temperature can be determined. People have applied this method to research Hawking tunneling radiation of several static, stationary, and dynamical black holes.

However, since the Hamilton-Jacobi equation is derived from Klein-Gordon equation, original Hamilton-Jacobi method just can be valid for scalar particles in principle. Therefore, Kerner and Mann studied fermions tunneling of black hole by a new method [22–32], which assumes the wave function of Dirac equation Ψ as spin-up and spin-down and then calculates the fermions tunneling, respectively. Nevertheless, this method is still impossible to apply in arbitrary dimensional spacetime. Our work in 2009 showed that the Hamilton-Jacobi equation can also be derived from Dirac equation via semiclassical approximation, so we proved that the Hamilton-Jacobi equation can be used to study the fermions tunneling directly [33–35].

On the other hand, as the basis of general relativity and quantum field theory, Lorentz invariance is proposed to be spontaneously violated at higher energy scales. A possible deformed dispersion relation is given by [36–44]

$$p_0^2 = \vec{p}^{\,2} + m^2 - (Lp_0)^{\alpha}\,\vec{p}^{\,2}, \tag{1}$$

where p_0 and $\vec{p}$ are the energy and momentum of particle and L is "minimal length" with the order of the Plank length. The work of spacetime foam Liouville string models has introduced this relation with $\alpha = 1$, and people also proposed quantum equation of spineless particles by using this relation. Recently, Kruglov considers the deformed dispersion relation with $\alpha = 2$ and proposes modified Dirac equation [45]:

$$\left[\overline{\gamma}^{\mu}\partial_{\mu} + m - iL\left(\overline{\gamma}^{t}\partial_{t}\right)\left(\overline{\gamma}^{j}\partial_{j}\right)\right]\psi = 0, \tag{2}$$

where $\overline{\gamma}^{a}$ is ordinary gamma matrix and j is space coordinate, while μ is spacetime coordinate. The effect of the correctional term would be observed in higher energy experiment.

In this paper, we try to generalize the modified Dirac equation in curved spacetime and then study the correction of Hawking tunneling radiation. In Section 2, the modified Dirac equation in curved spacetime is constructed and then the modified Hamilton-Jacobi equation is derived via semiclassical approximation. We apply the modified Hamilton-Jacobi equation to the fermions tunneling radiation of 2 + 1-dimensional black string and higher dimensional BTZ-like black strings in Sections 3 and 4, respectively, and Section 5 includes some conclusion and the discussion about the correction of black hole's entropy.

2. Modified Dirac Equation and Hamilton-Jacobi Equation in Curved Spacetime

As we all know, the gamma matrix and partial derivative should become gamma matrix in curved spacetime γ^{a} and covariant $\mathscr{D}_{a}$ derivative, respectively, namely,

$$\begin{aligned} &\overline{\gamma}^{a} \longrightarrow \gamma^{a}, \\ &\partial_{a} \longrightarrow \mathscr{D}_{a} = \partial_{a} + \Omega_{a} + \frac{i}{\hbar}eA_{a}, \end{aligned} \tag{3}$$

where γ^{a} satisfy the relationship $\{\gamma^{a}, \gamma^{b}\} = \gamma^{a}\gamma^{b} + \gamma^{b}\gamma^{a} = 2g^{ab}I$, eA_{a} is charged term of Dirac equation, and $\Omega_{\mu} = (1/8)(\gamma^{a}\gamma^{b} - \gamma^{b}\gamma^{a})e_{a}^{\nu}(\partial_{\mu}e_{b\nu} - \Gamma_{\mu\nu}^{c}e_{bc})$ is spin connection. According to this transformation, we can construct the modified Dirac equation in curved spacetime as

$$\left[\gamma^{\mu}\mathscr{D}_{\mu} + \frac{m}{\hbar} - \sigma\hbar\left(\gamma^{t}\mathscr{D}^{t}\right)\left(\gamma^{j}\mathscr{D}^{j}\right)\right]\Psi = 0, \tag{4}$$

where we choose $c = 1$ but $\hbar \neq 1$, while $c = \hbar = 1$ in (1) and (2). It is assumed that $\sigma \ll 1$, so that the correctional term $\sigma\hbar(\gamma^{t}\mathscr{D}^{t})(\gamma^{j}\mathscr{D}^{j})$ is very small.

Now let us use the modified Dirac equation to derive the modified Hamilton-Jacobi equation. Firstly, we rewrite the wave function of Dirac equation as [33–35]

$$\Psi = \zeta\left(t, x^{j}\right)\exp\left[\frac{i}{\hbar}S\left(t, x^{j}\right)\right], \tag{5}$$

where $\zeta(t, x^{j})$ and Ψ are $m \times 1$ matrices and $\partial_{t}S = -\omega$. In semiclassical approximation, we can consider that $\hbar$ is very small, so that we can neglect the terms with $\hbar$ after dividing by the exponential terms and multiplying by $\hbar$. Therefore, (4) is rewritten as

$$\begin{aligned} &\left[i\gamma^{\mu}\left(\partial_{\mu}S + eA_{\mu}\right) + m - \sigma\gamma^{t}\left(\omega - eA_{t}\right)\gamma^{j}\left(\partial_{j}S + eA_{j}\right)\right] \\ &\quad \cdot \zeta\left(t, x^{j}\right) = 0. \end{aligned} \tag{6}$$

Considering the relationship

$$\gamma^{\mu}\left(\partial_{\mu}S + eA_{\mu}\right) = -\gamma^{t}\left(\omega - eA_{t}\right) + \gamma^{j}\left(\partial_{j}S + eA_{j}\right), \tag{7}$$

we can get

$$\left[i\Gamma^{\mu}\left(\partial_{\mu}S + eA_{\mu}\right) + M\right]\zeta\left(t, x^{j}\right) = 0, \tag{8}$$

where

$$\begin{aligned} \Gamma^{\mu} &= \left[1 + i\sigma\left(\omega - eA_{t}\right)\gamma^{t}\right]\gamma^{\mu}, \\ M &= m - \sigma g^{tt}\left(\omega - eA_{t}\right)^{2}. \end{aligned} \tag{9}$$

Now, multiplying both sides of (9) by the matrix $-i\Gamma^{\nu}(\partial_{\nu}S + eA_{\nu})$, we can obtain

$$\begin{aligned} &\Gamma^{\nu}\left(\partial_{\nu}S + eA_{\nu}\right)\Gamma^{\mu}\left(\partial_{\mu}S + eA_{\mu}\right)\zeta \\ &\quad - iM\Gamma^{\nu}\left(\partial_{\nu}S + eA_{\nu}\right)\zeta = 0. \end{aligned} \tag{10}$$

The second term of the above equation could be simplified again by (8), so the above equation can be rewritten as

$$\Gamma^{\nu}\Gamma^{\mu}\left(\partial_{\nu}S + eA_{\nu}\right)\left(\partial_{\mu}S + eA_{\mu}\right)\zeta + M^{2}\zeta = 0, \tag{11}$$

where we can prove the relation

$$\Gamma^{\nu}\Gamma^{\mu} = \gamma^{\nu}\gamma^{\mu} + 2i\sigma\left(\omega - eA_{t}\right)g^{t\nu}\gamma^{\mu} + \mathscr{O}\left(\sigma^{2}\right). \tag{12}$$

We always ignore $\mathscr{O}(\sigma^{2})$ terms because σ is very small. Now, let us exchange the position of μ and ν in (11) and consider the relation of gamma matrices $\{\gamma^{a}, \gamma^{b}\} = 2g^{ab}I$; then we can obtain

$$\begin{aligned} &\left\{\frac{\gamma^{\alpha}\gamma^{\beta} + \gamma^{\alpha}\gamma^{\beta}}{2}\left(\partial_{\alpha}S + eA_{\alpha}\right)\left(\partial_{\beta}S + eA_{\beta}\right) + m^{2}\right. \\ &\quad - 2\sigma m g^{tt}\left(\omega - eA_{t}\right)^{2} \\ &\quad \left. + 2i\sigma\left(\omega - eA_{t}\right)g^{t\rho}\left(\partial_{\rho}S + eA_{\rho}\right)\gamma^{\mu}\left(\partial_{\mu}S + eA_{\mu}\right)\right\} \\ &\quad \cdot \zeta + \mathscr{O}\left(\sigma^{2}\right) = \left\{g^{\alpha\beta}\left(\partial_{\alpha}S + eA_{\alpha}\right)\left(\partial_{\beta}S + eA_{\beta}\right)\right. \\ &\quad + m^{2} - 2\sigma m g^{tt}\left(\omega - eA_{t}\right)^{2} \\ &\quad \left. + 2i\sigma\left(\omega - eA_{t}\right)g^{t\rho}\left(\partial_{\rho}S + eA_{\rho}\right)\gamma^{\mu}\left(\partial_{\mu}S + eA_{\mu}\right)\right\}\zeta \\ &\quad + \mathscr{O}\left(\sigma^{2}\right) = 0. \end{aligned} \tag{13}$$

Namely,

$$\left[i\sigma\gamma^{\mu}\left(\partial_{\mu}S + eA_{\mu}\right) + \mathscr{M}\right]\zeta\left(t, x^{j}\right) = 0, \tag{14}$$

where

$$\mathscr{M} = \frac{g^{\alpha\beta}\left(\partial_\alpha S + eA_\alpha\right)\left(\partial_\beta S + eA_\beta\right) + m^2 - 2\sigma m g^{tt}\left(\omega - eA_t\right)^2}{2g^{t\rho}\left(\partial_\rho S + eA_\rho\right)\left(\omega - eA_t\right)}. \quad (15)$$

Using the idea of (10)-(11) again, we can multiply both sides of (15) by the matrix $-i\gamma^\nu(\partial_\nu S + eA_\nu)$, so that the equation becomes

$$\sigma\gamma^\nu\left(\partial_\nu S + eA_\nu\right)\gamma^\mu\left(\partial_\mu S + eA_\mu\right)\zeta - i\mathscr{M}\gamma^\nu\left(\partial_\nu S + eA_\nu\right)\zeta = 0. \quad (16)$$

The second term of the above equation could be simplified again by (15). Then, exchange μ and ν and use the relationship $\{\gamma^a, \gamma^b\} = 2g^{ab}I$, so the above equation can be rewritten as

$$\begin{aligned}&\left[\frac{\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu}{2}\sigma^2\left(\partial_\nu S + eA_\nu\right)\left(\partial_\mu S + eA_\mu\right) + \mathscr{M}^2\right]\\ &\cdot\zeta\left(t, x^j\right)\\ &= \left[\sigma^2 g^{\mu\nu}\left(\partial_\nu S + eA_\nu\right)\left(\partial_\mu S + eA_\mu\right) + \mathscr{M}^2\right]\zeta\left(t, x^j\right)\\ &= 0.\end{aligned} \quad (17)$$

The condition that (17) has nontrivial solution required the determinant of coefficient in (17) should vanish, so we can directly get the equation

$$\sigma^2 g^{\mu\nu}\left(\partial_\nu S + eA_\nu\right)\left(\partial_\mu S + eA_\mu\right) + \mathscr{M}^2 = 0. \quad (18)$$

Consider the square root for left side of (18) and ignore all $\mathscr{O}(\sigma^2)$ terms, so we can directly get the modified Hamilton-Jacobi equation:

$$g^{\mu\nu}\left(\partial_\nu S + eA_\nu\right)\left(\partial_\mu S + eA_\mu\right) + m^2 - 2\sigma m g^{tt}\left(\omega - eA_t\right)^2 = 0. \quad (19)$$

Therefore, we find that the modified Dirac equation from Lorentz invariance violation could lead to the modified Hamilton-Jacobi equation, and the correction of Hamilton-Jacobi equation depends on the energy and mass of radiation fermions. Using the modified Hamilton-Jacobi equation, we then investigate the fermions Hawking tunneling radiation of 2 + 1-dimensional black string and n + 1-dimensional BTZ-like string in the following two sections.

3. Fermions Tunneling of 2 + 1-Dimensional Black String

The research of gravity in 2 + 1 dimension can help people further understand the properties of gravity, and it is also important to construct the quantum gravity. Recently, Murata et al. have researched the 2 + 1-dimensional gravity with dilaton field, whose action is given by [46]

$$I = M_3\int d^3x\sqrt{-g}\left(BR + \frac{\lambda^2}{B}\right), \quad (20)$$

where B, M_3, and λ are, respectively, the dilaton field, 3-dimensional Planck mass, and the parameter with mass dimension. The static black string solution is given by

$$\begin{aligned}ds^2 &= -\ln\left(\frac{r}{r_H}\right)dt^2 + \ln\left(\frac{r}{r_H}\right)^{-1}dr^2 + dy^2,\\ B &= \lambda r.\end{aligned} \quad (21)$$

It is evident that the horizon of this black hole is r_H, but the black string is unstable as $r_H \lesssim \mathscr{L}$ (where $\mathscr{L}$ is scale of compactification), so it is assumed that $r_H \gg \mathscr{L}$.

Now we research the fermions tunneling of this black hole, so the modified Hamilton-Jacobi equation in this spacetime is given by

$$-(1 - 2\sigma m)\ln\left(\frac{r}{r_H}\right)^{-1}\omega^2 + \ln\left(\frac{r}{r_H}\right)\left(\frac{dR}{dr}\right)^2 + \left(\frac{dY}{dy}\right)^2 + m^2 = 0, \quad (22)$$

where we have set $S = -\omega t + R(r) + Y(y)$, yielding the radial Hamilton-Jacobi equation as

$$-(1 - 2\sigma m)\ln\left(\frac{r}{r_H}\right)^{-1}\omega^2 + \ln\left(\frac{r}{r_H}\right)\left(\frac{dR}{dr}\right)^2 + \lambda_0 + m^2 = 0, \quad (23)$$

where the constant λ_0 is from separation of variables, and (23) finally can be written as

$$R_\pm(r) = \pm\int\ln\left(\frac{r}{r_H}\right)^{-1}\cdot\sqrt{(1 - 2\sigma m)\omega^2 - \ln\left(\frac{r}{r_H}\right)(\lambda_0 + m^2)}. \quad (24)$$

At the horizon r_H of the black string, the above equation is integrated via residue theorem, and we can get

$$R_\pm(r) = \pm i\pi(1 - \sigma m)r_H\omega \quad (25)$$

and the fermions tunneling rate

$$\begin{aligned}\Gamma &= \exp\left(-\frac{2}{\hbar}\operatorname{Im} S\right) = \exp\left[-\frac{2}{\hbar}\left(\operatorname{Im} R_+ - \operatorname{Im} R_-\right)\right]\\ &= e^{-(4\pi/\hbar)(1-\sigma m)r_H\omega} = e^{-\omega/T_H}.\end{aligned} \quad (26)$$

The relationship between tunneling rate and Hawking temperature required

$$T_H = \hbar\frac{1 + \sigma m}{4\pi r_H} = (1 + \sigma m)T_0, \quad (27)$$

where T_H and T_0 are the modified and nonmodified Hawking temperature of the 2 + 1-dimensional black string, respectively, and σ term is the correction.

4. Fermions Tunneling of Higher Dimensional BTZ-Like Black Strings

As we all know that the linear Maxwell action fails to satisfy the conformal symmetry in higher dimensional spacetime [47, 48], so Hassaïne and Martínez proposed gravity theory with nonlinear Maxwell field in arbitrary dimensional spacetime:

$$I = -\frac{1}{16\pi}\int_M d^{n+1}x\sqrt{-g}\left[R + \frac{2}{l^2} - \beta\left(\alpha F_{\mu\nu}F^{\mu\nu}\right)^s\right], \quad (28)$$

where $\Lambda \equiv -l^{-2}$ is cosmological constant. Hendi researched $n+1$-dimensional static black strings solution with $\beta = 1$, $\alpha = -1$, and $s = n/2$, and it is charged BTZ-like solutions [49], whose metric is

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\sum_k\left(dx^k\right)^2, \quad (29)$$

where

$$f(r) = \frac{r^2}{l^2} - r^{2-n}\left(M + 2^{n/2}Q^{n-1}A_t\right), \quad (30)$$

and the electromagnetic potential is

$$A = A_t dt = Q\ln\left(\frac{r}{l}\right)dt. \quad (31)$$

As $n = 2$, this solution is no other than the static charged BTZ solution. We will study the Hawking radiation and black hole temperature at the event horizon r_H of this black string. In (19), we can set $S = -\omega t + R(r) + Y(x^k)$, where x^k are the space coordinates excluding the radial coordinate, so that the modified Hamilton-Jacobi equation is given by

$$\begin{aligned} &-(1-2\sigma m)f^{-1}(r)\left(\omega - eA_t\right)^2 + f(r)\left(\frac{dR}{dr}\right)^2 \\ &\quad + \frac{1}{r^2}\sum_k\left(\frac{dY}{dx^k}\right)^2 + m^2 = 0, \end{aligned} \quad (32)$$

and the radial equation with constant λ_0 is

$$\begin{aligned} &-(1-2\sigma m)f^{-1}(r)\left(\omega - eA_t\right)^2 + f(r)\left(\frac{dR}{dr}\right)^2 + \frac{\lambda_0}{r^2} \\ &\quad + m^2 = 0. \end{aligned} \quad (33)$$

Therefore, at the horizon r_H of the black string, $f(r_H) = 0$ and we finally get

$$\begin{aligned} R_\pm(r) &= \pm\int f(r)^{-1} \\ &\quad \cdot\sqrt{(1-2\sigma m)\left(\omega - eA_t\right)^2 - f(r)\left(\lambda_0 + m^2\right)} \\ &= \pm i\pi(1-\sigma m)\frac{\omega-\omega_0}{f'(r_H)}, \end{aligned} \quad (34)$$

where $\omega_0 = eA_t(r_H)$. This means that the fermions tunneling rate is

$$\begin{aligned} \Gamma &= \exp\left(-\frac{2}{\hbar}\operatorname{Im}S\right) = \exp\left[-\frac{2}{\hbar}\left(\operatorname{Im}R_+ - \operatorname{Im}R_-\right)\right] \\ &= e^{-(4\pi/\hbar)(1-\sigma m)((\omega-\omega_0)/f'(r_H))} = e^{-((\omega-\omega_0)/T_H)} \end{aligned} \quad (35)$$

and the Hawking temperature is

$$\begin{aligned} T_H &= \hbar\frac{1+\sigma m}{4\pi}f'(r_H) \\ &= \hbar\frac{1+\sigma m}{4\pi}\left(\frac{nr_H}{l^2} - 2^{n/2}Q^n r_H^{1-n}\right) = (1+\sigma m)T_0, \end{aligned} \quad (36)$$

where T_H and T_0 are the modified and nonmodified Hawking temperature of the $n+1$-dimensional BTZ-like black string, respectively, and σ term is the correction.

5. Conclusions

In this paper, we consider the deformed dispersion relation with Lorentz invariance violation and generalize the modified Dirac equation in curved spacetime. The fermions tunneling radiation of black strings is researched, and we find that the modified Dirac equation could lead to Hawking temperature's correction, which depends on the correction parameter σ and particle mass m in the modified Dirac equation. Next we will discuss the correction of black hole entropy in this theory.

The first law of black hole thermodynamics requires

$$dM = TdS + \Xi dJ + UdQ, \quad (37)$$

where Ξ and U are electromagnetic potential and rotating potential, so the nonmodified entropy of black hole is [1, 2, 50, 51]

$$dS_0 = \frac{dM - \Xi dJ - UdQ}{T_0}. \quad (38)$$

From the above results, we know that the relationship between modified and nonmodified Hawking temperature is $T_H = (1+\sigma m)T_0$, since the nonmodified black hole entropy is given by

$$\begin{aligned} S_H &= \int dS_H = \int\frac{dM - \Xi dJ - UdQ}{(1+\sigma m)T_0} \\ &= S_0 - m\int\sigma dS_0 + \mathcal{O}\left(\sigma^2\right), \end{aligned} \quad (39)$$

where we can ignore $\mathcal{O}(\sigma^2)$ because $\sigma \ll 1$. Equation (39) shows that the correction of black hole entropy depends on σ, which is independent from time and space coordinates. However, it is possible that σ depends on other parameters in curved spacetime, and it is very interesting that σ depends on S_0. In particular, as $\sigma = \sigma_0/S_0 + \cdots$, we can get the logarithmic correction of black hole entropy

$$S_H = S_0 - m\sigma_0\ln S_0 + \cdots. \quad (40)$$

In quantum gravity theory, the logarithmic correction has been researched in detail [52–69], and according to [70], it is required that the coefficient of logarithmic correction should be $-(n+1)/2(n-1)$ in $n+1$-dimensional spacetime, so it indicates that σ_0 could be $(n+1)/2m(n-1)$.

On the other hand, from the deformed dispersion relation (1) with $\alpha = 2$, it is implied that the Klein-Gordon equation could be given by

$$\left(-\partial_t^2 + \partial_j^2 + m^2 - \sigma^2\hbar^2\partial_t^2\partial_j^2\right)\Phi = 0, \tag{41}$$

so the generalized uncharged Klein-Gordon equation in static curved spacetime is

$$\left[g^{tt}\nabla_t^2 + g^{jj}\nabla_j^2 + m^2 + \sigma^2\hbar^2\left(g^{tt}\nabla_t^2\right)\left(g^{jj}\nabla_j^2\right)\right]\Phi = 0, \tag{42}$$

and using the semiclassical approximation with $\Phi = C\exp(iS/\hbar)$, the modified Hamilton-Jacobi equation in scalar field is given by

$$\left(1 + \sigma^2 g^{tt}\omega^2\right) g^{\mu\nu}\partial_\mu S\partial_\nu S + m^2 - \sigma^2\left(g^{tt}\right)^2\omega^4 = 0. \tag{43}$$

Namely,

$$\begin{aligned} &g^{\mu\nu}\partial_\mu S\partial_\nu S + m^2 - \sigma^2 g^{tt}\omega^2\left(m^2 + g^{tt}\omega^2\right) + \mathcal{O}\left(\sigma^4\right) \\ &\quad = 0. \end{aligned} \tag{44}$$

Contrasting (19) and (44) as uncharged case, we find that the correctional terms of Dirac field and scalar field are very different. The fact implies that the corrections of Hawking temperature and black hole entropy from Hawking tunneling radiation with different spin particles could be different, and this conclusion could be helpful to suggest a new idea to research the black hole information paradox. Work in these fields is currently in progress.

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

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Research Article

Thermodynamical Study of FRW Universe in Quasi-Topological Theory

H. Moradpour and R. Dehghani

Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), P.O. Box 55134-441, Maragha, Iran

Correspondence should be addressed to H. Moradpour; hn.moradpour@gmail.com

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By applying the unified first law of thermodynamics on the apparent horizon of FRW universe, we get the entropy relation for the apparent horizon in quasi-topological gravity theory. Throughout the paper, the results of considering the Hayward-Kodama and Cai-Kim temperatures are also addressed. Our study shows that whenever there is no energy exchange between the various parts of cosmos, we can get an expression for the apparent horizon entropy in quasi-topological gravity, which is in agreement with other attempts that followed different approaches. The effects of a mutual interaction between the various parts of cosmos on the apparent horizon entropy as well as the validity of second law of thermodynamics in quasi-topological gravity are perused.

1. Introduction

Since observational data indicates an accelerating universe [1–4], one should either consider a nontrivial fluid in the Einstein relativity [5–7] or modifying the Einstein theory [7–10]. Moreover, there are some observational evidences which permit a mutual interaction between the dark sides of cosmos [11–25], including the dark energy and dark matter [26]. The thermodynamic consequences of such interactions in the Einstein framework are addressed in [27]. Recently, it is argued that since the origin of dark energy is unknown, some dark energy candidates may affect the Bekenstein entropy of apparent horizon in a flat FRW universe [28–31]. In addition, by following [32], one can see that all of the dark energy candidates and their interaction with other parts of cosmos may affect the apparent horizon of a flat FRW universe in the Einstein framework. There are additional terms to the Einstein tensor in modified theories of gravity. Since these additional terms can be interpreted as a geometrical fluid, a mutual interaction between the dark sides of cosmos may be an interaction between these geometrical and material terms [10]. Therefore, one may expect that such modifications to Einstein theory and their interaction with other parts of cosmos may also affect the horizon entropy. Indeed, there are some new attempts in which authors give a positive answer to this expectation and show that such geometrical fluids and their interaction with other parts of cosmos may also affect the horizon entropy of FRW universe [33–36].

String theory together with AdS/CFT correspondence conjecture gives us the motivation to study space-times with dimensions more than 4 [37–39]. Moreover, brane scenarios also encourage us to study (4 + 1)-dimensional space-times [40, 41]. The backbone of Einstein-Hilbert action is producing the second-order field equations of motion. In looking for a suitable Lagrangian for higher dimensions, which also keeps the field equations of motion for metric of second-order, one reaches to Lovelock Lagrangian instead of the usual $(n + 1)$ version of Einstein-Hilbert action [42]. It is shown that the corresponding gravitational field equations of static spherically symmetric space-times in the Einstein and Lovelock theories are parallel to the availability of the first law of thermodynamics on the horizons of considered metric [43]. For a FRW universe, where the field equations govern the universe expansion history, one can also reach the corresponding Friedmann equations in the Einstein and Lovelock theories by applying the first law of thermodynamics on the apparent horizon of FRW universe [44]. Indeed, such generalization of Einstein-Hilbert action leads to additional terms besides the Einstein tensor in space-times with dimensions more than 4. Authors in [33] interpret

such additional terms as the geometrical fluids and show that these terms and their interactions with other parts of cosmos affect the horizon entropy.

By considering cubic and higher curvature interactions, some authors try to find new Lagrangian which keeps the field equation of motion for the metric of second-order [45–47]. Their theory is now called quasi-topological gravity which affects the gravitational field equations in space-time with dimensions more than 4. Thermodynamics of static black holes are studied in this theory which leads to an expression for the horizon entropy [45–50]. Moreover, it is shown that if one assumes that the black hole entropy relation, derived in [46–50], is available for the apparent horizon of FRW universe, then, by applying the unified first law of thermodynamics on the apparent horizon of FRW universe, one can get the corresponding Friedmann equations in quasi-topological gravity [51]. Thereinafter, authors point out the second law of thermodynamics and its generalized form [51]. In fact, this scheme proposes that, in quasi-topological gravity, one may generalize the black hole entropy to the cosmological setup by inserting the apparent horizon radii instead of the event horizon radii. Is it the only entropy relation for the apparent horizon? Does a mutual interaction between the geometrical fluid, arising from the terms besides the Einstein tensor in the field equations, and other parts of cosmos affect the horizon entropy?

Our aim in this paper is to study the thermodynamics of apparent horizon of FRW universe in the quasi-topological gravity theory to provide proper answers for the above-mentioned questions. In this investigation, we point to the results of considering the Hayward-Kodama and Cai-Kim temperatures for the apparent horizon. In order to achieve this goal, by looking at the higher-order curvature terms, arising in quasi-topological gravity, as a geometrical fluid and applying the unified first law of thermodynamics on the apparent horizon, we get an expression for the apparent horizon entropy in this theory, while the geometrical fluid does not interact with other parts of cosmos. Moreover, we focus on a FRW universe in which the geometrical fluid interacts with other parts of cosmos. This study signals us to the effects of such mutual interaction on the apparent horizon entropy of a FRW universe with arbitrary curvature for quasi-topological gravity theory. As the noninteracting case, the second law of thermodynamics is also studied in an interacting universe.

The paper is organized as follows. In the next section, we give an introductory note about the quasi-topological gravity, the corresponding Friedmann equations, and some properties of FRW universe, including its apparent horizon radii, surface gravity, and thus the corresponding Hayward-Kodama temperature. Section 3 is devoted to a brief discussion about the Unified First Law (UFL) of thermodynamics. In Section 4, bearing the Hayward-Kodama temperature together with the unified first law of thermodynamics in mind, we study the effect of a geometrical (curvature) fluid on the apparent horizon entropy in a FRW universe with arbitrary curvature parameter for quasi-topological theory. The results of considering the Cai-Kim temperature are also addressed. Thereinafter, we generalize our study to a universe in which the geometrical fluid interacts with other parts of cosmos. The results of attributing the Cai-Kim temperature to the apparent horizon in an interacting universe are also pointed out in Section 4. Section 5 includes some notes about the availability of second law of thermodynamics in interacting cosmos described by quasi-topological gravity theory. The final section is devoted to summary and concluding remarks.

2. A Brief Overview of Quasi-Topological Gravity

A natural generalization of the Einstein-Hilbert action to higher dimensional space-time, and higher-order gravity with second-order equation of motion, is the Lovelock action

$$I_G = \frac{1}{16\pi G_{n+1}} \int d^{n+1}x\sqrt{-g}\left(\sum_{i=1}^{m} c_i \mathscr{L}_i + \mathscr{L}_M\right), \quad (1)$$

where c_i's are Lovelock coefficients, $\mathscr{L}_i$'s are dimensionally extended Euler densities, and $\mathscr{L}_M$ is the matter Lagrangian. In the above action because of the topological origin of the Lovelock terms, the term proportional to c_i contributes to the equations of motion in dimensions with $n \geq 2m$, where m is the order of Lovelock theory. Generally, although the equations of motion of kth-order Lovelock gravity are second-order differential equations, the kth-order Lovelock term has no contribution to the field equations in $2k$ and lower dimensions. For example, the cubic term $\mathscr{L}_3$ does not have any dynamical effect in five dimensions. Recently, a modification of Lovelock gravity called quasi-topological gravity has been introduced, which has contribution to the field equations in five dimensions from the mth-order ($m = 3$) term in Riemann tensor. Several aspects of mth-order quasi-topological terms which have at most second-order derivatives of the metric in the field equations for spherically symmetric space-times in five and higher dimensions except $2p$ dimensions have been investigated.

The action of 4th-order quasi-topological gravity in $(n+1)$ dimensions can be written as follows:

$$I_G = \frac{1}{16\pi G_{n+1}} \int d^{n+1}x\sqrt{-g}\left[\mu_1\mathscr{L}_1 + \mu_2\mathscr{L}_2 + \mu_3\mathscr{X}_3 + \mu_4\mathscr{X}_4 + \mathscr{L}_{\text{matter}}\right], \quad (2)$$

which not only works in five dimensions but also yields second-order equations of motion for spherically symmetric space-times.

In action (2), $\mathscr{L}_1 = R$ is just the Einstein-Hilbert Lagrangian, $\mathscr{L}_2 = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$ is the second-order Lovelock (Gauss-Bonnet) Lagrangian, and $\mathscr{L}_{\text{matter}}$ is the Lagrangian of the matter field. $\mathscr{X}_3$ is the curvature-cubed Lagrangian given by [46]

$$\begin{aligned}\mathscr{X}_3 &= R^{cd}_{ab}R^{ef}_{cd}R^{ab}_{ef} \\ &\quad + \frac{1}{(2n-1)(n-3)}\left(\frac{3(3n-5)}{8}R_{abcd}R^{abcd}R\right. \\ &\quad \left. - 3(n-1)R_{abcd}R^{abc}_{e}R^{de} + 3(n+1)R_{abcd}R^{ac}R^{bd}\right.\end{aligned}$$

$$+ 6(n-1) R_a^b R_b^c R_c^a - \frac{3(3n-1)}{2} R_a^b R_b^a R + \frac{3(n+1)}{8} R^3 \Big), \tag{3}$$

and $\mathcal{X}_4$ is the fourth-order term of quasi-topological gravity [47]:

$$\begin{aligned}\mathcal{X}_4 = {} & c_1 R_{abcd} R^{cdef} R_{ef}^{hg} R_{hg}^{ab} + c_2 R_{abcd} R^{abcd} R_{ef} R^{ef} \\ & + c_3 R R_{ab} R^{ac} R_c^b + c_4 \left(R_{abcd} R^{abcd}\right)^2 \\ & + c_5 R_{ab} R^{ac} R_{cd} R^{db} + c_6 R R_{abcd} R^{ac} R^{db} \\ & + c_7 R_{abcd} R^{ac} R^{be} R_e^d + c_8 R_{abcd} R^{acef} R_e^b R_f^d \\ & + c_9 R_{abcd} R^{ac} R_{ef} R^{bedf} + c_{10} R^4 + c_{11} R^2 R_{abcd} R^{abcd} \\ & + c_{12} R^2 R_{ab} R^{ab} + c_{13} R_{abcd} R^{abef} R_{efg}^c R^{dg} \\ & + c_{14} R_{abcd} R^{aecf} R_{gehf} R^{gbhd}.\end{aligned} \tag{4}$$

The coefficients c_i in the above term are given by

$$\begin{aligned}c_1 &= -(n-1)\left(n^7 - 3n^6 - 29n^5 + 170n^4 - 349n^3 + 348n^2 - 180n + 36\right), \\ c_2 &= -4(n-3)\left(2n^6 - 20n^5 + 65n^4 - 81n^3 + 13n^2 + 45n - 18\right), \\ c_3 &= -64(n-1)\left(3n^2 - 8n + 3\right)\left(n^2 - 3n + 3\right), \\ c_4 &= -\left(n^8 - 6n^7 + 12n^6 - 22n^5 + 114n^4 - 345n^3 + 468n^2 - 270n + 54\right), \\ c_5 &= 16(n-1)\left(10n^4 - 51n^3 + 93n^2 - 72n + 18\right), \\ c_6 &= -32(n-1)^2(n-3)^2\left(3n^2 - 8n + 3\right), \\ c_7 &= 64(n-2)(n-1)^2\left(4n^3 - 18n^2 + 27n - 9\right), \\ c_8 &= -96(n-1)(n-2)\left(2n^4 - 7n^3 + 4n^2 + 6n - 3\right), \\ c_9 &= 16(n-1)^3\left(2n^4 - 26n^3 + 93n^2 - 117n + 36\right), \\ c_{10} &= n^5 - 31n^4 + 168n^3 - 360n^2 + 330n - 90, \\ c_{11} &= 2\left(6n^6 - 67n^5 + 311n^4 - 742n^3 + 936n^2 - 576n + 126\right), \\ c_{12} &= 8\left(7n^5 - 47n^4 + 121n^3 - 141n^2 + 63n - 9\right), \\ c_{13} &= 16n(n-1)(n-2)(n-3)\left(3n^2 - 8n + 3\right), \\ c_{14} &= 8(n-1)\left(n^7 - 4n^6 - 15n^5 + 122n^4 - 287n^3 + 297n^2 - 126n + 18\right).\end{aligned} \tag{5}$$

In the context of universal thermodynamics, our universe should be a nonstationary gravitational system while, from the cosmological point of view, it should be homogeneous and isotropic. Therefore, the natural choice is the FRW universe, a dynamical spherically symmetric space-time, having only inner trapping horizon (the apparent horizon), which is described by the line element

$$ds^2 = h_{ab} dx^a dx^b + \tilde{r}^2 d\Omega^2, \tag{6}$$

where $x^0 = t$, $x^1 = r$, $\tilde{r} = a(t) r$, $a(t)$ is the scale factor of the universe with the curvature parameter k with values $-1, 0, 1$ corresponding to the open, flat, and closed universes respectively, $h_{ab} = \mathrm{diag}(-1, a(t)^2/(1-kr^2))$, and $d\Omega^2$ is the metric of $(n-1)$-dimensional unit sphere.

Varying the action (2) with respect to metric leads to [51]

$$\sum_{i=1}^{m} \widehat{\mu}_i l^{2i-2} \left(H^2 + \frac{k}{a^2}\right)^i = \frac{16\pi G_{n+1}}{n(n-1)} \rho. \tag{7}$$

This is the Friedmann equation of arbitrary-order quasi-topological cosmology where H is the Hubble parameter. From now on, we set $G_{n+1} = 1$ for simplicity. Moreover, $\widehat{\mu}_i$'s are dimensionless parameters as follows:

$$\begin{aligned}\widehat{\mu}_1 &= 1, \\ \widehat{\mu}_2 &= \frac{(n-2)(n-3)}{l^2} \mu_2, \\ \widehat{\mu}_3 &= \frac{(n-2)(n-5)\left(3n^2 - 9n + 4\right)}{8(2n-1) l^4} \mu_3, \\ \widehat{\mu}_4 &= \frac{n(n-1)(n-2)^2(n-3)(n-7)\left(n^5 - 15n^4 + 72n^3 - 156n^2 + 150n - 42\right)}{l^6} \mu_4.\end{aligned} \tag{8}$$

The dynamical apparent horizon is determined by the relation $h^{ab}\partial_a\tilde{r}\partial_b\tilde{r} = 0$. It is a matter of calculation to show that the radius of the apparent horizon for the FRW universe is [52]

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}}. \tag{9}$$

In cosmological context, various definitions of temperature are used to get the corresponding Friedmann equations on the apparent horizon. First, we use the original definition of temperature (Hayward-Kodama temperature) together with the Clausius relation ($TdS_A = dQ^m$) as well as the unified form of the first law of thermodynamics to extract an expression for the entropy of apparent horizon in the quasi-topological cosmology. The Hayward-Kodama temperature associated with the apparent horizon is defined as $T_h = \kappa/2\pi$, where κ is the surface gravity which can be evaluated by using $\kappa = (1/2\sqrt{-h})\partial_a(\sqrt{-h}h^{ab}\partial_b\tilde{r})$ [53–58]. Therefore, the surface gravity at the apparent horizon of the FRW universe has the following form:

$$\kappa = -\frac{1}{\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right), \tag{10}$$

which leads to

$$T_h = \frac{\kappa}{2\pi} = -\frac{1}{2\pi\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right), \tag{11}$$

for the Hayward-Kodama temperature of apparent horizon. Then, using the Cai-Kim temperature and the Clausius relation, we get the entropy of apparent horizon. In this approach, the horizon temperature is [44, 59]

$$T = \frac{1}{2\pi\tilde{r}_A}. \tag{12}$$

In obtaining the above relation, we consider an infinitesimal time interval, so the horizon radius will have a small change and we can use the $d\tilde{r}_A = 0$ approximation [44, 59].

3. Unified First Law of Thermodynamics and Horizon Entropy

The unified first law of thermodynamics can be expressed as [52, 60, 61]

$$dE = A\Psi + WdV, \tag{13}$$

where E is the total baryonic energy content of the universe inside an n-sphere of volume V, while A is the area of the horizon. The energy flux Ψ is termed as the energy supply vector and W is the work function which are, respectively, defined as

$$A\Psi = A\left(T_a^b\partial_b\tilde{r}_A + W\partial_a\tilde{r}_A\right), \tag{14}$$

$$W = -\frac{1}{2}T^{ab}h_{ab}. \tag{15}$$

The term WdV in the first law comes from the fact that we have a volume change for the total system enveloped by the apparent horizon. For a pure de Sitter space, $\rho = p$, and the work term reduces to the standard pdV; thus we obtain exactly the standard first law of thermodynamics, $dE = TdS - pdV$. It is a matter of calculation to show that (14) can be written as

$$A\Psi = -AH\tilde{r}_A\left(\frac{\rho + p}{2}\right)dt + Aa\left(\frac{\rho + p}{2}\right)dr_A, \tag{16}$$

and thus

$$\begin{aligned} A\Psi = &-\frac{3V(\rho + p)H}{2}dt \\ &+ \frac{A(\rho + p)}{2}\left(d\tilde{r}_A - \tilde{r}_A Hdt\right), \end{aligned} \tag{17}$$

on the apparent horizon of FRW universe. In obtaining the last equation, we used $A\tilde{r}_A = 3V$ relation.

4. Thermodynamics of Apparent Horizon in Quasi-Topological Gravity Theory

Our universe is undergoing an accelerating expansion which represents a new imbalance in the governing Friedmann equations. Physicists have addressed such imbalances either by introducing new sources or by changing the governing equations. The standard cosmology model addresses this imbalance by introducing a new source (dark energy) in the Friedmann equations. On the contrary, a group of physicists have explored the second route, that is, a modified gravity approach, that, at large scales, Einstein theory of general relativity breaks down and a more general action describes the gravitational field. In this study, we follow the second approach.

The Friedmann equation of FRW universe in quasi-topological gravity is represented by (7). In modified gravity theories, the modified Friedmann equations can be written as

$$H^2 + \frac{k}{a^2} = \frac{16\pi}{n(n-1)}\rho_t, \tag{18}$$

$$\dot{H} - \frac{k}{a^2} = -\frac{8\pi}{(n-1)}(\rho_t + p_t). \tag{19}$$

In the above equations ρ_t is the total energy density which includes two noninteracting fluid systems: one is the usual fluid of energy density ρ and thermodynamic pressure p, while the second one is termed as effective energy density ρ_e due to curvature contributions and its corresponding pressure p_e. So, we have

$$\begin{aligned} \rho_t &= \rho + \rho_e, \\ \rho_t + p_t &= (\rho + p) + (\rho_e + p_e), \end{aligned} \tag{20}$$

where

$$\rho_e = -\frac{n(n-1)}{16\pi}\sum_{i=2}^{m}\frac{\widehat{\mu}_i l^{2i-2}}{\widetilde{r}_A^{2i}}, \tag{21}$$

$$\rho_e + p_e = -\frac{\varepsilon(n-1)}{4\pi}\sum_{i=2}^{m}\frac{i\widehat{\mu}_i l^{2i-2}}{\widetilde{r}_A^{2i}}, \tag{22}$$

where we have defined $\varepsilon = \dot{\widetilde{r}}_A/2H\widetilde{r}_A$, in which $\dot{\widetilde{r}}_A = d\widetilde{r}_A/dt$, and, in terms of the horizon radius $\widetilde{r}_A$, we have [54]

$$\dot{H} - \frac{k}{a^2} = -\frac{2\varepsilon}{\widetilde{r}_A^2}. \tag{23}$$

Finally, we should note that, by combining (21) and (22), we get $\dot{\rho}_e + nH(\rho_e + p_e) = 0$. It means that (21) and (22) are only valid if the $\dot{\rho}_e + nH(\rho_e + p_e) = 0$ and $\dot{\rho} + nH(\rho + p) = 0$ conditions are simultaneously available. These results demonstrate that there is no energy exchange between the geometrical and material fluids.

4.1. Noninteracting Case. The terms due to the curvature formally play the role of a further source term in the field equations whose effect is the same as that of an effective fluid of purely geometrical origin. Indeed, since such terms modify the Einstein theory, they lead to changing the Bekenstein relation for the entropy of black hole horizon [46–49], which is in agreement with the result obtained by applying the first law of thermodynamics on the black hole horizon [50]. In order to study the thermodynamics of apparent horizon of FRW universe in quasi-topological gravity, authors in [51] assumed that the black hole entropy expression, derived in [46–50], is also valid for the apparent horizon of FRW universe. In addition, by applying the UFL of thermodynamics on the apparent horizon, they could get the Friedmann equation in the quasi-topological gravity [51]. In fact, their recipe is a way of proposing an expression for the apparent horizon entropy instead of a way for deriving the corresponding entropy. Therefore, the inverse of their recipe may indeed be considered theoretically as a more acceptable way of getting the apparent horizon entropy. Finally, based on approach [51], it seems that their proposal for the apparent horizon entropy is independent of any interaction between the cosmos sectors. It is also worthy to note that the mutual interaction between the geometrical ($T^e_{\mu\nu}$) and nongeometrical ($T^m_{\mu\nu}$) sectors of cosmos in various modified theories of gravity affects the apparent horizon entropy [33–36]. In fact, depending on the scales, it is such a curvature fluid which may play the role of dark energy. In Einstein general relativity framework, a dark energy candidate and its interaction with other parts of cosmos affect the Bekenstein entropy [28–32]. Therefore, the results obtained in [33–36] are in line with those of [28–32]. In order to get the effects of mutual interaction between the geometrical and nongeometrical parts of cosmos, we need to establish a recipe to find out the horizon entropy by starting from the Friedmann equations and applying the UFL of thermodynamics on the apparent horizon of FRW universe. The energy conservation law leads to the continuity equation in the form

$$\dot{\rho}_t + nH(\rho_t + p_t) = 0. \tag{24}$$

For a noninteracting system it breaks down to

$$\begin{aligned}\dot{\rho} + nH(\rho + p) &= 0,\\ \dot{\rho}_e + nH(\rho_e + p_e) &= 0,\end{aligned} \tag{25}$$

which implies that (21) and (22) are available. Differentiating (18), we reach

$$-\frac{2}{\widetilde{r}_A^3}d\widetilde{r}_A - \frac{16\pi}{n(n-1)}d\rho_e = \frac{16\pi}{n(n-1)}d\rho. \tag{26}$$

Bearing (25) in mind, we have

$$\frac{d\widetilde{r}_A}{\widetilde{r}_A^3} + \frac{8\pi}{n(n-1)}d\rho_e = \frac{8\pi}{n-1}H(\rho + p)\,dt. \tag{27}$$

Multiplying both sides of the above equation by $(-T)$, one obtains

$$\begin{aligned}&(-T)\left(\frac{d\widetilde{r}_A}{\widetilde{r}_A^3} + \frac{8\pi}{n(n-1)}d\rho_e\right)\\ &\quad= \frac{8\pi}{n-1}H(\rho + p)\,dt\left[\frac{1}{2\pi\widetilde{r}_A}\left(1 - \frac{\dot{\widetilde{r}}_A}{2H\widetilde{r}_A}\right)\right].\end{aligned} \tag{28}$$

Assume that the total energy content of the universe inside an n-sphere of radius $\widetilde{r}_A$ is $E = \rho V$, where $V = \Omega_n\widetilde{r}_A^n$ is the volume enveloped by an n-dimensional sphere. Taking differential form of the total energy, after using the continuity equation (25), we obtain

$$dE = n\Omega_n\rho\widetilde{r}_A^{n-1}d\widetilde{r}_A - nH\Omega_n\widetilde{r}_A^n(\rho + p)\,dt, \tag{29}$$

which leads to

$$(\rho + p)\,dt = -\frac{dE}{nH\Omega_n\widetilde{r}_A^n} + \frac{\rho d\widetilde{r}_A}{H\widetilde{r}_A}. \tag{30}$$

Substituting (30) into (28), simple calculations lead to

$$\begin{aligned}&T\left(\frac{d\widetilde{r}_A}{\widetilde{r}_A^3} + \frac{8\pi}{n(n-1)}d\rho_e\right)\\ &\quad= \frac{4dE}{n(n-1)\Omega_n\widetilde{r}_A^{n+1}} - \frac{2(\rho - p)\,d\widetilde{r}_A}{(n-1)\widetilde{r}_A^2},\end{aligned} \tag{31}$$

and consequently

$$\begin{aligned}&T\left(\frac{d\widetilde{r}_A}{\widetilde{r}_A^3} + \frac{8\pi}{n(n-1)}d\rho_e\right)\\ &\quad= \frac{4}{n(n-1)\Omega_n\widetilde{r}_A^{n+1}}\left[dE - WdV\right],\end{aligned} \tag{32}$$

where we have used $dV = n\Omega_n\widetilde{r}_A^{n-1}d\widetilde{r}_A$. In this equation the work density $W = (\rho - p)/2$ is regarded as the work done

when the apparent horizon radius changes from $\tilde{r}_A$ to $\tilde{r}_A + d\tilde{r}_A$. The Clausius relation is [32]

$$TdS_A = \delta Q^m = A\Psi, \tag{33}$$

where δQ^m is the energy flux crossing the horizon during the universe expansion. Considering (13), (32), and (33), we have

$$dS_A = \frac{n(n-1)\Omega_n \tilde{r}_A^{n-2} d\tilde{r}_A}{4} + 2\pi\Omega_n \tilde{r}_A^{n+1} d\rho_e. \tag{34}$$

Using (25) leads to

$$\begin{aligned} dS_A = {} & \frac{n(n-1)\Omega_n \tilde{r}_A^{n-2} d\tilde{r}_A}{4} \\ & - 2nH\pi\Omega_n \tilde{r}_A^{n+1}(\rho_e + p_e)\, dt. \end{aligned} \tag{35}$$

Now, by either inserting (21) into (34) or (22) into (35) and integrating the result, we get

$$S_A = \frac{A}{4}\sum_{i=1}^{m} i \frac{(n-1)}{(n-2i+1)} \frac{\hat{\mu}_i l^{2i-2}}{\tilde{r}_A^{2i-2}} + S_0, \tag{36}$$

for the horizon entropy. This result is in full agreement with previous proposal in which authors assumed that the event horizon entropy is extendable to the apparent horizon of FRW universe [51]. S_0 is an integration constant and we can consider it zero without loss of generality. It is also useful to mention here that we considered the Clausius relation in the form $TdS_A = \delta Q^m$ to obtain this result, while authors in [51] took into account the $TdS_A = -\delta Q^m$ form of the Clausius relation. This discrepancy is due to our different way of defining the horizon temperature. We have used the Hayward-Kodama definition of temperature (11), while authors in [51] used its absolute value to avoid attributing negative temperatures on the apparent horizon and thus the Hawking radiation.

The Cai-Kim Approach. Here, we focus on the Cai-Kim approach to get the effects of geometrical fluid on the horizon entropy. When one applies the first law on the apparent horizon to calculate the surface gravity and thereby the temperature and considers an infinitesimal amount of energy crossing the apparent horizon, the apparent horizon radius $\tilde{r}_A$ should be regarded to have a fixed value ($dV = 0$) [59]. Bearing (17) together with $d\tilde{r}_A = 0$ approximation in mind, by using (25) and the Clausius relation in the form $TdS_A = -\delta Q^m = -A\Psi$, where δQ^m is the energy flux crossing the horizon during the infinitesimal time interval dt, we get

$$TdS_A = -Vd\rho. \tag{37}$$

Substituting $d\rho$ from (26) into the above equation, we are led to

$$dS_A = -\frac{V}{T}d\rho = 2\pi\Omega_n \tilde{r}_A^{n+1}\left[\frac{n(n-1)\, d\tilde{r}_A}{8\pi\tilde{r}_A^3} + d\rho_e\right], \tag{38}$$

where we have used $V = \Omega_n \tilde{r}_A^n$ and the Cai-Kim temperature ($T = 1/2\pi\tilde{r}_A$) [44, 59]. Finally, we obtain

$$dS_A = \frac{n(n-1)\Omega_n \tilde{r}_A^{n-2} d\tilde{r}_A}{4} + 2\pi\Omega_n \tilde{r}_A^{n+1} d\rho_e. \tag{39}$$

Consequently the modified entropy on the event horizon has the explicit form

$$S_A = \frac{A}{4}\sum_{i=1}^{m} i \frac{(n-1)}{(n-2i+1)} \frac{\hat{\mu}_i l^{2i-2}}{\tilde{r}_A^{2i-2}} + S_0, \tag{40}$$

which is compatible with the results obtained by considering the Hayward-Kodama temperature (36). As before, since entropy is not an absolute quantity, without loss of generality, we can set S_0 to zero. Setting $n = 3$ and $i = 1$ in the above result leads to the Bekenstein-Hawking entropy formula.

4.2. Interacting Universe. In order to take the interaction into account, for the quasi-topological gravity, consider the energy-momentum conservation law in the forms

$$\dot{\rho} + nH(\rho + p) = Q, \tag{41}$$

$$\dot{\rho}_e + nH(\rho_e + p_e) = -Q, \tag{42}$$

where Q is the mutual interaction between different parts of the cosmos. The latter means that (21) and (22) are not simultaneously valid for an interacting universe. Using (26) and (41), we get

$$\frac{d\tilde{r}_A}{\tilde{r}_A^3} + \frac{8\pi}{n(n-1)}(d\rho_e + Qdt) = \frac{8\pi H}{n-1}(\rho + p)\, dt. \tag{43}$$

Multiplying both sides of the above equation by $(-T)$, one obtains

$$\begin{aligned} & T\left(\frac{d\tilde{r}_A}{\tilde{r}_A^3} + \frac{8\pi}{n(n-1)}(d\rho_e + Qdt)\right) \\ & \quad = -\frac{2H(\rho + p)}{(n-1)\tilde{r}_A}dt + \frac{2(\rho + p)}{(n-1)\tilde{r}_A^2}adr_A, \end{aligned} \tag{44}$$

where we have used $\dot{\tilde{r}}_A dt = adr_A + r_A Hdt$. Multiplying the result by the factor $n(n-1)\Omega_n \tilde{r}_A^{n+1}/4$, we arrive at

$$\begin{aligned} & T\left(\frac{d\tilde{r}_A}{\tilde{r}_A^3} + \frac{8\pi}{n(n-1)}(d\rho_e + Qdt)\right) \\ & \quad \cdot \left(\frac{n(n-1)\Omega_n \tilde{r}_A^{n+1}}{4}\right) \\ & \quad = n\Omega_n \tilde{r}_A^n \left[-\frac{\rho + p}{2}Hdt + \frac{\rho + p}{2\tilde{r}_A}adr_A\right]. \end{aligned} \tag{45}$$

Simple calculations lead to

$$\begin{aligned} & T\left(\frac{n(n-1)\Omega_n \tilde{r}_A^{n-2} d\tilde{r}_A}{4} + 2\pi\Omega_n \tilde{r}_A^{n+1}(d\rho_e + Qdt)\right) \\ & \quad = -AH\frac{\rho + p}{2}\tilde{r}_A dt + Aa\frac{\rho + p}{2}dr_A. \end{aligned} \tag{46}$$

The right-hand side of (46) is nothing but the energy flux (δQ^m) crossing the apparent horizon (16). Bearing the Clausius relation (33) in mind, we get

$$dS_A = \frac{n(n-1)\Omega_n \tilde{r}_A^{n-2} d\tilde{r}_A}{4} + 2\pi\Omega_n \tilde{r}_A^{n+1}(d\rho_e + Qdt). \quad (47)$$

Now, use (42) to obtain

$$dS_A = \frac{n(n-1)\Omega_n \tilde{r}_A^{n-2} d\tilde{r}_A}{4} - 2\pi n H \Omega_n \tilde{r}_A^{n+1}(\rho_e + p_e)\, dt, \quad (48)$$

which leads to

$$S_A = \frac{A}{4} - 2\pi n \Omega_n \int H \tilde{r}_A^{n+1}(\rho_e + p_e)\, dt + S_0, \quad (49)$$

which is in agreement with the results obtained by authors [33] for the Lovelock theory but in a different way. In fact, since in the interacting universes ρ_e and $\rho_e + p_e$ cannot simultaneously meet (21) and (22), respectively, we can not use them to integrate with the RHS of (47) and (48) to get an expression for the horizon entropy. Now, consider a special situation in which ρ_e satisfies (21); from (47) we get

$$dS_A = \frac{n(n-1)\Omega_n \tilde{r}_A^{n-2} d\tilde{r}_A}{4} + 2\pi\Omega_n \tilde{r}_A^{n+1} d\rho_e + 2\pi\Omega_n \tilde{r}_A^{n+1} Q dt. \quad (50)$$

In the above equation, the first two terms of the right-hand side are exactly the same as (39). In other words,

$$(S_A)_{\text{interaction}} = (S_A)_{\text{non-interaction}} + 2\pi\Omega_n \int \tilde{r}_A^{n+1} Q dt. \quad (51)$$

It is crystal clear that the second term of RHS of this equation counts the effects of mutual interaction between the geometrical and nongeometrical fluids on the horizon entropy. Loosely speaking, the above expression for horizon entropy is no longer the usual Bekenstein entropy formula; rather there are two correction terms which are due to higher-order curvature terms, arising from the quasi-topological nature of gravity theory, along with the mutual interaction between the geometrical and nongeometrical parts. Note that, in this situation, due to (42), the $\rho_e + p_e$ term does not meet (22). As another example, consider a situation in which $\rho_e + p_e$ satisfies (22), which also means that, due to (42), ρ_e does not obey (21). For this case, by combining (42) and (47) and integrating the result, we obtain

$$(S_A)_{\text{interaction}} = (S_A)_{\text{non-interaction}}, \quad (52)$$

which means that the mutual interaction Q does not affect the horizon entropy in this special case.

The Cai-Kim Approach. The rest of this section is devoted to achieving (48) by using the Cai-Kim approach [44, 59]. For an interacting universe the UFL is in the form [31]

$$dS_A = -\frac{V}{T}(d\rho - Qdt). \quad (53)$$

Using (26) and (42), we have

$$d\rho - Qdt = -\frac{n(n-1)\, d\tilde{r}_A}{8\pi \tilde{r}_A^3} + nH(\rho_e + p_e)\, dt. \quad (54)$$

Substituting it into (53) leads to

$$dS_A = \frac{n(n-1)\Omega_n \tilde{r}_A^{n-2} d\tilde{r}_A}{4} - 2\pi n H \Omega_n \tilde{r}_A^{n+1}(\rho_e + p_e)\, dt, \quad (55)$$

which is in full agreement with (48).

5. Second Law of Thermodynamics

The time evolution of the entropy in a noninteracting universe governed by quasi-topological gravity is already considered by authors in [51]. Now, we are interested in examining the validity of the second law of thermodynamics in an interacting universe. This law states that the horizon entropy should meet the $dS_A/dt \geq 0$ condition [62]. For this propose, from (48), we have

$$\frac{dS_A}{dt} = \frac{n(n-1)\Omega_n \tilde{r}_A^{n-2} \dot{\tilde{r}}_A}{4} - 2\pi n H \Omega_n \tilde{r}_A^{n+1}(\rho_e + p_e). \quad (56)$$

However, using (42) and (43), one gets

$$\frac{\dot{\tilde{r}}_A}{\tilde{r}_A^3} - \frac{8\pi H}{n-1}(\rho_e + p_e) = \frac{8\pi H}{n-1}(\rho + p), \quad (57)$$

which leads to the Raychaudhuri equation

$$\dot{\tilde{r}}_A = \frac{8\pi H}{n-1}(\rho + p + \rho_e + p_e)\tilde{r}_A^3. \quad (58)$$

Substituting it into (56), we arrive at

$$\frac{dS_A}{dt} = 2\pi n H \Omega_n \tilde{r}_A^{n+1}(\rho + p). \quad (59)$$

The above result indicates that the second law of thermodynamics ($dS_A/dt \geq 0$) is valid for the apparent horizon under the condition $(\rho + p) > 0$ and if we define the state parameter $\omega = p/\rho$, this law is valid whenever the state parameter obeys $\omega \geq -1$.

6. Summary and Conclusions

After giving a brief review of the quasi-topological gravity theory, by considering a FRW universe, we pointed out to the corresponding Friedmann equation in this modified gravity theory. In Section 2, we mentioned the apparent horizon of FRW universe as the proper causal boundary, its surface gravity, and the Hayward-Kodama temperature of this hypersurface. Motivated by recent works on the thermodynamics of horizon of FRW universe in some modified gravity theories [33–36], we considered the terms other than Einstein tensor as a fluid with energy density ρ_e and pressure p_e. In fact, since

these terms may play the role of dark energy and because a dark energy candidate may also affect the horizon entropy [28–32], such justification at least is not forbidden. Then, we showed that if we use either the Hayward-Kodama or the Cai-Kim temperature, the same result for the horizon entropy is obtainable ((36) and (40)) which is in agreement with previous work [50], in which authors, by using different definitions for the horizon temperature and generalizing the black hole entropy to the cosmological horizon setup, could propose the same expression as ours for the apparent horizon entropy in quasi-topological gravity. Moreover, it is useful to note that (36) is valid only if there is no energy-momentum exchange between the geometrical ($T^{e}_{\mu\nu}$) and nongeometrical ($T^{m}_{\mu\nu}$) fluids.

Finally, we generalized our investigation to an interacting FRW universe with arbitrary curvature parameter (k) in quasi-topological gravity. We got a relation for the horizon entropy (see (49)) which showed the effects of an energy-momentum exchange between the geometrical and nongeometrical fluids on the apparent horizon entropy. Thereinafter, we studied the special case in which ρ_e meets (21) to perceive clearly the effects of such mutual interaction on the horizon entropy. We got (51) for the horizon entropy, which showed two terms besides the Bekenstein entropy due to higher-order curvature terms, arising from the quasi-topological nature of model, and mutual interaction between the geometrical and nongeometrical fluids. Our investigation also showed that whenever $\rho_e + p_e$ satisfies (22), the mutual interaction Q does not affect the horizon entropy and therefore the result of noninteracting case is obtainable. We have also pointed out the validity of second law of thermodynamics in an interacting case and found out that, independent of curvature parameter (k), whenever $\rho + p \geq 0$, the second law is obtainable.

Our study shows that, in quasi-topological gravity theory, a geometrical fluid with energy density ρ_e and pressure p_e, independent of its nature, affects the apparent horizon entropy as

$$S_A = \frac{A}{4} - 2\pi n \Omega_n \int H \tilde{r}_A^{n+1} \left(\rho_e + p_e\right) dt, \qquad (60)$$

which is in full agreement with other studies [28–36]. Since such correction in quasi-topological gravity has geometrical nature, it is indeed inevitable. Moreover, the effects of mutual interaction between the dark energy candidate and other parts of cosmos are stored in the second term of RHS of this equation. Indeed, based on our approach, one may find the apparent horizon entropy of FRW universe with arbitrary curvature parameter (k) in modified theories of gravity by interpreting the terms other than Einstein tensor in the corresponding Friedmann equation as a geometrical fluid and applying the UFL of thermodynamics on the apparent horizon. At the end, we should stress that, in the absence of a mutual interaction, the results of interacting universes in quasi-topological theory converge to those of the noninteracting case.

Competing Interests

The authors declare that they have no competing interests.

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Research Article

The Minimal Length and the Shannon Entropic Uncertainty Relation

Pouria Pedram

Department of Physics, Islamic Azad University, Science and Research Branch, Tehran 1477893855, Iran

Correspondence should be addressed to Pouria Pedram; p.pedram@srbiau.ac.ir

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In the framework of the generalized uncertainty principle, the position and momentum operators obey the modified commutation relation $[X, P] = i\hbar(1 + \beta P^2)$, where β is the deformation parameter. Since the validity of the uncertainty relation for the Shannon entropies proposed by Beckner, Bialynicki-Birula, and Mycielski (BBM) depends on both the algebra and the used representation, we show that using the formally self-adjoint representation, that is, $X = x$ and $P = \tan(\sqrt{\beta}p)/\sqrt{\beta}$, where $[x, p] = i\hbar$, the BBM inequality is still valid in the form $S_x + S_p \geq 1 + \ln \pi$ as well as in ordinary quantum mechanics. We explicitly indicate this result for the harmonic oscillator in the presence of the minimal length.

1. Introduction

The existence of a minimal observable length proportional to the Planck length $\ell_{\mathrm{P}} = \sqrt{G\hbar/c^3} \approx 10^{-35}$ m is motivated by various proposals of quantum gravity such as string theory, loop quantum gravity, noncommutative geometry, and black-hole [1–3]. Indeed, several schemes have been established to investigate the effects of the minimal length range from astronomical observations [4, 5] to table-top experiments [6]. In particular, a measurement method is proposed recently to detect this fundamental length scale which is based on the possible deviations from ordinary quantum commutation relation at the Planck scale within the current technology [6].

Deformed commutation relations have attracted much attention in recent years and several problems range from classical to quantum mechanical systems have been studied exactly or approximately in the context of the generalized (gravitational) uncertainty principle (GUP). Among these investigations in quantum domain, we can mention the harmonic oscillator [7–9], Coulomb potential [10–12], singular inverse square potential [13], coherent states [14], Dirac oscillator [15], Lamb's shift, Landau levels, tunneling current in scanning tunneling microscope [16], ultracold neutrons in gravitational field [17, 18], Casimir effect [19], relativistic quantum mechanics [20–22], and cosmological problems [23–26]. On the other hand, in the classical domain, deformed classical systems in phase space [27, 28], Keplerian orbits [29], composite systems [30], and the thermostatistics [31, 32] have been investigated in the presence of the minimal length.

In the last decade, many applications of information theoretic measures such as entropic uncertainty relations, as alternatives to Heisenberg uncertainty relation, appeared in various quantum mechanical systems [33–48]. The concept of statistical complexity was introduced by Shannon in 1948 [49], where the term uncertainty can be considered as a measure of the missing information. In particular, it is shown that the information entropies such as Shannon entropy may be used to replace the well-known quantum mechanical uncertainty relation. The first entropic uncertainty relation for position and momentum observables was proposed by Hirschmann [50] and is later improved by Beckner, Bialynicki-Birula, and Mycielski (BBM) [51–54].

In the context of the generalized uncertainty principle, there exists two questions. (i) Are the wave functions in position space and momentum space related by the Fourier transform in arbitrary GUP algebra in the form $[X, P] = i\hbar f(X, P)$? (ii) If in a particular algebra these wave functions

are not related by the Fourier transform, is the BBM inequality still valid? Note that the validity of the BBM entropic uncertainty relation depends on both the algebra and the used representation. For instance, consider the modified commutation relation in the form $[X, P] = i\hbar(1 + \beta P^2 + \alpha X^2)$ which implies a minimal length and a minimal momentum proportional to $\hbar\sqrt{\beta}$ and $\hbar\sqrt{\alpha}$, respectively. This form of GUP has no formally self-adjoint representation in the form $X = x$ and $P = f(p)$. So the momentum space and coordinate space wave functions are not related by the Fourier transform in any representation and the BBM uncertainty relation does not hold in this framework.

The well-known Robertson uncertainty principle for two noncommuting observables is given by

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle \psi | [A, B] | \psi \rangle \right|, \tag{1}$$

where $[A, B]$ denotes the commutator of A and B and ΔA and ΔB are their dispersions. However, this uncertainty relation suffers from two serious shortcomings [55, 56]. (i) For two noncommuting observables of a finite N-dimensional Hilbert space, since the right-hand side of (1) depends on the wave function ψ, it is not a fixed lower bound. Indeed, if ψ is the eigenstate of the observable A or B, the right-hand side of (1) vanishes and there is no restriction on ΔA or ΔB by this uncertainty relation. (ii) The dispersions cannot be considered as suitable measures for the uncertainty of two complementary observables with continuous probability densities. This problem is more notable when their corresponding probability densities contain several sharp peaks. Among various proposals for the uncertainty relations that are not suffered from these shortcomings, we can mention the information-theoretical entropy instead of the dispersions which is a proper measure of the uncertainty.

In this paper, we study the effects of the minimal length on the entropic uncertainty relation. In this scenario, the position and momentum operators obey the modified commutation relation $[X, P] = i\hbar(1 + \beta P^2)$, where β is the deformation parameter. Using formally self-adjoint representation of the algebra, we show that the coordinate space and momentum space wave functions are related by the Fourier transform and consequently the BBM inequality is preserved. However, as we will show, in the quasiposition representation the momentum space and quasiposition space wave functions are not related by the Fourier transformation and the BBM inequality does not hold. As an application, we obtain the generalized Schrödinger equation for the harmonic oscillator and exactly solve the corresponding differential equation in momentum space. Then, we find information entropies for the two lowest energy eigenstates and explicitly show the validity of the BMM inequality in the presence of the minimal length.

2. The Generalized Uncertainty Principle

In one-dimension, the deformed commutation relation reads [7]

$$[X, P] = i\hbar\left(1 + \beta P^2\right), \tag{2}$$

which results in $\Delta X \Delta P \geq (\hbar/2)[1 + \beta(\Delta P)^2]$ (generalized uncertainty principle) and for $\beta \to 0$ the well-known commutation relation in ordinary quantum mechanics is recovered. Notice that, ΔX cannot take arbitrarily small values and the absolutely smallest uncertainty in position is given by $(\Delta X)_{\min} = \hbar\sqrt{\beta}$.

Now, consider the *formally* self-adjoint representation [9]

$$X = x, \qquad P = \frac{\tan\left(\sqrt{\beta}p\right)}{\sqrt{\beta}}, \tag{3}$$

where $[x, p] = i\hbar$, $-\pi/2\sqrt{\beta} < p < \pi/2\sqrt{\beta}$, and it exactly satisfies (2). In this representation, the ordinary nature of the position operator is preserved and the inner product of states takes the following form:

$$\langle \psi \mid \phi \rangle = \int_{-\pi/2\sqrt{\beta}}^{+\pi/2\sqrt{\beta}} \psi^*(p)\,\phi(p)\,dp. \tag{4}$$

Note that although the position operator obeys $X^\dagger = X = i\hbar\partial/\partial p$ in momentum space, we have $\mathscr{D}(X) \subset \mathscr{D}(X^\dagger)$. So X is merely symmetric and it is not a true self-adjoint operator. However, based on the von Neumann's theorem, for the momentum operator we obtain $P = P^\dagger$ and $\mathscr{D}(P) = \mathscr{D}(P^\dagger) = \{\phi \in \mathscr{D}_{\max}(\mathbb{R})\}$ [9]. Thus, P is indeed a self-adjoint operator. Moreover, the scalar product and the completeness relation read

$$\left\langle p' \mid p \right\rangle = \delta\left(p - p'\right), \tag{5}$$

$$\int_{-\pi/2\sqrt{\beta}}^{+\pi/2\sqrt{\beta}} |p\rangle \langle p|\, dp = 1. \tag{6}$$

In momentum space, the eigenfunctions of the position operator are given by the solutions of the eigenvalue equation

$$X u_x(p) = x u_x(p). \tag{7}$$

Here, $u_x(p) = \langle p \mid x \rangle$ which can be expressed as

$$u_x(p) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{ip}{\hbar}x\right). \tag{8}$$

Now, using (6) and (8), coordinate space wave function can be written as

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi/2\sqrt{\beta}}^{+\pi/2\sqrt{\beta}} e^{ipx/\hbar}\phi(p)\,dp. \tag{9}$$

Moreover, $\phi(p)$ is given by the inverse Fourier transform of the coordinate space wave function; namely,

$$\phi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar}\psi(x)\,dx. \tag{10}$$

To this end, by taking $\phi(p) = 0$ for $|p| > \pi/2\sqrt{\beta}$ we can formally extend the domain of the momentum integral (9) to $-\infty < p < \infty$ without changing the coordinate space

wave function $\psi(x)$. Therefore, $\psi(x)$ is the Fourier transform of $\phi(p)$. Now, using the Babenko-Beckner inequality [51, 52] and following Bialynicki-Birula and Mycielski [53] we obtain (see [54] for details)

$$S_x + S_p \geq 1 + \ln \pi, \tag{11}$$

where

$$\begin{aligned} S_x &= -\int_{-\infty}^{+\infty} |\psi(x)|^2 \ln |\psi(x)|^2 \, dx, \\ S_p &= -\int_{-\infty}^{+\infty} |\phi(p)|^2 \ln |\phi(p)|^2 \, dp, \end{aligned} \tag{12}$$

subject to $\phi(p) = 0$ for $|p| > \pi/2\sqrt{\beta}$. Note that, in this representation, the expression for the entropic uncertainty relation is similar to the ordinary quantum mechanics. However, as we will see in the next section, the presence of the minimal length modifies the Hamiltonian, its solutions, and the values of S_x and S_p. But since $\psi(x)$ and $\phi(p)$ are still related by the Fourier transform, the lower bound for the entropic uncertainty relation will not be modified.

3. Quasiposition Representation

Another possible representation that exactly satisfies (2) is [7]

$$\begin{aligned} X\phi(p) &= i\hbar\left(1 + \beta p^2\right)\partial_p \phi(p), \\ P\phi(p) &= p\phi(p). \end{aligned} \tag{13}$$

The corresponding scalar product and completeness relations read

$$\left\langle p' \mid p \right\rangle = \left(1 + \beta p^2\right)\delta\left(p - p'\right), \tag{14}$$

$$\int_{-\infty}^{+\infty} \frac{1}{1 + \beta p^2} |p\rangle \langle p| \, dp = 1. \tag{15}$$

Now, since the measure in the integral (15) is not flat, the momentum space entropy

$$S_p = -\int_{-\infty}^{+\infty} \frac{1}{1 + \beta p^2} |\phi(p)|^2 \ln |\phi(p)|^2 \, dp, \tag{16}$$

has no proper form of the continuous Shannon entropy relation in this representation.

The quasiposition wave function is defined as [7]

$$\phi(\xi) \equiv \left\langle \psi_\xi^{ml} \mid \phi \right\rangle, \tag{17}$$

where

$$\psi_\xi^{ml}(p) = \sqrt{\frac{2\sqrt{\beta}}{\pi}} \left(1 + \beta p^2\right)^{-1/2} e^{-i(\xi \tan^{-1}(\sqrt{\beta}p)/\hbar\sqrt{\beta})} \tag{18}$$

denotes the maximal localization states. These states satisfy $\langle \psi_\xi^{ml} | X | \psi_\xi^{ml} \rangle = \xi$ and $(\Delta X)_{|\psi_\xi^{ml}\rangle} = \hbar\sqrt{\beta}$ and are not mutually orthogonal; that is, $\langle \psi_{\xi'}^{ml} \mid \psi_\xi^{ml} \rangle \neq \delta(\xi - \xi')$. In this representation, the relation between the momentum space wave functions and quasiposition wave functions is given by

$$\begin{aligned} &\psi(\xi) \\ &= \sqrt{\frac{2\sqrt{\beta}}{\pi}} \int_{-\infty}^{+\infty} \frac{1}{\left(1 + \beta p^2\right)^{3/2}} e^{i\xi \tan^{-1}(\sqrt{\beta}p)/\hbar\sqrt{\beta}} \psi(p) \, dp. \end{aligned} \tag{19}$$

Thus, the quasiposition wave functions are not Fourier transform of the momentum space wave functions and the quasiposition entropy

$$\begin{aligned} S_\xi = &-\left(8\pi\hbar^2\sqrt{\beta}\right)^{-1} \\ &\cdot \iiint_{-\infty}^{+\infty} e^{i(\xi-\xi')(\tan^{-1}(\sqrt{\beta}p)/\hbar\sqrt{\beta})} \psi^*(\xi)\psi(\xi') \\ &\cdot \ln \left|\psi(\xi')\right|^2 dp \, d\xi \, d\xi' \end{aligned} \tag{20}$$

does not represent the continuous Shannon entropy and contains plenty of overcounting (because of the nonorthogonality of $|\psi_\xi^{ml}\rangle$) that should be avoided. These results show that the information entropies S_p (16) and S_ξ (20) are not proper measures of uncertainty. However, the information entropies (12) based on formally self-adjoint representation do not suffer from these shortcomings and they can be considered as proper measures of uncertainty in the presence of the minimal length.

4. Quantum Oscillator

The Hamiltonian of the harmonic oscillator is given by $H = P^2/2m + (1/2)m\omega^2 X^2$. So the generalized Schrödinger equation in momentum space using the representation (3) reads

$$-\frac{1}{2}m\hbar^2\omega^2 \frac{d^2\phi(p)}{dp^2} + \frac{\tan^2\left(\sqrt{\beta}p\right)}{2m\beta}\phi(p) = E\phi(p). \tag{21}$$

Using the new variable $\xi = \sqrt{\beta}p$, the above equation can be written as

$$\frac{d^2\phi(\xi)}{d\xi^2} + \left(\epsilon - \frac{V}{\cos^2\xi}\right)\phi(\xi) = 0, \tag{22}$$

where $V = (m\beta\hbar\omega)^{-2}$ and $\epsilon = V(1 + 2m\beta E)$. Now, taking

$$\phi_n(\xi) = P_n(s)\cos^\lambda \xi \tag{23}$$

results in

$$\begin{aligned} &\left(1 - s^2\right)\frac{d^2 P_n(s)}{ds^2} - s(1 + 2\lambda)\frac{dP_n(s)}{ds} \\ &\quad + \left(\epsilon - \lambda^2\right)P_n(s) = 0, \end{aligned} \tag{24}$$

where $s = \sin\xi$ and

$$\begin{aligned} V &= \lambda(\lambda - 1), \\ \lambda &= \frac{1}{2}\left[1 + \sqrt{1 + \frac{4}{m^2\beta^2\hbar^2\omega^2}}\right]. \end{aligned} \tag{25}$$

It is known that the solutions of the above equation for $\epsilon = (n+\lambda)^2$ are given by the Gegenbauer polynomials $C_n^\lambda(s)$. Therefore, the exact solutions read

$$\phi_n(p) = N_n C_n^\lambda\left(\sin\left(\sqrt{\beta}p\right)\right)\cos^\lambda\left(\sqrt{\beta}p\right), \tag{26}$$

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right)\left(\sqrt{1+\frac{\eta^2}{4}}+\frac{\eta}{2}\right)+\frac{1}{2}\hbar\omega\eta n^2, \quad n = 0,1,2,\ldots, \tag{27}$$

where $\eta = m\beta\hbar\omega$, N_n is the normalization coefficient, and the Gegenbauer polynomials are defined as [57]

$$C_n^\lambda(s) = \sum_{k=0}^{[n/2]}(-1)^k\frac{\Gamma(n-k+\lambda)}{\Gamma(\lambda)\,k!\,(n-2k)!}(2s)^{n-2k}. \tag{28}$$

Note that for $\beta \to 0$ we obtain the ordinary energy spectrum of the harmonic oscillator; that is, $E_n = \hbar\omega(n+1/2)$. The Gegenbauer polynomials also satisfy the following useful formula [57]:

$$\int_{-1}^{1}\left(1-x^2\right)^{\nu-1/2}\left[C_n^\nu(x)\right]^2dx = \frac{\pi 2^{1-2\nu}\Gamma(n+2\nu)}{n!\,(n+\nu)\left[\Gamma(\nu)\right]^2}, \quad \operatorname{Re}\nu > -\frac{1}{2}. \tag{29}$$

Since $s = \sin\xi$ and N_n is given by the normalization condition $\int_{-\pi/2\sqrt{\beta}}^{+\pi/2\sqrt{\beta}}|\phi(\xi)|^2dp = 1$, we find

$$N_n = \sqrt{\frac{\sqrt{\beta}n!\,(n+\lambda)\left[\Gamma(\lambda)\right]^2}{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}}. \tag{30}$$

The solutions can be also written in terms of relativistic Hermite polynomials using the relation [58]

$$H_n^\lambda\left(\sqrt{\lambda}u\right) = \frac{n!}{\lambda^{n/2}}\left(1+u^2\right)^{n/2}C_n^\lambda\left(\frac{u}{\sqrt{1+u^2}}\right), \tag{31}$$

where $H_n^\lambda(z)$ denotes relativistic Hermite polynomials. Thus, we obtain $C_n^\lambda(\sin(\sqrt{\beta}p)) = (\lambda^{n/2}/n!)\cos^n(\sqrt{\beta}p)H_n^\lambda(\sqrt{\lambda}\tan(\sqrt{\beta}p))$ which results in

$$\phi_n(p) = \frac{N_n\lambda^{n/2}}{n!}\cos^{\lambda+n}\left(\sqrt{\beta}p\right)H_n^\lambda\left(\sqrt{\lambda}\tan\left(\sqrt{\beta}p\right)\right). \tag{32}$$

For the small values of the deformation parameter we have ($m = \hbar = 1$)

$$\begin{gathered}\lim_{\beta\to 0}\lambda = \frac{1}{\omega\beta}, \quad \beta \longrightarrow 0 \Longleftrightarrow \lambda \longrightarrow \infty,\\ \lim_{\lambda\to\infty}\cos^{\lambda+n}\left(\sqrt{\beta}p\right) = \exp\left(-\frac{p^2}{2\omega}\right),\\ \lim_{\lambda\to\infty}\sqrt{\lambda}\tan\left(\sqrt{\beta}p\right) = \frac{p}{\sqrt{\omega}},\\ \lim_{\lambda\to\infty}H_n^\lambda(z) = H_n(z), \quad \text{Ref. [58]},\end{gathered} \tag{33}$$

where H_n denotes Hermite polynomials. So in this limit the solutions read

$$\lim_{\beta\to 0}\phi_n(p) = \frac{e^{-p^2/2\omega}H_n\left(p/\sqrt{\omega}\right)}{\sqrt{2^n n!\sqrt{\pi\omega}}}, \tag{34}$$

which are normalized eigenstates of the ordinary harmonic oscillator as we have expected.

5. Information Entropy

The information entropies for the position and momentum spaces can be now calculated for the harmonic oscillator in the GUP framework using (12). In ordinary quantum mechanics and in the position space, the information entropy can be obtained analytically for some quantum mechanical systems. However, since the momentum wave functions are derived from the Fourier transform, the corresponding momentum information entropies are rather difficult to obtain. For our case, as we will show, we find S_p analytically for the two lowest energy states and obtain S_x numerically. Thus, because of the difficulty in calculating S_x and S_p, we only consider the two lowest energy eigenstates.

First consider the Fourier transform of the momentum space ground state (26) which gives the following state in the position space ($\hbar = 1$):

$$\begin{aligned}\psi_0(x) = {}&\sqrt{\frac{\lambda\left[\Gamma(\lambda)\right]^2}{\pi^2\sqrt{\beta}\Gamma(2\lambda)}}\frac{\sin\left[(\pi/2)\left(x/\sqrt{\beta}-\lambda\right)\right]}{x/\sqrt{\beta}-\lambda}\\ &\cdot {}_2F_1\left[\frac{1}{2}\left(\frac{x}{\sqrt{\beta}}-\lambda\right),\right.\\ &\left.-\lambda,\frac{1}{2}\left(\frac{x}{\sqrt{\beta}}-(\lambda-2)\right),1\right].\end{aligned} \tag{35}$$

Also, for the first excited state ($n = 1$) we have

$$\begin{aligned}&\psi_1(x)\\ &= i\lambda\sqrt{\frac{(\lambda+1)\left[\Gamma(\lambda)\right]^2}{\pi^2\sqrt{\beta}\Gamma(2\lambda+1)}}\left\{\frac{\sin\left[(\pi/2)\left(x/\sqrt{\beta}-(\lambda+1)\right)\right]}{x/\sqrt{\beta}-(\lambda+1)}\right.\\ &\cdot {}_2F_1\left[\frac{1}{2}\left(\frac{x}{\sqrt{\beta}}-(\lambda+1)\right),\right.\\ &\left.-\lambda,\frac{1}{2}\left(\frac{x}{\sqrt{\beta}}-(\lambda-1)\right),1\right]\\ &-\frac{\sin\left[(\pi/2)\left(x/\sqrt{\beta}-(\lambda-1)\right)\right]}{x/\sqrt{\beta}-(\lambda-1)}\\ &\cdot {}_2F_1\left[\frac{1}{2}\left(\frac{x}{\sqrt{\beta}}-(\lambda-1)\right),\right.\\ &\left.\left.-\lambda,\frac{1}{2}\left(\frac{x}{\sqrt{\beta}}-(\lambda-3)\right),1\right]\right\}.\end{aligned} \tag{36}$$

Figure 1 shows the resulting ground state and first excited state in position space for $\beta = \{0, 0.5, 1\}$. Notice that,

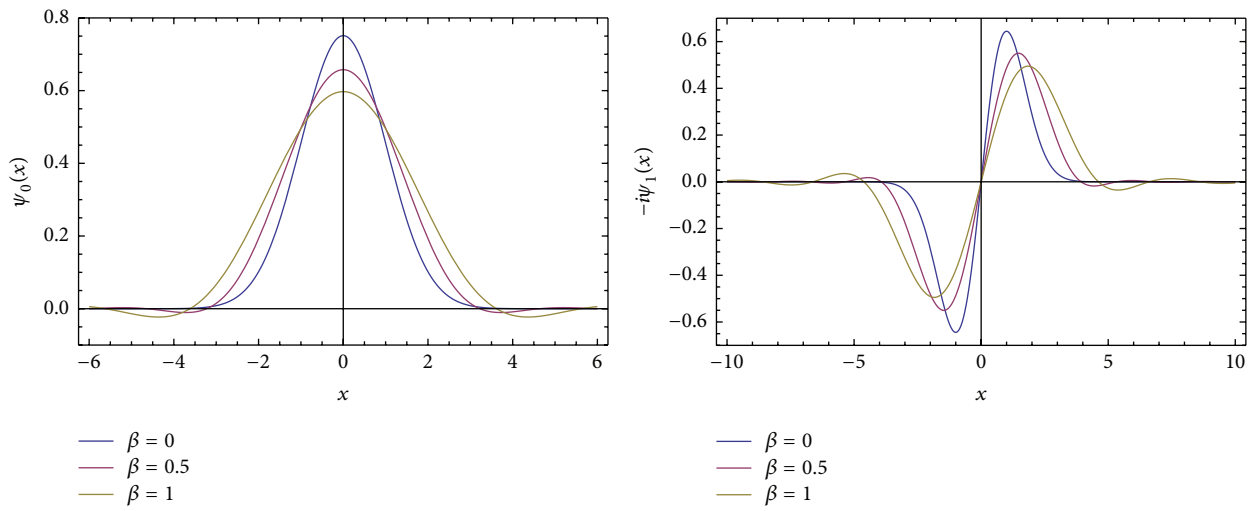

FIGURE 1: Plots of the position space wave functions for $m = \hbar = \omega = 1$ and $n = 0, 1$.

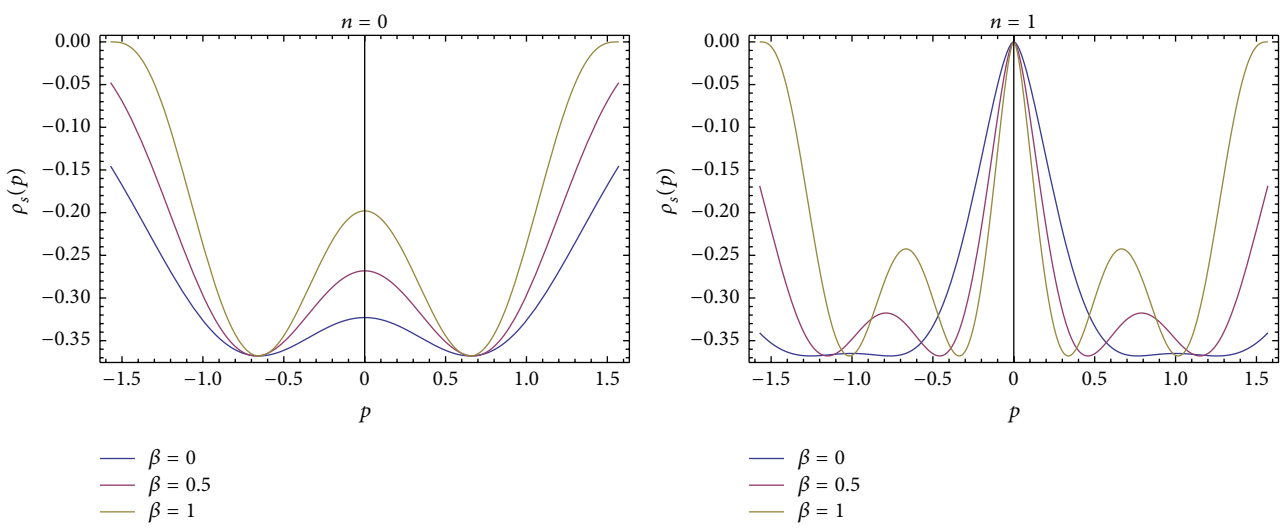

FIGURE 2: Plots of the momentum space entropy densities for $m = \hbar = \omega = 1$ and $n = 0, 1$.

for $\beta \to 0$, the solutions tend to the simple harmonic oscillator wave functions; that is, $\lim_{\beta\to 0}\psi_n(x) = \sqrt{\sqrt{\omega}/2^n n!\sqrt{\pi}}e^{-\omega x^2/2}H_n(\sqrt{\omega}x)$.

For the ground state, the analytical expression for S_p reads

$$S_p^0 = \lambda H_\lambda - \lambda H_{\lambda-1/2} + \ln\sqrt{\pi} - \ln\left(\frac{\sqrt{\beta}\lambda\Gamma(\lambda)}{\Gamma(\lambda+1/2)}\right), \quad (37)$$

where H_λ denotes the harmonic number $H_n = \sum_{k=1}^{n} 1/k$. It is easy to check that, for the small values of β, the momentum information entropy tends to the ordinary harmonic oscillator information entropy; namely,

$$\lim_{\beta\to 0} S_p^0 = \lim_{\beta\to 0} S_x^0 = \frac{1}{2}(1 + \ln\pi). \quad (38)$$

For the first excited state, S_p is given by

$$S_p^1 = (1+\lambda)H_{\lambda+1} - \lambda H_{\lambda-1/2} + \ln\sqrt{\pi} - 2 - \ln\left(\frac{\sqrt{\beta}\Gamma(\lambda+2)}{2\Gamma(\lambda+1/2)}\right), \quad (39)$$

and for $\beta \to 0$ it reads

$$\lim_{\beta\to 0} S_p^1 = \lim_{\beta\to 0} S_x^1 = \frac{1}{2}\ln\pi + \ln 2 + \gamma - \frac{1}{2}, \quad (40)$$

where $\gamma \approx 0.5772$ is the Euler constant.

The information entropy densities are defined as $\rho_s(x) = |\psi(x)|^2 \ln|\psi(x)|^2$ and $\rho_s(p) = |\phi(p)|^2 \ln|\phi(p)|^2$. The behavior of $\rho_s(x)$ and $\rho_s(p)$ is illustrated in Figures 2 and 3 for $n = 0, 1$ and several values of the deformation parameter. Now, using the numerical values for S_x, we obtain the left-hand

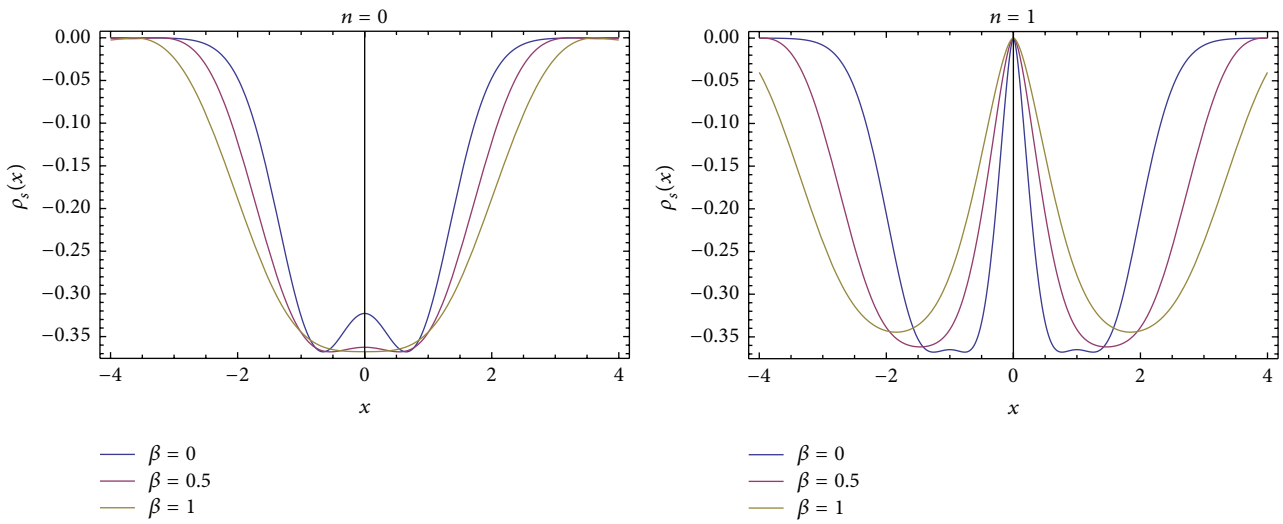

Figure 3: Plots of the position space entropy densities for $m = \hbar = \omega = 1$ and $n = 0, 1$.

Table 1: The numerical results that establish the BBM entropic uncertainty relation for $m = \hbar = \omega = 1$ and various values of β.

n	β	S_x	S_p	$S_x + S_p$	$1 + \ln \pi$
0	0.1	1.12153	1.02361	2.14515	2.14473
	0.5	1.30251	0.85220	2.15471	2.14473
	1.0	1.49095	0.68153	2.17248	2.14473
1	0.1	1.40656	1.24992	2.65648	2.14473
	0.5	1.62672	0.97566	2.60238	2.14473
	1.0	1.84423	0.74892	2.59315	2.14473

side of (11). In Table 1, we have reported the position and momentum space information entropies for $\beta = \{0.1, 0.5, 1\}$ and showed that they obey the BBM inequality. These results indicate that the position space information entropy increases with the GUP parameter β and vice versa for the momentum space information entropy but their sum stays above the value $1 + \ln \pi$.

6. Conclusions

In this paper, we studied the Shannon entropic uncertainty relation in the presence of a minimal measurable length proportional to the Planck length. We showed that using the formally self-adjoint representation, since the coordinate space and momentum space wave functions are related by the Fourier transformation, the measure in the information entropic integral is flat and the lower bound that is predicted by the BBM inequality is guaranteed in the GUP framework. It is worth mentioning that the BBM inequality does not hold for all wave functions. In fact, its validity depends on both the deformed algebra and its representation. As we have indicated, this inequality does not hold in quasiposition representation of our algebra $[X, P] = i\hbar(1 + \beta P^2)$. As another example, we mentioned the algebra $[X, P] = i\hbar(1 + \alpha X^2 + \beta P^2)$ that implies both a minimal length and a minimal momentum. Since this algebra has no formally self-adjoint representation in the form $X = x$ and $P = f(p)$, the momentum space and coordinate space wave functions are not related by the Fourier transform and the BBM uncertainty relation is not valid for this form of GUP. For the case of the harmonic oscillator, we exactly solved the generalized Schrödinger equation in momentum space and found the solutions in terms of the Gegenbauer polynomials. Also, for the two lowest energy eigenstates, we obtained the solutions in position space in terms of the hypergeometric functions. Then, the analytical expressions for the information entropies are found in the momentum space with proper limiting values for $\beta \to 0$. Using the numerical values for the position information entropy, we explicitly showed that the BBM inequality holds for various values of the deformation parameter. To check the validity of the BBM inequality for other potentials, we need to solve the generalized Schrödinger equation which contains higher order differential terms. However, for the small anharmonic potential terms, the perturbation theory can be used to find the approximate solutions. Also, For other types of GUPs, if a formally self-adjoint representation is viable, the BBM inequality is still valid.

Competing Interests

The author declares that he has no competing interests.

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Research Article

An Attempt for an Emergent Scenario with Modified Chaplygin Gas

Sourav Dutta, Sudeshna Mukerji, and Subenoy Chakraborty

Department of Mathematics, Jadavpur University, Kolkata, West Bengal 700032, India

Correspondence should be addressed to Sourav Dutta; sduttaju@gmail.com

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The present work is an attempt for emergent universe scenario with modified Chaplygin gas. The universe is chosen as spatially flat FRW space-time with modified Chaplygin gas as the only cosmic substratum. It is found that emergent scenario is possible for some specific (unrealistic) choice of the parameters in the equation of state for modified Chaplygin gas.

1. Introduction

The origin of the universe is a controversial issue in cosmology. It may start from the big bang singularity or there are proposals for nonsingular model of the universe. The inability of Einstein's general theory of relativity at zero volume leads to the well known big bang singularity in standard cosmology. To overrule this initial discomfortable situation various cosmological scenarios have been proposed and are classified as bouncing universes or the emergent universes. Here, we will choose the second option which results from searching for singularity-free inflationary scenario in the background of classical general relativity. In a word, emergent universe is a model universe, ever existing with almost static behavior in the infinite past ($t \to -\infty$) (gradually evolves into inflationary stage) and having no time-like singularity. Also, the modern and extended version of the original Lemaitre-Eddington universe can be identified as the emergent universe scenario.

Long back in 1967, Harrison [1] showed a model of the closed universe containing radiation, which approaches the state of an Einstein static model asymptotically (i.e., $t \to -\infty$). This kind of model was again reinvestigated after a long gap by Ellis and collaborators [2, 3]. Although they were not able to obtain exact solutions, they presented closed universes with a minimally coupled scalar field ϕ having typical self-interacting potential and possibly some ordinary matter with equation of state $p = w\rho$, $(-1/3 \leq w \leq 1)$, whose behavior similar to that of an emergent universe was highlighted. Then, in starobinsky model, Mukherjee et al. [4] derived solutions for flat FRW space-time having emergent character in infinite past. Subsequently, Mukherjee and associates [5] presented a general framework for an emergent universe model with an ad hoc equation of state connecting the pressure and density, having exotic nature in some cases. These models are interesting as they can be cited as specific examples of nonsingular (i.e., geometrically complete) inflationary universes. Also, it is worth mentioning here that entropy considerations favour the Einstein static model as the initial state for our universe [6, 7]. Thereafter, a series of works [8–16] have been done to formulate emergent universe in different gravity models and also for various types of matter. Very recently, emergent scenario has been formulated with some interesting physical aspects. The idea of quantum tunneling [17] has been used for the decay of a scalar field having initial static state as false vacuum to a state of true vacuum. Secondly, a model of an emergent universe has been formulated in the background of nonequilibrium thermodynamical prescription with dissipation due to particle creation mechanism [18]. Very recently, Paul and Majumdar [19] have formulated emergent universe with interacting fields. Finally, Pavon et al. [20, 21] have studied the emergent scenario from thermodynamical view point. They have examined the validity of the generalized second law of thermodynamics

during the transition from a generic initial Einstein static phase to the inflationary phase and also during the transition from inflationary era to the standard radiation dominated era.

2. Chaplygin Gas and Possible Solution

Mixed exotic fluid known as modified Chaplygin gas [21] has the equation of state [22, 23]

$$p = A\rho - \frac{B}{\rho^n}, \quad 0 < n \leq 1. \tag{1}$$

This equation of state shows barotropic perfect fluid $p = A\rho$, at very early phase (when the scale factor $a(t)$ is vanishingly small), while it approaches ΛCDM model when the scale factor is infinitely large. It shows a mixture at all stages. Note that at some intermediate stage the pressure vanishes and the matter content is equivalent to pure dust. Further, this typical model is equivalent to a self-interacting scalar field from field theoretic point of view. It should be noted that the Chaplygin gas was introduced in the context of aerodynamics. In the present paper, we will examine whether emergent scenario is possible for FRW model of the universe with matter content as modified Chaplygin gas (MCG).

For homogeneous and isotropic flat FRW model of the universe, the Einstein field equations are (choosing $8\pi G = 1$)

$$\begin{aligned} 3H^2 &= \rho, \\ 2\dot{H} &= -(\rho + p), \end{aligned} \tag{2}$$

with energy conservation relation:

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{3}$$

Using (1) in (3), one can integrate ρ as

$$\rho = \left[\frac{B}{1+A} + \frac{c}{a^{3\mu}}\right]^{1/(n+1)}, \tag{4}$$

with $c > 0$, a constant of integration.

Now, using this ρ in the first Friedmann equation in (2), one can integrate to obtain cosmic time as a function of the scale factor as

$$\begin{aligned} &\frac{\sqrt{3}}{2}(1+A)c^{\alpha}(t - t_0) \\ &\quad = a^{3(1+A)/2}\, {}_2F_1\left[\alpha, \alpha, 1+\alpha, -\frac{B}{C(1+A)}a^{3(1+A)/2\alpha}\right], \end{aligned} \tag{5}$$

where $\alpha = 1/2(1+n)$ and ${}_2F_1$ is the usual hypergeometric function.

3. Asymptotic Analysis and Equivalent Two Fluid Systems

We will now analyze the two asymptotic cases.

(i) When the Scale Factor "a" Is Very Small. For small "a", ρ can be approximated from (4) and p can be approximated from (1) as follows:

$$\begin{aligned} \rho &\cong \left(\frac{\rho_0}{A+1}\right)^{1/(n+1)} a^{-3(A+1)} \\ &\quad + \frac{B}{(n+1)(A+1)^{1/(n+1)}\rho_0^{(n/(n+1))}} a^{3(1+A)n} \\ &\equiv \rho_{1i} + \rho_{2i}, \\ p &\cong \frac{A\rho_0^{1/(n+1)}}{(A+1)^{1/(n+1)} a^{3(A+1)}} \\ &\quad - \frac{B[1+n(A+1)]}{(n+1)(A+1)^{1/(n+1)}\rho_0^{n/(n+1)}} a^{3n(1+A)} \\ &\equiv p_{1i} + p_{2i}. \end{aligned} \tag{6}$$

(ii) When the Scale Factor "a" Has Infinitely Large Value. Similarly, for large "a", ρ and p are approximated from (4) and (1), respectively, as follows:

$$\begin{aligned} \rho &\cong \left(\frac{B}{A+1}\right)^{1/(n+1)} + \frac{\rho_0}{(n+1)B}\left(\frac{B}{A+1}\right)^{1/(n+1)} a^{-3\mu} \\ &\equiv \rho_{1f} + \rho_{2f}, \\ p &\cong -\frac{1}{(A+1)^{1/(n+1)}} + \frac{n+(n+1)A}{(A+1)^{1/(n+1)}}\frac{\rho_0}{(n+1)B} a^{-3\mu} \\ &\equiv p_{1f} + p_{2f}. \end{aligned} \tag{7}$$

Thus, in the asymptotic limits, the components of energy density and pressure can be expressed as sum of two noninteracting barotropic fluids having equation of states:

$$\begin{aligned} w_{1i} &= A, \\ w_{2i} &= -[1+n(A+1)], \\ w_{1f} &= -B^{-(1/(n+1))}, \\ w_{2f} &= \frac{n+(n+1)A}{B^{(1/(n+1))}}. \end{aligned} \tag{8}$$

Thus, MCG can be considered in the asymptotic limits as two barotropic fluids of constant equation of state of which one is exotic in nature. However, one can consider that the two fluids in question (in the asymptotic limit) may be interacting with separate equation of state as

$$\begin{aligned} \dot{\rho_{1i}} + 3(\rho_{1i} + p_{1i})H &= Q_i, \\ \dot{\rho_{2i}} + 3(\rho_{2i} + p_{2i})H &= -Q_i, \\ \dot{\rho_{1f}} + 3(\rho_{1f} + p_{1f})H &= Q_f, \\ \dot{\rho_{2f}} + 3(\rho_{2f} + p_{2f})H &= -Q_f, \end{aligned} \tag{9}$$

where Q_i and Q_f represent the interaction term.

$Q_i > 0$ indicates a flow of energy from fluid 2 (having energy density ρ_{2i}) to fluid 1 (having energy density ρ_{1i}) and similarly for Q_f also. Further, one can rewrite the conservation equations (9) as

$$\begin{aligned} \dot{\rho_{1i}} + 3H\left(1 + w_{1i}^{\text{eff}}\right)\rho_{1i} &= 0, \\ \dot{\rho_{2i}} + 3H\left(1 + w_{2i}^{\text{eff}}\right)\rho_{2i} &= 0, \\ \dot{\rho_{1f}} + 3H\left(1 + w_{1f}^{\text{eff}}\right)\rho_{1f} &= 0, \\ \dot{\rho_{2f}} + 3H\left(1 + w_{2f}^{\text{eff}}\right)\rho_{2f} &= 0, \end{aligned} \tag{10}$$

with

$$\begin{aligned} w_{1i}^{\text{eff}} &= w_{1i} - \frac{Q_i}{3H\rho_{1i}}, \\ w_{2i}^{\text{eff}} &= w_{2i} + \frac{Q_i}{3H\rho_{2i}}, \\ w_{1f}^{\text{eff}} &= w_{1f} - \frac{Q_f}{3H\rho_{1f}}, \\ w_{2f}^{\text{eff}} &= w_{2f} + \frac{Q_f}{3H\rho_{2f}}. \end{aligned} \tag{11}$$

The above conservation equations show that the fluids may be considered as noninteracting at the cost of variable equation of state.

4. Emergent Scenario and Thermodynamical Analysis

One should note that in integrating (3) to have (4) we assume that $A \neq -1$. Now, we will discuss the situation when $A = -1$.

The expression for energy density now becomes

$$\rho = \left[3(n+1)B\ln\left(\frac{a}{a_0}\right)\right]^{1/(n+1)}, \tag{12}$$

which from the first Friedmann equation gives

$$\begin{aligned} a &= a_0 \exp\left[b_0 (t - t_0)^{(1/(1-\alpha))}\right], \\ b_0 &= \left(\frac{\sqrt{3}}{2} B(2n+1)\right)^{(1/(1-\alpha))}. \end{aligned} \tag{13}$$

From the solutions (5) and (13), we see (Figures 1 and 2) that $a \to 0$ as $t \to -\infty$, so it is not possible to have emergent scenario with the usual modified Chaplygin gas. However, if we choose $-1 < n < -1/2$, then $\alpha > 1$ and we have from solution (13) $a \to a_0$ as $t \to -\infty$ (see Figure 3). Hence, it is possible to have emergent scenario with this revised form of MCG.

We will now discuss the thermodynamics of the emergent scenario with this revised form of MCG as the cosmic substratum.

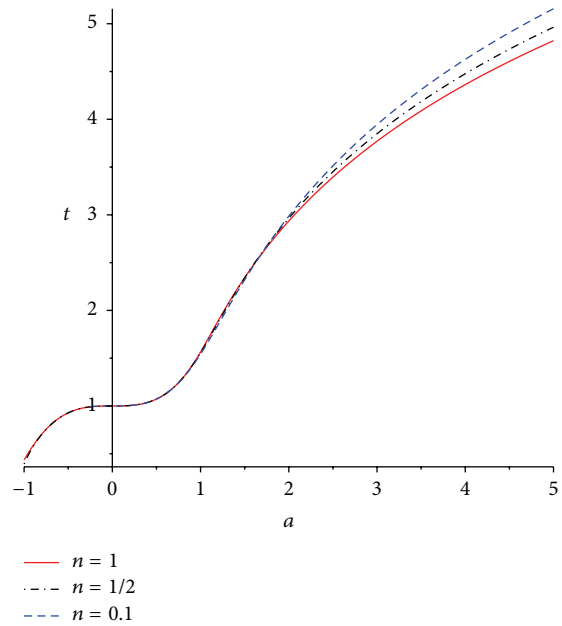

Figure 1: The Figure represents the scale factor $a(t)$ against t for $A \neq -1$.

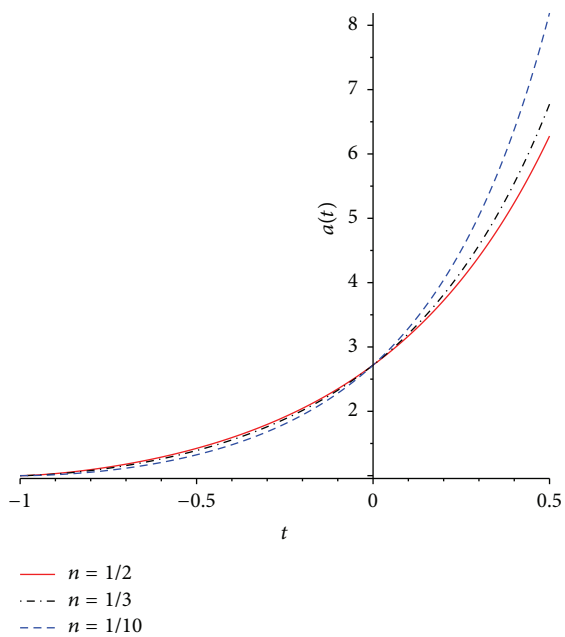

Figure 2: The Figure shows the graphical representation of the scale factor $a(t)$ against t for $A = -1$.

Assuming the validity of the first law of thermodynamics at the horizon (having area radius R_h), we have the Clausius relation:

$$-dE_h = T_h dS_h, \tag{14}$$

where T_h is the temperature of the horizon and s_h is the entropy of the horizon. In the above, E_h is the amount of

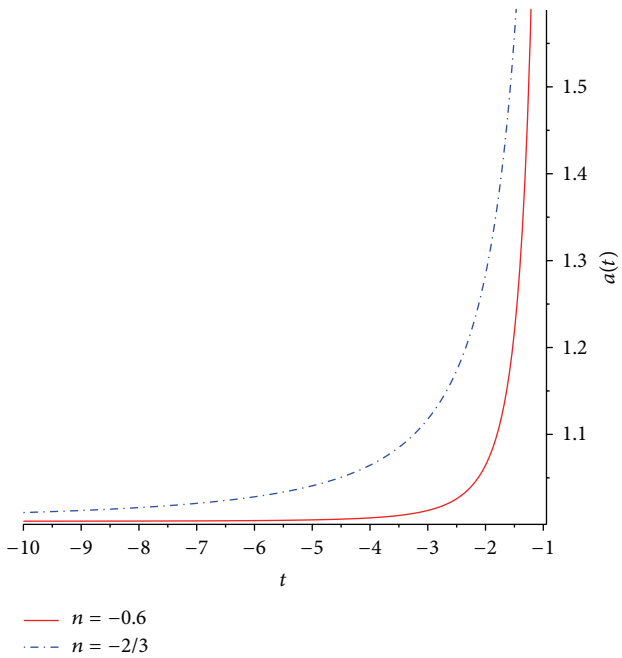

FIGURE 3: The Figure shows the graphical representation of the scale factor $a(t)$ against t for $A = -1$.

energy crossing the horizon during time dt and is given by [24–26]

$$-dE_h = 4\pi R_h^3 H(\rho + p)\, dt. \tag{15}$$

So, using (15) in (14), we have the rate of change of the horizon entropy as

$$\frac{dS_h}{dt} = \frac{4\pi R_h^3 H(\rho + p)}{T_h}. \tag{16}$$

To obtain the entropy of the inside fluid, we start with the Gibbs equation [26, 27]

$$T_h dS_f = dE_f + p dV, \tag{17}$$

where S_f is the entropy of the fluid bounded by the horizon and E_F is the energy of the matter distribution. Here, for thermodynamical equilibrium, the temperature of the fluid is taken as that of the horizon, that is, T_h.

Now, using $V = 4\pi R_h^3/3$, $E_f = (4\pi R_h^3/3)\rho$, and the Friedmann equations, the entropy variation of the fluid is given by

$$\frac{dS_f}{dt} = \frac{4\pi R_h^2}{T_h}(\rho + p)\left(\dot{R}_h - HR_h\right). \tag{18}$$

Thus, combining (16) and (18), the variation of the total entropy (S_T) is given by

$$\frac{dS_T}{dt} = \frac{d}{dt}\left(S_h + S_f\right) = \frac{4\pi R_h^2}{T_h}(\rho + p)\dot{R}_h. \tag{19}$$

Case 1 (apparent horizon). The area radius for apparent horizon is given by

$$R_A = \frac{1}{H}, \tag{20}$$

so that

$$\dot{R}_A = -\frac{\dot{H}}{H^2} = \frac{4\pi G(\rho + p)}{H^2}. \tag{21}$$

Hence,

$$\frac{dS_T}{dt} = \frac{(4\pi)^2 G(\rho + p)^2}{T_A H^4} > 0. \tag{22}$$

Thus, generalised second law of thermodynamics (GSLT) is always true at the apparent horizon.

Case 2 (event horizon). The area radius for event horizon is given by

$$R_E = a\int_t^\infty \frac{dt}{a}. \tag{23}$$

The above improper integral converges for accelerating phase of the FRW model. Hence, in the present scenario, it is very much relevant. From the above definition

$$\dot{R}_E = HR_E - 1, \tag{24}$$

so from (19)

$$\begin{aligned}\frac{dS_T}{dt} &= \frac{(4\pi) R_E^2(\rho + p)}{T_E}(HR_E - 1)\\ &= \frac{(4\pi) R_E^2 H}{T_E}\left[(A+1)\rho - \frac{B}{\rho^n}\right](R_E - R_A)\\ &= \frac{(4\pi) R_E^2 H}{T_E}\frac{c(1+a)}{a^{3\mu}\cdot\rho^n}(R_E - R_A).\end{aligned} \tag{25}$$

Hence, the validity of GSLT is possible if $R_E > R_A$ (as $\alpha > 1$). In the above, temperature is chosen as the hawking temperature on the horizon as [28, 29]

$$\begin{aligned}T_A &= \frac{1}{2\pi R_A},\\ T_E &= \frac{R_E}{2\pi R_A^2}.\end{aligned} \tag{26}$$

Additional Points

In the present work, we have examined the cosmology of the emergent scenario for modified Chaplygin gas as the cosmic fluid. It is found that for both the solutions (with $A \neq -1$ and $A = -1$) the model does not exhibit emergent scenario at early epochs. So, one can conclude that it is not possible to have emergent scenario with MCG. However, if n is chosen to be negative, that is, $-1 < n < -1/2$, then $a \to a_0$ as $t \to -\infty$; that is, initial big bang singularity is avoided.

Finally, thermodynamical analysis of the emergent scenario has been presented.

Competing Interests

The authors declare that they have no competing interests.

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