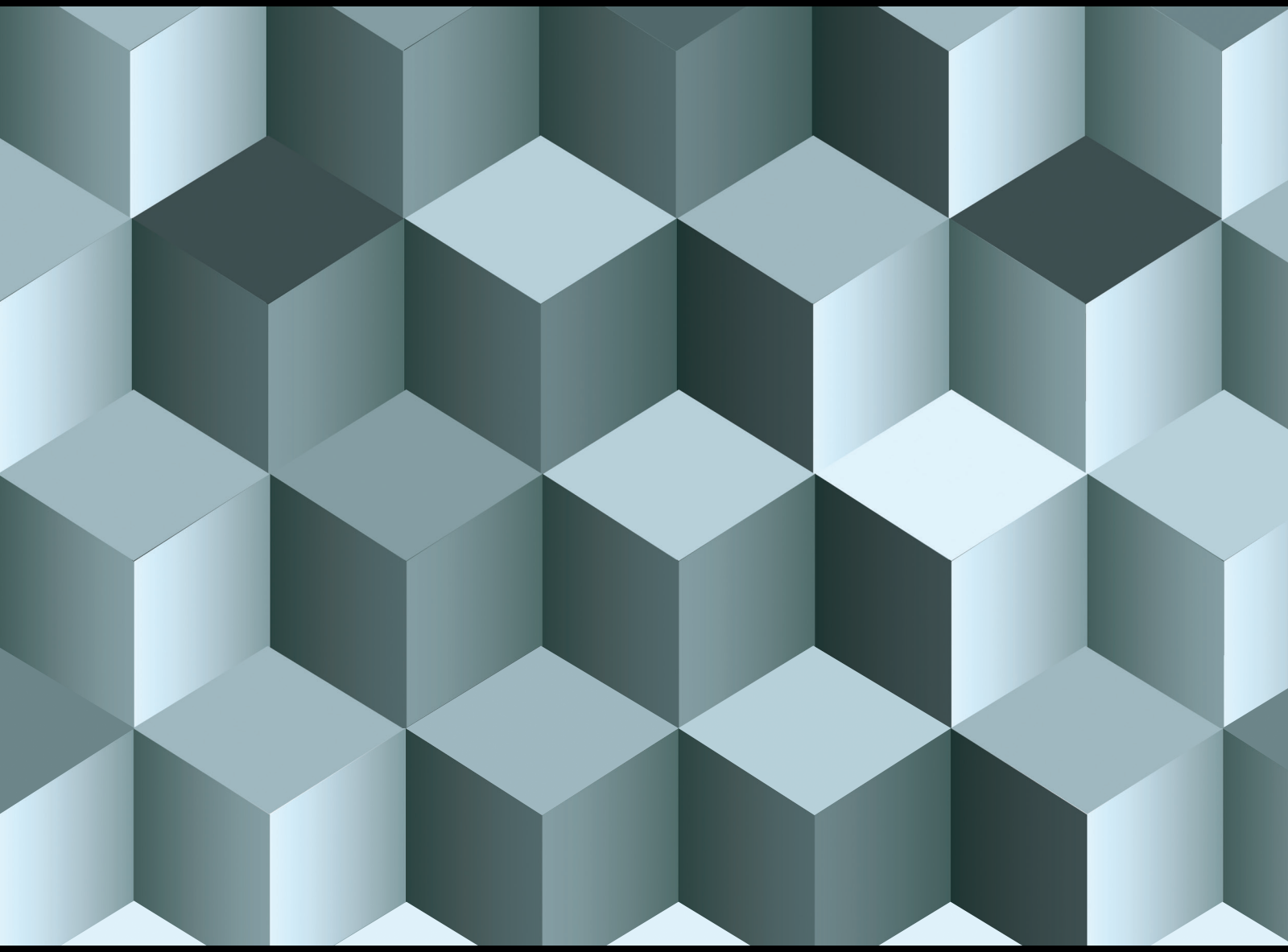
Journal of Function Spaces

Fixed Point Theorems with Applications to the Solvability of Operator Equations and Inclusions on Function Spaces

Guest Editors: Nawab Hussain, Jesus Garcia-Falset, Mohamed-Aziz Taoudi, and Calogero Vetro



Fixed Point Theorems with Applications to the Solvability of Operator Equations and Inclusions on Function Spaces

Journal of Function Spaces

Fixed Point Theorems with Applications to the Solvability of Operator Equations and Inclusions on Function Spaces

Guest Editors: Nawab Hussain, Jesus Garcia-Falset, Mohamed-Aziz Taoudi, and Calogero Vetro

Copyright © 2015 Hindawi Publishing Corporation. All rights reserved.

This is a special issue published in "Journal of Function Spaces." All articles are open access articles distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Editorial Board

John R. Akeroyd, USA
Gerassimos Barbatis, Greece
Ismat Beg, Pakistan
Bjorn Birnir, USA
Messaoud Bounkhel, KSA
Huy Qui Bui, New Zealand
Victor I. Burenkov, Italy
Jiecheng Chen, China
J. Chung, Republic of Korea
S. Dhompongsa, Thailand
Luisa Di Piazza, Italy
Lars Diening, Germany
Dragan Djordjevic, Serbia
M. Engliš, Czech Republic
Jose A. Ezquerro, Spain
Dashan Fan, USA
Xiang Fang, USA
Hans G. Feichtinger, Austria
Alberto Fiorenza, Italy
Ajda Fošner, Slovenia
Eva A. G. Gutiérrez, Spain
Aurelian Gheondea, Turkey
Antonio S. Granero, Spain
Yongsheng S. Han, USA
Seppo Hassi, Finland
Stanislav Hencl, Czech Republic
Norimichi Hirano, Japan
Henryk Hudzik, Poland
Pankaj Jain, India
Krzysztof Jarosz, USA
Anna Kamińska, USA
H. T. Kaptanoglu, Turkey
V. Keyantuo, Puerto Rico
V. M. Kokilashvili, Georgia
Alois Kufner, Czech Republic
David R. Larson, USA
Yuri Latushkin, USA
Y. J. Lee, Republic of Korea
Hugo Leiva, Venezuela
Guozhen Lu, USA
Dag Lukkassen, Norway
Qiaozhen Ma, China
Mark A. McKibben, USA
Mihail Megan, Romania
A. Montes-Rodriguez, Spain
Dumitru Motreanu, France
Sivaram K. Narayan, USA
Renxing Ni, China
Kasso A. Okoudjou, USA
Gestur Ólafsson, USA
Josip E. Pečarić, Croatia
José Áfh Peláez, Spain
Lars E. Persson, Sweden
Adrian Petrusel, Romania
Konrad Podczeck, Austria
José Rodríguez, Spain
Natasha Samko, Portugal
Simone Secchi, Italy
Naseer Shahzad, KSA
Mitsuru Sugimoto, Japan
Rodolfo H. Torres, USA
Wilfredo Urbina, USA
Nikolai L. Vasilevski, Mexico
P. Veeramani, India
Igor E. Verbitsky, USA
Dragan Vukotic, Spain
Bruce A. Watson, South Africa
Anthony Weston, USA
Quanhua Xu, France
Gen-QI Xu, China
Dachun Yang, China
Kari Ylinen, Finland
Chengbo Zhai, China
Ruhan Zhao, USA
Kehe Zhu, USA
William P. Ziemer, USA

Contents

Fixed Point Theorems with Applications to the Solvability of Operator Equations and Inclusions on Function Spaces, Nawab Hussain, Jesus Garcia-Falset, Mohamed-Aziz Taoudi, and Calogero Vetro
Volume 2015, Article ID 415950, 1 page

Hybrid Steepest-Descent Methods for Triple Hierarchical Variational Inequalities, L. C. Ceng, A. Latif, C. F. Wen, and A. E. Al-Mazrooei
Volume 2015, Article ID 980352, 22 pages

A Lyapunov-Type Inequality for a Fractional Differential Equation under a Robin Boundary Condition, Mohamed Jleli, Lakhdar Ragoub, and Bessem Samet
Volume 2015, Article ID 468536, 5 pages

Two Positive Solutions of Third-Order BVP with Integral Boundary Condition and Sign-Changing Green's Function, Bing-Wei Niu, Jian-Ping Sun, and Qiu-Yan Ren
Volume 2015, Article ID 491423, 8 pages

Upper Semicontinuity of Solution Mappings to Parametric Generalized Vector Quasiequilibrium Problems, Shu-qiang Shan, Yu Han, and Nan-jing Huang
Volume 2015, Article ID 764187, 6 pages

Existence of Positive Solutions for a Fourth-Order m-Point Boundary Value Problem, Liu Yang and Chunfang Shen
Volume 2015, Article ID 928105, 8 pages

Boundary Value Problems of Fractional Order Differential Equation with Integral Boundary Conditions and Not Instantaneous Impulses, Peiluan Li and Changjin Xu
Volume 2015, Article ID 954925, 9 pages

Common Fixed Point of Two R-Weakly Commuting Mappings in b-Metric Spaces, Poom Kumam, Wutiphol Sintunavarat, Shaban Sedghi, and Nabi Shobkolaei
Volume 2015, Article ID 350840, 5 pages

Some Generalized Difference Sequence Spaces Defined by a Sequence of Moduli in n-Normed Spaces, Abdullah Alotaibi, Kuldip Raj, and S. A. Mohiuddine
Volume 2015, Article ID 413850, 8 pages

On a New Faster Implicit Fixed Point Iterative Scheme in Convex Metric Spaces, Renu Chugh, Preety Malik, and Vivek Kumar
Volume 2015, Article ID 905834, 11 pages

$(\alpha - \psi)$-Contractive Mappings on Generalized Quasimetric Spaces, Erdal Karapınar and Hosein Lakzian
Volume 2014, Article ID 914398, 7 pages

Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 415950, 1 page
http://dx.doi.org/10.1155/2015/415950

Editorial

Fixed Point Theorems with Applications to the Solvability of Operator Equations and Inclusions on Function Spaces

Nawab Hussain,[1] Jesus Garcia-Falset,[2] Mohamed-Aziz Taoudi,[3,4] and Calogero Vetro[5]

[1] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[2] *Department of Mathematical Analysis, University of Valencia, Spain*
[3] *Centre Universitaire Polydisciplinaire, Kelaa des Sraghna, Morocco*
[4] *Université Cadi Ayyad, Laboratoire de Mathématiques et de Dynamique de Populations, Marrakech, Morocco*
[5] *Department of Mathematics and Computer Science, University of Palermo, Via Archirafi 34, 90123 Palermo, Italy*

Correspondence should be addressed to Nawab Hussain; nhusain@kau.edu.sa

Received 14 April 2015; Accepted 14 April 2015

Copyright © 2015 Nawab Hussain et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Fixed point theory is an elegant mathematical theory which is a beautiful mixture of analysis, topology, and geometry. It is an interdisciplinary theory which provides powerful tools for the solvability of central problems in many areas of current interest in mathematics and other quantitative sciences, such as physics, engineering, biology, and economy. In fact, the existence of linear and nonlinear problems is frequently transformed into fixed point problems, for example, the existence of solutions to partial differential equations, the existence of solutions to integral equations, and the existence of periodic orbits in dynamical systems. This makes fixed point theory a topical area and a subject of active scientific research, constantly evolving and growing and in a perpetual progress.

Fixed point theory has two main branches: on one hand, we can consider the results obtained by using metric properties; on the other hand, we can consider the results deduced from topological properties. Among the classical results, which are the basis for the metric branch, we retain the Banach contraction principle in complete metric spaces established by Banach in 1922. With respect to the topology branch, the main theorems are those of Brouwer and his infinite dimensional version, the Schauder fixed point theorem, which were proved in 1912 and 1930, respectively. In both, the compactness plays an essential role. A hybrid result combining metric and topological properties was established by Krasnoselskii in 1955. This result is exciting and has a very wide scope. In recent years, a number of authors have found this theorem a very satisfactory outcome for the study of stability and asymptotic behavior of solutions for certain differential equations which resisted the direct Lyapunov method.

The development of such fixed point theorems remains a constant concern for many investigators who have continued to enrich this field by new quite interesting contributions following constantly the progress of applications to fully meet the needs of this dynamic and active field. Thus, fixed point theorems were developed for single-valued and multivalued mappings on topological vector spaces, metric spaces, Banach spaces, Banach algebras, posets, lattices, Banach lattices, and so forth.

Due to the importance of fixed point theory and its applications, it is worthwhile to publish a special issue on this topic to highlight recent advances made by mathematicians actively working in this area.

This special issue was originally elaborated to report the latest advances in fixed point theory in abstract spaces and their applications. It includes works on single-valued and multivalued mappings in normed and metric spaces and various applications to boundary value problems, equilibrium problems, and variational inequalities.

Nawab Hussain
Jesus Garcia-Falset
Mohamed-Aziz Taoudi
Calogero Vetro

Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 980352, 22 pages
http://dx.doi.org/10.1155/2015/980352

Research Article

Hybrid Steepest-Descent Methods for Triple Hierarchical Variational Inequalities

L. C. Ceng,[1] A. Latif,[2] C. F. Wen,[3] and A. E. Al-Mazrooei[4]

[1] *Department of Mathematics, Shanghai Normal University, and Scientific Computing Key Laboratory of Shanghai Universities, Shanghai 200234, China*
[2] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[3] *Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung 807, Taiwan*
[4] *Department of Mathematics, College of Science, University of Jeddah, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Correspondence should be addressed to A. Latif; alatif@kau.edu.sa

Received 28 October 2014; Accepted 5 January 2015

Academic Editor: Mohamed-Aziz Taoudi

Copyright © 2015 L. C. Ceng et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We introduce and analyze a relaxed iterative algorithm by combining Korpelevich's extragradient method, hybrid steepest-descent method, and Mann's iteration method. We prove that, under appropriate assumptions, the proposed algorithm converges strongly to a common element of the fixed point set of infinitely many nonexpansive mappings, the solution set of finitely many generalized mixed equilibrium problems (GMEPs), the solution set of finitely many variational inclusions, and the solution set of general system of variational inequalities (GSVI), which is just a unique solution of a triple hierarchical variational inequality (THVI) in a real Hilbert space. In addition, we also consider the application of the proposed algorithm for solving a hierarchical variational inequality problem with constraints of finitely many GMEPs, finitely many variational inclusions, and the GSVI. The results obtained in this paper improve and extend the corresponding results announced by many others.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. Let $S : C \to H$ be a nonlinear mapping on C. We denote by Fix(S) the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping $S : C \to H$ is called L-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|Sx - Sy\| \le L\|x - y\|, \quad \forall x, y \in C. \tag{1}$$

In particular, if $L = 1$, then S is called a nonexpansive mapping; if $L \in (0, 1)$, then S is called a contraction.

Let $A : C \to H$ be a nonlinear mapping on C. We consider the following variational inequality problem (VIP): find a point $x \in C$ such that

$$\langle Ax, y - x\rangle \ge 0, \quad \forall y \in C. \tag{2}$$

The solution set of VIP (2) is denoted by VI(C, A).

The VIP (2) was first discussed by Lions [1]. In 1976, Korpelevich [2] proposed an iterative algorithm for solving the VIP (2) in Euclidean space $\mathbf{R}^n$:

$$\begin{aligned} y_n &= P_C(x_n - \tau A x_n), \\ x_{n+1} &= P_C(x_n - \tau A y_n), \quad \forall n \ge 0, \end{aligned} \tag{3}$$

with $\tau > 0$ a given number, which is known as the extragradient method.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function, let $A : H \to H$ be a nonlinear mapping, and let $\Theta : C \times C \to \mathbf{R}$ be a bifunction. In 2008, Peng and Yao [3] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \ge 0, \quad \forall y \in C. \tag{4}$$

We denote the set of solutions of GMEP (4) by GMEP(Θ, φ, A).

It was assumed in [3] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)–(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$;

(A3) Θ is upper-hemicontinuous; that is, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta\left(tz + (1-t)x, y\right) \leq \Theta\left(x, y\right); \tag{5}$$

(A4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that, for any $z \in C \setminus D_x$,

$$\Theta\left(z, y_x\right) + \varphi\left(y_x\right) - \varphi(z) + \frac{1}{r}\left\langle y_x - z, z - x\right\rangle < 0; \tag{6}$$

(B2) C is a bounded set.

For a given positive number r, let $T_r^{(\Theta,\varphi)} : H \to C$ be the solution set of the auxiliary mixed equilibrium problem; that is, for each $x \in H$,

$$\begin{aligned} T_r^{(\Theta,\varphi)}(x) := \Big\{ & y \in C : \Theta\left(y, z\right) + \varphi(z) - \varphi\left(y\right) \\ & + \frac{1}{r}\left\langle y - x, z - y\right\rangle \geq 0,\ \forall z \in C \Big\}. \end{aligned} \tag{7}$$

On the other hand, let B be a single-valued mapping of C into H and let R be a multivalued mapping with $D(R) = C$. Consider the following variational inclusion: find a point $x \in C$ such that

$$0 \in Bx + Rx. \tag{8}$$

We denote by $I(B, R)$ the solution set of the variational inclusion (8). It is known that problem (8) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity problems, variational inequalities, optimal control, mathematical economics, equilibria and game theory. Let $R : D(R) \subset H \to 2^H$ be a maximal monotone set-valued mapping. We define the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}, \quad \forall x \in H, \tag{9}$$

where λ is a positive number.

Let $F_1, F_2 : C \to H$ be two mappings. Consider the following general system of variational inequalities (GSVI) of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\left\langle \nu_1 F_1 y^* + x^* - y^*, x - x^*\right\rangle \geq 0, \quad \forall x \in C, \\ &\left\langle \nu_2 F_2 x^* + y^* - x^*, x - y^*\right\rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{10}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. It was considered and studied in [4–6]. In particular, if $F_1 = F_2 = A$, then the GSVI (10) reduces to the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\left\langle \nu_1 A y^* + x^* - y^*, x - x^*\right\rangle \geq 0, \quad \forall x \in C, \\ &\left\langle \nu_2 A x^* + y^* - x^*, x - y^*\right\rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{11}$$

which is defined by Verma [7] and it is called a new system of variational inequalities (NSVI). Further, if $x^* = y^*$, additionally, then the NSVI reduces to the classical VIP (2). In 2008, Ceng et al. [6] transformed the GSVI (10) into a fixed point problem in the following way.

Proposition CWY (see [6]). *For given $\overline{x}, \overline{y} \in C$, $(\overline{x}, \overline{y})$ is a solution of the GSVI (10) if and only if $\overline{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$Gx = P_C\left(I - \nu_1 F_1\right) P_C\left(I - \nu_2 F_2\right) x, \quad \forall x \in C, \tag{12}$$

where $\overline{y} = P_C(I - \nu_2 F_2)\overline{x}$.

In particular, if the mapping $F_j : C \to H$ is ζ_j-inverse strongly monotone for $j = 1, 2$, then the mapping G is nonexpansive provided $\nu_j \in (0, 2\zeta_j]$ for $j = 1, 2$. We denote by GSVI(G) the fixed point set of the mapping G.

Let S and T be two nonexpansive mappings. In 2009, Yao et al. [8] considered the following hierarchical VIP: find hierarchically a fixed point of T, which is a solution to the VIP for monotone mapping $I - S$; namely, find $\widetilde{x} \in \mathrm{Fix}(T)$, such that

$$\left\langle (I - S)\widetilde{x}, p - \widetilde{x}\right\rangle \geq 0, \quad \forall p \in \mathrm{Fix}\left(T\right). \tag{13}$$

The solution set of the hierarchical VIP (13) is denoted by Λ. It is not hard to check that solving the hierarchical VIP (13) is equivalent to the fixed point problem of the composite mapping $P_{\mathrm{Fix}(T)}S$; that is, find $\widetilde{x} \in C$, such that $\widetilde{x} = P_{\mathrm{Fix}(T)}S\widetilde{x}$. The authors [8] introduced and analyzed the following iterative algorithm for solving the hierarchical VIP (13):

$$\begin{aligned} y_n &= \beta_n S x_n + \left(1 - \beta_n\right) x_n, \\ x_{n+1} &= \alpha_n V x_n + \left(1 - \alpha_n\right) T y_n, \quad \forall n \geq 0. \end{aligned} \tag{14}$$

Theorem YLM (see [8, Theorem 3.2]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S and T be two nonexpansive mappings of C into itself. Let $V : C \to C$ be a fixed contraction with $\alpha \in (0, 1)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0, 1)$. For any given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by (14). Assume that the sequence $\{x_n\}$ is bounded and that*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty}(1/\alpha_n)|1/\beta_n - 1/\beta_{n-1}| = 0$, $\lim_{n\to\infty}(1/\beta_n)|1 - \alpha_{n-1}/\alpha_n| = 0$;

(iii) $\lim_{n\to\infty}\beta_n = 0$, $\lim_{n\to\infty}(\alpha_n/\beta_n) = 0$ *and* $\lim_{n\to\infty}(\beta_n^2/\alpha_n) = 0$;

(iv) $\mathrm{Fix}(T) \cap \operatorname{int} C \neq \emptyset$;

(v) *there exists a constant* $k > 0$ *such that* $\|x - Tx\| \geq k\mathrm{Dist}(x, \mathrm{Fix}(T))$ *for each* $x \in C$, *where* $\mathrm{Dist}(x, \mathrm{Fix}(T)) = \inf_{y \in \mathrm{Fix}(T)} \|x - y\|$. *Then,* $\{x_n\}$ *converges strongly to* $\widetilde{x} = P_\Lambda V\widetilde{x}$ *which solves the hierarchical VIP*

$$\left\langle (I - S)\,\widetilde{x}, p - \widetilde{x} \right\rangle \leq 0, \quad \forall p \in \mathrm{Fix}\,(T)\,. \tag{15}$$

Very recently, Iiduka [9, 10] considered a variational inequality with a variational inequality constraint over the set of fixed points of a nonexpansive mapping. Since this problem has a triple structure in contrast with bilevel programming problems or hierarchical constrained optimization problems or hierarchical fixed point problem, it is refereed as triple hierarchical constrained optimization problem (THCOP). He presented some examples of THCOP and developed iterative algorithms to find the solution of such a problem. The convergence analysis of the proposed algorithms was also studied in [9, 10]. Since the original problem is a variational inequality, in this paper, we call it a triple hierarchical variational inequality (THVI). Subsequently, Zeng et al. [11] introduced and considered the following triple hierarchical variational inequality (THVI).

Problem I. Assume that

(i) each $T_i : H \to H$ is a nonexpansive mapping with $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \neq \emptyset$;

(ii) $A_1 : H \to H$ is α-inverse strongly monotone;

(iii) $A_2 : H \to H$ is β-strongly monotone and L-Lipschitz continuous;

(iv) $\mathrm{IV}(\bigcap_{i=1}^{N} \mathrm{Fix}(T_i), A_1) \neq \emptyset$.

Then, the objective is to

$$\begin{aligned} \text{find } x^* \in \mathrm{VI}&\left(\mathrm{VI}\left(\bigcap_{i=1}^{N} \mathrm{Fix}\,(T_i), A_1\right), A_2\right) \\ := \Bigg\{ x^* \in \mathrm{VI}&\left(\bigcap_{i=1}^{N} \mathrm{Fix}\,(T_i), A_1\right) : \\ &\left\langle A_2 x^*, v - x^* \right\rangle \geq 0, \\ &\forall v \in \mathrm{VI}\left(\bigcap_{i=1}^{N} \mathrm{Fix}\,(T_i), A_1\right)\Bigg\}. \end{aligned} \tag{16}$$

In [11], the authors proposed the following algorithm for solving Problem I.

Algorithm ZWY ([11, Algorithm 3.2]). Let $T_i : H \to H$ $(i = 1, 2, \ldots, N)$ and $A_i : H \to H$ $(i = 1, 2)$ satisfy assumptions (i)–(iv) in Problem I. The following steps are presented for solving Problem I.

Step 0. Take $\{\alpha_n\}_{n=0}^{\infty} \subset (0, 1]$, $\{\rho_n\}_{n=0}^{\infty} \subset (0, 2\alpha]$, and $\mu \in (0, 2\beta/L^2)$, choose $x_0 \in H$ arbitrarily, and let $n := 0$.

Step 1. Given $x_n \in H$, compute $x_{n+1} \in H$ as

$$\begin{aligned} y_n &= T_{[n+1]}\left(x_n - \rho_n A_1 x_n\right), \\ x_{n+1} &= y_n - \mu\alpha_n A_2 y_n, \end{aligned} \tag{17}$$

where $T_{[k]} := T_{k \bmod N}$, for integer $k \geq 1$, with the mod function taking values in the set $\{1, 2, \ldots, N\}$; that is, if $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_{[k]} = T_N$ if $q = 0$ and $T_{[k]} = T_q$ if $1 \leq q < N$.

The following convergence analysis was presented in [11].

Theorem ZWY ([11, Theorem 3.2]). *Let* $\mu \in (0, 2\beta/L^2)$, $\{\alpha_n\}_{n=0}^{\infty} \subset (0, 1]$, *and* $\{\rho_n\}_{n=0}^{\infty} \subset (0, 2\alpha]$, *such that (i)* $\lim_{n\to\infty}\alpha_n = 0$, *(ii)* $\sum_{n=0}^{\infty}\alpha_n = \infty$, *(iii)* $\lim_{n\to\infty}(\alpha_n - \alpha_{n+N})/\alpha_{n+N} = 0$ *or* $\sum_{n=0}^{\infty}|\alpha_{n+N} - \alpha_n| < \infty$, *(iv)* $\lim_{n\to\infty}(\rho_n - \rho_{n+N})/\rho_{n+N} = 0$ *or* $\sum_{n=0}^{\infty}|\rho_{n+N} - \rho_n| < \infty$, *and (v)* $\rho_n \leq \alpha_n$ *for all* $n \geq 0$. *Assume in addition that*

$$\begin{aligned} \bigcap_{i=1}^{N} \mathrm{Fix}\,(T_i) &= \mathrm{Fix}\,(T_1 T_2 \cdots T_N) \\ &= \mathrm{Fix}\,(T_N T_1 \cdots T_{N-1}) \\ &= \cdots = \mathrm{Fix}\,(T_2 T_3 \cdots T_N T_1)\,. \end{aligned} \tag{18}$$

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by Algorithm ZWY satisfies the following properties:

(a) $\{x_n\}_{n=0}^{\infty}$ is bounded;

(b) $\lim_{n\to\infty}\|x_{n+N} - x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - T_{n+N}\cdots T_{n+1}x_n\| = 0$;

(c) $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of Problem I provided $\|x_n - y_n\| = o(\rho_n)$.

In this paper, we introduce and study the following triple hierarchical variational inequality (THVI) with constraints of finitely many GMEPs, finitely many variational inclusions, and general system of variational inequalities.

Problem II. Let M, N be two integers. Assume that

(i) $\{T_n\}_{n=1}^{\infty}$ is a sequence of nonexpansive self-mappings on C and $F_j : C \to H$ is ζ_j-inverse strongly monotone for $j = 1, 2$;

(ii) $\widetilde{A}_1 : H \to H$ is α-inverse strongly monotone and $\widetilde{A}_2 : H \to H$ is β-strongly monotone and L-Lipschitz continuous;

(iii) Θ_k is a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ is a proper lower semicontinuous and convex function, where $k \in \{1, 2, \ldots, M\}$;

(iv) $R_i : C \to 2^H$ is a maximal monotone mapping, and $A_k : H \to H$ and $B_i : C \to H$ are μ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $k \in \{1, 2, \ldots, M\}$ and $i \in \{1, 2, \ldots, N\}$;

(v) $\mathrm{VI}(\Omega, \widetilde{A}_1) \neq \emptyset$, where $\Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \bigcap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, A_k) \cap \bigcap_{i=1}^{N} I(B_i, R_i) \cap \mathrm{GSVI}(G)$.

Then, the objective is to

$$\begin{aligned} &\text{find } x^* \in \mathrm{VI}\left(\mathrm{VI}\left(\Omega, \widetilde{A}_1\right), \widetilde{A}_2\right) \\ &:= \left\{x^* \in \mathrm{VI}\left(\Omega, \widetilde{A}_1\right) : \langle \widetilde{A}_2 x^*, v - x^* \rangle \geq 0, \quad \forall v \in \mathrm{VI}\left(\Omega, \widetilde{A}_1\right)\right\}. \end{aligned} \tag{19}$$

Motivated and inspired by the above facts, we introduce and analyze a relaxed iterative algorithm by combining Korpelevich's extragradient method, hybrid steepest-descent method, and Mann's iteration method. We prove that under mild conditions, the proposed algorithm converges strongly to a common element $x^* \in \Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \bigcap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, A_k) \cap \bigcap_{i=1}^{N} I(B_i, R_i) \cap \mathrm{GSVI}(G)$ of the fixed point set of infinitely many nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$, the solution set of finitely many GMEPs, the solution set of finitely many variational inclusions, and the solution set of GSVI (10), which is just a unique solution of the THVI (19). In addition, we also consider the application of the proposed algorithm to solve a hierarchical variational inequality problem with constraints of finitely many GMEPs, finitely many variational inclusions, and GSVI (10). That is, under appropriate conditions, we prove that the proposed algorithm converges strongly to a unique solution $u^* \in \Omega$ of the VIP: $\langle \widetilde{A}_2 u^*, p - u^* \rangle \geq 0$, $\forall p \in \Omega$; equivalently, $P_\Omega(I - \widetilde{A}_2)u^* = u^*$. The results obtained in this paper improve and extend the corresponding results announced by many others including [12, 13]. A comprehensive survey on triple hierarchical variational inequalities can be found in [14].

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$; that is,

$$\begin{aligned} \omega_w(x_n) := \big\{x \in H : x_{n_i} \rightharpoonup x \\ \text{for some subsequence } \left\{x_{n_i}\right\} \text{ of } \{x_n\}\big\}. \end{aligned} \tag{20}$$

Recall that a mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C; \tag{21}$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in C; \tag{22}$$

(iii) α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{23}$$

It is obvious that if A is α-inverse strongly monotone, then A is monotone and $(1/\alpha)$-Lipschitz continuous. Moreover, we also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned} &\|(I - \lambda A)u - (I - \lambda A)v\|^2 \\ &\quad = \|(u - v) - \lambda(Au - Av)\|^2 \\ &\quad = \|u - v\|^2 - 2\lambda\langle Au - Av, u - v\rangle + \lambda^2 \|Au - Av\|^2 \\ &\quad \leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2. \end{aligned} \tag{24}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$, satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C). \tag{25}$$

Some important properties of projections are gathered in the following proposition.

Proposition 1. *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z \rangle \leq 0$, $\forall y \in C$;

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, $\forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2$, $\forall y \in H$.

Consequently, P_C is nonexpansive and monotone.

Definition 2. A mapping $T : H \to H$ is said to be

(a) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H; \tag{26}$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive or, equivalently, if T is 1-inverse strongly monotone (1-ism),

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in H; \tag{27}$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S), \tag{28}$$

where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that a projection P_C is 1-ism. Inverse strongly monotone (also referred to as cocoercive) operators have been applied widely in solving practical problems in various fields.

Next, we list some elementary conclusions for the mixed equilibrium problem (MEP) where $\mathrm{MEP}(\Theta, \varphi)$ is the solution set.

Proposition 3 (see [15]). *Assume that $\Theta : C \times C \to \mathbf{R}$ satisfies (A1)–(A4) and let $\varphi : C \to \mathbf{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$T_r^{(\Theta,\varphi)}(x) = \left\{ z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\} \tag{29}$$

for all $x \in H$. Then, the following hold:

(i) *for each $x \in H$, $T_r^{(\Theta,\varphi)}(x)$ is nonempty and single-valued;*

(ii) *$T_r^{(\Theta,\varphi)}$ is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\left\| T_r^{(\Theta,\varphi)} x - T_r^{(\Theta,\varphi)} y \right\|^2 \leq \langle T_r^{(\Theta,\varphi)} x - T_r^{(\Theta,\varphi)} y, x - y\rangle; \tag{30}$$

(iii) $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = MEP(\Theta, \varphi)$;

(iv) *$MEP(\Theta, \varphi)$ is closed and convex;*

(v) $\|T_s^{(\Theta,\varphi)} x - T_t^{(\Theta,\varphi)} x\|^2 \leq ((s - t)/s)\langle T_s^{(\Theta,\varphi)} x - T_t^{(\Theta,\varphi)} x, T_s^{(\Theta,\varphi)} x - x\rangle$ *for all $s, t > 0$ and $x \in H$.*

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 4. *Let X be a real inner product space. Then, there holds the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X. \tag{31}$$

Lemma 5. *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem, the characterization of the projection (see Proposition 1(i)) implies*

$$u \in \mathrm{VI}(C, A) \Longleftrightarrow u = P_C(u - \lambda A u), \quad \lambda > 0. \tag{32}$$

Lemma 6 (see [16, demiclosedness principle]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive self-mapping on C. Then, $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here, I is the identity operator of H.*

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0, 1]$. For any $n \geq 0$, define a mapping W_n on C as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1 - \lambda_n) I,\\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1}) I,\\
&\vdots\\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1 - \lambda_k) I,\\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1}) I,\\
&\vdots\\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1 - \lambda_2) I,\\
W_n &= U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1) I.
\end{aligned} \tag{33}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

Lemma 7 (see [17, Lemma 3.2]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Then, for every $x \in C$ and $k \geq 1$, the limit $\lim_{n\to\infty} U_{n,k} x$ exists where $U_{n,k}$ is defined as in (33).*

Lemma 8 (see [17, Lemma 3.3]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$, and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Then, $\mathrm{Fix}(W) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.*

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0, 1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to H$, we define the mapping $T^{\lambda} : C \to H$ by

$$T^{\lambda} x := Tx - \lambda\mu F(Tx), \quad \forall x \in C, \tag{34}$$

where $F : H \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on H; that is, F satisfies the conditions:

$$\begin{aligned}
\|Fx - Fy\| &\leq \kappa \|x - y\|,\\
\langle Fx - Fy, x - y\rangle &\geq \eta \|x - y\|^2,
\end{aligned} \tag{35}$$

for all $x, y \in H$.

Lemma 9 (see [18, Lemma 3.1]). *T^{λ} is a contraction provided $0 < \mu < 2\eta/\kappa^2$; that is,*

$$\left\| T^{\lambda} x - T^{\lambda} y \right\| \leq (1 - \lambda\tau) \|x - y\|, \quad \forall x, y \in C, \tag{36}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Lemma 10 (see [18]). *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the conditions*

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n\beta_n, \quad \forall n \ge 1, \tag{37}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$ *or, equivalently,*

$$\prod_{n=1}^{\infty}(1-\alpha_n) := \lim_{n\to\infty}\prod_{k=1}^{n}(1-\alpha_k) = 0; \tag{38}$$

(ii) $\limsup_{n\to\infty}\beta_n \le 0$, *or* $\sum_{n=1}^{\infty}|\alpha_n\beta_n| < \infty$.

Then, $\lim_{n\to\infty}s_n = 0$.

Lemma 11 (see [16]). *Let H be a real Hilbert space. Then, the following hold:*

(a) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x, y \in H$;

(b) $\|\lambda x+\mu y\|^2 = \lambda\|x\|^2+\mu\|y\|^2-\lambda\mu\|x-y\|^2$ *for all* $x, y \in H$ *and* $\lambda, \mu \in [0,1]$ *with* $\lambda+\mu = 1$;

(c) *If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that*

$$\begin{aligned}&\limsup_{n\to\infty}\|x_n - y\|^2\\ &\quad = \limsup_{n\to\infty}\|x_n - x\|^2 + \|x-y\|^2, \quad \forall y \in H.\end{aligned} \tag{39}$$

Finally, recall that a set-valued mapping $T : D(T) \subset H \to 2^H$ is called monotone if, for all $x, y \in D(T)$, $f \in Tx$ and $g \in Ty$ imply

$$\langle f-g, x-y\rangle \ge 0. \tag{40}$$

A set-valued mapping T is called maximal monotone if T is monotone and $(I+\lambda T)D(T) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(T)$ the graph of T. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H\times H$, $\langle f-g, x-y\rangle \ge 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let $A : C \to H$ be a monotone, k-Lipschitz-continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$; that is,

$$N_C v = \{u \in H : \langle v-p, u\rangle \ge 0,\ \forall p \in C\}. \tag{41}$$

Define

$$\widetilde{T}v = \begin{cases} Av + N_C v, & \text{if } v \in C,\\ \emptyset, & \text{if } v \notin C.\end{cases} \tag{42}$$

Then, $\widetilde{T}$ is maximal monotone (see [19]) such that

$$0 \in \widetilde{T}v \Longleftrightarrow v \in \mathrm{VI}(C, A). \tag{43}$$

Let $R : D(R) \subset H \to 2^H$ be a maximal monotone mapping. Let $\lambda, \mu > 0$ be two positive numbers.

Lemma 12 (see [20]). *There holds the resolvent identity*

$$J_{R,\lambda}x = J_{R,\mu}\left(\frac{\mu}{\lambda}x + \left(1-\frac{\mu}{\lambda}\right)J_{R,\lambda}x\right), \quad \forall x \in H. \tag{44}$$

Remark 13. For $\lambda, \mu > 0$, there holds the following relation:

$$\begin{aligned}&\left\|J_{R,\lambda}x - J_{R,\mu}y\right\|\\ &\quad \le \|x-y\| + |\lambda-\mu|\left(\frac{1}{\lambda}\left\|J_{R,\lambda}x - y\right\| + \frac{1}{\mu}\left\|x - J_{R,\mu}y\right\|\right),\\ &\qquad \forall x, y \in H.\end{aligned} \tag{45}$$

In terms of Huang [21], there holds the following property for the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$.

Lemma 14. *$J_{R,\lambda}$ is single-valued and firmly nonexpansive; that is,*

$$\langle J_{R,\lambda}x - J_{R,\lambda}y, x-y\rangle \ge \left\|J_{R,\lambda}x - J_{R,\lambda}y\right\|^2, \quad \forall x, y \in H. \tag{46}$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 15 (see [22]). *Let R be a maximal monotone mapping with $D(R) = C$. Then, for any given $\lambda > 0$, $u \in C$ is a solution of problem (12) if and only if $u \in C$ satisfies*

$$u = J_{R,\lambda}(u - \lambda Bu). \tag{47}$$

Lemma 16 (see [23]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a strongly monotone, continuous, and single-valued mapping. Then, for each $z \in H$, the equation $z \in (B+\lambda R)x$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 17 (see [22]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a monotone, continuous, and single-valued mapping. Then, $(I+\lambda(R+B))C = H$ for each $\lambda > 0$. In this case, $R + B$ is maximal monotone.*

3. Main Results

In this section, we will introduce and analyze a relaxed iterative algorithm for finding a solution of the THVI (19) with constraints of several problems: finitely many GMEPs, finitely many variational inclusions, and GSVI (10) in a real Hilbert space. This algorithm is based on Korpelevich's extragradient method, hybrid steepest-descent method, and Mann's iteration method. We prove the strong convergence of the proposed algorithm to a unique solution of THVI (19) under suitable conditions. In addition, we also consider the application of the proposed algorithm to solve a hierarchical VIP with the same constraints.

We are now in a position to state and prove our first main result.

Theorem 18. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let M, N be two integers. Let Θ_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let*

$\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function, where* $k \in \{1, 2, \ldots, M\}$. *Let* $R_i : C \to 2^H$ *be a maximal monotone mapping and let* $A_k : H \to H$ *and* $B_i : C \to H$ *be* μ_k*-inverse strongly monotone and* η_i*-inverse strongly monotone, respectively, where* $k \in \{1, 2, \ldots, M\}$, $i \in \{1, 2, \ldots, N\}$. *Let* $\{T_n\}_{n=1}^{\infty}$ *be a sequence of nonexpansive self-mappings on* C *and* $\{\lambda_n\}_{n=1}^{\infty}$ *be a sequence in* $(0, b]$ *for some* $b \in (0, 1)$. *Let* $F_j : C \to H$ *be* ζ_j*-inverse strongly monotone for* $j = 1, 2$. *Let* $\widetilde{A}_1 : H \to H$ *be* α*-inverse strongly monotone with* $(I - \widetilde{A}_1)C \subset C$ *and let* $\widetilde{A}_2 : H \to H$ *be* β*-strongly monotone and* L*-Lipschitz continuous. Assume that either (B1) or (B2) holds and that* $\mathrm{VI}(\Omega, \widetilde{A}_1) \neq \emptyset$ *where* $\Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \bigcap_{k=1}^{M} GMEP(\Theta_k, \varphi_k, A_k) \cap \bigcap_{i=1}^{N} I(B_i, R_i) \cap GSVI(G)$. *Let* $\mu \in (0, 2\beta/L^2)$ *and* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\rho_n\}_{n=1}^{\infty}$ *be sequences in* $(0, 1]$. *For arbitrarily given* $x_1 \in H$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} u_n &= T_{r_{M,n}}^{(\Theta_M, \varphi_M)} \left(I - r_{M,n} A_M\right) T_{r_{M-1,n}}^{(\Theta_{M-1}, \varphi_{M-1})} \\ &\quad \cdot \left(I - r_{M-1,n} A_{M-1}\right) \cdots T_{r_{1,n}}^{(\Theta_1, \varphi_1)} \left(I - r_{1,n} A_1\right) x_n, \\ v_n &= J_{R_N, \lambda_{N,n}} \left(I - \lambda_{N,n} B_N\right) J_{R_{N-1}, \lambda_{N-1,n}} \\ &\quad \cdot \left(I - \lambda_{N-1,n} B_{N-1}\right) \cdots J_{R_1, \lambda_{1,n}} \left(I - \lambda_{1,n} B_1\right) u_n, \\ y_n &= W_n G\left(v_n - \rho_n \widetilde{A}_1 v_n\right), \\ x_{n+1} &= y_n - \mu \alpha_n \widetilde{A}_2 y_n, \quad \forall n \geq 1, \end{aligned} \tag{48}$$

where G *is defined as in (12) with* $\nu_j \in (0, 2\zeta_j)$ *for* $j = 1, 2$, *and* W_n *is the* W*-mapping defined by (33). Suppose that the following conditions are satisfied:*

(H1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} (1/\alpha_n)|1 - \rho_{n-1}/\rho_n| = 0$;

(H2) $\lim_{n\to\infty} (1/\alpha_n)|1/\rho_n - 1/\rho_{n-1}| = 0$ *and* $\lim_{n\to\infty} (1/\rho_n)|1 - \alpha_{n-1}/\alpha_n| = 0$;

(H3) $\lim_{n\to\infty} (\rho_n/\alpha_n) = 0$ *and* $\lim_{n\to\infty} (b^n/\alpha_n\rho_n) = 0$;

(H4) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ *and* $\lim_{n\to\infty} (|\lambda_{i,n} - \lambda_{i,n-1}|/\alpha_n\rho_n) = 0$ *for all* $i \in \{1, 2, \ldots, N\}$;

(H5) $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ *and* $\lim_{n\to\infty} (|r_{k,n} - r_{k,n-1}|/\rho_n\alpha_n) = 0$ *for all* $k \in \{1, 2, \ldots, M\}$.

Then, there hold the following:

(i) $\lim_{n\to\infty} (\|x_{n+1} - x_n\|/\rho_n) = 0$;

(ii) $\omega_w(x_n) \subset \Omega$;

(iii) $\omega_w(x_n) \subset \mathrm{VI}(\Omega, \widetilde{A}_1)$ *provided* $\|x_n - y_n\| = o(\rho_n)$ *additionally.*

Proof. Let $\{x^*\} = \mathrm{VI}(\mathrm{VI}(\Omega, \widetilde{A}_1), \widetilde{A}_2)$. Taking into account that $\lim_{n\to\infty} (\rho_n/\alpha_n) = 0$, we may assume, without loss of generality, that $\rho_n \leq \alpha_n$ and $\rho_n \leq 2\alpha$ for all $n \geq 1$. Since $\widetilde{A}_2$ is L-Lipschitz continuous, we get

$$\left\|\widetilde{A}_2 y_n - \widetilde{A}_2 x^*\right\| \leq L \left\|y_n - x^*\right\|, \quad \forall n \geq 1. \tag{49}$$

Putting $\widetilde{v}_n = v_n - \rho_n \widetilde{A}_1 v_n$ for all $n \geq 1$, we have

$$\begin{aligned} x_{n+1} &= y_n - \mu \alpha_n \widetilde{A}_2 y_n \\ &= W_n G \widetilde{v}_n - \mu \alpha_n \widetilde{A}_2 W_n G \widetilde{v}_n \\ &= \left(I - \mu \alpha_n \widetilde{A}_2\right) W_n G \widetilde{v}_n, \quad \forall n \geq 1. \end{aligned} \tag{50}$$

Moreover, observe that

$$\widetilde{v}_n = v_n - \rho_n \widetilde{A}_1 v_n = \left(1 - \rho_n\right) v_n + \rho_n \left(I - \widetilde{A}_1\right) v_n. \tag{51}$$

(This is Mann's iteration method.) Since $(I - \widetilde{A}_1)C \subset C$ and $\{v_n\}_{n=1}^{\infty} \subset C$, we know from the convexity of C that $\{\widetilde{v}_n\}_{n=1}^{\infty} \subset C$. Put

$$\begin{aligned} \Delta_n^k &= T_{r_{k,n}}^{(\Theta_k, \varphi_k)} \left(I - r_{k,n} A_k\right) T_{r_{k-1,n}}^{(\Theta_{k-1}, \varphi_{k-1})} \\ &\quad \cdot \left(I - r_{k-1,n} A_{k-1}\right) \cdots T_{r_{1,n}}^{(\Theta_1, \varphi_1)} \left(I - r_{1,n} A_1\right) x_n \end{aligned} \tag{52}$$

for all $k \in \{1, 2, \ldots, M\}$ and $n \geq 1$,

$$\begin{aligned} \Lambda_n^i &= J_{R_i, \lambda_{i,n}} \left(I - \lambda_{i,n} B_i\right) J_{R_{i-1}, \lambda_{i-1,n}} \\ &\quad \cdot \left(I - \lambda_{i-1,n} B_{i-1}\right) \cdots J_{R_1, \lambda_{1,n}} \left(I - \lambda_{1,n} B_1\right) \end{aligned} \tag{53}$$

for all $i \in \{1, 2, \ldots, N\}$, $\Delta_n^0 = I$, and $\Lambda_n^0 = I$, where I is the identity mapping on H. Then, we have that $u_n = \Delta_n^M x_n$ and $v_n = \Lambda_n^N u_n$.

We divide the rest of the proof into several steps.

Step 1. We prove that $\{x_n\}$ is bounded.

Indeed, utilizing (24) and Proposition 3(ii), we have

$$\begin{aligned} \left\|u_n - x^*\right\| &= \left\|T_{r_{M,n}}^{(\Theta_M, \varphi_M)} \left(I - r_{M,n} B_M\right) \Delta_n^{M-1} x_n \right. \\ &\qquad \left. - T_{r_{M,n}}^{(\Theta_M, \varphi_M)} \left(I - r_{M,n} B_M\right) \Delta_n^{M-1} x^* \right\| \\ &\leq \left\|\left(I - r_{M,n} B_M\right) \Delta_n^{M-1} x_n - \left(I - r_{M,n} B_M\right) \Delta_n^{M-1} x^*\right\| \\ &\leq \left\|\Delta_n^{M-1} x_n - \Delta_n^{M-1} x^*\right\| \\ &\vdots \\ &\leq \left\|\Delta_n^0 x_n - \Delta_n^0 x^*\right\| \\ &= \left\|x_n - x^*\right\|. \end{aligned} \tag{54}$$

Utilizing (24) and Lemma 14, we have

$$\begin{aligned}\left\|v_n - x^*\right\| &= \left\|J_{R_N,\lambda_{N,n}}\left(I-\lambda_{N,n}A_N\right)\Lambda_n^{N-1}u_n\right.\\ &\qquad \left.-J_{R_N,\lambda_{N,n}}\left(I-\lambda_{N,n}A_N\right)\Lambda_n^{N-1}x^*\right\|\\ &\le \left\|\left(I-\lambda_{N,n}A_N\right)\Lambda_n^{N-1}u_n-\left(I-\lambda_{N,n}A_N\right)\Lambda_n^{N-1}x^*\right\|\\ &\le \left\|\Lambda_n^{N-1}u_n-\Lambda_n^{N-1}x^*\right\|\\ &\vdots\\ &\le \left\|\Lambda_n^{0}u_n-\Lambda_n^{0}x^*\right\|\\ &= \left\|u_n-x^*\right\|.\end{aligned} \tag{55}$$

Combining (54) and (55), we have

$$\left\|v_n - x^*\right\| \le \left\|x_n - x^*\right\|. \tag{56}$$

Since $x^* = Gx^* = P_C(I-\nu_1F_1)P_C(I-\nu_2F_2)x^*$, F_j is ζ_j-inverse strongly monotone for $j = 1, 2$, and $0 \le \nu_j \le 2\zeta_j$ for $j = 1, 2$, we deduce that, for any $n \ge 1$,

$$\begin{aligned}&\left\|G\widetilde{v}_n - x^*\right\|^2\\ &\quad= \left\|P_C\left(I-\nu_1F_1\right)P_C\left(I-\nu_2F_2\right)\widetilde{v}_n\right.\\ &\qquad\left.-P_C\left(I-\nu_1F_1\right)P_C\left(I-\nu_2F_2\right)x^*\right\|^2\\ &\quad\le \left\|\left(I-\nu_1F_1\right)P_C\left(I-\nu_2F_2\right)\widetilde{v}_n\right.\\ &\qquad\left.-\left(I-\nu_1F_1\right)P_C\left(I-\nu_2F_2\right)x^*\right\|^2\\ &\quad= \left\|\left[P_C\left(I-\nu_2F_2\right)\widetilde{v}_n-P_C\left(I-\nu_2F_2\right)x^*\right]\right.\\ &\qquad\left.-\nu_1\left[F_1P_C\left(I-\nu_2F_2\right)v_n-F_1P_C\left(I-\nu_2F_2\right)x^*\right]\right\|^2\\ &\quad\le \left\|P_C(I-\nu_2F_2)\widetilde{v}_n-P_C(I-\nu_2F_2)x^*\right\|^2\\ &\qquad+\nu_1\left(\nu_1-2\zeta_1\right)\\ &\qquad\cdot\left\|F_1P_C(I-\nu_2F_2)v_n-F_1P_C(I-\nu_2F_2)p\right\|^2\\ &\quad\le \left\|P_C(I-\nu_2F_2)\widetilde{v}_n-P_C(I-\nu_2F_2)x^*\right\|^2\\ &\quad\le \left\|(I-\nu_2F_2)\widetilde{v}_n-(I-\nu_2F_2)x^*\right\|^2\\ &\quad= \left\|(\widetilde{v}_n-p)-\nu_2(F_2\widetilde{v}_n-F_2x^*)\right\|^2\\ &\quad\le \left\|\widetilde{v}_n-x^*\right\|^2+\nu_2\left(\nu_2-2\zeta_2\right)\left\|F_2\widetilde{v}_n-F_2x^*\right\|^2\\ &\quad\le \left\|\widetilde{v}_n-x^*\right\|^2.\end{aligned} \tag{57}$$

Since $\widetilde{A}_1$ is α-inverse strongly monotone and $\{\rho_n\}_{n=1}^{\infty} \subset (0, 2\alpha]$, we have

$$\begin{aligned}&\left\|v_n - x^* - \rho_n(\widetilde{A}_1v_n-\widetilde{A}_1x^*)\right\|^2\\ &\quad= \left\|v_n-x^*\right\|^2-2\rho_n\langle\widetilde{A}_1v_n-\widetilde{A}_1x^*,x_n-x^*\rangle\\ &\qquad+\rho_n^2\left\|\widetilde{A}_1v_n-\widetilde{A}_1x^*\right\|^2\\ &\quad\le \left\|v_n-x^*\right\|^2-\rho_n\left(2\alpha-\rho_n\right)\left\|\widetilde{A}_1v_n-\widetilde{A}_1x^*\right\|^2\\ &\quad\le \left\|v_n-x^*\right\|^2.\end{aligned} \tag{58}$$

Utilizing Lemma 9, the nonexpansivity of W_n and the one of G (due to Proposition CWY), we obtain from (24) and (56) that

$$\begin{aligned}&\left\|x_{n+1}-x^*\right\|\\ &\quad= \left\|\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nG\widetilde{v}_n-x^*\right\|\\ &\quad\le \left\|\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nG\widetilde{v}_n-\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nGx^*\right\|\\ &\qquad+\left\|\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nGx^*-W_nGx^*\right\|\\ &\quad\le \left(1-\alpha_n\tau\right)\left\|G\widetilde{v}_n-Gx^*\right\|+\alpha_n\mu\left\|\widetilde{A}_2W_nGx^*\right\|\\ &\quad\le \left(1-\alpha_n\tau\right)\left\|\widetilde{v}_n-x^*\right\|+\alpha_n\mu\left\|\widetilde{A}_2x^*\right\|\\ &\quad= \left(1-\alpha_n\tau\right)\left\|v_n-x^*-\rho_n\widetilde{A}_1v_n\right\|+\alpha_n\mu\left\|\widetilde{A}_2x^*\right\|\\ &\quad= \left(1-\alpha_n\tau\right)\left\|v_n-x^*-\rho_n\left(\widetilde{A}_1v_n-\widetilde{A}_1x^*\right)-\rho_n\widetilde{A}_1x^*\right\|\\ &\qquad+\alpha_n\mu\left\|\widetilde{A}_2x^*\right\|\\ &\quad\le \left(1-\alpha_n\tau\right)\left[\left\|v_n-x^*-\rho_n\left(\widetilde{A}_1v_n-\widetilde{A}_1x^*\right)\right\|\right.\\ &\qquad\left.+\rho_n\left\|\widetilde{A}_1x^*\right\|\right]+\alpha_n\mu\left\|\widetilde{A}_2x^*\right\|\\ &\quad\le \left(1-\alpha_n\tau\right)\left[\left\|v_n-x^*\right\|+\rho_n\left\|\widetilde{A}_1x^*\right\|\right]+\alpha_n\mu\left\|\widetilde{A}_2x^*\right\|\\ &\quad\le \left(1-\alpha_n\tau\right)\left[\left\|x_n-x^*\right\|+\rho_n\left\|\widetilde{A}_1x^*\right\|\right]+\alpha_n\mu\left\|\widetilde{A}_2x^*\right\|\\ &\quad\le \left(1-\alpha_n\tau\right)\left\|x_n-x^*\right\|+\rho_n\left\|\widetilde{A}_1x^*\right\|+\alpha_n\mu\left\|\widetilde{A}_2x^*\right\|\\ &\quad\le \left(1-\alpha_n\tau\right)\left\|x_n-x^*\right\|+\alpha_n\left(\left\|\widetilde{A}_1x^*\right\|+\mu\left\|\widetilde{A}_2x^*\right\|\right)\\ &\quad= \left(1-\alpha_n\tau\right)\left\|x_n-x^*\right\|+\alpha_n\tau\cdot\frac{\left\|\widetilde{A}_1x^*\right\|+\mu\left\|\widetilde{A}_2x^*\right\|}{\tau}\\ &\quad\le \max\left\{\left\|x_n-x^*\right\|,\frac{\left\|\widetilde{A}_1x^*\right\|+\mu\left\|\widetilde{A}_2x^*\right\|}{\tau}\right\},\end{aligned} \tag{59}$$

where $\tau := 1 - \sqrt{1-\mu(2\beta-\mu L^2)}$. By induction, we find that

$$\begin{aligned}\left\|x_{n+1}-x^*\right\| &\le \max\left\{\left\|x_1-x^*\right\|, \frac{\left\|\widetilde{A}_1x^*\right\|+\mu\left\|\widetilde{A}_2x^*\right\|}{\tau}\right\}, \quad \forall n\ge 1.\end{aligned} \tag{60}$$

Thus, $\{x_n\}_{n=1}^{\infty}$ is bounded and so are the sequences $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$, $\{\widetilde{v}_n\}_{n=1}^{\infty}$, and $\{y_n\}_{n=1}^{\infty}$.

Step 2. We prove that $\lim_{n\to\infty}(\|x_{n+1}-x_n\|/\rho_n)=0$.

Indeed, utilizing (24) and (45), we obtain that

$$\begin{aligned}\left\|v_{n+1}-v_n\right\| &= \left\|\Lambda_{n+1}^N u_{n+1}-\Lambda_n^N u_n\right\| \\ &= \left\|J_{R_N,\lambda_{N,n+1}}\left(I-\lambda_{N,n+1}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right. \\ &\quad \left.-J_{R_N,\lambda_{N,n}}\left(I-\lambda_{N,n}B_N\right)\Lambda_n^{N-1}u_n\right\| \\ &\le \left\|J_{R_N,\lambda_{N,n+1}}\left(I-\lambda_{N,n+1}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right. \\ &\quad \left.-J_{R_N,\lambda_{N,n+1}}\left(I-\lambda_{N,n}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right\| \\ &\quad +\left\|J_{R_N,\lambda_{N,n+1}}\left(I-\lambda_{N,n}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right. \\ &\quad \left.-J_{R_N,\lambda_{N,n}}\left(I-\lambda_{N,n}B_N\right)\Lambda_n^{N-1}u_n\right\| \\ &\le \left\|\left(I-\lambda_{N,n+1}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right. \\ &\quad \left.-\left(I-\lambda_{N,n}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right\| \\ &\quad +\left\|\left(I-\lambda_{N,n}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}-\left(I-\lambda_{N,n}B_N\right)\Lambda_n^{N-1}u_n\right\| \\ &\quad +\left|\lambda_{N,n+1}-\lambda_{N,n}\right| \\ &\quad \times\left(\frac{1}{\lambda_{N,n+1}}\left\|J_{R_N,\lambda_{N,n+1}}\left(I-\lambda_{N,n}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right.\right. \\ &\quad \left.-\left(I-\lambda_{N,n}B_N\right)\Lambda_n^{N-1}u_n\right\| \\ &\quad +\frac{1}{\lambda_{N,n}}\left\|\left(I-\lambda_{N,n}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right. \\ &\quad \left.\left.-J_{R_N,\lambda_{N,n}}\left(I-\lambda_{N,n}B_N\right)\Lambda_n^{N-1}u_n\right\|\right) \\ &\le \left|\lambda_{N,n+1}-\lambda_{N,n}\right|\left(\left\|B_N\Lambda_{n+1}^{N-1}u_{n+1}\right\|+\widetilde{M}\right) \\ &\quad +\left\|\Lambda_{n+1}^{N-1}u_{n+1}-\Lambda_n^{N-1}u_n\right\| \\ &\le \left|\lambda_{N,n+1}-\lambda_{N,n}\right|\left(\left\|B_N\Lambda_{n+1}^{N-1}u_{n+1}\right\|+\widetilde{M}\right) \\ &\quad +\left|\lambda_{N-1,n+1}-\lambda_{N-1,n}\right|\left(\left\|B_{N-1}\Lambda_{n+1}^{N-2}u_{n+1}\right\|+\widetilde{M}\right) \\ &\quad +\left\|\Lambda_{n+1}^{N-2}u_{n+1}-\Lambda_n^{N-2}u_n\right\| \\ &\quad \vdots \\ &\le \left|\lambda_{N,n+1}-\lambda_{N,n}\right|\left(\left\|B_N\Lambda_{n+1}^{N-1}u_{n+1}\right\|+\widetilde{M}\right) \\ &\quad +\left|\lambda_{N-1,n+1}-\lambda_{N-1,n}\right|\left(\left\|B_{N-1}\Lambda_{n+1}^{N-2}u_{n+1}\right\|+\widetilde{M}\right) \\ &\quad +\cdots+\left|\lambda_{1,n+1}-\lambda_{1,n}\right|\left(\left\|B_1\Lambda_{n+1}^{0}u_{n+1}\right\|+\widetilde{M}\right) \\ &\quad +\left\|\Lambda_{n+1}^{0}u_{n+1}-\Lambda_n^{0}u_n\right\| \\ &\le \widetilde{M}_0\sum_{i=1}^{N}\left|\lambda_{i,n+1}-\lambda_{i,n}\right|+\left\|u_{n+1}-u_n\right\|,\end{aligned} \tag{61}$$

where

$$\begin{aligned}\sup_{n\ge1}\left\{\frac{1}{\lambda_{N,n+1}}\left\|J_{R_N,\lambda_{N,n+1}}\left(I-\lambda_{N,n}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right.\right. \\ \left.-\left(I-\lambda_{N,n}B_N\right)\Lambda_n^{N-1}u_n\right\| \\ +\frac{1}{\lambda_{N,n}}\left\|\left(I-\lambda_{N,n}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right. \\ \left.\left.-J_{R_N,\lambda_{N,n}}\left(I-\lambda_{N,n}B_N\right)\Lambda_n^{N-1}u_n\right\|\right\}\le\widetilde{M},\end{aligned} \tag{62}$$

for some $\widetilde{M}>0$ and $\sup_{n\ge1}\{\sum_{i=1}^{N}\|B_i\Lambda_{n+1}^{i-1}u_{n+1}\|+\widetilde{M}\}\le\widetilde{M}_0$ for some $\widetilde{M}_0>0$. Hence, it follows from (24) and $\{\rho_n\}_{n=1}^{\infty}\subset(0,2\alpha]$ that

$$\begin{aligned}\left\|\widetilde{v}_{n+1}-\widetilde{v}_n\right\| &= \left\|\left(v_{n+1}-\rho_{n+1}\widetilde{A}_1v_{n+1}\right)-\left(v_n-\rho_n\widetilde{A}_1v_n\right)\right\| \\ &\le \left\|\left(v_{n+1}-\rho_{n+1}\widetilde{A}_1v_{n+1}\right)-\left(v_n-\rho_{n+1}\widetilde{A}_1v_n\right)\right\| \\ &\quad +\left\|\left(v_n-\rho_{n+1}\widetilde{A}_1v_n\right)-\left(v_n-\rho_n\widetilde{A}_1v_n\right)\right\| \\ &\le \left\|v_{n+1}-v_n\right\|+\left|\rho_{n+1}-\rho_n\right|\left\|\widetilde{A}_1v_n\right\| \\ &\le \widetilde{M}_0\sum_{i=1}^{N}\left|\lambda_{i,n+1}-\lambda_{i,n}\right|+\left\|u_{n+1}-u_n\right\| \\ &\quad +\left|\rho_{n+1}-\rho_n\right|\left\|\widetilde{A}_1v_n\right\|.\end{aligned} \tag{63}$$

Also, utilizing Proposition 3(ii), (v), we deduce that

$$
\begin{aligned}
&\left\|u_{n+1}-u_n\right\| \\
&\quad=\left\|\Delta_{n+1}^{M}x_{n+1}-\Delta_{n}^{M}x_n\right\| \\
&\quad=\left\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right. \\
&\qquad\left.-T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}A_M\right)\Delta_{n}^{M-1}x_n\right\| \\
&\quad\le\left\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right. \\
&\qquad\left.-T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right\| \\
&\qquad+\left\|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right. \\
&\qquad\left.-T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}A_M\right)\Delta_{n}^{M-1}x_n\right\| \\
&\quad\le\left\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right. \\
&\qquad\left.-T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right\| \\
&\qquad+\left\|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right. \\
&\qquad\left.-T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right\| \\
&\qquad+\left\|\left(I-r_{M,n}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}-\left(I-r_{M,n}A_M\right)\Delta_{n}^{M-1}x_n\right\| \\
&\quad\le\frac{\left|r_{M,n+1}-r_{M,n}\right|}{r_{M,n+1}}\left\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right. \\
&\qquad\left.-\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right\| \\
&\qquad+\left|r_{M,n+1}-r_{M,n}\right|\left\|A_M\Delta_{n+1}^{M-1}x_{n+1}\right\| \\
&\qquad+\left\|\Delta_{n+1}^{M-1}x_{n+1}-\Delta_{n}^{M-1}x_n\right\| \\
&\quad=\left|r_{M,n+1}-r_{M,n}\right| \\
&\qquad\cdot\left[\left\|A_M\Delta_{n+1}^{M-1}x_{n+1}\right\|\right. \\
&\qquad\quad+\frac{1}{r_{M,n+1}}\left\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right. \\
&\qquad\qquad\left.\left.-\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right\|\right] \\
&\qquad+\left\|\Delta_{n+1}^{M-1}x_{n+1}-\Delta_{n}^{M-1}x_n\right\| \\
&\qquad\vdots \\
&\quad\le\left|r_{M,n+1}-r_{M,n}\right| \\
&\qquad\cdot\left[\left\|A_M\Delta_{n+1}^{M-1}x_{n+1}\right\|\right. \\
&\qquad\quad+\frac{1}{r_{M,n+1}}\left\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right. \\
&\qquad\qquad\left.\left.-\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right\|\right] \\
&\qquad+\cdots+\left|r_{1,n+1}-r_{1,n}\right| \\
&\qquad\cdot\left[\left\|A_1\Delta_{n+1}^{0}x_{n+1}\right\|\right. \\
&\qquad\quad+\frac{1}{r_{1,n+1}}\left\|T_{r_{1,n+1}}^{(\Theta_1,\varphi_1)}\left(I-r_{1,n+1}A_1\right)\Delta_{n+1}^{0}x_{n+1}\right. \\
&\qquad\qquad\left.\left.-\left(I-r_{1,n+1}A_1\right)\Delta_{n+1}^{0}x_{n+1}\right\|\right] \\
&\qquad+\left\|\Delta_{n+1}^{0}x_{n+1}-\Delta_{n}^{0}x_n\right\| \\
&\quad\le\widetilde{M}_1\sum_{k=1}^{M}\left|r_{k,n+1}-r_{k,n}\right|+\left\|x_{n+1}-x_n\right\|,
\end{aligned}
\tag{64}
$$

where $\widetilde{M}_1>0$ is a constant such that, for each $n\ge 1$,

$$
\begin{aligned}
&\sum_{k=1}^{M}\left[\left\|A_k\Delta_{n+1}^{k-1}x_{n+1}\right\|+\frac{1}{r_{k,n+1}}\right. \\
&\qquad\cdot\left\|T_{r_{k,n+1}}^{(\Theta_k,\varphi_k)}\left(I-r_{k,n+1}A_k\right)\Delta_{n+1}^{k-1}x_{n+1}\right. \\
&\qquad\left.\left.-\left(I-r_{k,n+1}A_k\right)\Delta_{n+1}^{k-1}x_{n+1}\right\|\right]\le\widetilde{M}_1.
\end{aligned}
\tag{65}
$$

In the meantime, from (33), since W_n, T_n, and $U_{n,i}$ are all nonexpansive, we have

$$
\begin{aligned}
&\left\|W_{n+1}G\widetilde{\nu}_n-W_nG\widetilde{\nu}_n\right\| \\
&\quad=\left\|\lambda_1T_1U_{n+1,2}G\widetilde{\nu}_n-\lambda_1T_1U_{n,2}G\widetilde{\nu}_n\right\| \\
&\quad\le\lambda_1\left\|U_{n+1,2}G\widetilde{\nu}_n-U_{n,2}G\widetilde{\nu}_n\right\| \\
&\quad=\lambda_1\left\|\lambda_2T_2U_{n+1,3}G\widetilde{\nu}_n-\lambda_2T_2U_{n,3}G\widetilde{\nu}_n\right\| \\
&\quad\le\lambda_1\lambda_2\left\|U_{n+1,3}G\widetilde{\nu}_n-U_{n,3}G\widetilde{\nu}_n\right\| \\
&\quad\vdots \\
&\quad\le\lambda_1\lambda_2\cdots\lambda_n\left\|U_{n+1,n+1}G\widetilde{\nu}_n-U_{n,n+1}G\widetilde{\nu}_n\right\| \\
&\quad\le\widetilde{M}_2\prod_{i=1}^{n}\lambda_i,
\end{aligned}
\tag{66}
$$

where $\widetilde{M}_2$ is a constant such that $\|U_{n+1,n+1}G\widetilde{\nu}_n\|+\|U_{n,n+1}G\widetilde{\nu}_n\|\le\widetilde{M}_2$ for each $n\ge 1$. So, utilizing (24),

(63)–(66), from $\{\lambda_n\} \subset (0,b] \subset (0,1)$, $\{\alpha_n\} \subset (0,1]$, $\{\rho_n\} \subset (0,2\alpha]$, and $\mu \in (0, 2\beta/L^2)$, we deduce that

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\quad = \|W_{n+1}G\widetilde{v}_{n+1} - W_nG\widetilde{v}_n\| \\
&\quad \le \|W_{n+1}G\widetilde{v}_{n+1} - W_{n+1}G\widetilde{v}_n\| + \|W_{n+1}G\widetilde{v}_n - W_nG\widetilde{v}_n\| \\
&\quad \le \|\widetilde{v}_{n+1} - \widetilde{v}_n\| + \widetilde{M}_2\prod_{i=1}^{n}\lambda_i \\
&\quad \le \|\widetilde{v}_{n+1} - \widetilde{v}_n\| + \widetilde{M}_2 b^n \\
&\quad \le \widetilde{M}_0\sum_{i=1}^{N}|\lambda_{i,n+1} - \lambda_{i,n}| + \|u_{n+1} - u_n\| \\
&\qquad + |\rho_{n+1} - \rho_n|\left\|\widetilde{A}_1 v_n\right\| + \widetilde{M}_2 b^n \\
&\quad \le \widetilde{M}_0\sum_{i=1}^{N}|\lambda_{i,n+1} - \lambda_{i,n}| \\
&\qquad + \widetilde{M}_1\sum_{k=1}^{M}|r_{k,n+1} - r_{k,n}| + \|x_{n+1} - x_n\| \\
&\qquad + |\rho_{n+1} - \rho_n|\left\|\widetilde{A}_1 v_n\right\| + \widetilde{M}_2 b^n,
\end{aligned} \tag{67}$$

and, by Lemma 9,

$$\begin{aligned}
&\|x_{n+2} - x_{n+1}\| \\
&\quad = \left\|\left(I - \mu\alpha_{n+1}\widetilde{A}_2\right)W_{n+1}G\widetilde{v}_{n+1} - \left(I - \mu\alpha_n\widetilde{A}_2\right)W_nG\widetilde{v}_n\right\| \\
&\quad \le \left\|\left(I - \mu\alpha_{n+1}\widetilde{A}_2\right)W_{n+1}G\widetilde{v}_{n+1} - \left(I - \mu\alpha_{n+1}\widetilde{A}_2\right)W_nG\widetilde{v}_n\right\| \\
&\qquad + \left\|\left(I - \mu\alpha_{n+1}\widetilde{A}_2\right)W_nG\widetilde{v}_n - \left(I - \mu\alpha_n\widetilde{A}_2\right)W_nG\widetilde{v}_n\right\| \\
&\quad \le (1 - \alpha_{n+1}\tau)\|W_{n+1}G\widetilde{v}_{n+1} - W_nG\widetilde{v}_n\| \\
&\qquad + |\alpha_{n+1} - \alpha_n|\,\mu\left\|\widetilde{A}_2W_nG\widetilde{v}_n\right\| \\
&\quad \le (1 - \alpha_{n+1}\tau) \\
&\qquad \cdot \left[\widetilde{M}_0\sum_{i=1}^{N}|\lambda_{i,n+1} - \lambda_{i,n}|\right. \\
&\qquad\quad + \widetilde{M}_1\sum_{k=1}^{M}|r_{k,n+1} - r_{k,n}| + \|x_{n+1} - x_n\| \\
&\qquad\quad \left. + |\rho_{n+1} - \rho_n|\left\|\widetilde{A}_1 v_n\right\| + \widetilde{M}_2 b^n\right] \\
&\qquad + |\alpha_{n+1} - \alpha_n|\,\mu\left\|\widetilde{A}_2W_nG\widetilde{v}_n\right\| \\
&\quad \le (1 - \alpha_{n+1}\tau)\|x_{n+1} - x_n\| \\
&\qquad + \widetilde{M}_0\sum_{i=1}^{N}|\lambda_{i,n+1} - \lambda_{i,n}| + \widetilde{M}_1\sum_{k=1}^{M}|r_{k,n+1} - r_{k,n}| \\
&\qquad + |\rho_{n+1} - \rho_n|\left\|\widetilde{A}_1 v_n\right\| + \widetilde{M}_2 b^n \\
&\qquad + |\alpha_{n+1} - \alpha_n|\,\mu\left\|\widetilde{A}_2W_nG\widetilde{v}_n\right\| \\
&\quad \le (1 - \alpha_{n+1}\tau)\|x_{n+1} - x_n\| \\
&\qquad + \widetilde{M}_3\left(\sum_{i=1}^{N}|\lambda_{i,n+1} - \lambda_{i,n}| + \sum_{k=1}^{M}|r_{k,n+1} - r_{k,n}|\right. \\
&\qquad\quad \left. + |\rho_{n+1} - \rho_n| + |\alpha_{n+1} - \alpha_n| + b^n\right),
\end{aligned} \tag{68}$$

where $\sup_{n\ge1}\{\widetilde{M}_0 + \widetilde{M}_1 + \widetilde{M}_2 + \|\widetilde{A}_1 v_n\| + \mu\|\widetilde{A}_2W_nG\widetilde{v}_n\|\} \le \widetilde{M}_3$ for some $\widetilde{M}_3 > 0$. Consequently,

$$\begin{aligned}
&\frac{\|x_{n+1} - x_n\|}{\rho_n} \\
&\quad \le (1 - \alpha_n\tau)\frac{\|x_n - x_{n-1}\|}{\rho_n} \\
&\qquad + \widetilde{M}_3\left(\sum_{i=1}^{N}\frac{|\lambda_{i,n} - \lambda_{i,n-1}|}{\rho_n} + \sum_{k=1}^{M}\frac{|r_{k,n} - r_{k,n-1}|}{\rho_n}\right. \\
&\qquad\quad \left. + \frac{|\rho_n - \rho_{n-1}|}{\rho_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\rho_n} + \frac{b^{n-1}}{\rho_n}\right) \\
&\quad = (1 - \tau\alpha_n)\frac{\|x_n - x_{n-1}\|}{\rho_{n-1}} \\
&\qquad + (1 - \tau\alpha_n)\|x_n - x_{n-1}\|\left(\frac{1}{\rho_n} - \frac{1}{\rho_{n-1}}\right) \\
&\qquad + \widetilde{M}_3\left(\sum_{i=1}^{N}\frac{|\lambda_{i,n} - \lambda_{i,n-1}|}{\rho_n} + \sum_{k=1}^{M}\frac{|r_{k,n} - r_{k,n-1}|}{\rho_n}\right. \\
&\qquad\quad \left. + \frac{|\rho_n - \rho_{n-1}|}{\rho_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\rho_n} + \frac{b^{n-1}}{\rho_n}\right) \\
&\quad \le (1 - \tau\alpha_n)\frac{\|x_n - x_{n-1}\|}{\rho_{n-1}} \\
&\qquad + \tau\alpha_n\cdot\frac{\widetilde{M}_4}{\tau}\left\{\frac{1}{\alpha_n}\left|\frac{1}{\rho_n} - \frac{1}{\rho_{n-1}}\right|\right. \\
&\qquad\qquad + \sum_{i=1}^{N}\frac{|\lambda_{i,n} - \lambda_{i,n-1}|}{\alpha_n\rho_n} \\
&\qquad\qquad + \sum_{k=1}^{M}\frac{|r_{k,n} - r_{k,n-1}|}{\alpha_n\rho_n} + \frac{1}{\alpha_n}\left|1 - \frac{\rho_{n-1}}{\rho_n}\right| \\
&\qquad\qquad \left. + \frac{1}{\rho_n}\left|1 - \frac{\alpha_{n-1}}{\alpha_n}\right| + \frac{b^{n-1}}{\alpha_n\rho_n}\right\},
\end{aligned} \tag{69}$$

where $\sup_{n\geq 1}\{\|x_{n+1}-x_n\|+\widetilde{M}_3\}\leq \widetilde{M}_4$ for some $\widetilde{M}_4>0$. From (H1)–(H5), it follows that $\sum_{n=1}^{\infty}\tau\alpha_n=\infty$ and

$$\lim_{n\to\infty}\frac{\widetilde{M}_4}{\tau}\left\{\frac{1}{\alpha_n}\left|\frac{1}{\rho_n}-\frac{1}{\rho_{n-1}}\right|+\sum_{i=1}^{N}\frac{\left|\lambda_{i,n}-\lambda_{i,n-1}\right|}{\alpha_n\rho_n}+\sum_{k=1}^{M}\frac{\left|r_{k,n}-r_{k,n-1}\right|}{\alpha_n\rho_n}+\frac{1}{\alpha_n}\left|1-\frac{\rho_{n-1}}{\rho_n}\right|+\frac{1}{\rho_n}\left|1-\frac{\alpha_{n-1}}{\alpha_n}\right|+\frac{b^{n-1}}{\alpha_n\rho_n}\right\}=0. \tag{70}$$

Thus, applying Lemma 10 to (69), we immediately conclude that

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\rho_n}=0. \tag{71}$$

So, from (H3), it follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{72}$$

Step 3. We prove that $\lim_{n\to\infty}\|x_n-u_n\|=0$, $\lim_{n\to\infty}\|x_n-v_n\|=0$, $\lim_{n\to\infty}\|\widetilde{v}_n-G\widetilde{v}_n\|=0$, and $\lim_{n\to\infty}\|\widetilde{v}_n-W\widetilde{v}_n\|=0$, where $\widetilde{v}_n=v_n-\rho_n\widetilde{A}_1v_n$.

Indeed, utilizing Lemma 4, from (48), we get

$$\begin{aligned}
\|x_{n+1}-x^*\|^2&=\left\|(I-\mu\alpha_n\widetilde{A}_2)W_nG\widetilde{v}_n-x^*\right\|^2\\
&=\left\|\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nG\widetilde{v}_n-\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nGx^*\right.\\
&\quad\left.+\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nGx^*-W_nGx^*\right\|^2\\
&\leq\left\|\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nG\widetilde{v}_n-\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nGx^*\right\|^2\\
&\quad-2\mu\alpha_n\langle\widetilde{A}_2x^*,x_{n+1}-x^*\rangle\\
&\leq(1-\alpha_n\tau)\|G\widetilde{v}_n-x^*\|^2-2\mu\alpha_n\langle\widetilde{A}_2x^*,x_{n+1}-x^*\rangle\\
&\leq(1-\alpha_n\tau)\|\widetilde{v}_n-x^*\|^2-2\mu\alpha_n\langle\widetilde{A}_2x^*,x_{n+1}-x^*\rangle\\
&=(1-\alpha_n\tau)\left\|v_n-x^*-\rho_n\widetilde{A}_1v_n\right\|^2\\
&\quad-2\mu\alpha_n\langle\widetilde{A}_2x^*,x_{n+1}-x^*\rangle\\
&\leq(1-\alpha_n\tau)\left(\|v_n-x^*\|^2-2\rho_n\left\langle\widetilde{A}_1v_n,\widetilde{v}_n-x^*\right\rangle\right)\\
&\quad-2\mu\alpha_n\langle\widetilde{A}_2x^*,x_{n+1}-x^*\rangle\\
&\leq\|v_n-x^*\|^2-2(1-\alpha_n\tau)\rho_n\langle\widetilde{A}_1v_n,\widetilde{v}_n-x^*\rangle\\
&\quad-2\mu\alpha_n\langle\widetilde{A}_2x^*,x_{n+1}-x^*\rangle.
\end{aligned} \tag{73}$$

On the other hand, observe that

$$\begin{aligned}
\left\|\Delta_n^kx_n-x^*\right\|^2&=\left\|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}A_k)\Delta_n^{k-1}x_n-T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}A_k)x^*\right\|^2\\
&\leq\left\|(I-r_{k,n}A_k)\Delta_n^{k-1}x_n-(I-r_{k,n}A_k)x^*\right\|^2\\
&\leq\left\|\Delta_n^{k-1}x_n-x^*\right\|^2+r_{k,n}(r_{k,n}-2\mu_k)\left\|A_k\Delta_n^{k-1}x_n-A_kx^*\right\|^2\\
&\leq\|x_n-x^*\|^2+r_{k,n}(r_{k,n}-2\mu_k)\left\|A_k\Delta_n^{k-1}x_n-A_kx^*\right\|^2,
\end{aligned} \tag{74}$$

$$\begin{aligned}
\left\|\Lambda_n^iu_n-x^*\right\|^2&=\left\|J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)\Lambda_n^{i-1}u_n-J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)x^*\right\|^2\\
&\leq\left\|(I-\lambda_{i,n}B_i)\Lambda_n^{i-1}u_n-(I-\lambda_{i,n}B_i)x^*\right\|^2\\
&\leq\left\|\Lambda_n^{i-1}u_n-x^*\right\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\left\|B_i\Lambda_n^{i-1}u_n-B_ix^*\right\|^2\\
&\leq\|u_n-x^*\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\left\|B_i\Lambda_n^{i-1}u_n-B_ix^*\right\|^2\\
&\leq\|x_n-x^*\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\left\|B_i\Lambda_n^{i-1}u_n-B_ix^*\right\|^2.
\end{aligned} \tag{75}$$

Combining (73)–(75), we get

$$\begin{aligned}
\|x_{n+1}-x^*\|^2&\leq\|v_n-x^*\|^2-2(1-\alpha_n\tau)\rho_n\left\langle\widetilde{A}_1v_n,\widetilde{v}_n-x^*\right\rangle\\
&\quad-2\mu\alpha_n\langle\widetilde{A}_2x^*,x_{n+1}-x^*\rangle\\
&\leq\left\|\Lambda_n^iu_n-x^*\right\|^2+2\rho_n\left\|\widetilde{A}_1v_n\right\|\left\|\widetilde{v}_n-x^*\right\|\\
&\quad+2\mu\alpha_n\left\|\widetilde{A}_2x^*\right\|\left\|x_{n+1}-x^*\right\|\\
&\leq\|u_n-x^*\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\left\|B_i\Lambda_n^{i-1}u_n-B_ix^*\right\|^2\\
&\quad+2\rho_n\left\|\widetilde{A}_1v_n\right\|\left\|\widetilde{v}_n-x^*\right\|\\
&\quad+2\mu\alpha_n\left\|\widetilde{A}_2x^*\right\|\left\|x_{n+1}-x^*\right\|\\
&\leq\left\|\Delta_n^kx_n-x^*\right\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\left\|B_i\Lambda_n^{i-1}u_n-B_ix^*\right\|^2\\
&\quad+2\rho_n\left\|\widetilde{A}_1v_n\right\|\left\|\widetilde{v}_n-x^*\right\|\\
&\quad+2\mu\alpha_n\left\|\widetilde{A}_2x^*\right\|\left\|x_{n+1}-x^*\right\|\\
&\leq\|x_n-x^*\|^2+r_{k,n}(r_{k,n}-2\mu_k)\left\|A_k\Delta_n^{k-1}x_n-A_kx^*\right\|^2\\
&\quad+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\left\|B_i\Lambda_n^{i-1}u_n-B_ix^*\right\|^2\\
&\quad+2\rho_n\left\|\widetilde{A}_1v_n\right\|\left\|\widetilde{v}_n-x^*\right\|+2\mu\alpha_n\left\|\widetilde{A}_2x^*\right\|\left\|x_{n+1}-x^*\right\|,
\end{aligned} \tag{76}$$

which leads to

$$\begin{aligned}
&r_{k,n}(2\mu_k - r_{k,n})\left\|A_k\Delta_n^{k-1}x_n - A_kx^*\right\|^2 \\
&\quad + \lambda_{i,n}(2\eta_i - \lambda_{i,n})\left\|B_i\Lambda_n^{i-1}u_n - B_ix^*\right\|^2 \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\rho_n\left\|\widetilde{A}_1v_n\right\|\|\widetilde{v}_n - x^*\| \\
&\quad + 2\mu\alpha_n\left\|\widetilde{A}_2x^*\right\|\|x_{n+1} - x^*\| \\
&\le \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\quad + 2\rho_n\left\|\widetilde{A}_1v_n\right\|\|\widetilde{v}_n - x^*\| + 2\mu\alpha_n\left\|\widetilde{A}_2x^*\right\|\|x_{n+1} - x^*\|.
\end{aligned} \tag{77}$$

Since $\alpha_n \to 0$, $\rho_n \to 0$, $\|x_{n+1} - x_n\| \to 0$, and $\{x_n\}, \{v_n\}, \{\widetilde{v}\}$ are bounded sequences, it follows from $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ that

$$\begin{aligned}
\lim_{n\to\infty}\left\|A_k\Delta_n^{k-1}x_n - A_kx^*\right\| &= 0, \\
\lim_{n\to\infty}\left\|B_i\Lambda_n^{i-1}u_n - B_ix^*\right\| &= 0,
\end{aligned} \tag{78}$$

for all $k \in \{1, 2, \ldots, M\}$ and $i \in \{1, 2, \ldots, N\}$.

Furthermore, by Proposition 3(ii) and Lemma 11 (a), we have

$$\begin{aligned}
&\left\|\Delta_n^kx_n - x^*\right\|^2 \\
&= \left\|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}A_k)\Delta_n^{k-1}x_n - T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}A_k)x^*\right\|^2 \\
&\le \left\langle(I - r_{k,n}A_k)\Delta_n^{k-1}x_n - (I - r_{k,n}A_k)x^*, \Delta_n^kx_n - x^*\right\rangle \\
&= \frac{1}{2}\left(\left\|(I - r_{k,n}A_k)\Delta_n^{k-1}x_n - (I - r_{k,n}A_k)x^*\right\|^2\right. \\
&\qquad + \left\|\Delta_n^kx_n - x^*\right\|^2 \\
&\qquad - \left\|(I - r_{k,n}A_k)\Delta_n^{k-1}x_n - (I - r_{k,n}A_k)x^*\right. \\
&\qquad\qquad \left.\left. - \left(\Delta_n^kx_n - x^*\right)\right\|^2\right) \\
&\le \frac{1}{2}\left(\left\|\Delta_n^{k-1}x_n - x^*\right\|^2 + \left\|\Delta_n^kx_n - x^*\right\|^2\right. \\
&\qquad \left. - \left\|\Delta_n^{k-1}x_n - \Delta_n^kx_n - r_{k,n}\left(A_k\Delta_n^{k-1}x_n - A_kx^*\right)\right\|^2\right),
\end{aligned} \tag{79}$$

which implies that

$$\begin{aligned}
&\left\|\Delta_n^kx_n - x^*\right\|^2 \\
&\le \left\|\Delta_n^{k-1}x_n - x^*\right\|^2 \\
&\quad - \left\|\Delta_n^{k-1}x_n - \Delta_n^kx_n - r_{k,n}(A_k\Delta_n^{k-1}x_n - A_kx^*)\right\|^2 \\
&= \left\|\Delta_n^{k-1}x_n - x^*\right\|^2 - \left\|\Delta_n^{k-1}x_n - \Delta_n^kx_n\right\|^2 \\
&\quad - r_{k,n}^2\left\|A_k\Delta_n^{k-1}x_n - A_kx^*\right\|^2 \\
&\quad + 2r_{k,n}\langle\Delta_n^{k-1}x_n - \Delta_n^kx_n, A_k\Delta_n^{k-1}x_n - A_kx^*\rangle \\
&\le \left\|\Delta_n^{k-1}x_n - x^*\right\|^2 - \left\|\Delta_n^{k-1}x_n - \Delta_n^kx_n\right\|^2 \\
&\quad + 2r_{k,n}\left\|\Delta_n^{k-1}x_n - \Delta_n^kx_n\right\|\left\|A_k\Delta_n^{k-1}x_n - A_kx^*\right\| \\
&\le \|x_n - x^*\|^2 - \left\|\Delta_n^{k-1}x_n - \Delta_n^kx_n\right\|^2 \\
&\quad + 2r_{k,n}\left\|\Delta_n^{k-1}x_n - \Delta_n^kx_n\right\|\left\|A_k\Delta_n^{k-1}x_n - A_kx^*\right\|.
\end{aligned} \tag{80}$$

By Lemma 11 (a) and Lemma 14, we obtain

$$\begin{aligned}
&\left\|\Lambda_n^iu_n - x^*\right\|^2 \\
&= \left\|J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)x^*\right\|^2 \\
&\le \left\langle(I - \lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - (I - \lambda_{i,n}B_i)x^*, \Lambda_n^iu_n - x^*\right\rangle \\
&= \frac{1}{2}\left(\left\|(I - \lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - (I - \lambda_{i,n}B_i)x^*\right\|^2\right. \\
&\qquad + \left\|\Lambda_n^iu_n - x^*\right\|^2 \\
&\qquad - \left\|(I - \lambda_{i,n}B_i)\Lambda_n^{i-1}u_n\right. \\
&\qquad\qquad \left.\left. -(I - \lambda_{i,n}B_i)x^* - (\Lambda_n^iu_n - x^*)\right\|^2\right) \\
&\le \frac{1}{2}\left(\left\|\Lambda_n^{i-1}u_n - x^*\right\|^2 + \left\|\Lambda_n^iu_n - x^*\right\|^2\right. \\
&\qquad \left. - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n - \lambda_{i,n}(B_i\Lambda_n^{i-1}u_n - B_ix^*)\right\|^2\right) \\
&\le \frac{1}{2}\left(\|u_n - x^*\|^2 + \left\|\Lambda_n^iu_n - x^*\right\|^2\right. \\
&\qquad \left. - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n - \lambda_{i,n}(B_i\Lambda_n^{i-1}u_n - B_ix^*)\right\|^2\right) \\
&\le \frac{1}{2}\left(\|x_n - x^*\|^2 + \left\|\Lambda_n^iu_n - x^*\right\|^2\right. \\
&\qquad \left. - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n - \lambda_{i,n}(B_i\Lambda_n^{i-1}u_n - B_ix^*)\right\|^2\right),
\end{aligned} \tag{81}$$

which implies

$$\begin{aligned}
&\left\|\Lambda_n^i u_n - x^*\right\|^2 \\
&\quad\le \left\|x_n - x^*\right\|^2 \\
&\qquad - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}(B_i\Lambda_n^{i-1}u_n - B_i x^*)\right\|^2 \\
&\quad= \left\|x_n - x^*\right\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad - \lambda_{i,n}^2\left\|B_i\Lambda_n^{i-1}u_n - B_i x^*\right\|^2 \\
&\qquad + 2\lambda_{i,n}\langle \Lambda_n^{i-1}u_n - \Lambda_n^i u_n, B_i\Lambda_n^{i-1}u_n - B_i x^*\rangle \\
&\quad\le \left\|x_n - x^*\right\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|\left\|B_i\Lambda_n^{i-1}u_n - B_i x^*\right\|.
\end{aligned} \tag{82}$$

Combining (73) and (82), we conclude that

$$\begin{aligned}
&\left\|x_{n+1} - x^*\right\|^2 \\
&\quad\le \left\|v_n - x^*\right\|^2 - 2\left(1-\alpha_n\tau\right)\rho_n\langle \widetilde{A}_1 v_n, \widetilde{v}_n - x^*\rangle \\
&\qquad - 2\mu\alpha_n\langle \widetilde{A}_2 x^*, x_{n+1} - x^*\rangle \\
&\quad\le \left\|\Lambda_n^i u_n - x^*\right\|^2 + 2\rho_n\left\|\widetilde{A}_1 v_n\right\|\left\|\widetilde{v}_n - x^*\right\| \\
&\qquad + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\left\|x_{n+1} - x^*\right\| \\
&\quad\le \left\|x_n - x^*\right\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|\left\|B_i\Lambda_n^{i-1}u_n - B_i x^*\right\| \\
&\qquad + 2\rho_n\left\|\widetilde{A}_1 v_n\right\|\left\|\widetilde{v}_n - x^*\right\| + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\left\|x_{n+1} - x^*\right\|,
\end{aligned} \tag{83}$$

which yields

$$\begin{aligned}
&\left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2 \\
&\quad\le \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 \\
&\qquad + 2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|\left\|B_i\Lambda_n^{i-1}u_n - B_i x^*\right\| \\
&\qquad + 2\rho_n\left\|\widetilde{A}_1 v_n\right\|\left\|\widetilde{v}_n - x^*\right\| + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\left\|x_{n+1} - x^*\right\| \\
&\quad\le \left\|x_n - x_{n+1}\right\|\left(\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|\right) \\
&\qquad + 2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|\left\|B_i\Lambda_n^{i-1}u_n - B_i x^*\right\| \\
&\qquad + 2\rho_n\left\|\widetilde{A}_1 v_n\right\|\left\|\widetilde{v}_n - x^*\right\| + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\left\|x_{n+1} - x^*\right\|.
\end{aligned} \tag{84}$$

Since $\alpha_n \to 0$, $\rho_n \to 0$, and $\|x_{n+1} - x_n\| \to 0$ and $\{x_n\}$, $\{v_n\}$, and $\{\widetilde{v}\}$ are bounded sequences, it follows from (78) and $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ that

$$\lim_{n\to\infty}\left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\| = 0, \quad \forall i \in \{1, 2, \ldots, N\}. \tag{85}$$

Also, combining (55), (73), and (80), we deduce that

$$\begin{aligned}
&\left\|x_{n+1} - x^*\right\|^2 \\
&\quad\le \left\|v_n - x^*\right\|^2 - 2\left(1-\alpha_n\tau\right)\rho_n\left\langle \widetilde{A}_1 v_n, \widetilde{v}_n - x^*\right\rangle \\
&\qquad - 2\mu\alpha_n\langle \widetilde{A}_2 x^*, x_{n+1} - x^*\rangle \\
&\quad\le \left\|u_n - x^*\right\|^2 + 2\rho_n\left\|\widetilde{A}_1 v_n\right\|\left\|\widetilde{v}_n - x^*\right\| \\
&\qquad + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\left\|x_{n+1} - x^*\right\| \\
&\quad\le \left\|\Delta_n^k x_n - x^*\right\|^2 + 2\rho_n\left\|\widetilde{A}_1 v_n\right\|\left\|\widetilde{v}_n - x^*\right\| \\
&\qquad + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\left\|x_{n+1} - x^*\right\| \\
&\quad\le \left\|x_n - x^*\right\|^2 - \left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|^2 \\
&\qquad + 2r_{k,n}\left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|\left\|A_k\Delta_n^{k-1}x_n - A_k x^*\right\| \\
&\qquad + 2\rho_n\left\|\widetilde{A}_1 v_n\right\|\left\|\widetilde{v}_n - x^*\right\| + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\left\|x_{n+1} - x^*\right\|,
\end{aligned} \tag{86}$$

which leads to

$$\begin{aligned}
&\left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|^2 \\
&\quad\le \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 \\
&\qquad + 2r_{k,n}\left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|\left\|A_k\Delta_n^{k-1}x_n - A_k x^*\right\| \\
&\qquad + 2\rho_n\left\|\widetilde{A}_1 v_n\right\|\left\|\widetilde{v}_n - x^*\right\| + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\left\|x_{n+1} - x^*\right\| \\
&\quad\le \left\|x_n - x_{n+1}\right\|\left(\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|\right) \\
&\qquad + 2r_{k,n}\left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|\left\|A_k\Delta_n^{k-1}x_n - A_k x^*\right\| \\
&\qquad + 2\rho_n\left\|\widetilde{A}_1 v_n\right\|\left\|\widetilde{v}_n - x^*\right\| + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\left\|x_{n+1} - x^*\right\|.
\end{aligned} \tag{87}$$

Since $\alpha_n \to 0$, $\rho_n \to 0$, and $\|x_{n+1} - x_n\| \to 0$ and $\{x_n\}$, $\{v_n\}$ and, $\{\widetilde{v}\}$ are bounded sequences, it follows from (78) and $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ that

$$\lim_{n\to\infty}\left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\| = 0, \quad \forall k \in \{1, 2, \ldots, M\}. \tag{88}$$

Hence, from (85) and (88), we get

$$\begin{aligned}
\left\|x_n - u_n\right\| &= \left\|\Delta_n^0 x_n - \Delta_n^M x_n\right\| \\
&\le \left\|\Delta_n^0 x_n - \Delta_n^1 x_n\right\| + \left\|\Delta_n^1 x_n - \Delta_n^2 x_n\right\| \\
&\quad + \cdots + \left\|\Delta_n^{M-1}x_n - \Delta_n^M x_n\right\| \\
&\longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{89}$$

$$\begin{aligned}\|u_n - v_n\| &= \left\|\Lambda_n^0 u_n - \Lambda_n^N u_n\right\| \\ &\le \left\|\Lambda_n^0 u_n - \Lambda_n^1 u_n\right\| + \left\|\Lambda_n^1 u_n - \Lambda_n^2 u_n\right\| \\ &\quad + \cdots + \left\|\Lambda_n^{N-1} u_n - \Lambda_n^N u_n\right\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{90}$$

respectively. Thus, from (89) and (90), we obtain

$$\begin{aligned}\|x_n - v_n\| &\le \|x_n - u_n\| + \|u_n - v_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{91}$$

Next, for simplicity, we write $y^* = P_C(I - \nu_2 F_2)x^*$, $w_n = P_C(I - \nu_2 F_2)\widetilde{\nu}_n$, and $\widetilde{\omega}_n = G\widetilde{\nu}_n = P_C(I - \nu_1 F_1)w_n$ for all $n \ge 1$. Then,

$$\begin{aligned}x^* = Gx^* &= P_C\left(I - \nu_1 F_1\right) y^* \\ &= P_C\left(I - \nu_1 F_1\right) P_C\left(I - \nu_2 F_2\right) x^*.\end{aligned} \tag{92}$$

We now show that $\lim_{n\to\infty}\|G\widetilde{\nu}_n - \widetilde{\nu}_n\| = 0$; that is, $\lim_{n\to\infty}\|\widetilde{\omega}_n - \widetilde{\nu}_n\| = 0$. As a matter of fact, from (56), (57), and (73) it follows that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad \le (1 - \alpha_n\tau)\|G\widetilde{\nu}_n - x^*\|^2 - 2\mu\alpha_n\langle \widetilde{A}_2 x^*, x_{n+1} - x^*\rangle \\ &\quad \le \|\widetilde{\omega}_n - x^*\|^2 + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\|x_{n+1} - x^*\| \\ &\quad \le \|w_n - y^*\|^2 + \nu_1(\nu_1 - 2\zeta_1)\|F_1 w_n - F_1 y^*\|^2 \\ &\qquad + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\|x_{n+1} - x^*\| \\ &\quad \le \|\widetilde{\nu}_n - x^*\|^2 + \nu_2(\nu_2 - 2\zeta_2)\|F_2\widetilde{\nu}_n - F_2 x^*\|^2 \\ &\qquad + \nu_1(\nu_1 - 2\zeta_1)\|F_1 w_n - F_1 y^*\|^2 \\ &\qquad + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\|x_{n+1} - x^*\| \\ &\quad = \left\|\nu_n - x^* - \rho_n\widetilde{A}_1\nu_n\right\|^2 \\ &\qquad + \nu_2(\nu_2 - 2\zeta_2)\|F_2\widetilde{\nu}_n - F_2 x^*\|^2 \\ &\qquad + \nu_1(\nu_1 - 2\zeta_1)\|F_1 w_n - F_1 y^*\|^2 \\ &\qquad + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\|x_{n+1} - x^*\| \\ &\quad \le \|\nu_n - x^*\|^2 + 2\rho_n\left\|\widetilde{A}_1\nu_n\right\|\|\widetilde{\nu}_n - x^*\| \\ &\qquad + \nu_2(\nu_2 - 2\zeta_2)\|F_2\widetilde{\nu}_n - F_2 x^*\|^2 \\ &\qquad + \nu_1(\nu_1 - 2\zeta_1)\|F_1 w_n - F_1 y^*\|^2 \\ &\qquad + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\|x_{n+1} - x^*\| \\ &\quad \le \|x_n - x^*\|^2 + 2\rho_n\left\|\widetilde{A}_1\nu_n\right\|\|\widetilde{\nu}_n - x^*\| \\ &\qquad + \nu_2(\nu_2 - 2\zeta_2)\|F_2\widetilde{\nu}_n - F_2 x^*\|^2 \\ &\qquad + \nu_1(\nu_1 - 2\zeta_1)\|F_1 w_n - F_1 y^*\|^2 \\ &\qquad + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\|x_{n+1} - x^*\|,\end{aligned} \tag{93}$$

which immediately yields

$$\begin{aligned}&\nu_2(2\zeta_2 - \nu_2)\|F_2\widetilde{\nu}_n - F_2 x^*\|^2 + \nu_1(2\zeta_1 - \nu_1)\|F_1 w_n - F_1 y^*\|^2 \\ &\quad \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\rho_n\left\|\widetilde{A}_1\nu_n\right\|\|\widetilde{\nu}_n - x^*\| \\ &\qquad + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\|x_{n+1} - x^*\| \\ &\quad \le \|x_n - x_{n+1}\|\left(\|x_n - x^*\| + \|x_{n+1} - x^*\|\right) \\ &\qquad + 2\rho_n\left\|\widetilde{A}_1\nu_n\right\|\|\widetilde{\nu}_n - x^*\| + 2\mu\alpha_n\left\|\widetilde{A}_2 x^*\right\|\|x_{n+1} - x^*\|.\end{aligned} \tag{94}$$

Since $\alpha_n \to 0$, $\rho_n \to 0$, and $\|x_{n+1} - x_n\| \to 0$ and $\{x_n\}$, $\{\nu_n\}$, and $\{\widetilde{\nu}_n\}$ are bounded sequences, it follows from $\nu_j \in (0, 2\zeta_j)$, $j = 1, 2$, that

$$\lim_{n\to\infty}\|F_2\widetilde{\nu}_n - F_2 x^*\| = 0, \qquad \lim_{n\to\infty}\|F_1 w_n - F_1 y^*\| = 0. \tag{95}$$

In the meantime, in terms of the firm nonexpansivity of P_C and the ζ_j-inverse strong monotonicity of F_j for $j = 1, 2$, we obtain from $\nu_j \in (0, 2\zeta_j)$, $j = 1, 2$, and (57) that

$$\begin{aligned}&\|w_n - y^*\|^2 \\ &\quad = \|P_C(I - \nu_2 F_2)\widetilde{\nu}_n - P_C(I - \nu_2 F_2)x^*\|^2 \\ &\quad \le \langle\left(I - \nu_2 F_2\right)\widetilde{\nu}_n - \left(I - \nu_2 F_2\right)x^*, w_n - y^*\rangle \\ &\quad = \frac{1}{2}\left[\|(I - \nu_2 F_2)\widetilde{\nu}_n - (I - \nu_2 F_2)x^*\|^2 + \|w_n - y^*\|^2\right. \\ &\qquad \left. - \|(I - \nu_2 F_2)\widetilde{\nu}_n - (I - \nu_2 F_2)x^* - (w_n - y^*)\|^2\right] \\ &\quad \le \frac{1}{2}\left[\|\widetilde{\nu}_n - x^*\|^2 + \|w_n - y^*\|^2\right. \\ &\qquad \left. - \|(\widetilde{\nu}_n - w_n) - \nu_2(F_2\widetilde{\nu}_n - F_2 x^*) - (x^* - y^*)\|^2\right] \\ &\quad = \frac{1}{2}\left[\|\widetilde{\nu}_n - x^*\|^2 + \|w_n - y^*\|^2 - \|(\widetilde{\nu}_n - w_n) - (x^* - y^*)\|^2\right. \\ &\qquad + 2\nu_2\langle(\widetilde{\nu}_n - w_n) - (x^* - y^*), F_2\widetilde{\nu}_n - F_2 x^*\rangle \\ &\qquad \left. -\nu_2^2\|F_2\widetilde{\nu}_n - F_2 x^*\|^2\right] \\ &\quad \le \frac{1}{2}\left[\|\widetilde{\nu}_n - x^*\|^2 + \|w_n - y^*\|^2 - \|(\widetilde{\nu}_n - w_n) - (x^* - y^*)\|^2\right. \\ &\qquad \left. + 2\nu_2\|(\widetilde{\nu}_n - w_n) - (x^* - y^*)\|\|F_2\widetilde{\nu}_n - F_2 x^*\|\right],\end{aligned}$$

$$
\begin{aligned}
&\left\|\widetilde{w}_n - x^*\right\|^2 \\
&\quad = \left\|P_C\left(I - \nu_1 F_1\right) w_n - P_C\left(I - \nu_1 F_1\right) y^*\right\|^2 \\
&\quad \le \left\langle \left(I - \nu_1 F_1\right) \widetilde{v}_n - \left(I - \nu_1 F_1\right) \widetilde{p}, \widetilde{w}_n - x^* \right\rangle \\
&\quad = \frac{1}{2}\left[\left\|(I - \nu_1 F_1) w_n - (I - \nu_1 F_1) y^*\right\|^2 + \left\|\widetilde{w}_n - x^*\right\|^2 \right. \\
&\qquad \left. - \left\|(I - \nu_1 F_1) w_n - (I - \nu_1 F_1) y^* - (\widetilde{w}_n - x^*)\right\|^2\right] \\
&\quad \le \frac{1}{2}\left[\left\|w_n - y^*\right\|^2 + \left\|\widetilde{w}_n - x^*\right\|^2 \right. \\
&\qquad - \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\|^2 \\
&\qquad + 2\nu_1 \left\langle F_1 w_n - F_1 y^*, (w_n - \widetilde{w}_n) + (x^* - y^*)\right\rangle \\
&\qquad \left. -\nu_1^2 \left\|F_1 w_n - F_1 y^*\right\|^2\right] \\
&\quad \le \frac{1}{2}\left[\left\|\widetilde{v}_n - x^*\right\|^2 + \left\|\widetilde{w}_n - x^*\right\|^2 \right. \\
&\qquad - \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\|^2 \\
&\qquad \left. + 2\nu_1 \left\langle F_1 w_n - F_1 y^*, (w_n - \widetilde{w}_n) + (x^* - y^*)\right\rangle\right].
\end{aligned} \tag{96}
$$

Thus, we have

$$
\begin{aligned}
\left\|w_n - y^*\right\|^2 &\le \left\|\widetilde{v}_n - x^*\right\|^2 - \left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\|^2 \\
&\quad + 2\nu_2 \left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\| \left\|F_2 \widetilde{v}_n - F_2 x^*\right\|,
\end{aligned} \tag{97}
$$

$$
\begin{aligned}
\left\|\widetilde{w}_n - x^*\right\|^2 &\le \left\|\widetilde{v}_n - x^*\right\|^2 - \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\|^2 \\
&\quad + 2\nu_1 \left\|F_1 w_n - F_1 y^*\right\| \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\|,
\end{aligned} \tag{98}
$$

respectively. Consequently, from (56), (73), and (98), it follows that

$$
\begin{aligned}
&\left\|x_{n+1} - x^*\right\|^2 \\
&\quad \le \left(1 - \alpha_n \tau\right) \left\|G\widetilde{v}_n - x^*\right\|^2 - 2\mu\alpha_n \langle \widetilde{A}_2 x^*, x_{n+1} - x^*\rangle \\
&\quad \le \left\|\widetilde{w}_n - x^*\right\|^2 + 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\| \\
&\quad \le \left\|\widetilde{v}_n - x^*\right\|^2 - \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\|^2 \\
&\qquad + 2\nu_1 \left\|F_1 w_n - F_1 y^*\right\| \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\| \\
&\qquad + 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\| \\
&\quad = \left\|v_n - x^* - \rho_n \widetilde{A}_1 v_n\right\|^2 - \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\|^2 \\
&\qquad + 2\nu_1 \left\|F_1 w_n - F_1 y^*\right\| \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\| \\
&\qquad + 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\| \\
&\quad \le \left\|v_n - x^*\right\|^2 + 2\rho_n \left\|\widetilde{A}_1 v_n\right\| \left\|\widetilde{v}_n - x^*\right\| \\
&\qquad - \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\|^2 \\
&\qquad + 2\nu_1 \left\|F_1 w_n - F_1 y^*\right\| \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\| \\
&\qquad + 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\| \\
&\quad \le \left\|x_n - x^*\right\|^2 + 2\rho_n \left\|\widetilde{A}_1 v_n\right\| \left\|\widetilde{v}_n - x^*\right\| \\
&\qquad - \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\|^2 \\
&\qquad + 2\nu_1 \left\|F_1 w_n - F_1 y^*\right\| \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\| \\
&\qquad + 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\|,
\end{aligned} \tag{99}
$$

which yields

$$
\begin{aligned}
&\left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\|^2 \\
&\quad \le \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 + 2\rho_n \left\|\widetilde{A}_1 v_n\right\| \left\|\widetilde{v}_n - x^*\right\| \\
&\qquad + 2\nu_1 \left\|F_1 w_n - F_1 y^*\right\| \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\| \\
&\qquad + 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\| \\
&\quad \le \left\|x_n - x_{n+1}\right\| \left(\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|\right) \\
&\qquad + 2\rho_n \left\|\widetilde{A}_1 v_n\right\| \left\|\widetilde{v}_n - x^*\right\| \\
&\qquad + 2\nu_1 \left\|F_1 w_n - F_1 y^*\right\| \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\| \\
&\qquad + 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\|.
\end{aligned} \tag{100}
$$

Since $\alpha_n \to 0$, $\rho_n \to 0$, and $\|x_{n+1} - x_n\| \to 0$ and $\{x_n\}$, $\{v_n\}$, $\{\widetilde{v}_n\}$, $\{w_n\}$, and $\{\widetilde{w}_n\}$ are bounded sequences, it follows from (95) that

$$
\lim_{n \to \infty} \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\| = 0. \tag{101}
$$

Also, from (56), (57), (73), and (97), it follows that

$$
\begin{aligned}
&\left\|x_{n+1} - x^*\right\|^2 \\
&\quad \le \left(1 - \alpha_n \tau\right) \left\|G\widetilde{v}_n - x^*\right\|^2 - 2\mu\alpha_n \langle \widetilde{A}_2 x^*, x_{n+1} - x^*\rangle \\
&\quad \le \left\|\widetilde{w}_n - x^*\right\|^2 + 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\| \\
&\quad \le \left\|w_n - y^*\right\|^2 + 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\| \\
&\quad \le \left\|\widetilde{v}_n - x^*\right\|^2 - \left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\|^2 \\
&\qquad + 2\nu_2 \left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\| \left\|F_2 \widetilde{v}_n - F_2 x^*\right\|
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\| \\
&= \left\|v_n - x^* - \rho_n \widetilde{A}_1 v_n\right\|^2 - \left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\|^2 \\
&\quad + 2\nu_2 \left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\| \left\|F_2\widetilde{v}_n - F_2 x^*\right\| \\
&\quad + 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\| \\
&\le \left\|v_n - x^*\right\|^2 + 2\rho_n \left\|\widetilde{A}_1 v_n\right\| \left\|\widetilde{v}_n - x^*\right\| \\
&\quad - \left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\|^2 \\
&\quad + 2\nu_2 \left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\| \left\|F_2\widetilde{v}_n - F_2 x^*\right\| \\
&\quad + 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\| \\
&\le \left\|x_n - x^*\right\|^2 + 2\rho_n \left\|\widetilde{A}_1 v_n\right\| \left\|\widetilde{v}_n - x^*\right\| \\
&\quad - \left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\|^2 \\
&\quad + 2\nu_2 \left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\| \left\|F_2\widetilde{v}_n - F_2 x^*\right\| \\
&\quad + 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\|,
\end{aligned} \tag{102}
$$

which leads to

$$
\begin{aligned}
&\left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\|^2 \\
&\le \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 + 2\rho_n \left\|\widetilde{A}_1 v_n\right\| \left\|\widetilde{v}_n - x^*\right\| \\
&\quad + 2\nu_2 \left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\| \left\|F_2\widetilde{v}_n - F_2 x^*\right\| \\
&\quad + 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\| \\
&\le \left\|x_n - x_{n+1}\right\| \left(\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|\right) \\
&\quad + 2\rho_n \left\|\widetilde{A}_1 v_n\right\| \left\|\widetilde{v}_n - x^*\right\| \\
&\quad + 2\nu_2 \left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\| \left\|F_2\widetilde{v}_n - F_2 x^*\right\| \\
&\quad + 2\mu\alpha_n \left\|\widetilde{A}_2 x^*\right\| \left\|x_{n+1} - x^*\right\|.
\end{aligned} \tag{103}
$$

Since $\alpha_n \to 0$, $\rho_n \to 0$, and $\|x_{n+1} - x_n\| \to 0$ and $\{x_n\}$, $\{v_n\}$, $\{\widetilde{v}_n\}$, and $\{w_n\}$ are bounded sequences, it follows from (95) that

$$
\lim_{n\to\infty} \left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\| = 0. \tag{104}
$$

Note that

$$
\begin{aligned}
\left\|\widetilde{v}_n - \widetilde{w}_n\right\| \le\ & \left\|(\widetilde{v}_n - w_n) - (x^* - y^*)\right\| \\
& + \left\|(w_n - \widetilde{w}_n) + (x^* - y^*)\right\|.
\end{aligned} \tag{105}
$$

Hence, from (101) and (104), we get

$$
\lim_{n\to\infty} \left\|\widetilde{v}_n - G\widetilde{v}_n\right\| = \lim_{n\to\infty} \left\|\widetilde{v}_n - \widetilde{w}_n\right\| = 0. \tag{106}
$$

Also, observe that $\|\widetilde{v}_n - x_n\| \le \|v_n - x_n\| + \rho_n\|\widetilde{A}_1 v_n\|$ and

$$
\left\|y_n - x_n\right\| \le \left\|x_{n+1} - x_n\right\| + \mu\alpha_n \left\|\widetilde{A}_2 y_n\right\|. \tag{107}
$$

Hence, from (91), $\alpha_n \to 0$, $\rho_n \to 0$, and $\|x_{n+1} - x_n\| \to 0$, we obtain that

$$
\lim_{n\to\infty} \left\|\widetilde{v}_n - x_n\right\| = 0, \qquad \lim_{n\to\infty} \left\|y_n - x_n\right\| = 0. \tag{108}
$$

So, from (106) and (108), we deduce that

$$
\begin{aligned}
&\left\|W_n\widetilde{v}_n - \widetilde{v}_n\right\| \\
&\quad \le \left\|W_n\widetilde{v}_n - y_n\right\| + \left\|y_n - x_n\right\| + \left\|x_n - \widetilde{v}_n\right\| \\
&\quad = \left\|W_n\widetilde{v}_n - W_n G\widetilde{v}_n\right\| + \left\|y_n - x_n\right\| + \left\|x_n - \widetilde{v}_n\right\| \\
&\quad \le \left\|\widetilde{v}_n - G\widetilde{v}_n\right\| + \left\|y_n - x_n\right\| + \left\|x_n - \widetilde{v}_n\right\| \longrightarrow 0 \\
&\qquad\qquad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{109}
$$

In addition, it is clear that

$$
\left\|\widetilde{v}_n - W\widetilde{v}_n\right\| \le \left\|\widetilde{v}_n - W_n\widetilde{v}_n\right\| + \left\|W_n\widetilde{v}_n - W\widetilde{v}_n\right\|. \tag{110}
$$

Thus, we conclude from [24, Remark 3.2], (3.28,) and the boundedness of $\{\widetilde{v}_n\}$ that

$$
\lim_{n\to\infty} \left\|\widetilde{v}_n - W\widetilde{v}_n\right\| = 0. \tag{111}
$$

Step 4. We prove that $\omega_w(x_n) \subset \Omega$.

Indeed, since H is reflexive and $\{x_n\}$ is bounded, there exists at least a weak convergence subsequence of $\{x_n\}$. Hence, it is known that $\omega_w(x_n) \neq \emptyset$. Now, take an arbitrary $w \in \omega_w(x_n)$. Then, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. From (85)–(89), (91), and (108), we have that $u_{n_i} \rightharpoonup w$, $v_{n_i} \rightharpoonup w$, $\widetilde{v}_{n_i} \rightharpoonup w$, $\Lambda^m_{n_i} u_{n_i} \rightharpoonup w$, and $\Delta^k_{n_i} x_{n_i} \rightharpoonup w$, where $m \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$. Utilizing Lemma 6, we deduce from $\widetilde{v}_{n_i} \rightharpoonup w$, (106), and (111) that $w \in \mathrm{GSVI}(G)$ and $w \in \mathrm{Fix}(W) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ (due to Lemma 8). Next, we prove that $w \in \bigcap_{m=1}^{N} I(B_m, R_m)$. As a matter of fact, since B_m is η_m-inverse strongly monotone, B_m is a monotone and Lipschitz continuous mapping. It follows from Lemma 17 that $R_m + B_m$ is maximal monotone. Let $(v, g) \in G(R_m + B_m)$; that is, $g - B_m v \in R_m v$. Again, since $\Lambda^m_n u_n = J_{R_m, \lambda_{m,n}}(I - \lambda_{m,n} B_m)\Lambda^{m-1}_n u_n$, $n \ge 1$, $m \in \{1, 2, \ldots, N\}$, we have

$$
\Lambda^{m-1}_n u_n - \lambda_{m,n} B_m \Lambda^{m-1}_n u_n \in \left(I + \lambda_{m,n} R_m\right) \Lambda^m_n u_n. \tag{112}
$$

That is,

$$
\frac{1}{\lambda_{m,n}} \left(\Lambda^{m-1}_n u_n - \Lambda^m_n u_n - \lambda_{m,n} B_m \Lambda^{m-1}_n u_n\right) \in R_m \Lambda^m_n u_n. \tag{113}
$$

In terms of the monotonicity of R_m, we get

$$
\begin{aligned}
&\left\langle v - \Lambda^m_n u_n, g - B_m v \right. \\
&\qquad \left. - \frac{1}{\lambda_{m,n}} \left(\Lambda^{m-1}_n u_n - \Lambda^m_n u_n - \lambda_{m,n} B_m \Lambda^{m-1}_n u_n\right) \right\rangle \ge 0
\end{aligned} \tag{114}
$$

and hence

$$\begin{aligned}
&\langle v-\Lambda_n^m u_n, g\rangle \\
&\quad\geq \left\langle v-\Lambda_n^m u_n, B_m v\right. \\
&\qquad \left.+\frac{1}{\lambda_{m,n}}\left(\Lambda_n^{m-1}u_n-\Lambda_n^m u_n-\lambda_{m,n}B_m\Lambda_n^{m-1}u_n\right)\right\rangle \\
&\quad= \left\langle v-\Lambda_n^m u_n, B_m v-B_m\Lambda_n^m u_n+B_m\Lambda_n^m u_n\right. \\
&\qquad \left.-B_m\Lambda_n^{m-1}u_n+\frac{1}{\lambda_{m,n}}\left(\Lambda_n^{m-1}u_n-\Lambda_n^m u_n\right)\right\rangle \\
&\quad\geq \left\langle v-\Lambda_n^m u_n, B_m\Lambda_n^m u_n-B_m\Lambda_n^{m-1}u_n\right\rangle \\
&\qquad+\left\langle v-\Lambda_n^m u_n, \frac{1}{\lambda_{m,n}}\left(\Lambda_n^{m-1}u_n-\Lambda_n^m u_n\right)\right\rangle.
\end{aligned} \tag{115}$$

In particular,

$$\begin{aligned}
&\left\langle v-\Lambda_{n_i}^m u_{n_i}, g\right\rangle \\
&\quad\geq \left\langle v-\Lambda_{n_i}^m u_{n_i}, B_m\Lambda_{n_i}^m u_{n_i}-B_m\Lambda_{n_i}^{m-1}u_{n_i}\right\rangle \\
&\qquad+\left\langle v-\Lambda_{n_i}^m u_{n_i}, \frac{1}{\lambda_{m,n_i}}\left(\Lambda_{n_i}^{m-1}u_{n_i}-\Lambda_{n_i}^m u_{n_i}\right)\right\rangle.
\end{aligned} \tag{116}$$

Since $\|\Lambda_n^m u_n-\Lambda_n^{m-1}u_n\|\to 0$ (due to (85)) and $\|B_m\Lambda_n^m u_n-B_m\Lambda_n^{m-1}u_n\|\to 0$ (due to the Lipschitz continuity of B_m), we conclude from $\Lambda_{n_i}^m u_{n_i}\rightharpoonup w$ and $\{\lambda_{i,n}\}\subset[a_i,b_i]\subset(0,2\eta_i)$ that

$$\lim_{i\to\infty}\left\langle v-\Lambda_{n_i}^m u_{n_i}, g\right\rangle=\langle v-w, g\rangle\geq 0. \tag{117}$$

It follows from the maximal monotonicity of B_m+R_m that $0\in(R_m+B_m)w$; that is, $w\in I(B_m,R_m)$. Therefore, $w\in\bigcap_{m=1}^N I(B_m,R_m)$. Next, we prove that $w\in\bigcap_{k=1}^M \mathrm{GMEP}(\Theta_k,\varphi_k,A_k)$. Since $\Delta_n^k x_n=T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}A_k)\Delta_n^{k-1}x_n$, $n\geq 1$, $k\in\{1,2,\ldots,M\}$, we have

$$\begin{aligned}
&\Theta_k\left(\Delta_n^k x_n, y\right)+\varphi_k(y)-\varphi_k\left(\Delta_n^k x_n\right) \\
&\quad+\left\langle A_k\Delta_n^{k-1}x_n, y-\Delta_n^k x_n\right\rangle \\
&\quad+\frac{1}{r_{k,n}}\left\langle y-\Delta_n^k x_n, \Delta_n^k x_n-\Delta_n^{k-1}x_n\right\rangle\geq 0.
\end{aligned} \tag{118}$$

By (A2), we have

$$\begin{aligned}
&\varphi_k(y)-\varphi_k\left(\Delta_n^k x_n\right)+\left\langle A_k\Delta_n^{k-1}x_n, y-\Delta_n^k x_n\right\rangle \\
&\quad+\frac{1}{r_{k,n}}\left\langle y-\Delta_n^k x_n, \Delta_n^k x_n-\Delta_n^{k-1}x_n\right\rangle\geq\Theta_k\left(y,\Delta_n^k x_n\right).
\end{aligned} \tag{119}$$

Let $z_t=ty+(1-t)w$ for all $t\in(0,1]$ and $y\in C$. This implies that $z_t\in C$. Then, we have

$$\begin{aligned}
&\left\langle z_t-\Delta_n^k x_n, A_k z_t\right\rangle \\
&\quad\geq\varphi_k\left(\Delta_n^k x_n\right)-\varphi_k(z_t)+\left\langle z_t-\Delta_n^k x_n, A_k z_t\right\rangle \\
&\qquad-\left\langle z_t-\Delta_n^k x_n, A_k\Delta_n^{k-1}x_n\right\rangle \\
&\qquad-\left\langle z_t-\Delta_n^k x_n, \frac{\Delta_n^k x_n-\Delta_n^{k-1}x_n}{r_{k,n}}\right\rangle+\Theta_k\left(z_t,\Delta_n^k x_n\right) \\
&\quad=\varphi_k\left(\Delta_n^k x_n\right)-\varphi_k(z_t)+\left\langle z_t-\Delta_n^k x_n, A_k z_t-A_k\Delta_n^k x_n\right\rangle \\
&\qquad+\left\langle z_t-\Delta_n^k x_n, A_k\Delta_n^k x_n-A_k\Delta_n^{k-1}x_n\right\rangle \\
&\qquad-\left\langle z_t-\Delta_n^k x_n, \frac{\Delta_n^k x_n-\Delta_n^{k-1}x_n}{r_{k,n}}\right\rangle+\Theta_k\left(z_t,\Delta_n^k x_n\right).
\end{aligned} \tag{120}$$

By (88), we have $\|A_k\Delta_n^k x_n-A_k\Delta_n^{k-1}x_n\|\to 0$ as $n\to\infty$. Furthermore, by the monotonicity of A_k, we obtain $\langle z_t-\Delta_n^k x_n, A_k z_t-A_k\Delta_n^k x_n\rangle\geq 0$. Then, by (A4), we obtain

$$\langle z_t-w, A_k z_t\rangle\geq\varphi_k(w)-\varphi_k(z_t)+\Theta_k(z_t,w). \tag{121}$$

Utilizing (A1), (A4), and (121), we obtain

$$\begin{aligned}
0&=\Theta_k(z_t,z_t)+\varphi_k(z_t)-\varphi_k(z_t) \\
&\leq t\Theta_k(z_t,y)+(1-t)\Theta_k(z_t,w) \\
&\quad+t\varphi_k(y)+(1-t)\varphi_k(w)-\varphi_k(z_t) \\
&\leq t\left[\Theta_k(z_t,y)+\varphi_k(y)-\varphi_k(z_t)\right]+(1-t)\langle z_t-w, A_k z_t\rangle \\
&=t\left[\Theta_k(z_t,y)+\varphi_k(y)-\varphi_k(z_t)\right] \\
&\quad+(1-t)t\langle y-w, A_k z_t\rangle,
\end{aligned} \tag{122}$$

and hence

$$0\leq\Theta_k(z_t,y)+\varphi_k(y)-\varphi_k(z_t)+(1-t)\langle y-w, A_k z_t\rangle. \tag{123}$$

Letting $t\to 0$, we have, for each $y\in C$,

$$0\leq\Theta_k(w,y)+\varphi_k(y)-\varphi_k(w)+\langle y-w, A_k w\rangle. \tag{124}$$

This implies that $w\in\mathrm{GMEP}(\Theta_k,\varphi_k,A_k)$ and hence $w\in\bigcap_{k=1}^M\mathrm{GMEP}(\Theta_k,\varphi_k,A_k)$. Thus, $w\in\Omega=\bigcap_{n=1}^\infty\mathrm{Fix}(T_n)\cap\bigcap_{k=1}^M\mathrm{GMEP}(\Theta_k,\varphi_k,A_k)\cap\bigcap_{m=1}^N I(B_m,R_m)$. Consequently, $w\in\bigcap_{n=1}^\infty\mathrm{Fix}(T_n)\cap\bigcap_{k=1}^M\mathrm{GMEP}(\Theta_k,\varphi_k,A_k)\cap\bigcap_{m=1}^N I(B_m,R_m)\cap\mathrm{GSVI}(G)=:\Omega$. This shows that $\omega_w(x_n)\subset\Omega$.

Step 5. We prove that $\omega_w(x_n)\subset\mathrm{VI}(\Omega,\widetilde{A}_1)$ provided $\|x_n-y_n\|=o(\rho_n)$ additionally.

Indeed, take an arbitrary $w \in \omega_w(x_n)$. Then, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. Utilizing the arguments similar to those of (56) and (57), we can readily obtain that for all $p \in \Omega$

$$\|v_n - p\| \le \|x_n - p\|, \qquad \|G\widetilde{v}_n - p\| \le \|\widetilde{v}_n - p\|. \tag{125}$$

Since A_1 is α-inverse strongly monotone, from (48) and (125) we conclude that for all $p \in \Omega$

$$\begin{aligned}
&\|y_n - p\|^2 \\
&= \|W_n G\widetilde{v}_n - p\|^2 \\
&\le \|\widetilde{v}_n - p\|^2 \\
&= \left\|v_n - p - \rho_n \widetilde{A}_1 v_n\right\|^2 \\
&\le \|v_n - p\|^2 - 2\rho_n \langle \widetilde{A}_1 v_n, \widetilde{v}_n - p\rangle \\
&= \|v_n - p\|^2 - 2\rho_n \left(\left\langle \widetilde{A}_1 v_n, v_n - p\right\rangle + \left\langle \widetilde{A}_1 v_n, \widetilde{v}_n - v_n\right\rangle\right) \\
&= \|v_n - p\|^2 - 2\rho_n \left(\left\langle \widetilde{A}_1 v_n - \widetilde{A}_1 p, v_n - p\right\rangle \right. \\
&\qquad \left. + \left\langle \widetilde{A}_1 p, v_n - p\right\rangle + \left\langle \widetilde{A}_1 v_n, \widetilde{v}_n - v_n\right\rangle\right) \\
&\le \|x_n - p\|^2 - 2\rho_n \left\langle \widetilde{A}_1 p, v_n - p\right\rangle \\
&\quad + 2\rho_n \left\|\widetilde{A}_1 v_n\right\| \|\widetilde{v}_n - v_n\|,
\end{aligned} \tag{126}$$

which implies that

$$\begin{aligned}
&\langle \widetilde{A}_1 p, v_n - p\rangle \\
&\quad \le \frac{1}{2\rho_n}\left(\|x_n - p\|^2 - \|y_n - p\|^2\right) + \left\|\widetilde{A}_1 v_n\right\| \|\widetilde{v}_n - v_n\| \\
&\quad \le \frac{\|x_n - y_n\|}{2\rho_n}\left(\|x_n - p\| + \|y_n - p\|\right) + \rho_n \left\|\widetilde{A}_1 v_n\right\|^2.
\end{aligned} \tag{127}$$

So, from $\rho_n \to 0$ and the assumption $\|x_n - y_n\| = o(\rho_n)$, we get

$$\limsup_{n\to\infty} \langle \widetilde{A}_1 p, v_n - p\rangle \le 0. \tag{128}$$

Thus, it follows from (91) that for all $p \in \Omega$

$$\begin{aligned}
\left\langle \widetilde{A}_1 p, w - p\right\rangle &= \lim_{i\to\infty} \left\langle \widetilde{A}_1 p, x_{n_i} - p\right\rangle \\
&\le \limsup_{n\to\infty} \left\langle \widetilde{A}_1 p, x_n - p\right\rangle \\
&= \limsup_{n\to\infty} \left(\left\langle \widetilde{A}_1 p, v_n - p\right\rangle + \left\langle \widetilde{A}_1 p, x_n - v_n\right\rangle\right) \\
&= \limsup_{n\to\infty} \left\langle \widetilde{A}_1 p, v_n - p\right\rangle \\
&\le 0.
\end{aligned} \tag{129}$$

That is,

$$\left\langle \widetilde{A}_1 p, p - w\right\rangle \ge 0, \quad \forall p \in \Omega. \tag{130}$$

Since $\widetilde{A}_1$ is α-inverse strongly monotone, by Minty's Lemma [16], we know that (130) is equivalent to the VIP

$$\left\langle \widetilde{A}_1 w, p - w\right\rangle \ge 0, \quad \forall p \in \Omega. \tag{131}$$

This shows that $w \in \mathrm{VI}(\Omega, \widetilde{A}_1)$. Therefore, $\omega_w(x_n) \subset \mathrm{VI}(\Omega, \widetilde{A}_1)$. □

Theorem 19. *Assume that there hold all the conditions in Theorem 18. Then, we have*

(i) $\{x_n\}$ *converges strongly to a point* $u^* \in \Omega$, *which is a unique solution of the VIP*

$$\langle \widetilde{A}_2 u^*, p - u^*\rangle \ge 0, \quad \forall p \in \Omega; \tag{132}$$

(ii) $\{x_n\}$ *converges strongly to a unique solution of THVI (19) provided* $\|x_n - y_n\| = o(\theta_n)$ *additionally.*

Proof. Since $\widetilde{A}_2$ is β-strongly monotone and L-Lipschitz continuous, there exists a unique solution $u^* \in \Omega$ of the VIP

$$\langle \widetilde{A}_2 u^*, p - u^*\rangle \ge 0, \quad \forall p \in \Omega. \tag{133}$$

Now, let us show that

$$\limsup_{n\to\infty} \langle \widetilde{A}_2 u^*, u^* - x_n\rangle \le 0. \tag{134}$$

Since $\{x_n\}$ is bounded, we may assume, without loss of generality, that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$ and

$$\begin{aligned}
&\limsup_{n\to\infty} \left\langle \widetilde{A}_2 u^*, u^* - x_n\right\rangle \\
&\quad = \lim_{i\to\infty} \langle \widetilde{A}_2 u^*, u^* - x_{n_i}\rangle = \langle \widetilde{A}_2 u^*, u^* - w\rangle.
\end{aligned} \tag{135}$$

In terms of Theorem 18 (ii), we know that $w \in \omega_w(x_n) \subset \Omega$. So, from (133), it follows that

$$\limsup_{n\to\infty} \left\langle \widetilde{A}_2 u^*, u^* - x_n\right\rangle = \langle \widetilde{A}_2 u^*, u^* - w\rangle \le 0. \tag{136}$$

Next, let us show that $\lim_{n\to\infty}\|x_n - u^*\| = 0$. In fact, utilizing Lemma 4, from (48) and (125) with $p = u^*$, we get

$$\begin{aligned}
&\|x_{n+1} - u^*\|^2 \\
&\quad = \left\|(I - \mu\alpha_n \widetilde{A}_2) W_n G\widetilde{v}_n - u^*\right\|^2 \\
&\quad = \left\|\left(I - \mu\alpha_n \widetilde{A}_2\right) W_n G\widetilde{v}_n - \left(I - \mu\alpha_n \widetilde{A}_2\right) W_n G u^* \right. \\
&\qquad \left. + (I - \mu\alpha_n \widetilde{A}_2) W_n G u^* - W_n G u^*\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\le \left\|\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nG\widetilde{v}_n-\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nGu^*\right\|^2\\
&\quad-2\mu\alpha_n\langle\widetilde{A}_2u^*,x_{n+1}-u^*\rangle\\
&\le(1-\alpha_n\tau)\left\|G\widetilde{v}_n-u^*\right\|^2-2\mu\alpha_n\left\langle\widetilde{A}_2u^*,x_{n+1}-u^*\right\rangle\\
&\le(1-\alpha_n\tau)\left\|\widetilde{v}_n-u^*\right\|^2-2\mu\alpha_n\left\langle\widetilde{A}_2u^*,x_{n+1}-u^*\right\rangle\\
&=(1-\alpha_n\tau)\left\|v_n-u^*-\rho_n\widetilde{A}_1v_n\right\|^2\\
&\quad-2\mu\alpha_n\langle\widetilde{A}_2u^*,x_{n+1}-u^*\rangle\\
&\le(1-\alpha_n\tau)\left(\left\|v_n-u^*\right\|^2-2\rho_n\left\langle\widetilde{A}_1v_n,\widetilde{v}_n-u^*\right\rangle\right)\\
&\quad-2\mu\alpha_n\left\langle\widetilde{A}_2u^*,x_{n+1}-u^*\right\rangle\\
&\le(1-\alpha_n\tau)\left\|x_n-u^*\right\|^2+2\rho_n\left\|\widetilde{A}_1v_n\right\|\left\|\widetilde{v}_n-u^*\right\|\\
&\quad+2\mu\alpha_n\langle\widetilde{A}_2u^*,u^*-x_{n+1}\rangle\\
&=(1-\alpha_n\tau)\left\|x_n-u^*\right\|^2\\
&\quad+\alpha_n\tau\cdot\frac{2}{\tau}\left\{\frac{\rho_n}{\alpha_n}\left\|\widetilde{A}_1v_n\right\|\left\|\widetilde{v}_n-u^*\right\|\right.\\
&\qquad\left.+\mu\left\langle\widetilde{A}_2u^*,u^*-x_{n+1}\right\rangle\right\},
\end{aligned}
\tag{137}
$$

where $\tau=1-\sqrt{1-\mu(2\beta-\mu L^2)}$. Since $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\lim_{n\to\infty}(\rho_n/\alpha_n)=0$, and $\limsup_{n\to\infty}\langle\widetilde{A}_2u^*,u^*-x_{n+1}\rangle\le 0$ (due to (136)), we deduce that $\sum_{n=1}^{\infty}\alpha_n\tau=\infty$ and

$$
\begin{aligned}
&\limsup_{n\to\infty}\frac{2}{\tau}\left\{\frac{\rho_n}{\alpha_n}\left\|\widetilde{A}_1v_n\right\|\left\|\widetilde{v}_n-u^*\right\|+\mu\left\langle\widetilde{A}_2u^*,u^*-x_{n+1}\right\rangle\right\}\\
&\le 0.
\end{aligned}
\tag{138}
$$

Therefore, applying Lemma 10 to (137), we infer that $\lim_{n\to\infty}\|x_n-u^*\|=0$.

Finally, we prove that $\lim_{n\to\infty}\|x_n-x^*\|=0$ provided $\|x_n-y_n\|=o(\alpha_n)$ additionally, where $\{x^*\}=\mathrm{VI}(\mathrm{VI}(\Omega,\widetilde{A}_1),\widetilde{A}_2)$.

Indeed, first of all, let us show that $\omega_w(x_n)=\{x^*\}$. As a matter of fact, take an arbitrary $w\in\omega_w(x_n)$. Then, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}\rightharpoonup w$. Moreover, by Theorem 18 (iii), we know that $w\in\omega_w(x_n)\subset\mathrm{VI}(\Omega,\widetilde{A}_1)$. Utilizing the arguments similar to those of (56) and (57), we can readily obtain that for all $p\in\mathrm{VI}(\Omega,\widetilde{A}_1)$

$$
\left\|v_n-p\right\|\le\left\|x_n-p\right\|,\qquad\left\|G\widetilde{v}_n-p\right\|\le\left\|\widetilde{v}_n-p\right\|.
\tag{139}
$$

Utilizing Lemma 4, from (48) and (139), we deduce that for all $p\in\mathrm{VI}(\Omega,\widetilde{A}_1)$

$$
\begin{aligned}
&\left\|x_{n+1}-p\right\|^2\\
&\quad=\left\|(I-\mu\alpha_n\widetilde{A}_2)W_nG\widetilde{v}_n-p\right\|^2\\
&\quad=\left\|\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nG\widetilde{v}_n-\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nGp\right.\\
&\qquad\left.+(I-\mu\alpha_n\widetilde{A}_2)W_nGp-W_nGp\right\|^2\\
&\quad\le\left\|\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nG\widetilde{v}_n-\left(I-\mu\alpha_n\widetilde{A}_2\right)W_nGp\right\|^2\\
&\qquad-2\mu\alpha_n\langle\widetilde{A}_2p,x_{n+1}-p\rangle\\
&\quad\le(1-\alpha_n\tau)\left\|G\widetilde{v}_n-p\right\|^2-2\mu\alpha_n\langle\widetilde{A}_2p,x_{n+1}-p\rangle\\
&\quad\le(1-\alpha_n\tau)\left\|\widetilde{v}_n-p\right\|^2-2\mu\alpha_n\langle\widetilde{A}_2p,x_{n+1}-p\rangle\\
&\quad=(1-\alpha_n\tau)\left\|v_n-p-\rho_n\widetilde{A}_1v_n\right\|^2-2\mu\alpha_n\langle\widetilde{A}_2p,x_{n+1}-p\rangle\\
&\quad\le(1-\alpha_n\tau)\left(\left\|v_n-p\right\|^2-2\rho_n\left\langle\widetilde{A}_1v_n,\widetilde{v}_n-p\right\rangle\right)\\
&\qquad-2\mu\alpha_n\langle\widetilde{A}_2p,x_{n+1}-p\rangle\\
&\quad\le\left\|v_n-p\right\|^2-2(1-\alpha_n\tau)\rho_n\langle\widetilde{A}_1v_n,\widetilde{v}_n-p\rangle\\
&\qquad-2\mu\alpha_n\langle\widetilde{A}_2p,x_{n+1}-p\rangle\\
&\quad\le\left\|x_n-p\right\|^2-2(1-\alpha_n\tau)\rho_n\langle\widetilde{A}_1v_n,\widetilde{v}_n-p\rangle\\
&\qquad-2\mu\alpha_n\langle\widetilde{A}_2p,x_{n+1}-p\rangle,
\end{aligned}
\tag{140}
$$

where $\tau=1-\sqrt{1-\mu(2\beta-\mu L^2)}$. So, it follows that

$$
\begin{aligned}
&\langle\widetilde{A}_2p,x_{n+1}-p\rangle\\
&\quad\le\frac{1}{2\mu\alpha_n}\left(\left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2\right)\\
&\qquad-\frac{(1-\alpha_n\tau)\rho_n}{\mu\alpha_n}\langle\widetilde{A}_1v_n,\widetilde{v}_n-p\rangle\\
&\quad\le\frac{\left\|x_n-x_{n+1}\right\|}{2\mu\alpha_n}\left(\left\|x_n-p\right\|+\left\|x_{n+1}-p\right\|\right)\\
&\qquad+\frac{(1-\alpha_n\tau)\rho_n}{\mu\alpha_n}\left\|\widetilde{A}_1v_n\right\|\left\|\widetilde{v}_n-p\right\|.
\end{aligned}
\tag{141}
$$

Since $\alpha_n\to 0$, $\rho_n=o(\alpha_n)$, and $\|x_{n+1}-x_n\|=o(\rho_n)$, we find that

$$
\lim_{n\to\infty}\frac{\left\|x_{n+1}-x_n\right\|}{\alpha_n}=\lim_{n\to\infty}\frac{\left\|x_{n+1}-x_n\right\|}{\rho_n}\cdot\frac{\rho_n}{\alpha_n}=0.
\tag{142}
$$

Hence, we conclude from (141) that for all $p \in \mathrm{VI}(\Omega, \widetilde{A}_1)$

$$\begin{aligned}
&\langle \widetilde{A}_2 p, w - p\rangle \\
&\quad = \lim_{j\to\infty} \left\langle \widetilde{A}_2 p, x_{n_j} - p \right\rangle \\
&\quad \le \limsup_{n\to\infty} \left\langle \widetilde{A}_2 p, x_n - p \right\rangle \\
&\quad = \limsup_{n\to\infty} \left(\left\langle \widetilde{A}_2 p, x_n - x_{n+1} \right\rangle + \left\langle \widetilde{A}_2 p, x_{n+1} - p \right\rangle \right) \\
&\quad = \limsup_{n\to\infty} \left\langle \widetilde{A}_2 p, x_{n+1} - p \right\rangle \\
&\quad \le 0.
\end{aligned} \tag{143}$$

That is,

$$\langle \widetilde{A}_2 p, p - w\rangle \ge 0, \quad \forall p \in \mathrm{VI}\left(\Omega, \widetilde{A}_1\right). \tag{144}$$

Since $\widetilde{A}_2$ is β-strongly monotone and L-Lipschitz continuous, by Minty's Lemma [16], we know that (144) is equivalent to the VIP

$$\langle \widetilde{A}_2 w, p - w\rangle \ge 0, \quad \forall p \in \mathrm{VI}\left(\Omega, \widetilde{A}_1\right). \tag{145}$$

This shows that $w \in \mathrm{VI}(\mathrm{VI}(\Omega, \widetilde{A}_1), \widetilde{A}_2)$. Taking into account $\{x^*\} = \mathrm{VI}(\mathrm{VI}(\Omega, \widetilde{A}_1), \widetilde{A}_2)$, we know that $w = x^*$. Thus, $\omega_w(x_n) = \{x^*\}$; that is, $x_n \rightharpoonup x^*$.

Next we prove that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. As a matter of fact, utilizing (139) and (140) with $p = x^*$, we get

$$\begin{aligned}
&\|x_{n-1} - x^*\|^2 \\
&\quad \le (1 - \alpha_n\tau)\left(\|v_n - x^*\|^2 - 2\rho_n \left\langle \widetilde{A}_1 v_n, \widetilde{v}_n - x^* \right\rangle\right) \\
&\qquad - 2\mu\alpha_n \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle \\
&\quad \le (1 - \alpha_n\tau)\|x_n - x^*\|^2 - 2\rho_n \left\|\widetilde{A}_1 v_n\right\| \|\widetilde{v}_n - x^*\| \\
&\qquad + 2\mu\alpha_n \left\langle \widetilde{A}_2 x^*, x^* - x_{n+1} \right\rangle \\
&\quad = (1 - \alpha_n\tau)\|x_n - x^*\|^2 \\
&\qquad + \alpha_n\tau \cdot \frac{2}{\tau}\left\{ \frac{\rho_n}{\alpha_n} \left\|\widetilde{A}_1 v_n\right\| \|\widetilde{v}_n - x^*\| \right. \\
&\qquad\qquad \left. + \mu \left\langle \widetilde{A}_2 x^*, x^* - x_{n+1} \right\rangle \right\}.
\end{aligned} \tag{146}$$

Since $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}(\rho_n/\alpha_n) = 0$, and $\lim_{n\to\infty}\langle \widetilde{A}_2 u^*, u^* - x_{n+1}\rangle = 0$ (due to $x_n \rightharpoonup x^*$), we deduce that $\sum_{n=1}^{\infty}\alpha_n\tau = \infty$ and

$$\lim_{n\to\infty}\frac{2}{\tau}\left\{ \frac{\rho_n}{\alpha_n} \left\|\widetilde{A}_1 v_n\right\| \|\widetilde{v}_n - u^*\| + \mu \left\langle \widetilde{A}_2 u^*, u^* - x_{n+1} \right\rangle \right\} = 0. \tag{147}$$

Therefore, applying Lemma 10 to (146), we infer that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. This completes the proof. □

Remark 20. It is obvious that iterative scheme (48) is very different from Yao et al. iterative one (14) and Zeng et al. iterative one in Algorithm ZWY. Here, the two-step iterative scheme in [11, Algorithm 3.2] is extended to develop four-step iterative scheme (48) for the THVI (19) by combining Korpelevich's extragradient method, hybrid steepest-descent method, and Mann's iteration method. It is worth pointing out that under the lack of the assumptions similar to those in [8, Theorem 3.2], for example, $\{x_n\}$ is bounded, $\mathrm{Fix}(T) \cap \mathrm{int}\, C \ne \emptyset$, and $\|x - Tx\| \ge k\mathrm{Dist}(x, \mathrm{Fix}(T))$, $\forall x \in C$, for some $k > 0$, the sequence $\{x_n\}$ generated by (48) converges strongly to a point $u^* \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \bigcap_{k=1}^{M}\mathrm{GMEP}(\Theta_k, \varphi_k, A_k) \cap \bigcap_{i=1}^{N} I(B_i, R_i) \cap \mathrm{GSVI}(G) =: \Omega$, which is a unique solution of the VIP: $\langle \widetilde{A}_2 u^*, p - u^*\rangle \ge 0$, $\forall p \in \Omega$.

Remark 21. Theorems 18 and 19 improve, extend, supplement, and develop Yao et al. [8, Theorems 3.1 and 3.2] and Zeng et al. [11, Theorem 3.2] in the following aspects.

(a) THVI (19) with the unique solution $x^* \in \Omega$ satisfying

$$\begin{aligned}
x^* = {} & P_{\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \bigcap_{k=1}^{M}\mathrm{GMEP}(\Theta_k,\varphi_k,A_k) \cap \bigcap_{i=1}^{N} I(B_i,R_i) \cap \mathrm{GSVI}(G)} \\
& \cdot \left(I - \widetilde{A}_1\right) x^*
\end{aligned} \tag{148}$$

is more general than the problem of finding a point $\widetilde{x} \in C$ satisfying $\widetilde{x} = P_{\mathrm{Fix}(T)} S\widetilde{x}$ in [8] and than the problem of finding a point $x^* \in \bigcap_{i=1}^{N}\mathrm{Fix}(T_i)$ satisfying $x^* = P_{\bigcap_{i=1}^{N}\mathrm{Fix}(T_i)}(I - A_1)x^*$ in [11, Theorem 3.2]. It is worth pointing out that S is nonexpansive if and only if the complement $I - S$ is $(1/2)$-inverse strongly monotone; see [25].

(b) Four-step iterative scheme (48) for THVI (19) is more flexible, more advantageous, and more subtle than Zeng et al. two-step iterative one in [11, Algorithm 3.2] and than Yao et al. two-step iterative one (14) because it can be used to solve several kinds of problems, for example, the THVI, the hierarchical VIP, and the problem of finding a common point of four sets: $\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$, $\bigcap_{k=1}^{M}\mathrm{GMEP}(\Theta_k, \varphi_k, A_k)$, $\bigcap_{i=1}^{N} I(B_i, R_i)$, and $\mathrm{GSVI}(G)$. In addition, it also drops the crucial requirements that $\mathrm{Fix}(T) \cap \mathrm{int}\, C \ne \emptyset$ and $\|x - Tx\| \ge k\mathrm{Dist}(x, \mathrm{Fix}(T))$, $\forall x \in C$, for some $k > 0$ in [8, Theorem 3.2 (v)].

(c) The argument techniques in Theorems 18 and 19 are very different from the argument ones in [8, Theorems 3.1 and 3.2] and from the argument ones in [11, Theorem 3.2] because we use the W-mapping approach to fixed points of infinitely many nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ (see Lemmas 7 and 8), the properties of resolvent operators and maximal monotone mappings, the fixed point equation $x^* = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x^*$ equivalent to the GSVI (10) (see Proposition CWY), and the contractive coefficient estimates for the contractions associated with nonexpansive mappings (see Lemma 9).

(d) Compared with the restrictions on the parameter sequences of [8, Theorem 19] and [11, Theorem 3.2], respectively, the hypotheses (H3)–(H5) in Theorem 18 are additionally added because Theorem 18 involves the quite complex problem, that is, the THVI (19) (over the set $\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$)

with constraints of several problems: finitely many GMEPs, finitely many variational inclusions, and GSVI (10).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR for the technical and financial support.

References

[1] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Paris, Farnce, 1969.

[2] G. M. Korpelevich, "The extragradient method for finding saddle points and other problems," *Matecon*, vol. 12, pp. 747–756, 1976.

[3] J. W. Peng and J. C. Yao, "A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems," *Taiwanese Journal of Mathematics*, vol. 12, no. 6, pp. 1401–1432, 2008.

[4] L. C. Ceng, S. M. Guu, and J. C. Yao, "Hybrid viscosity CQ method for finding a common solution of a variational inequality, a general system of variational inequalities, and a fixed point problem," *Fixed Point Theory and Applications*, vol. 2013, article 313, 2013.

[5] Y. Yao, Y. C. Liou, and S. M. Kang, "Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3472–3480, 2010.

[6] L.-C. Ceng, C.-Y. Wang, and J.-C. Yao, "Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities," *Mathematical Methods of Operations Research*, vol. 67, no. 3, pp. 375–390, 2008.

[7] R. U. Verma, "On a new system of nonlinear variational inequalities and associated iterative algorithms," *Mathematical Sciences Research Hot-Line*, vol. 3, no. 8, pp. 65–68, 1999.

[8] Y. Yao, Y.-C. Liou, and G. Marino, "Two-step iterative algorithms for hierarchical fixed point problems and variational inequality problems," *Journal of Applied Mathematics and Computing*, vol. 31, no. 1-2, pp. 433–445, 2009.

[9] H. Iiduka, "Strong convergence for an iterative method for the triple-hierarchical constrained optimization problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 12, pp. e1292–e1297, 2009.

[10] H. Iiduka, "Iterative algorithm for solving triple-hierarchical constrained optimization problem," *Journal of Optimization Theory and Applications*, vol. 148, no. 3, pp. 580–592, 2011.

[11] L. C. Zeng, M. M. Wong, and J. C. Yao, "Strong convergence of relaxed hybrid steepest-descent methods for triple hierarchical constrained optimization," *Fixed Point Theory and Applications*, vol. 2012, article 29, 24 pages, 2012.

[12] A. Latif, A. E. Al-Mazrooei, B. A. B. Dehaish, and J. C. Yao, "Hybrid viscosity approximation methods for general systems of variational inequalities in Banach spaces," *Fixed Point Theory and Applications*, vol. 2013, article 258, 2013.

[13] L. C. Ceng, A. Latif, Q. H. Ansari, and J.-C. Yao, "Hybrid extragradient method for hierarchical variational inequalities," *Fixed Point Theory and Applications*, vol. 2014, article 222, 2014.

[14] Q. H. Ansari, L.-C. Ceng, and H. Gupta, *Triple Hierarchical Variational Inequalities in Nonlinear Analysis: Approximation Theory, Optimization and Applications*, Springer, New York, NY, USA, 2014.

[15] L.-C. Ceng and J.-C. Yao, "A hybrid iterative scheme for mixed equilibrium problems and fixed point problems," *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 186–201, 2008.

[16] K. Goebel and W. A. Kirk, *Topics on Metric Fixed-Point Theory*, Cambridge University Press, Cambridge, UK, 1990.

[17] J. G. O'Hara, P. Pillay, and H. K. Xu, "Iterative approaches to convex feasibility problems in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 9, pp. 2022–2042, 2006.

[18] H. K. Xu and T. H. Kim, "Convergence of hybrid steepest-descent methods for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 119, no. 1, pp. 185–201, 2003.

[19] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization*, vol. 14, no. 5, pp. 877–898, 1976.

[20] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Groningen, The Netherlands, 1976.

[21] N.-J. Huang, "A new completely general class of variational inclusions with noncompact valued mappings," *Computers & Mathematics with Applications*, vol. 35, no. 10, pp. 9–14, 1998.

[22] L.-C. Ceng, Q. H. Ansari, M. M. Wong, and J.-C. Yao, "Mann type hybrid extragradient method for variational inequalities, variational inclusions and fixed point problems," *Fixed Point Theory*, vol. 13, no. 2, pp. 403–422, 2012.

[23] L.-C. Zeng, S.-M. Guu, and J.-C. Yao, "Characterization of H-monotone operators with applications to variational inclusions," *Computers & Mathematics with Applications*, vol. 50, no. 3-4, pp. 329–337, 2005.

[24] Y. Yao, Y.-C. Liou, and J.-C. Yao, "Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2007, Article ID 64363, 12 pages, 2007.

[25] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.

Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 468536, 5 pages
http://dx.doi.org/10.1155/2015/468536

Research Article

A Lyapunov-Type Inequality for a Fractional Differential Equation under a Robin Boundary Condition

Mohamed Jleli,[1] Lakhdar Ragoub,[2] and Bessem Samet[1]

[1] *Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia*
[2] *Department of Mathematics, College of Computing and Information Systems, Al Yamamah University, Riyadh, Saudi Arabia*

Correspondence should be addressed to Bessem Samet; bessem.samet@gmail.com

Received 13 July 2014; Accepted 31 August 2014

Academic Editor: Calogero Vetro

Copyright © 2015 Mohamed Jleli et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We establish a new Lyapunov-type inequality for a class of fractional differential equations under Robin boundary conditions. The obtained inequality is used to obtain an interval where a linear combination of certain Mittag-Leffler functions has no real zeros.

1. Introduction and Preliminaries

Let the following $\chi : [a,b] \to \mathbb{R}$ be a continuous function. The well-known Lyapunov inequality [1] states that a necessary condition for the boundary value problem

$$\begin{aligned} &u''(t) + \chi(t)\,u(t) = 0, \quad a < t < b \\ &u(a) = u(b) = 0 \end{aligned} \tag{1}$$

to have nontrivial solutions is that

$$\int_a^b |\chi(t)|\,dt > \frac{4}{b-a}. \tag{2}$$

This result found many practical applications in differential and difference equations (oscillation theory, disconjugacy, eigenvalue problems, etc.); see, for instance, [2–7]. On the other hand, many improvements of (2) have been carried out, and similar inequalities have been obtained for other types of differential equations; compare to the Pachpatte monograph [8]. The search for Lyapunov-type inequalities in which the starting differential equation is constructed via fractional differential operators has begun very recently. For example, in [9], a Lyapunov-type inequality was obtained for differential equations depending on the Riemann-Liouville fractional derivative; that is, for the boundary value problem

$$\begin{aligned} &\left({}_aD^{\alpha} u\right)(t) + \chi(t)\,u(t) = 0, \quad a < t < b, 1 < \alpha \le 2, \\ &u(a) = u(b) = 0, \end{aligned} \tag{3}$$

where ${}_aD^{\alpha}$ denotes the Riemann-Liouville fractional derivative of order α. Precisely, the author proved that if (3) has a nontrivial solution, then we have

$$\int_a^b |\chi(t)|\,dt > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{4}$$

Clearly, if we let $\alpha = 2$ in (4), one obtains Lyapunov's classical inequality (2).

In [10], a Lyapunov-type inequality was obtained for the Caputo fractional boundary value problem

$$\begin{aligned} &\left({}_a^{C}D^{\alpha} u\right)(t) + \chi(t)\,u(t) = 0, \quad a < t < b, 1 < \alpha \le 2, \\ &u(a) = u(b) = 0, \end{aligned} \tag{5}$$

where ${}_a^{C}D^{\alpha}$ is the Caputo fractional derivative of order α. It was proved that if (5) has a nontrivial solution, then we have

$$\int_a^b |\chi(t)|\,dt > \frac{\Gamma(\alpha)\,\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{6}$$

Similarly, if we let $\alpha = 2$ in (6), one obtains Lyapunov's classical inequality (2).

Motivated by the above works, we consider in this paper a Caputo fractional differential equation under Robin boundary conditions. More precisely, we consider the boundary value problem

$$\begin{aligned} &\left({}_a^C D^{\alpha} u\right)(t) + \chi(t)\, u(t) = 0, \quad a < t < b, 1 < \alpha \le 2, \\ &u(a) - u'(a) = u(b) + u'(b) = 0, \end{aligned} \tag{7}$$

and we get a corresponding Lyapunov-type inequality. This result is then used to obtain a real interval where a linear combination of certain Mittag-Leffler functions has no (real) zeros.

Before presenting the main results, let us start by recalling the concepts of the Riemann-Liouville fractional integral and the Caputo fractional derivative of order $\alpha \ge 0$. For more details, we refer to [11].

Definition 1. Let $\alpha \ge 0$ and let f be a real function defined on $[a, b]$. The Riemann-Liouville fractional integral of order α is defined by $({}_aI^0 f) \equiv f$ and

$$\left({}_aI^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \quad \alpha > 0, t \in [a, b]. \tag{8}$$

Definition 2. The Caputo fractional derivative of order $\alpha \ge 0$ is defined by $({}_a^C D^0 f) \equiv f$ and $({}_a^C D^{\alpha} f)(t) = ({}_aI^{m-\alpha} D^m f)(t)$ for $\alpha > 0$, where m is the smallest integer greater than or equal to α.

The following result is standard within the fractional calculus theory involving the Caputo differential operator (see [7]).

Lemma 3. *One has that $u \in C[a, b]$ is a solution to (7) if and only if*

$$u(t) = c_0 + c_1 (t-a) - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \chi(s)\, u(s)\, ds, \tag{9}$$

where c_0 and c_1 are some real constants.

Now, we are ready to state and prove our main results.

2. Main Results

2.1. A Lyapunov-Type Inequality. At first, we consider the following notations:

$$\begin{aligned} \gamma &= b - a + 2, \\ h_1(t, s) &= 1 + t - a - \frac{\gamma (t-s)^{\alpha-1}}{(b-s)^{\alpha-1} + (\alpha-1)(b-s)^{\alpha-2}}, \\ h_2(t, s) &= 1 + t - a. \end{aligned} \tag{10}$$

Now, let us write problem (7) in its equivalent integral form.

Lemma 4. *One has that $u \in C[a, b]$ is a solution to (7) if and only if u satisfies the integral equation*

$$u(t) = \int_a^b G(t, s)\, \chi(s)\, u(s)\, ds, \tag{11}$$

where

$$G(t, s) = \frac{(b-s)^{\alpha-2} (b - s + \alpha - 1)}{\gamma \Gamma(\alpha)} H(t, s), \tag{12}$$

$$H(t, s) = \begin{cases} h_1(t, s), & a \le s \le t \le b, \\ h_2(t, s), & a \le t \le s \le b. \end{cases} \tag{13}$$

Proof. From Lemma 3, we have

$$u(t) = c_0 + c_1 (t-a) - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \chi(s)\, u(s)\, ds, \tag{14}$$

where c_0 and c_1 are some real constants. Then,

$$u'(t) = c_1 - \frac{1}{\Gamma(\alpha)} \int_a^t (\alpha-1)(t-s)^{\alpha-2} \chi(s)\, u(s)\, ds. \tag{15}$$

Since $u(a) - u'(a) = 0$, we obtain

$$c_0 = c_1. \tag{16}$$

From the boundary condition $u(b) + u'(b) = 0$, we get

$$c_1 = \frac{1}{\gamma \Gamma(\alpha)} \int_a^b \left((\alpha-1)(b-s)^{\alpha-2} + (b-s)^{\alpha-1}\right) \chi(s)\, u(s)\, ds. \tag{17}$$

Using (14), (16), and (17), we obtain the desired result. □

Lemma 5. *For all $(t, s) \in [a, b] \times (a, b)$, one has*

$$|H(t, s)| \le \max\left\{1 + b - a, \frac{2-\alpha}{\alpha-1}(b-a) - 1\right\}. \tag{18}$$

Proof. It is easy to see that, for $a \le t \le s \le b$, we have

$$0 \le h_2(t, s) \le 1 + b - a. \tag{19}$$

On the other hand, for $a \le s < t \le b$, we have

$$\frac{\partial h_1}{\partial t}(t, s) = 1 - \frac{(\alpha-1)\gamma (t-s)^{\alpha-2}}{(b-s)^{\alpha-1} + (\alpha-1)(b-s)^{\alpha-2}}. \tag{20}$$

Hence,

$$\lim_{t \to s^+} \frac{\partial h_1}{\partial t}(t, s) \in [-\infty, 0). \tag{21}$$

Now, for fixed s in (a, b), we want to study the variation of the function $t \mapsto h_1(t, s)$ for t in $[s, b]$. First, we have

$$\left.\frac{\partial h_1}{\partial t}(t, s)\right|_{t=b} = 1 - \frac{(\alpha-1)\gamma}{\alpha - 1 + b - s}. \tag{22}$$

Let

$$a^* = b + (1-\gamma)(\alpha-1). \tag{23}$$

We distinguish two eventual cases according to the value of a^*. □

Case 1. If $a^* \le a$, in this case, we have

$$\frac{\partial h_1}{\partial t}(t,s)\Big|_{t=b} \le 0, \quad s \in (a,b). \tag{24}$$

From (20), (21), and (24), we deduce

$$\frac{\partial h_1}{\partial t}(t,s) \le 0, \quad s < t. \tag{25}$$

This yields

$$\begin{aligned} h_1(b,s) &= 1 + b - a - \frac{\gamma(b-s)}{b-s+\alpha-1} \\ &\le h_1(t,s) \le h_1(s,s) \le 1 + b - a. \end{aligned} \tag{26}$$

Observe that, in this case, we have

$$\frac{2-\alpha}{\alpha-1} \le \frac{1}{b-a}. \tag{27}$$

Using the above inequality, we obtain

$$\begin{aligned} &(1+b-a)(b-s) + (\alpha-1)(1+b-a) - \gamma(b-s) \\ &\quad = s - b + (\alpha-1)(1+b-a) \\ &\quad \ge (b-a)(\alpha-2) + \alpha - 1 \\ &\quad \ge 0, \end{aligned} \tag{28}$$

which implies that

$$h_1(b,s) \ge 0. \tag{29}$$

From (26) and (29), we deduce

$$0 \le h_1(t,s) \le 1 + b - a. \tag{30}$$

Case 2. If $a < a^* \le b$, in this case, we have two possibilities.

(i) If $a^* \le s < b$, in this case, we have

$$\frac{\partial h_1}{\partial t}(t,s)\Big|_{t=b} \le 0. \tag{31}$$

Therefore, we conclude that

$$h_1(b,s) \le h_1(t,s) \le h_1(s,s) \le 1 + b - a. \tag{32}$$

On the other hand, we have

$$\begin{aligned} &(1+b-a)(b-s+\alpha-1) - \gamma(b-s) \\ &\quad = \left(1+b-a-\gamma\right)(b-s) + (\alpha-1)(1+b-a) \\ &\quad = s - b + (\alpha-1)(1+b-a) \\ &\quad \ge \left(2-\gamma+b-a\right)(\alpha-1) \ge 0, \end{aligned} \tag{33}$$

which implies that

$$h_1(b,s) \ge 0. \tag{34}$$

From (32) and (34), we deduce

$$0 \le h_1(t,s) \le 1 + b - a. \tag{35}$$

(ii) If $a < s < a^*$, in this case, we have

$$\frac{\partial h_1}{\partial t}(t,s)\Big|_{t=b} > 0. \tag{36}$$

Hence, there would exist $t^* \in (s,b)$ such that

$$\frac{\partial h_1}{\partial t}(t,s)\Big|_{t=t^*} = 0. \tag{37}$$

As mentioned above, it is easy to verify that $h_1(s,s) \ge 0$ and $h_1(b,s) \le 0$. This yields

$$h_1(t^*,s) \le h_1(b,s) \le h_1(s,s) \le \frac{r}{p} + b - a. \tag{38}$$

Then,

$$\left|h_1(t,s)\right| \le \max\left\{-h_1(t^*,s), 1+b-a\right\}. \tag{39}$$

Observe that $(\partial h_1/\partial t)(t,s)|_{t=t^*} = 0$ is equivalent to

$$(\alpha-1)\gamma(t^*-s)^{\alpha-2} = (b-s)^{\alpha-1} + (\alpha-1)(b-s)^{\alpha-2}. \tag{40}$$

Therefore, we get

$$\begin{aligned} h_1(t^*,s) &= \frac{\alpha-2}{\alpha-1}t^* + \frac{s}{\alpha-1} + 1 - a \\ &\ge \frac{\alpha-2}{\alpha-1}b + \frac{a}{\alpha-1} + 1 - a \\ &= (b-a)\frac{\alpha-2}{\alpha-1} + 1. \end{aligned} \tag{41}$$

Finally, using the above inequality and (39), we obtain

$$\left|h_1(t,s)\right| \le \max\left\{1+b-a, \frac{2-\alpha}{\alpha-1}(b-a) - 1\right\}, \tag{42}$$

which makes end to the proof.

Our first main result is as follows.

Theorem 6. *If (7) admits a nontrivial continuous solution, then*

$$\begin{aligned} &\int_a^b (b-s)^{\alpha-2}(b-s+\alpha-1)\left|\chi(s)\right| ds \\ &\quad \ge \frac{\gamma\Gamma(\alpha)}{\max\left\{b-a+1, \left((2-\alpha)/(\alpha-1)\right)(b-a)-1\right\}}. \end{aligned} \tag{43}$$

Proof. Let $B = C[a,b]$ be the Banach space endowed with norm

$$\|x\|_\infty = \max_{a\le t\le b} |x(t)|, \quad x \in B. \tag{44}$$

From Lemma 4, for all $t \in [a,b]$, we have

$$u(t) = \frac{1}{\gamma\Gamma(\alpha)} \times \int_a^b (b-s)^{\alpha-2} (b-s+\alpha-1) H(t,s) \chi(s) u(s)\, ds. \tag{45}$$

Now, an application of Lemma 3 yields

$$\|u\|_\infty \le \|u\|_\infty \frac{\max\{b-a+1, ((2-\alpha)/(\alpha-1))(b-a)-1\}}{\gamma\Gamma(\alpha)} \times \int_a^b (b-s)^{\alpha-2} (b-s+\alpha-1) \left|\chi(s)\right| ds, \tag{46}$$

from which the above theorem follows. □

2.2. Nonexistence Result of Real Zeros for a Linear Combination of Certain Mittag-Leffler Functions. Let $\alpha, \beta > 0$ be fixed. The complex function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad \alpha > 0, \beta > 0, z \in \mathbb{C} \tag{47}$$

is analytic in the whole complex plane; it will be referred to [12–14] as the Mittag-Leffler function with parameters (α, β).

At this stage, using the above Lyapunov-type inequality, we give an interval where a linear combination of Mittag-Leffler functions has no real zeros. More precisely, we prove

Theorem 7. *Let* $1 < \alpha \le 2$. *Then* $E_{\alpha,2}(z) + E_{\alpha,1}(z) + zE_{\alpha,\alpha}(z)$ *has no real zeros for*

$$z \in \left(\frac{-3\alpha\Gamma(\alpha)}{(1+\alpha)\max\{2, ((2-\alpha)/(\alpha-1))-1\}}, 0 \right]. \tag{48}$$

Proof. Let $(a,b) = (0,1)$, and consider the fractional Sturm-Liouville eigenvalue problem

$$\begin{aligned} \left({}_0^C D^\alpha u\right)(t) + \lambda u(t) &= 0, \quad 0 < t < 1, \\ u(a) - u'(a) &= u(b) + u'(b) = 0. \end{aligned} \tag{49}$$

The real values of λ, for which there exists a non-trivial solution to (49), are called eigenvalues of (49); and the corresponding solutions are called eigenfunctions.

As established in [15], the eigenvalues of (49) must be positive; moreover, the positive number λ is an eigenvalue of (49) if and only if

$$E_{\alpha,2}(-\lambda) + E_{\alpha,1}(-\lambda) - \lambda E_{\alpha,\alpha}(-\lambda) = 0. \tag{50}$$

Thanks to Theorem 6, if a positive real eigenvalue λ of (49) exists, then

$$\lambda \int_0^1 (1-s)^{\alpha-2} (\alpha-s)\, ds \ge \frac{3\Gamma(\alpha)}{\max\{2, ((2-\alpha)/(\alpha-1))-1\}}. \tag{51}$$

Hence,

$$\lambda\left(\frac{\alpha+1}{\alpha}\right) \ge \frac{3\Gamma(\alpha)}{\max\{2, ((2-\alpha)/(\alpha-1))-1\}}, \tag{52}$$

which concludes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All the authors contributed equally and significantly to writing this paper. All authors read and approved the final paper.

Acknowledgment

The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for its funding of this research through the Research Group Project no. RG-1435-034.

References

[1] A. M. Liapunov, "Problème général de la stabilité du mouvement," *Annals of Mathematics Studies*, vol. 2, pp. 203–407, 1907.

[2] D. Cakmak, "Lyapunov-type integral inequalities for certain higher order differential equations," *Applied Mathematics and Computation*, vol. 216, no. 2, pp. 368–373, 2010.

[3] S. Clark and D. Hinton, "A Liapunov inequality for linear Hamiltonian systems," *Mathematical Inequalities & Applications*, vol. 1, no. 2, pp. 201–209, 1998.

[4] N. Parhi and S. Panigrahi, "Liapunov-type inequality for higher order differential equations," *Mathematica Slovaca*, vol. 52, no. 1, pp. 31–46, 2002.

[5] X. Yang, "On Liapunov-type inequality for certain higher-order differential equations," *Applied Mathematics and Computation*, vol. 134, no. 2-3, pp. 307–317, 2003.

[6] X. Yang and K. Lo, "Lyapunov-type inequality for a class of even-order differential equations," *Applied Mathematics and Computation*, vol. 215, no. 11, pp. 3884–3890, 2010.

[7] Q.-M. Zhang and X. H. Tang, "Lyapunov inequalities and stability for discrete linear Hamiltonian systems," *Applied Mathematics and Computation*, vol. 218, no. 2, pp. 574–582, 2011.

[8] B. G. Pachpatte, *Mathematical Inequalities*, North Holland Mathematical Library, Elsevier, 2005.

[9] R. A. C. Ferreira, "A Lyapunov-type inequality for a fractional boundary value problem," *Fractional Calculus and Applied Analysis*, vol. 16, no. 4, pp. 978–984, 2013.

[10] R. A. Ferreira, "On a Lyapunov-type inequality and the zeros of a certain Mittag-LEFfler function," *Journal of Mathematical Analysis and Applications*, vol. 412, no. 2, pp. 1058–1063, 2014.

[11] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, The Netherlands, 2006.

[12] F. Mainardi, "Fractional relaxation-oscillation and fractional diffusion-wave phenomena," *Chaos, Solitons and Fractals*, vol. 7, no. 9, pp. 1461–1477, 1996.

[13] F. Mainardi and R. Gorenflo, "On Mittag-Leffler-type functions in fractional evolution processes," *Journal of Computational and Applied Mathematics*, vol. 118, no. 1-2, pp. 283–299, 2000.

[14] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[15] J.-S. Duan, Z. Wang, Y.-L. Liu, and X. Qiu, "Eigenvalue problems for fractional ordinary differential equations," *Chaos, Solitons & Fractals*, vol. 46, pp. 46–53, 2013.

Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 491423, 8 pages
http://dx.doi.org/10.1155/2015/491423

Research Article

Two Positive Solutions of Third-Order BVP with Integral Boundary Condition and Sign-Changing Green's Function

Bing-Wei Niu,[1] Jian-Ping Sun,[1] and Qiu-Yan Ren[2]

[1]*Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, China*
[2]*College of Technology and Engineering, Lanzhou University of Technology, Lanzhou, Gansu 730050, China*

Correspondence should be addressed to Jian-Ping Sun; jpsun2012@163.com

Received 25 August 2014; Accepted 19 November 2014

Academic Editor: Mohamed-Aziz Taoudi

Copyright © 2015 Bing-Wei Niu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We are concerned with the following third-order boundary value problem with integral boundary condition: $u'''(t) = f(t, u(t))$, $t \in [0, 1]$, $u'(0) = u(1) = 0$, $u''(\eta) + \int_\alpha^\beta u(t)dt = 0$, where $1/2 < \alpha \le \beta \le 1, \alpha + \beta \le 4/3$, and $\eta \in (1/2, \alpha]$. Although the corresponding Green's function is sign-changing, we still obtain the existence of at least two positive and decreasing solutions under some suitable conditions on f by using the two-fixed-point theorem due to Avery and Henderson. An example is also included to illustrate the main results obtained.

1. Introduction

Third-order differential equations arise from a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant or varying cross-section, a three-layer beam, electromagnetic waves or gravity driven flows, and so on [1].

Recently, the existence of single or multiple positive solutions for some third-order three-point boundary value problems (BVPs for short) and for some third-order BVPs with integral boundary conditions has received much attention from many authors; see [2–15] and the references therein. It is necessary to point out that all the abovementioned papers are achieved when the corresponding Green's functions are nonnegative, which is a very important condition.

In 2008, Palamides and Smyrlis [16] studied the existence of at least one positive solution to the singular third-order three-point BVP with an indefinitely signed Green's function:

$$u'''(t) = a(t) f(t, u(t)), \quad t \in (0, 1), \qquad u(0) = u(1) = u''(\eta) = 0, \tag{1}$$

where $\eta \in (17/24, 1)$. Their technique was a combination of the Guo-Krasnoselskii fixed point theorem and properties of the corresponding vector field.

Inspired greatly by [16], Sun and Zhao discussed the third-order three-point BVP with sign-changing Green's function:

$$u'''(t) = f(t, u(t)), \quad t \in [0, 1], \qquad u'(0) = u(1) = u''(\eta) = 0. \tag{2}$$

They proved the existence of at least one or three positive solutions when $\eta \in (1/2, 1)$ by using the Guo-Krasnoselskii and Leggett-Williams fixed point theorems in [17, 18] and obtained the existence of a positive solution when $\eta \in [2 - \sqrt{2}, 1)$ by using iterative technique in [19]. For similar results, one can refer to [20, 21].

It is worth mentioning that there are other types of works on sign-changing Green's functions which prove the existence of sign-changing solutions, positive in some cases; see Infante and Webb's papers [22–24].

In this paper, we consider the following third-order BVP involving integral boundary condition and sign-changing Green's function:

$$u'''(t) = f(t, u(t)), \quad t \in [0, 1], \qquad u'(0) = u(1) = 0, \quad u''(\eta) + \int_\alpha^\beta u(t)\,dt = 0. \tag{3}$$

Throughout this paper, we always assume that $1/2 < \alpha \le \beta \le 1, \alpha + \beta \le 4/3$, and $\eta \in (1/2, \alpha]$. By imposing suitable conditions on f, we obtain the existence of at least two positive and decreasing solutions for the BVP (3). The main tool used is the well-known Avery-Henderson two-fixed-point theorem.

To end this section, we state some fundamental definitions and the two-fixed-point theorem due to Avery and Henderson [25].

Let P be a cone in a real Banach space E.

Definition 1. A functional $\psi : P \to R$ is said to be increasing on P provided $\psi(x) \le \psi(y)$ for all $x, y \in P$ with $x \le y$, where $x \le y$ if and only if $y - x \in P$.

Definition 2. Let $\gamma : P \to [0, +\infty)$ be continuous. For each $d > 0$, one defines the set

$$P(\gamma, d) = \{u \in P : \gamma(u) < d\}. \tag{4}$$

Theorem 3. *Let ψ and γ be increasing, nonnegative, and continuous functionals on P, and let ω be a nonnegative continuous functional on P with $\omega(0) = 0$ such that, for some $c > 0$ and $M > 0$,*

$$\gamma(u) \le \omega(u) \le \psi(u), \qquad \|u\| \le M\gamma(u) \tag{5}$$

for all $u \in \overline{P(\gamma, c)}$. Suppose there exist a completely continuous operator $A : \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that

$$\omega(\lambda u) \le \lambda \omega(u) \quad \text{for } 0 \le \lambda \le 1,\ u \in \partial P(\omega, b), \tag{6}$$

and

(1) *$\gamma(Au) > c$ for all $u \in \partial P(\gamma, c)$;*

(2) *$\omega(Au) < b$ for all $u \in \partial P(\omega, b)$;*

(3) *$P(\psi, a) \ne \emptyset$, and $\psi(Au) > a$ for all $u \in \partial P(\psi, a)$.*

Then A has at least two fixed points u_1 and u_2 in $\overline{P(\gamma, c)}$ such that

$$\begin{aligned} &a < \psi(u_1) \quad \text{with } \omega(u_1) < b, \\ &b < \omega(u_2) \quad \text{with } \gamma(u_2) < c. \end{aligned} \tag{7}$$

2. Main Results

For any $y \in C[0, 1]$, we consider the BVP

$$\begin{aligned} &u'''(t) = y(t), \quad t \in [0, 1], \\ &u'(0) = u(1) = 0, \qquad u''(\eta) + \int_\alpha^\beta u(t)\, dt = 0. \end{aligned} \tag{8}$$

After a direct computation, one may obtain the expression of Green's function $G(t, s)$ of the BVP (8) as follows:

$$\begin{aligned} G(t, s) = &-\frac{3(1 - t^2)}{\rho} \int_\alpha^\beta K(\tau, s)\, d\tau - K(t, s) \\ &+ \begin{cases} 0, & s \ge \eta, \\ \dfrac{3(1 - t^2)}{\rho}, & s < \eta, \end{cases} \end{aligned} \tag{9}$$

where

$$\begin{aligned} \rho &= 6 - 3(\beta - \alpha) + (\beta^3 - \alpha^3), \\ K(t, s) &= \frac{1}{2} \begin{cases} (1 - t)(1 + t - 2s), & 0 \le s \le t \le 1, \\ (1 - s)^2, & 0 \le t \le s \le 1. \end{cases} \end{aligned} \tag{10}$$

Remark 4. Consider the following: $\rho > 0$.

Remark 5. Consider the following: $0 \le K(t, s) \le (1/2)(1 - s)^2$ for $(t, s) \in [0, 1] \times [0, 1]$.

Remark 6. $G(t, s)$ has the following properties:

$$G(t, s) \ge 0 \quad \text{for } 0 \le s < \eta, \qquad G(t, s) \le 0 \quad \text{for } \eta \le s \le 1. \tag{11}$$

Moreover, for $s \ge \eta$,

$$\begin{aligned} &\max\{G(t, s) : t \in [0, 1]\} = G(1, s) = 0, \\ &\min\{G(t, s) : t \in [0, 1]\} \\ &\quad = G(0, s) = -\frac{3}{\rho} \int_\alpha^\beta K(\tau, s)\, d\tau - \frac{1}{2}(1 - s)^2 \end{aligned} \tag{12}$$

and, for $s < \eta$,

$$\begin{aligned} &\max\{G(t, s) : t \in [0, 1]\} \\ &\quad = G(0, s) = \frac{3}{\rho} - \frac{3}{\rho} \int_\alpha^\beta K(\tau, s)\, d\tau - \frac{1}{2}(1 - s)^2, \\ &\min\{G(t, s) : t \in [0, 1]\} = G(1, s) = 0. \end{aligned} \tag{13}$$

In the remainder of this paper, we always assume that $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous and satisfies the following two conditions:

($C1$) for each $u \in [0, +\infty)$, the mapping $t \mapsto f(t, u)$ is decreasing;

($C2$) for each $t \in [0, 1]$, the mapping $u \mapsto f(t, u)$ is increasing.

Let real Banach space $C[0, 1]$ be equipped with the norm $\|u\| = \max_{t \in [0,1]} |u(t)|$ and

$$\begin{aligned} P = \Big\{ &u \in C[0, 1] : u(t) \text{ is nonnegative} \\ &\text{and decreasing on } [0, 1], \\ &\text{and } \min_{t \in [1-\theta, \theta]} u(t) \ge \theta^* \|u\| \Big\}, \end{aligned} \tag{14}$$

where $\theta \in (1/2, \eta)$ and $\theta^* = (\eta - \theta)/\eta$. Then P is a cone in $C[0, 1]$. Now, we define an operator A as follows:

$$(Au)(t) = \int_0^1 G(t, s) f(s, u(s))\, ds, \quad u \in P,\ t \in [0, 1]. \tag{15}$$

Obviously, if u is a fixed point of A in P, then u is a nonnegative and decreasing solution of the BVP (3).

Lemma 7. *$A : P \to P$ is completely continuous.*

Proof. Let $u \in P$. Then, for $t \in [0, \eta]$, we have

$$
\begin{aligned}
&(Au)(t)\\
&= \int_0^t \left[-\frac{3\left(1-t^2\right)}{\rho} \int_\alpha^\beta K(\tau, s)\, d\tau \right.\\
&\qquad \left. -\frac{1}{2}(1-t)(1+t-2s) + \frac{3\left(1-t^2\right)}{\rho} \right]\\
&\qquad \times f(s, u(s))\, ds\\
&\quad + \int_t^\eta \left[-\frac{3\left(1-t^2\right)}{\rho} \int_\alpha^\beta K(\tau, s)\, d\tau \right.\\
&\qquad \left. -\frac{1}{2}(1-s)^2 + \frac{3\left(1-t^2\right)}{\rho} \right]\\
&\qquad \times f(s, u(s))\, ds\\
&\quad + \int_\eta^1 \left[-\frac{3\left(1-t^2\right)}{\rho} \int_\alpha^\beta K(\tau, s)\, d\tau - \frac{1}{2}(1-s)^2 \right]\\
&\qquad \times f(s, u(s))\, ds.
\end{aligned}
\tag{16}
$$

In view of $u \in P$, $1/2 < \eta \le \alpha, \beta + \alpha \le 4/3$, Remark 5, and (C1) and (C2), we get

$$
\begin{aligned}
&(Au)'(t)\\
&= \int_0^t \left[\frac{6t}{\rho} \int_\alpha^\beta K(\tau, s)\, d\tau + t - s - \frac{6t}{\rho} \right] f(s, u(s))\, ds\\
&\quad + \frac{6t}{\rho} \int_t^\eta \left[\int_\alpha^\beta K(\tau, s)\, d\tau - 1 \right] f(s, u(s))\, ds\\
&\quad + \frac{6t}{\rho} \int_\eta^1 \int_\alpha^\beta K(\tau, s)\, d\tau f(s, u(s))\, ds\\
&\le \int_0^t \left[\frac{3t}{\rho} \int_\alpha^\beta (1-\tau)(1+\tau-2s)\, d\tau + t - s - \frac{6t}{\rho} \right]\\
&\qquad \times f(s, u(s))\, ds\\
&\quad + \frac{3t}{\rho} \int_t^\eta \left[\int_\alpha^\beta (1-\tau)(1+\tau-2s)\, d\tau - 2 \right] f(s, u(s))\, ds\\
&\quad + \frac{3t(\beta-\alpha)}{\rho} \int_\eta^1 (1-s)^2 f(s, u(s))\, ds\\
&= \frac{1}{\rho} \left\{ -\left[3t(\beta-\alpha)(2-\beta-\alpha) + \rho \right] \right.\\
&\qquad \times \int_0^t sf(s, u(s))\, ds\\
&\qquad - t \int_t^\eta \left[\rho + 3s(\beta-\alpha)(2-\beta-\alpha) \right] f(s, u(s))\, ds\\
&\qquad \left. +3t(\beta-\alpha) \int_\eta^1 (1-s)^2 f(s, u(s))\, ds \right\}\\
&\le \frac{f(\eta, u(\eta))}{\rho}\\
&\quad \times \left\{ -\left[3t(\beta-\alpha)(2-\beta-\alpha) + \rho \right] \right.\\
&\qquad \times \int_0^t s\, ds - t \int_t^\eta \left[\rho + 3s(\beta-\alpha)(2-\beta-\alpha) \right] ds\\
&\qquad \left. +\ 3t(\beta-\alpha) \int_\eta^1 (1-s)^2\, ds \right\}\\
&= \frac{tf(\eta, u(\eta))}{2\rho}\\
&\quad \times \left[-\rho(2\eta - t) - 3(\beta-\alpha)(2-\beta-\alpha)\eta^2 \right.\\
&\qquad \left. +2(\beta-\alpha)(1-\eta)^3 \right]\\
&\le \frac{tf(\eta, u(\eta))}{2\rho}\\
&\quad \times \left[-3(\beta-\alpha)(2-\beta-\alpha)\eta^2 + 2(\beta-\alpha)(1-\eta)^2 \right]\\
&= \frac{(\beta-\alpha)tf(\eta, u(\eta))}{2\rho}\\
&\quad \times \left\{ \left[3(\beta+\alpha) - 4 \right] \eta^2 + 2(1-2\eta) \right\}\\
&\le 0, \quad t \in [0, \eta],\\
&(Au)''(t)\\
&= \int_0^t \left[\frac{6}{\rho} \int_\alpha^\beta K(\tau, s)\, d\tau + 1 - \frac{6}{\rho} \right] f(s, u(s))\, ds\\
&\quad + \frac{6}{\rho} \int_t^\eta \left[\int_\alpha^\beta K(\tau, s)\, d\tau - 1 \right] f(s, u(s))\, ds\\
&\quad + \frac{6}{\rho} \int_\eta^1 \int_\alpha^\beta K(\tau, s)\, d\tau f(s, u(s))\, ds\\
&\le \int_0^t \left[\frac{3}{\rho} \int_\alpha^\beta (1-\tau)(1+\tau-2s)\, d\tau + 1 - \frac{6}{\rho} \right] f(s, u(s))\, ds\\
&\quad + \frac{3}{\rho} \int_t^\eta \left[\int_\alpha^\beta (1-\tau)(1+\tau-2s)\, d\tau - 2 \right] f(s, u(s))\, ds\\
&\quad + \frac{3(\beta-\alpha)}{\rho} \int_\eta^1 (1-s)^2 f(s, u(s))\, ds
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\rho}\left\{-3(\beta-\alpha)(2-\beta-\alpha)\int_0^t sf(s,u(s))\,ds\right.\\
&\quad -\int_t^\eta \left[\rho+3s(\beta-\alpha)(2-\beta-\alpha)\right] f(s,u(s))\,ds\\
&\quad \left.+\,3(\beta-\alpha)\int_\eta^1 (1-s)^2 f(s,u(s))\,ds\right\}\\
&\le \frac{f(\eta,u(\eta))}{\rho}\\
&\quad\times\left\{-3(\beta-\alpha)(2-\beta-\alpha)\int_0^t s\,ds\right.\\
&\quad -\int_t^\eta \left[\rho+3s(\beta-\alpha)(2-\beta-\alpha)\right] ds\\
&\quad \left.+\,3(\beta-\alpha)\int_\eta^1 (1-s)^2\,ds\right\}\\
&= \frac{f(\eta,u(\eta))}{2\rho}\\
&\quad\times\left[-2\rho(\eta-t)-3(\beta-\alpha)(2-\beta-\alpha)\eta^2\right.\\
&\quad \left.+2(\beta-\alpha)(1-\eta)^3\right]\\
&\le \frac{(\beta-\alpha)f(\eta,u(\eta))}{2\rho}\\
&\quad\times\left[-3(2-\beta-\alpha)\eta^2+2(1-\eta)^2\right]\\
&= \frac{(\beta-\alpha)f(\eta,u(\eta))}{2\rho}\\
&\quad\times\left\{\left[3(\beta+\alpha)-4\right]\eta^2+2(1-2\eta)\right\}\\
&\le 0, \quad t\in[0,\eta].
\end{aligned}
\tag{17}
$$

For $t\in(\eta,1]$, we have

$$
\begin{aligned}
&(Au)(t)\\
&= \int_0^\eta\left[-\frac{3\left(1-t^2\right)}{\rho}\int_\alpha^\beta K(\tau,s)\,d\tau-\frac{1}{2}(1-t)(1+t-2s)\right.\\
&\qquad \left.+\frac{3\left(1-t^2\right)}{\rho}\right] f(s,u(s))\,ds\\
&\quad +\int_\eta^t\left[-\frac{3\left(1-t^2\right)}{\rho}\int_\alpha^\beta K(\tau,s)\,d\tau\right.\\
&\qquad \left.-\frac{1}{2}(1-t)(1+t-2s)\right] f(s,u(s))\,ds\\
&\quad +\int_t^1\left[-\frac{3\left(1-t^2\right)}{\rho}\int_\alpha^\beta K(\tau,s)\,d\tau\right.\\
&\qquad \left.-\frac{1}{2}(1-s)^2\right] f(s,u(s))\,ds.
\end{aligned}
\tag{18}
$$

In view of $u\in P$, $1/2<\eta\le\alpha,\beta+\alpha\le 4/3$, Remark 5, and $(C1)$ and $(C2)$, we get

$$
\begin{aligned}
&(Au)'(t)\\
&= \int_0^\eta\left[\frac{6t}{\rho}\int_\alpha^\beta K(\tau,s)\,d\tau+t-s-\frac{6t}{\rho}\right] f(s,u(s))\,ds\\
&\quad +\int_\eta^t\left[\frac{6t}{\rho}\int_\alpha^\beta K(\tau,s)\,d\tau+t-s\right] f(s,u(s))\,ds\\
&\quad +\frac{6t}{\rho}\int_t^1\int_\alpha^\beta K(\tau,s)\,d\tau f(s,u(s))\,ds\\
&\le \int_0^\eta\left[\frac{3t}{\rho}\int_\alpha^\beta(1-\tau)(1+\tau-2s)\,d\tau\right.\\
&\qquad \left.+t-s-\frac{6t}{\rho}\right] f(s,u(s))\,ds\\
&\quad +\int_\eta^t\left[\frac{3t(\beta-\alpha)}{\rho}(1-s)^2+t-s\right] f(s,u(s))\,ds\\
&\quad +\frac{3t(\beta-\alpha)}{\rho}\int_t^1(1-s)^2 f(s,u(s))\,ds\\
&= \frac{1}{\rho}\left\{-\left[3t(\beta-\alpha)(2-\beta-\alpha)+\rho\right]\int_0^\eta sf(s,u(s))\,ds\right.\\
&\qquad +3t(\beta-\alpha)\int_\eta^1(1-s)^2 f(s,u(s))\,ds\\
&\qquad \left.+\rho\int_\eta^t(t-s)f(s,u(s))\,ds\right\}\\
&\le \frac{f(\eta,u(\eta))}{\rho}\\
&\quad\times\left\{-\left[3t(\beta-\alpha)(2-\beta-\alpha)+\rho\right]\int_0^\eta s\,ds\right.\\
&\qquad +3t(\beta-\alpha)\int_\eta^1(1-s)^2\,ds\\
&\qquad \left.+\rho\int_\eta^t(t-s)\,ds\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{tf(\eta, u(\eta))}{2\rho} \\
&\times \left[-3(\beta-\alpha)(2-\beta-\alpha)\eta^2 + 2(\beta-\alpha)(1-\eta)^3 \right. \\
&\qquad \left. +\rho(t-2\eta)\right] \\
&\le \frac{tf(\eta, u(\eta))}{2\rho} \\
&\times \left[-3(\beta-\alpha)(2-\beta-\alpha)\eta^2 + 2(\beta-\alpha)(1-\eta)^2 \right. \\
&\qquad \left. +\rho(1-2\eta)\right] \\
&= \frac{tf(\eta, u(\eta))}{2\rho} \\
&\times \left\{(\beta-\alpha)\left[3(\beta+\alpha)-4\right]\eta^2 + (1-2\eta)\left[2(\beta-\alpha)+\rho\right]\right\} \\
&\le 0, \quad t \in (\eta, 1].
\end{aligned} \tag{19}
$$

So, it follows from $(Au)'(t) \le 0$ for $t \in [0,1]$ that $(Au)(t)$ is decreasing on $[0,1]$, which together with $(Au)(1) = 0$ implies that $(Au)(t) \ge 0, t \in [0,1]$. At the same time, since $(Au)''(t) \le 0$ for $t \in [0,\eta]$, we know that $(Au)(t)$ is concave on $[0,\eta]$. Therefore,

$$
\begin{aligned}
(Au)(t) &\ge \left(1 - \frac{t}{\eta}\right)(Au)(0) + \frac{t}{\eta}(Au)(\eta) \\
&\ge \frac{\eta - t}{\eta}\|Au\|, \quad t \in [0,\eta].
\end{aligned} \tag{20}
$$

Consequently,

$$
\min_{t\in[1-\theta,\theta]}(Au)(t) = (Au)(\theta) \ge \frac{\eta-\theta}{\eta}\|Au\| = \theta^*\|Au\|. \tag{21}
$$

This shows that $A : P \to P$. Furthermore, although $G(t,s)$ is not continuous, it follows from known textbook results, for example, [26], that $A : P \to P$ is completely continuous. □

Lemma 8. *For any $u \in P$ and $t \in [1-\theta, \theta]$,*

$$
\begin{aligned}
&\int_0^{1-\theta} G(t,s) f(s,u(s))\,ds + \int_\theta^\eta G(t,s) f(s,u(s))\,ds \\
&\quad + \int_\eta^1 G(t,s) f(s,u(s))\,ds \ge 0.
\end{aligned} \tag{22}
$$

Proof. Let $u \in P$. Then it follows from $\theta < \eta \le \alpha$, $\beta + \alpha \le 4/3$, Remarks 5 and 6, and $(C1)$ and $(C2)$ that, for any $t \in [1-\theta, \theta]$,

$$
\begin{aligned}
&\int_0^{1-\theta} G(t,s) f(s,u(s))\,ds \\
&\quad + \int_\theta^\eta G(t,s) f(s,u(s))\,ds + \int_\eta^1 G(t,s) f(s,u(s))\,ds \\
&\ge \int_0^{1-\theta} G(t,s) f(s,u(s))\,ds + \int_\eta^1 G(t,s) f(s,u(s))\,ds \\
&\ge f(\eta,u(\eta))\left[\int_0^{1-\theta} G(t,s)\,ds + \int_\eta^1 G(t,s)\,ds\right] \\
&= f(\eta,u(\eta)) \\
&\times \left\{\int_0^{1-\theta}\left[-\frac{3\left(1-t^2\right)}{\rho}\int_\alpha^\beta K(\tau,s)\,d\tau \right.\right. \\
&\qquad \left. -\frac{1}{2}(1-t)(1+t-2s) + \frac{3\left(1-t^2\right)}{\rho}\right] ds \\
&\qquad \left. -\int_\eta^1\left[\frac{3\left(1-t^2\right)}{\rho}\int_\alpha^\beta K(\tau,s)\,d\tau + \frac{1}{2}(1-s)^2\right] ds\right\} \\
&\ge f(\eta,u(\eta)) \\
&\times \left\{\int_0^{1-\theta}\left[-\frac{3\left(1-t^2\right)}{2\rho}\int_\alpha^\beta (1-\tau)(1+\tau-2s)\,d\tau \right.\right. \\
&\qquad \left. -\frac{1}{2}(1-t)(1+t-2s) + \frac{3\left(1-t^2\right)}{\rho}\right] ds \\
&\qquad \left. -\frac{3\left(1-t^2\right)(\beta-\alpha)+\rho}{2\rho}\int_\eta^1 (1-s)^2\,ds\right\} \\
&= \frac{f(\eta,u(\eta))}{2\rho} \\
&\times \left\{\frac{3}{2}\left(1-t^2\right)(\beta-\alpha)(2-\beta-\alpha)(1-\theta)^2 \right. \\
&\qquad + \rho(1-t)(1-\theta)^2 \\
&\qquad \left. -\left[3\left(1-t^2\right)(\beta-\alpha)+\rho\right]\frac{(1-\eta)^3}{3}\right\} \\
&\ge \frac{f(\eta,u(\eta))}{2\rho} \\
&\times \left\{\frac{3}{2}\left(1-t^2\right)(\beta-\alpha)(2-\beta-\alpha)(1-\theta)^2 \right. \\
&\qquad \left. + \rho(1-t)(1-\theta)^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \left[3\left(1-t^2\right)(\beta-\alpha)+\rho\right]\frac{(1-\theta)^3}{3}\Bigg\} \\
&= \frac{(1-\theta)^2 f(\eta,u(\eta))}{2\rho} \\
&\times \left\{\left(1-t^2\right)(\beta-\alpha)\left[3-\frac{3}{2}(\beta+\alpha)-(1-\theta)\right]\right. \\
&\quad \left.+\rho(1-t)-\frac{\rho}{3}(1-\theta)\right\} \\
&\geq \frac{(1-\theta)^2 f(\eta,u(\eta))}{2\rho} \\
&\times \left\{\left(1-t^2\right)(\beta-\alpha)\left[2-\frac{3}{2}(\beta+\alpha)\right]+\frac{2\rho}{3}(1-\theta)\right\} \\
&\geq 0.
\end{aligned} \tag{23}
$$

□

In what follows, we impose conditions on f such that the BVP (3) has at least two positive and decreasing solutions. For convenience, we denote

$$
H_1 = \int_{1-\theta}^{\theta} G(\theta,s)\,ds, \qquad H_2 = \int_0^{\eta} G(\theta,s)\,ds,
$$
$$
H_3 = \int_{1-\theta}^{\theta} G(1-\theta,s)\,ds. \tag{24}
$$

Theorem 9. *Suppose that there exist numbers $a, b,$ and c with $0 < a < b < \theta^* c$ such that*

$$
f(\theta,c) > \frac{c}{H_1}, \tag{25}
$$

$$
f\left(0,\frac{b}{\theta^*}\right) < \frac{b}{H_2}, \tag{26}
$$

$$
f\left(\theta,\theta^* a\right) > \frac{a}{H_3}. \tag{27}
$$

Then the BVP (3) has at least two positive and decreasing solutions.

Proof. First, we define the increasing, nonnegative, and continuous functionals γ, ω, and ψ on P as follows:

$$
\begin{aligned}
\gamma(u) &= \min_{t\in[1-\theta,\theta]} u(t) = u(\theta), \\
\omega(u) &= \max_{t\in[\theta,1]} u(t) = u(\theta), \\
\psi(u) &= \max_{t\in[1-\theta,1]} u(t) = u(1-\theta).
\end{aligned} \tag{28}
$$

Obviously, for any $u \in P$, $\gamma(u) = \omega(u) \leq \psi(u)$. At the same time, for each $u \in P$, in view of $\gamma(u) = \min_{t\in[1-\theta,\theta]} u(t) \geq \theta^* \|u\|$, we have

$$
\|u\| \leq \frac{1}{\theta^*}\gamma(u) \quad \text{for } u \in P. \tag{29}
$$

In addition, we also note that

$$
\omega(\lambda u) = \lambda\omega(u) \quad \text{for } 0 \leq \lambda \leq 1,\ u \in P. \tag{30}
$$

Next, we assert that $\gamma(Au) > c$ for all $u \in \partial P(\gamma, c)$.

To prove this, let $u \in \partial P(\gamma, c)$; that is, $u \in P$ and $\gamma(u) = u(\theta) = c$. Then

$$
u(t) \geq u(\theta) = c, \quad t \in [1-\theta,\theta]. \tag{31}
$$

In view of Lemma 8, (25), (31), and($C1$) and ($C2$), we have

$$
\begin{aligned}
\gamma(Au) &= (Au)(\theta) \\
&= \int_0^1 G(\theta,s) f(s,u(s))\,ds \\
&\geq \int_{1-\theta}^{\theta} G(\theta,s) f(s,u(s))\,ds \\
&\geq \int_{1-\theta}^{\theta} G(\theta,s) f(\theta,c)\,ds \\
&> \frac{c}{H_1}\int_{1-\theta}^{\theta} G(\theta,s)\,ds \\
&= c.
\end{aligned} \tag{32}
$$

Then, we assert that $\omega(Au) < b$ for all $u \in \partial P(\omega, b)$.

To see this, suppose $u \in \partial P(\omega, b)$; that is, $u \in P$ and $\omega(u) = b$. Since $\|u\| \leq (1/\theta^*)\gamma(u) = (1/\theta^*)\omega(u)$, we have

$$
0 \leq u(t) \leq \|u\| \leq \frac{b}{\theta^*}, \quad t \in [0,\eta]. \tag{33}
$$

In view of Remark 6, (26), (33), and ($C1$) and ($C2$), we get

$$
\begin{aligned}
\omega(Au) &= (Au)(\theta) \\
&= \int_0^1 G(\theta,s) f(s,u(s))\,ds \\
&\leq \int_0^{\eta} G(\theta,s) f(s,u(s))\,ds \\
&\leq \int_0^{\eta} G(\theta,s) f\left(0,\frac{b}{\theta^*}\right)ds \\
&< \frac{b}{H_2}\int_0^{\eta} G(\theta,s)\,ds \\
&= b.
\end{aligned} \tag{34}
$$

Finally, we assert that $P(\psi, a) \neq \emptyset$, and $\psi(Au) > a$ for all $u \in \partial P(\psi, a)$.

In fact, the constant function $a/2 \in P(\psi, a)$. Moreover, for $u \in \partial P(\psi, a)$, that is, $u \in P$ and $\psi(u) = u(1-\theta) = a$. Then

$$
u(t) \geq \theta^* \|u\| \geq \theta^* u(1-\theta) = \theta^* a, \quad t \in [1-\theta,\theta]. \tag{35}
$$

In view of Lemma 8, (27), (35), and (C1) and (C2), we get

$$\begin{aligned}\psi(Au) &= (Au)(1-\theta)\\ &= \int_0^1 G(1-\theta,s)\, f(s,u(s))\, ds\\ &\geq \int_{1-\theta}^{\theta} G(1-\theta,s)\, f(s,u(s))\, ds\\ &\geq \int_{1-\theta}^{\theta} G(1-\theta,s)\, f(\theta,\theta^* a)\, ds\\ &> \frac{a}{H_3}\int_{1-\theta}^{\theta} G(1-\theta,s)\, ds\\ &= a.\end{aligned} \tag{36}$$

To sum up, all the hypotheses of Theorem 3 are satisfied. Hence A has at least two fixed points; that is, the BVP (3) has at least two positive and decreasing solutions u_1 and u_2 such that

$$\begin{aligned}&a < \max_{t\in[1-\theta,1]} u_1(t) \quad \text{with} \max_{t\in[\theta,1]} u_1(t) < b,\\ &b < \max_{t\in[\theta,1]} u_2(t) \quad \text{with} \min_{t\in[1-\theta,\theta]} u_2(t) < c.\end{aligned} \tag{37}$$

□

3. An Example

Example 1. Consider the BVP

$$\begin{aligned}&u'''(t) = f(t,u(t)), \quad t\in[0,1],\\ &u'(0) = u(1) = 0, \qquad u''\left(\frac{3}{5}\right) + \int_{3/5}^{11/15} u(t)\, dt = 0,\end{aligned} \tag{38}$$

where

$$f(t,u) = \begin{cases} \sqrt{u} + \dfrac{1-t^2}{2} + 85, & (t,u)\in[0,1]\times[0,144],\\ 4748u + \dfrac{1-t^2}{2} - 683615, & (t,u)\in[0,1]\times[144,145],\\ \dfrac{969u^2}{4205} + \dfrac{1-t^2}{2}, & (t,u)\in[0,1]\times[145,+\infty). \end{cases} \tag{39}$$

Since $\eta = \alpha = 3/5$ and $\beta = 11/15$, if we choose $\theta = 17/30$, then a simple calculation shows that

$$\begin{aligned}&\rho = \frac{19502}{3375}, \qquad \theta^* = \frac{1}{18}, \qquad H_1 = \frac{262691}{8775900},\\ &H_2 = \frac{5106323}{63186480}, \qquad H_3 = \frac{15256439}{394915500}.\end{aligned} \tag{40}$$

Thus, if we let $a = 3$, $b = 8$, and $c = 145$, then it is easy to verify that all the conditions of Theorem 9 are satisfied. So, it follows from Theorem 9 that the BVP (38) has at least two positive and decreasing solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] M. Greguš, *Third Order Linear Differential Equations*, vol. 22 of *Mathematics and its Applications (East European Series)*, D. Reidel, Dordrecht, The Netherlands, 1987.

[2] D. R. Anderson, "Green's function for a third-order generalized right focal problem," *Journal of Mathematical Analysis and Applications*, vol. 288, no. 1, pp. 1–14, 2003.

[3] Z. Bai and X. Fei, "Existence of triple positive solutions for a third order generalized right focal problem," *Mathematical Inequalities and Applications*, vol. 9, no. 3, pp. 437–444, 2006.

[4] X. Feng, H. Feng, and D. Bai, "Eigenvalue for a singular third-order three-point boundary value problem," *Applied Mathematics and Computation*, vol. 219, no. 18, pp. 9783–9790, 2013.

[5] J. R. Graef and J. R. L. Webb, "Third order boundary value problems with nonlocal boundary conditions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 5-6, pp. 1542–1551, 2009.

[6] J. R. Graef and B. Yang, "Positive solutions of a third order nonlocal boundary value problem," *Discrete and Continuous Dynamical Systems. Series S*, vol. 1, no. 1, pp. 89–97, 2008.

[7] L.-J. Guo, J.-P. Sun, and Y.-H. Zhao, "Existence of positive solutions for nonlinear third-order three-point boundary value problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 10, pp. 3151–3158, 2008.

[8] T. Jankowski, "Existence of positive solutions to third order differential equations with advanced arguments and nonlocal boundary conditions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 75, no. 2, pp. 913–923, 2012.

[9] J.-P. Sun and H.-B. Li, "Monotone positive solution of nonlinear third-order BVP with integral boundary conditions," *Boundary Value Problems*, vol. 2010, Article ID 874959, 12 pages, 2010.

[10] Y. Sun, "Positive solutions of singular third-order three-point boundary value problem," *Journal of Mathematical Analysis and Applications*, vol. 306, no. 2, pp. 589–603, 2005.

[11] Y. Sun, "Positive solutions for third-order three-point nonhomogeneous boundary value problems," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 45–51, 2009.

[12] F. J. Torres, "Positive solutions for a third-order three-point boundary-value problem," *Electronic Journal of Differential Equations*, vol. 2013, no. 147, pp. 1–11, 2013.

[13] Y. Wang and W. Ge, "Existence of solutions for a third order differential equation with integral boundary conditions," *Computers and Mathematics with Applications*, vol. 53, no. 1, pp. 144–154, 2007.

[14] Q. Yao, "The existence and multiplicity of positive solutions for a third-order three-point boundary value problem," *Acta Mathematicae Applicatae Sinica*, vol. 19, no. 1, pp. 117–122, 2003.

[15] Q. Yao, "Positive solutions of singular third-order three-point boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 354, no. 1, pp. 207–212, 2009.

[16] A. P. Palamides and G. Smyrlis, "Positive solutions to a singular third-order three-point boundary value problem with an indefinitely signed Green's function," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 7, pp. 2104–2118, 2008.

[17] J.-P. Sun and J. Zhao, "Positive solution for a third-order three-point boundary value problem with sign-changing Green's function," *Communications in Applied Analysis*, vol. 16, no. 2, pp. 219–228, 2012.

[18] J.-P. Sun and J. Zhao, "Multiple positive solutions for a third-order three-point BVP with sign-changing Green's function," *Electronic Journal of Differential Equations*, vol. 2012, no. 118, pp. 1–7, 2012.

[19] J.-P. Sun and J. Zhao, "Iterative technique for a third-order three-point BVP with sign-changing Green's function," *Electronic Journal of Differential Equations*, vol. 2013, no. 215, pp. 1–9, 2013.

[20] X.-L. Li, J.-P. Sun, and F.-D. Kong, "Existence of positive solution for a third-order three-point BVP with sign-changing Green's function," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 2013, no. 30, pp. 1–11, 2013.

[21] Y.-H. Zhao and X.-L. Li, "Iteration for a third-order three-point BVP with sign-changing Green's function," *Journal of Applied Mathematics*, vol. 2014, Article ID 541234, 6 pages, 2014.

[22] G. Infante and J. R. L. Webb, "Nonzero solutions of Hammerstein integral equations with discontinuous kernels," *Journal of Mathematical Analysis and Applications*, vol. 272, no. 1, pp. 30–42, 2002.

[23] G. Infante and J. R. L. Webb, "Three-point boundary value problems with solutions that change sign," *Journal of Integral Equations and Applications*, vol. 15, no. 1, pp. 37–57, 2003.

[24] G. Infante and J. R. L. Webb, "Loss of positivity in a nonlinear scalar heat equation," *NoDEA. Nonlinear Differential Equations and Applications*, vol. 13, no. 2, pp. 249–261, 2006.

[25] R. I. Avery and J. Henderson, "Two positive fixed points of nonlinear operators on ordered Banach spaces," *Communications on Applied Nonlinear Analysis*, vol. 8, no. 1, pp. 27–36, 2001.

[26] R. H. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, John Wiley & Sons, New York, NY, USA, 1976.

Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 764187, 6 pages
http://dx.doi.org/10.1155/2015/764187

Research Article

Upper Semicontinuity of Solution Mappings to Parametric Generalized Vector Quasiequilibrium Problems

Shu-qiang Shan, Yu Han, and Nan-jing Huang

Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China

Correspondence should be addressed to Nan-jing Huang; nanjinghuang@hotmail.com

Received 11 December 2014; Revised 16 March 2015; Accepted 25 March 2015

Academic Editor: Mohamed-Aziz Taoudi

Copyright © 2015 Shu-qiang Shan et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We establish the upper semicontinuity of solution mappings for a class of parametric generalized vector quasiequilibrium problems. As applications, we obtain the upper semicontinuity of solution mappings to several problems, such as parametric optimization problem, parametric saddle point problem, parametric Nash equilibria problem, parametric variational inequality, and parametric equilibrium problem.

1. Introduction

It is well known that the vector equilibrium problem provides a unified model of several problems, such as the vector optimization problem, the vector saddle point problem, the vector complementarity problem, and the vector variational inequality problem [1, 2]. In recent years, the existence of solutions for various types of vector equilibrium problems has been investigated intensively by many authors under different conditions (see, e.g., [3–8] and the references therein).

On the other hand, the stability analysis of the solution mapping to vector equilibrium problems is an important topic in vector optimization theory. In recent years, the lower semicontinuity and the upper semicontinuity of of the solution mappings to parametric optimization problems, parametric vector variational inequalities, and parametric vector equilibrium problems have been intensively studied in the literature; for instance, we refer the reader to [9–17]. Recently, Anh and Khanh [18] obtained the semicontinuity of the solution mapping to parametric vector quasiequilibrium problems. Khanh and Luu [19] discussed the upper semicontinuity of solution mapping to parametric vector quasivariational inequalities involving multifunctions without monotonicity assumptions. Fang and Huang [20] established upper semicontinuity of the solution maps to the vector homogeneous quasiequilibrium problems. By using a scalarization method, Cheng and Zhu [21] investigated the upper semicontinuity and lower semicontinuity of the solution mapping to a parametric weak vector variational inequality in finite-dimension Euclidean spaces. Li and Fang [22] studied the lower semicontinuity of the solution mappings to a parametric generalized Ky Fan inequality by using a key assumption that includes the information about the solutions set. By virtue of a density result and scalarization technique, Gong and Yao [23] first discussed the lower semicontinuity of the set of efficient solutions to parametric vector equilibrium problems. Li et al. [24] investigated the upper semicontinuity and lower semicontinuity of the solution mappings to a parametric generalized strong vector equilibrium problem. By using the Hölder relation, Zhang et al. [25] obtained the lower semicontinuity of the efficient solution mapping to a parametric vector equilibrium problem. Fan et al. [26] studied the continuity of the solution mapping concerned with a class of vector quasiequilibrium problems with an application to traffic network problems. Xu and Li [27] established the lower semicontinuity of solution mappings to a parametric generalized strong vector equilibrium problem by using a scalarization method. Very recently, by using a new proof method which is different from the ones used in the literature, Han and Gong [28] established the lower semicontinuity of the solution mappings to parametric generalized strong vector equilibrium problems without the assumptions of monotonicity and compactness.

The aim of this paper is to establish the upper semicontinuity of solution mappings for a class of parametric generalized vector quasiequilibrium problems under some suitable conditions. We provide a uniform method to deal with the upper semicontinuity of solution mappings for several problems, such as parametric optimization problem, parametric saddle point problem, parametric Nash equilibria problem, parametric variational inequality, and parametric equilibrium problem. The rest of the paper is organized as follows. In Section 2, we recall some basic concepts and known lemmas. In Section 3, we show a main result in connection with the upper semicontinuity of the solution mapping for the parametric generalized vector quasiequilibrium problem. Some applications of the main result are given in Section 4.

2. Preliminaries

Throughout this paper, unless otherwise specified, let Λ, W, X, and Y be four normed vector spaces. Let D be a nonempty closed subset of Y. Let $F : X \times X \times W \to 2^Y$ and $K : \Lambda \to 2^X$ be two set-valued mappings. For $(u, \lambda) \in W \times \Lambda$, we consider that the following parametric generalized vector quasiequilibrium problems consist of finding $x_0 \in K(\lambda)$ such that

$$F(x_0, y, u) \subseteq D, \quad \forall y \in K(\lambda). \tag{PGVQEP}$$

We define a solution mapping S to (PGVQEP) by

$$S(u, \lambda) = \{x \in K(\lambda) : F(x, y, u) \subseteq D,\ \forall y \in K(\lambda)\}, \quad (u, \lambda) \in W \times \Lambda. \tag{1}$$

Definition 1 (see [29]). Let T and Q be two topological vector spaces. A set-valued mapping $G : T \to 2^Q$ is said to be

(i) upper semicontinuous (u.s.c.) at $u_0 \in T$ if, for any neighborhood V of $G(u_0)$, there exists a neighborhood $U(u_0)$ of u_0 such that, for every $u \in U(u_0)$, $G(u) \subseteq V$;

(ii) lower semicontinuous (l.s.c.) at $u_0 \in T$ if, for any $x \in G(u_0)$ and any neighborhood V of x, there exists a neighborhood $U(u_0)$ of u_0 such that, for every $u \in U(u_0)$, $G(u) \cap V \neq \emptyset$.

A set-valued mapping G is said to be u.s.c. and l.s.c. on T, if it is u.s.c. and l.s.c. at each $u \in T$, respectively. We say that G is continuous on T, if it is both u.s.c and l.s.c on T.

Definition 2 (see [15]). Let T and Q be two topological vector spaces and let $C \subseteq Q$ be a cone. A set-valued mapping $G : T \to 2^Q$ is said to be C-lower semicontinuous (C-l.s.c.) at $u_0 \in T$ if, for any $x \in G(u_0)$ and any neighborhood V of x, there exists a neighborhood $U(u_0)$ of u_0 such that

$$G(u) \cap (V - C) \neq \emptyset, \quad \forall u \in U(u_0). \tag{2}$$

A set-valued mapping G is said to be C-l.s.c. on T, if it is C-l.s.c. at each $u \in T$.

Remark 3. It is easy to see that if G is l.s.c. at $u_0 \in T$, then it is C-l.s.c. at $u_0 \in T$. In fact, since $0 \in C$, we have $V \subseteq V - C$. It follows from $G(u) \cap V \neq \emptyset$ that

$$G(u) \cap (V - C) \neq \emptyset. \tag{3}$$

The following example shows that the reverse is not true in general.

Example 4. Let $C = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. We define a set-valued mapping $G : \mathbb{R} \to 2^{\mathbb{R}}$ as follows:

$$G(x) = \begin{cases} [\cos x, \sin x + 2], & x \neq 0, \\ [1, 3], & x = 0. \end{cases} \tag{4}$$

It is easy to see that G is C-l.s.c. at 0, but G is not l.s.c. at 0.

Lemma 5 (see [29]). *A set-valued mapping $G : T \to 2^Q$ is l.s.c. at $u_0 \in T$ if and only if, for any sequence $\{u_n\} \subseteq T$ with $u_n \to u_0$ and for any $x_0 \in G(u_0)$, there exists $x_n \in G(u_n)$ such that $x_n \to x_0$.*

Lemma 6 (see [30]). *Assume that $G : T \to 2^Q$ is a set-valued mapping. If $G(u_0)$ is compact for some $u_0 \in T$, then G is u.s.c. at $u_0 \in T$ if and only if, for any sequence $\{u_n\} \subseteq T$ with $u_n \to u_0$ and for any $x_n \in G(u_n)$, there exist $x_0 \in G(u_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$.*

3. Upper Semicontinuity of Solution Mapping to (PGVQEP)

In this section, we establish the upper semicontinuity of S at $(u_0, \lambda_0) \in W \times \Lambda$.

Theorem 7. *Let $(u_0, \lambda_0) \in W \times \Lambda$. Assume that $K(\lambda_0)$ is nonempty compact, $K(\cdot)$ is continuous at λ_0, and $F(\cdot, \cdot, \cdot)$ is l.s.c. on $K(\lambda_0) \times K(\lambda_0) \times \{u_0\}$. Then S is u.s.c. at (u_0, λ_0).*

Proof. Suppose to the contrary that $S(\cdot, \cdot)$ is not u.s.c. at (u_0, λ_0). Then there exists a neighborhood W_0 of $S(u_0, \lambda_0)$; for any neighborhood $U(u_0) \times U(\lambda_0)$ of (u_0, λ_0), there exists $(u', \lambda') \in U(u_0) \times U(\lambda_0)$ such that $S(u', \lambda') \not\subseteq W_0$. Hence, there exists a sequence $\{(u_n, \lambda_n)\}$ with $(u_n, \lambda_n) \to (u_0, \lambda_0)$ such that

$$S(u_n, \lambda_n) \not\subseteq W_0, \quad \forall n \in \mathbb{N}. \tag{5}$$

Then there exists

$$x_n \in S(u_n, \lambda_n) \tag{6}$$

such that

$$x_n \notin W_0, \quad \forall n \in \mathbb{N}. \tag{7}$$

Since $K(\cdot)$ is u.s.c. at λ_0 and $x_n \in K(\lambda_n)$, by Lemma 6, there exist $x_0 \in K(\lambda_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. Without loss of generality, we can assume that $x_n \to x_0$.

We claim that $x_0 \in S(u_0, \lambda_0)$. In fact, suppose that $x_0 \notin S(u_0, \lambda_0)$. Then there exists $y_0 \in K(\lambda_0)$ such that

$$F(x_0, y_0, u_0) \notin D. \tag{8}$$

Hence there exists $z_0 \in F(x_0, y_0, u_0)$ such that

$$z_0 \notin D. \tag{9}$$

Since $K(\cdot)$ is l.s.c. at λ_0 and $y_0 \in K(\lambda_0)$, by Lemma 5, there exists $y_n \in K(\lambda_n)$ such that $y_n \to y_0$. Since $F(\cdot,\cdot,\cdot)$ is l.s.c. at (x_0, y_0, u_0) and $z_0 \in F(x_0, y_0, u_0)$, by Lemma 5, there exists $z_n \in F(x_n, y_n, u_n)$ such that $z_n \to z_0$. Noting (9) and that D is closed, we know that $z_n \notin D$ for n large enough, which contradicts with (6). Therefore, $x_0 \in S(u_0, \lambda_0)$. It is easy to see that $x_n \to x_0 \in W_0$, which contradicts with (7). □

Remark 8. We would like to point out that the assumptions of Theorem 7 are quite natural and easy to be verified.

We give an example to illustrate Theorem 7.

Example 9. Let $Y = \mathbb{R}^2$, $D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le 0,\ x_2 \ge 0\}$, and $\Lambda = W = X = \mathbb{R}$. Let B be the closed unit ball of $\mathbb{R}^2$. Assume that $F : X \times X \times W \to 2^Y$ is defined by

$$F(x, y, u) = (f_1(x, y, u), f_2(x, y, u)) + B, \tag{10}$$

where

$$\begin{aligned} f_1(x, y, u) &= \ln x - xy - 2^y - y^2 \cos u \\ &\quad + 3y - x + 2\sin u - 3, \\ f_2(x, y, u) &= \ln x^2 + 2xy + y^2 + 2^x + \cos u. \end{aligned} \tag{11}$$

We define a set-valued mapping $K : \Lambda \to 2^X$ as follows:

$$K(\lambda) := \{x \in \mathbb{R} : -3 + \sin\lambda \le x \le 3 + \cos\lambda\}, \quad \lambda \in \Lambda. \tag{12}$$

Then it is easy to see that $1 \in S(0,0)$ and so $S(0,0) \neq \emptyset$. Moreover, it is easy to check that all the assumptions of Theorem 7 are satisfied. Thus, it follows from Theorem 7 that S is u.s.c. at $(0,0)$.

4. Some Applications

In this section, we give some applications of Theorem 7 to the optimization problem, the saddle point problem, the Nash equilibria problem, the variational inequality, the variational inequality with set-valued mappings, the equilibrium problem, the generalized strong vector equilibrium problem, and the generalized weak vector equilibrium problem.

Optimization Problem. Let $f : X \to R$ be a mapping. Let K be a nonempty subset of X. A point $x_0 \in K$ is called a solution of optimization problem if and only if

$$f(x_0) \le f(y), \quad \forall y \in K. \tag{13}$$

Let $f : X \times W \to R$ be a mapping and let $K : \Lambda \to 2^X$ be a set-valued mapping. For $(u, \lambda) \in W \times \Lambda$, we consider that the following parametric optimization problem consists of finding $x_0 \in K(\lambda)$ such that

$$f(x_0, u) \le f(y, u), \quad \forall y \in K(\lambda). \tag{POP}$$

Define a solution mapping $O : W \times \Lambda \to 2^X$ to (POP) by

$$O(u, \lambda) = \{x \in K(\lambda) : f(x, u) \le f(y, u),\ \forall y \in K(\lambda)\}. \tag{14}$$

From Theorem 7, we can get the following corollary.

Corollary 10. *Let $(u_0, \lambda_0) \in W \times \Lambda$. Assume that $K(\lambda_0)$ is nonempty compact, $K(\cdot)$ is continuous at λ_0, and $f(\cdot,\cdot)$ is continuous on $K(\lambda_0) \times \{u_0\}$. Then the mapping O is u.s.c. at (u_0, λ_0).*

Saddle Point Problem. Let $\varphi : X \times X \to R$ be a mapping. Let K be a nonempty subset of X. $(\overline{x}_1, \overline{x}_2)$ is called a saddle point on K if and only if

$$\begin{aligned} &(\overline{x}_1, \overline{x}_2) \in K \times K, \\ &\varphi(\overline{x}_1, y_2) \le \varphi(y_1, \overline{x}_2), \quad \forall (y_1, y_2) \in K \times K. \end{aligned} \tag{15}$$

Let $\varphi : X \times X \times W \to R$ be a mapping and let $K : \Lambda \to 2^X$ be a set-valued mapping. For $(u, \lambda) \in W \times \Lambda$, we consider that the following parametric saddle point problem consists of finding $(\overline{x}_1, \overline{x}_2) \in K(\lambda) \times K(\lambda)$ such that

$$\varphi(\overline{x}_1, y_2, u) \le \varphi(y_1, \overline{x}_2, u), \quad \forall (y_1, y_2) \in K(\lambda) \times K(\lambda). \tag{PSPP}$$

Define a solution mapping $P : W \times \Lambda \to 2^{X \times X}$ to (PSPP) by

$$\begin{aligned} P(u, \lambda) = \{&(x_1, x_2) \in K(\lambda) \times K(\lambda) : \varphi(x_1, y_2, u) \\ &\le \varphi(y_1, x_2, u),\ \forall (y_1, y_2) \in K(\lambda) \times K(\lambda)\}. \end{aligned} \tag{16}$$

From Theorem 7, we can get the following corollary.

Corollary 11. *Let $(u_0, \lambda_0) \in W \times \Lambda$. Assume that $K(\lambda_0)$ is nonempty compact, $K(\cdot)$ is continuous at λ_0, and $\varphi(\cdot,\cdot,\cdot)$ is continuous on $K(\lambda_0) \times K(\lambda_0) \times \{u_0\}$. Then the mapping P is u.s.c. at (u_0, λ_0).*

Nash Equilibria Problem. Let I be a finite index set. For every $i \in I$, let K_i be a nonempty subset of X and let $f_i : \prod_{i \in I} X \to R$ be a mapping. Let $K := \prod_{i \in I} K_i$. For $x = (x_i)_{i \in I} \in K$, we define $x^i = (x_j)_{j \in I, j \neq i}$. A point $\overline{x} = (\overline{x}_i)_{i \in I} \in K$ is called a Nash equilibrium if and only if, for any $i \in I$, we have

$$f_i(\overline{x}) \le f_i(\overline{x}^i, y_i), \quad \forall y_i \in K_i. \tag{17}$$

For every $i \in I$, let $f_i : (\prod_{i \in I} X) \times W \to R$ be a mapping and let $K_i : \Lambda \to 2^X$ be a set-valued mapping. Let $K(\lambda) := \prod_{i \in I} K_i(\lambda)$ for $\lambda \in \Lambda$. For $(u, \lambda) \in W \times \Lambda$, we consider that

the following parametric Nash equilibria problem consists of finding $\overline{x} = (\overline{x}_i)_{i\in I} \in K(\lambda)$ such that

$$\forall i \in I, \quad f_i(\overline{x}, u) \le f_i\left(\overline{x}^i, y_i, u\right), \quad \forall y_i \in K_i(\lambda). \tag{PNEP}$$

Define a solution mapping $N : W \times \Lambda \to 2^{\prod_{i\in I} X}$ to (PNEP) by

$$N(u,\lambda) = \left\{x = (x_i)_{i\in I} \in K(\lambda) : \forall i \in I, \; f_i(x,u) \le f_i\left(x^i, y_i, u\right), \; \forall y_i \in K_i(\lambda)\right\}. \tag{18}$$

From Theorem 7, we can get the following corollary.

Corollary 12. *Let $(u_0,\lambda_0) \in W \times \Lambda$. For every $i \in I$, assume that $K_i(\lambda_0)$ is nonempty compact, $K_i(\cdot)$ is continuous at λ_0, and f_i is continuous on $(\prod_{i\in I} K_i(\lambda_0)) \times \{u_0\}$. Then the mapping N is u.s.c. at (u_0,λ_0).*

Variational Inequality. Let X^* be the topological dual space of X. Let $T : X \to X^*$ be a mapping. Let K be a nonempty subset of X. A point $x_0 \in K$ is called a solution of variational inequality if and only if

$$\langle T(x_0), y - x_0\rangle \ge 0, \quad \forall y \in K. \tag{19}$$

Let $T : X \times W \to X^*$ be a mapping and let $K : \Lambda \to 2^X$ be a set-valued mapping. For $(u,\lambda) \in W \times \Lambda$, we consider that the following parametric variational inequality consists of finding $x_0 \in K(\lambda)$ such that

$$\langle T(x_0, u), y - x_0\rangle \ge 0, \quad \forall y \in K(\lambda). \tag{PVI}$$

Define a solution mapping $V : W \times \Lambda \to 2^X$ to (PVI) by

$$V(u,\lambda) = \left\{x \in K(\lambda) : \langle T(x,u), y - x\rangle \ge 0, \; \forall y \in K(\lambda)\right\}. \tag{20}$$

From Theorem 7, we can get the following corollary.

Corollary 13. *Let $(u_0,\lambda_0) \in W \times \Lambda$. Assume that $K(\lambda_0)$ is nonempty compact, $K(\cdot)$ is continuous at λ_0, and $T(\cdot,\cdot)$ is continuous on $K(\lambda_0) \times \{u_0\}$. Then the mapping V is u.s.c. at (u_0,λ_0).*

Variational Inequality with Set-Valued Mappings. Let X^* be the topological dual space of X. Let $T : X \to 2^{X^*}$ be a set-valued mapping. Let K be a nonempty subset of X. A point $x_0 \in K$ is called a solution of variational inequality with set-valued mappings if and only if

$$\exists \xi \in T(x_0), \quad \langle \xi, y - x_0\rangle \ge 0, \quad \forall y \in K. \tag{21}$$

Let $T : X \times W \to 2^{X^*}$ and $K : \Lambda \to 2^X$ be two set-valued mappings. For $(u,\lambda) \in W \times \Lambda$, we consider that the following parametric variational inequality with set-valued mappings consists of finding $x_0 \in K(\lambda)$ such that

$$\exists \xi \in T(x_0, u), \quad \langle \xi, y - x_0\rangle \ge 0, \quad \forall y \in K(\lambda). \tag{PVISM}$$

Define a solution mapping $V_s : W \times \Lambda \to 2^X$ to (PVISM) by

$$V_s(u,\lambda) = \{x \in K(\lambda) : \exists \xi \in T(x,u), \; \langle \xi, y - x\rangle \ge 0, \; \forall y \in K(\lambda)\}. \tag{22}$$

The proof of the following corollary is similar to the proof of Theorem 7. For the convenience of the readers, we also give the proof.

Corollary 14. *Let $(u_0,\lambda_0) \in W \times \Lambda$. Assume that $K(\lambda_0)$ is nonempty compact, $K(\cdot)$ is continuous at λ_0, and $T(\cdot,\cdot)$ is u.s.c. on $K(\lambda_0) \times \{u_0\}$ with nonempty compact values. Then the mapping V_s is u.s.c. at (u_0,λ_0).*

Proof. Suppose to the contrary that $V_s(\cdot,\cdot)$ is not u.s.c. at (u_0,λ_0). Then there exists a neighborhood W_0 of $V_s(u_0,\lambda_0)$; for any neighborhood $U(u_0) \times U(\lambda_0)$ of (u_0,λ_0), there exists $(u',\lambda') \in U(u_0) \times U(\lambda_0)$ such that $V_s(u',\lambda') \not\subset W_0$. Hence, there exists a sequence $\{(u_n,\lambda_n)\}$ with $(u_n,\lambda_n) \to (u_0,\lambda_0)$ such that

$$V_s(u_n,\lambda_n) \not\subset W_0, \quad \forall n \in \mathbb{N}. \tag{23}$$

Then there exists

$$x_n \in V_s(u_n,\lambda_n) \tag{24}$$

such that

$$x_n \notin W_0, \quad \forall n \in \mathbb{N}. \tag{25}$$

Since $K(\cdot)$ is u.s.c. at λ_0 and $x_n \in K(\lambda_n)$, by Lemma 6, there exist $x_0 \in K(\lambda_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. Without loss of generality, we can assume that $x_n \to x_0$.

By (24), there exists $\xi_n \in T(x_n,u_n)$ such that

$$\langle \xi_n, y - x_n\rangle \ge 0, \quad \forall y \in K(\lambda_n). \tag{26}$$

Since $\xi_n \in T(x_n,u_n)$, by Lemma 6, there exist $\xi_0 \in T(x_0,u_0)$ and a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\xi_{n_k} \to \xi_0$. Without loss of generality, we can assume that $\xi_n \to \xi_0$.

For any $\overline{y} \in K(\lambda_0)$, by Lemma 5, there exists $y_n \in K(\lambda_n)$ such that $y_n \to \overline{y}$. Noting (26), we have

$$\langle \xi_n, y_n - x_n\rangle \ge 0. \tag{27}$$

It follows from $\langle \xi_n, y_n - x_n\rangle \to \langle \xi_0, \overline{y} - x_0\rangle$ that

$$\langle \xi_0, \overline{y} - x_0\rangle \ge 0. \tag{28}$$

Therefore, $x_0 \in V_s(u_0,\lambda_0)$. We can see that $x_n \to x_0 \in W_0$, which contradicts with (25). □

Equilibrium Problem. Let $f : X \times X \to R$ be a mapping. Let K be a nonempty subset of X. A point $x_0 \in K$ is called a solution of equilibrium problem if and only if

$$f(x_0, y) \ge 0, \quad \forall y \in K. \tag{29}$$

Let $f : X \times X \times W \to R$ and $K : \Lambda \to 2^X$ be two set-valued mappings. For $(u, \lambda) \in W \times \Lambda$, we consider that the following parametric equilibrium problem consists of finding $x_0 \in K(\lambda)$ such that

$$f(x_0, y, u) \geq 0, \quad \forall y \in K(\lambda). \tag{PEP}$$

Define a solution mapping $E : W \times \Lambda \to 2^X$ to (PEP) by

$$E(u, \lambda) = \{x \in K(\lambda) : f(x, y, u) \geq 0, \ \forall y \in K(\lambda)\}. \tag{30}$$

From Theorem 7, we can get the following corollary.

Corollary 15. *Let $(u_0, \lambda_0) \in W \times \Lambda$. Assume that $K(\lambda_0)$ is nonempty compact, $K(\cdot)$ is continuous at λ_0, and $f(\cdot, \cdot, \cdot)$ is continuous on $K(\lambda_0) \times K(\lambda_0) \times \{u_0\}$. Then the mapping E is u.s.c. at (u_0, λ_0).*

Generalized Strong Vector Equilibrium Problem. Assume that $C \subseteq Y$ is a closed cone. Let $F : X \times X \to 2^Y$ be a set-valued mapping. Let K be a nonempty subset of X. A point $x_0 \in K$ is called a solution of generalized strong vector equilibrium problem if and only if

$$F(x_0, y) \subseteq C, \quad \forall y \in K. \tag{31}$$

Let $F : X \times X \times W \to 2^Y$ and $K : \Lambda \to 2^X$ be two set-valued mappings. For $(u, \lambda) \in W \times \Lambda$, we consider that the following parametric generalized strong vector equilibrium problem consists of finding $x_0 \in K(\lambda)$ such that

$$F(x_0, y, u) \subseteq C, \quad \forall y \in K(\lambda). \tag{PGSVEP}$$

Define a solution mapping $M : W \times \Lambda \to 2^X$ to (PGSVEP) by

$$M(u, \lambda) = \{x \in K(\lambda) : F(x, y, u) \subseteq C, \ \forall y \in K(\lambda)\}. \tag{32}$$

From Theorem 7, we can get the following corollary.

Corollary 16. *Let $(u_0, \lambda_0) \in W \times \Lambda$. Assume that $K(\lambda_0)$ is nonempty compact, $K(\cdot)$ is continuous at λ_0, and $F(\cdot, \cdot, \cdot)$ is l.s.c. on $K(\lambda_0) \times K(\lambda_0) \times \{u_0\}$. Then the mapping M is u.s.c. at (u_0, λ_0).*

Generalized Weak Vector Equilibrium Problem. Assume that $C \subseteq Y$ is a cone with nonempty interior. Let $F : X \times X \to 2^Y$ be a set-valued mapping. Let K be a nonempty subset of X. A point $x_0 \in K$ is called a solution of generalized weak vector equilibrium problem if and only if

$$F(x_0, y) \cap (-\operatorname{int} C) = \emptyset, \quad \forall y \in K. \tag{33}$$

Let $F : X \times X \times W \to 2^Y$ and $K : \Lambda \to 2^X$ be two set-valued mappings. For $(u, \lambda) \in W \times \Lambda$, we consider that the following parametric generalized weak vector equilibrium problem consists of finding $x_0 \in K(\lambda)$ such that

$$F(x_0, y, u) \cap (-\operatorname{int} C) = \emptyset, \quad \forall y \in K(\lambda). \tag{PGWVEP}$$

Define a solution mapping $G : W \times \Lambda \to 2^X$ to (PGWVEP) by

$$G(u, \lambda) = \{x \in K(\lambda) : F(x, y, u) \cap (-\operatorname{int} C) = \emptyset, \ \forall y \in K(\lambda)\}. \tag{34}$$

From Theorem 7, we can get the following corollary.

Corollary 17. *Let $(u_0, \lambda_0) \in W \times \Lambda$. Assume that $K(\lambda_0)$ is nonempty compact, $K(\cdot)$ is continuous at λ_0, and $F(\cdot, \cdot, \cdot)$ is l.s.c. on $K(\lambda_0) \times K(\lambda_0) \times \{u_0\}$. Then the mapping G is u.s.c. at (u_0, λ_0).*

Remark 18. Corollary 17 is similar to Theorem 3.4 of [18].

From Corollary 17, it is easy to get the following corollary.

Corollary 19. *Let $(u_0, \lambda_0) \in W \times \Lambda$. Assume that $K(\lambda_0)$ is nonempty compact, $K(\cdot)$ is continuous at λ_0, and $F(\cdot, \cdot, \cdot)$ is C-l.s.c. on $K(\lambda_0) \times K(\lambda_0) \times \{u_0\}$. Then G is u.s.c. at (u_0, λ_0).*

Remark 20. In the proof of upper semicontinuity of solution mapping, Corollary 19 improves Theorem 3.1 of [15].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are grateful to the editor and the referees for their valuable comments and suggestions. This work was supported by the National Natural Science Foundation of China (11171237, 11471230).

References

[1] G. Y. Chen, X. X. Huang, and X. Q. Yang, *Vector Optimization: Set-valued and Variational Analysis*, vol. 541 of *Lecture Notes in Economics and Mathematical Systems*, Springer, Berlin, Germany, 2005.

[2] F. Giannessi, Ed., *Vector Variational Inequalities and Vector Equilibria*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[3] G. Y. Chen, "Existence of solutions for a vector variational inequality: an extension of the Hartmann-Stampacchia theorem," *Journal of Optimization Theory and Applications*, vol. 74, no. 3, pp. 445–456, 1992.

[4] Q. H. Ansari, I. V. Konnov, and J. C. Yao, "Existence of a solution and variational principles for vector equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 110, no. 3, pp. 481–492, 2001.

[5] Q. H. Ansari, "Existence of solutions of systems of generalized implicit vector quasi-equilibrium problems," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 1271–1283, 2008.

[6] X. H. Gong, "Strong vector equilibrium problems," *Journal of Global Optimization*, vol. 36, no. 3, pp. 339–349, 2006.

[7] N.-J. Huang and Y.-P. Fang, "On vector variational inequalities in reflexive Banach spaces," *Journal of Global Optimization*, vol. 32, no. 4, pp. 495–505, 2005.

[8] N. J. Huang, J. Li, and J. C. Yao, "Gap functions and existence of solutions for a system of vector equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 133, no. 2, pp. 201–212, 2007.

[9] L. Q. Anh and P. Q. Khanh, "On the stability of the solution sets of general multivalued vector quasiequilibrium problems," *Journal of Optimization Theory and Applications*, vol. 135, no. 2, pp. 271–284, 2007.

[10] L. Q. Anh and P. Q. Khanh, "Continuity of solution maps of parametric quasiequilibrium problems," *Journal of Global Optimization*, vol. 46, no. 2, pp. 247–259, 2010.

[11] N. Dinh, M. A. Goberna, and M. A. López, "On the stability of the optimal value and the optimal set in optimization problems," *Journal of Convex Analysis*, vol. 19, no. 4, pp. 927–953, 2012.

[12] N. J. Huang, J. Li, and H. B. Thompson, "Stability for parametric implicit vector equilibrium problems," *Mathematical and Computer Modelling*, vol. 43, no. 11-12, pp. 1267–1274, 2006.

[13] X. H. Gong, "Continuity of the solution set to parametric weak vector equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 139, no. 1, pp. 35–46, 2008.

[14] C. R. Chen, S. J. Li, and K. L. Teo, "Solution semicontinuity of parametric generalized vector equilibrium problems," *Journal of Global Optimization*, vol. 45, no. 2, pp. 309–318, 2009.

[15] B. Chen and N. J. Huang, "Continuity of the solution mapping to parametric generalized vector equilibrium problems," *Journal of Global Optimization*, vol. 56, no. 4, pp. 1515–1528, 2013.

[16] R. Lucchetti and F. Patrone, "Closure and upper semicontinuity results in mathematical programming, Nash and economic equilibria," *Optimization*, vol. 17, no. 5, pp. 619–628, 1986.

[17] T. Tanino and Y. Sawaragi, "Stability of nondominated solutions in multicriteria decision-making," *Journal of Optimization Theory and Applications*, vol. 30, no. 2, pp. 229–253, 1980.

[18] L. Q. Anh and P. Q. Khanh, "Semicontinuity of the solution set of parametric multivalued vector quasiequilibrium problems," *Journal of Mathematical Analysis and Applications*, vol. 294, no. 2, pp. 699–711, 2004.

[19] P. Q. Khanh and L. M. Luu, "Upper semicontinuity of the solution set to parametric vector quasivariational inequalities," *Journal of Global Optimization*, vol. 32, no. 4, pp. 569–580, 2005.

[20] Y.-P. Fang and N.-J. Huang, "Upper semicontinuity of the solution maps in homogeneous vector quasi-equilibrium problems," *Optimization*, vol. 55, no. 3, pp. 231–239, 2006.

[21] Y. H. Cheng and D. L. Zhu, "Global stability results for the weak vector variational inequality," *Journal of Global Optimization*, vol. 32, no. 4, pp. 543–550, 2005.

[22] S. J. Li and Z. M. Fang, "Lower semicontinuity of the solution mappings to a parametric generalized Ky Fan inequality," *Journal of Optimization Theory and Applications*, vol. 147, no. 3, pp. 507–515, 2010.

[23] X. H. Gong and J. C. Yao, "Lower semicontinuity of the set of efficient solutions for generalized systems," *Journal of Optimization Theory and Applications*, vol. 138, no. 2, pp. 197–205, 2008.

[24] S. J. Li, H. M. Liu, Y. Zhang, and Z. M. Fang, "Continuity of the solution mappings to parametric generalized strong vector equilibrium problems," *Journal of Global Optimization*, vol. 55, no. 3, pp. 597–610, 2013.

[25] W. Y. Zhang, Z. M. Fang, and Y. Zhang, "A note on the lower semicontinuity of efficient solutions for parametric vector equilibrium problems," *Applied Mathematics Letters*, vol. 26, no. 4, pp. 469–472, 2013.

[26] X. D. Fan, C. Z. Cheng, and H. J. Wang, "Stability analysis for vector quasiequilibrium problems," *Positivity*, vol. 17, no. 3, pp. 365–379, 2013.

[27] Y. D. Xu and S. J. Li, "On the lower semicontinuity of the solution mappings to a parametric generalized strong vector equilibrium problem," *Positivity*, vol. 17, no. 2, pp. 341–353, 2013.

[28] Y. Han and X. H. Gong, "Lower semicontinuity of solution mapping to parametric generalized strong vector equilibrium problems," *Applied Mathematics Letters*, vol. 28, pp. 38–41, 2014.

[29] J.-P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley & Sons, New York, NY, USA, 1984.

[30] A. Göpfert, H. Riahi, C. Tammer, and C. Zălinescu, *Variational methods in partially ordered spaces*, Springer, New York, NY, USA, 2003.

Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 928105, 8 pages
http://dx.doi.org/10.1155/2015/928105

Research Article

Existence of Positive Solutions for a Fourth-Order m-Point Boundary Value Problem

Liu Yang and Chunfang Shen

Department of Mathematics, Hefei Normal University, Hefei, Anhui 230061, China

Correspondence should be addressed to Liu Yang; yliu722@163.com

Received 3 August 2014; Accepted 5 February 2015

Academic Editor: Jesús G. Falset

Copyright © 2015 L. Yang and C. Shen. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

By using Krasnosel'skii's fixed point theorem and the fixed point index theorem in the special function space, we obtain some sufficient conditions for the existence of positive solutions of fourth-order boundary value problem with multipoint boundary conditions. Applications of our results to some special problems are also discussed.

1. Introduction

In this paper, we study the existence of positive solutions of the boundary value problem consisting of the nonlinear fourth-order differential equation

$$u^{(4)}(t) + a(t) f(u(t)) = 0, \quad t \in [0,1] \tag{1}$$

and the multipoint boundary condition

$$u'''(0) = 0, \qquad u''(0) = 0, \qquad u'(0) = 0,$$
$$u(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i) \tag{2}$$

or

$$u'''(1) = 0, \qquad u''(1) = 0, \qquad u'(1) = 0,$$
$$u(0) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \tag{3}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 < \beta_i < 1$, $i = 1, 2, \ldots, m-2$, $\sum_{i=1}^{m-2} \beta_i < 1$ and $f \in C([0,+\infty), [0,+\infty))$, $a(t) \in C([0,1], [0,+\infty))$, and there exist $t_0 \in [0,1]$ such that $a(t_0) > 0$.

It is well known that boundary value problems of fourth-order ordinary differential equations are used to describe a large number of physical, biological, and chemical phenomena. For an example, (1) is often used to describe the deformation of an elastic beam under a variety of boundary conditions [1–3]. For this reason, there is a wide literature that deals with the existence and the multiplicity of solutions for fourth-order boundary value problems; see [4–19].

Krasnoselskiis theorem in a cone and fixed point index theorem have often been used to study the existence and multiplicity of positive solutions of nonlocal boundary value problems over the last several years. As to fourth-order Bvps, by using Krasnoselskiis fixed point theorem, Graef et al. [11] established the existence results of positive solutions for the nonlinear fourth-order ordinary differential equation

$$u^{(4)}(t) + \lambda g(t) f(u(t)) = 0, \quad t \in [0,1] \tag{4}$$

with the boundary conditions

$$u(0) = u'(1) = u''(0) = u''(p) - u''(1) = 0. \tag{5}$$

However, the aforementioned papers mainly considered the two or three point boundary conditions. There are very few works on the multipoint boundary value problem for fourth-order ordinary differential equations. As to m-point boundary value problem ((1), (2)) or ((1), (3)), there is no existence results of solution or positive solution. The goal of this paper is to fill this gap. In this paper, by constructing an available integral operator and combining fixed point theorem and fixed point index theorem, we establish some sufficient conditions of the existence of positive solutions for problems ((1), (2)) and ((1), (3)).

The rest of this paper is organized as follows. In Section 2, we given some preliminaries and lemmas for use later. Section 3 is devoted to the existence and multiplicity of positive solutions of problem ((1), (2)). In Section 4, we establish the existence results of positive solutions for problem ((1), (3)). In Section 5, we give some examples to demonstrate the main results of this paper.

2. Preliminaries

Definition 1. Let E be a real Banach space over the real numbers. A nonempty convex closed set $P \subset E$ is said to be a cone provided that

(i) $au \in P$, for all $u \in P$, $a \geq 0$,

(ii) $u, -u \in P$ implies $u = 0$.

Definition 2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Let E be the Banach space $C[0,1]$ endowed with the norm $\|u\| = \max_{0\leq t\leq 1}|u(t)|$. We denote

$$f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}. \tag{6}$$

Lemma 3 (see [20]). *Let E be a Banach space and $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \longrightarrow K \tag{7}$$

be a completely continuous operator such that

$$\begin{aligned} &\|Au\| \leq \|u\|, \quad u \in K \cap \partial\Omega_1, \\ &\|Au\| \geq \|u\|, \quad u \in K \cap \partial\Omega_2 \end{aligned} \tag{8}$$

or

$$\begin{aligned} &\|Au\| \geq \|u\|, \quad u \in K \cap \partial\Omega_1, \\ &\|Au\| \leq \|u\|, \quad u \in K \cap \partial\Omega_2; \end{aligned} \tag{9}$$

then, A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 4 (see [20]). *Let E be a Banach space and let $K \subset E$ be a cone. For $\rho > 0$, define $K_\rho = \{u \in K : \|u\| \leq \rho\}$ and assume that $A : K_\rho \to K$ is a completely continuous operator such that $Au \neq u$ for $x \in \partial K_\rho = \{u \in K : \|u\| = \rho\}$. Then,*

$$\begin{aligned} &\|Au\| \leq \|u\|, \quad u \in \partial K_\rho, \ \text{then } i\left(A, K_\rho, K\right) = 1, \\ &\|Au\| \geq \|u\|, \quad u \in \partial K_\rho, \ \text{then } i\left(A, K_\rho, K\right) = 0. \end{aligned} \tag{10}$$

3. Existence Results of Problems (1) and (2)

We begin with the fourth-order m-point boundary value problem

$$u^{(4)}(t) + y(t) = 0, \quad t \in [0,1], \tag{11}$$

$$u'''(0) = 0, \qquad u''(0) = 0, \qquad u'(0) = 0,$$
$$u(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i). \tag{12}$$

Lemma 5. *Denoting $\xi_0 = 0$, $\xi_{m-1} = 1$, $\beta_0 = \beta_{m-1} = 0$, and $y(t) \in C[0,1]$, problem (11), (12) has the unique solution*

$$u(t) = \int_0^1 G(t,s)\, y(s)\, ds, \tag{13}$$

where

$$G(t,s) = \begin{cases} -\frac{1}{6}s^3 \\ \quad + \left(\frac{1}{6} - \frac{1}{2}s + \frac{1}{2}s^2 - \frac{1}{6}\sum_{k=0}^{i-1}\beta_k s^3 \right. \\ \qquad \left. + \sum_{k=i}^{m-2}\beta_k\xi_k\left(-\frac{1}{6}\xi_k^2 + \frac{1}{2}\xi_k s - \frac{1}{2}s^2\right)\right) \\ \quad \cdot \left(1 - \sum_{k=0}^{m-1}\beta_k\right)^{-1}, & t \leq s, \\ -\frac{1}{6}t^3 + \frac{1}{2}st^2 - \frac{1}{2}s^2 t \\ \quad + \left(\frac{1}{6} - \frac{1}{2}s + \frac{1}{2}s^2 - \frac{1}{6}\sum_{k=0}^{i-1}\beta_k s^3 \right. \\ \qquad \left. + \sum_{k=i}^{m-2}\beta_k\xi_k\left(-\frac{1}{6}\xi_k^2 + \frac{1}{2}\xi_k s - \frac{1}{2}s^2\right)\right) \\ \quad \cdot \left(1 - \sum_{k=0}^{m-1}\beta_k\right)^{-1}, & t \geq s, \end{cases} \tag{14}$$

for $\xi_{i-1} \leq s \leq \xi_i$, $i = 1, 2, \ldots, m-1$.

Proof. Let $G(t,s)$ be Green's function of problem $-u^{(4)}(t) = 0$ with boundary condition (12). We can suppose

$$G(t,s) = \begin{cases} a_3t^3 + a_2t^2 + a_1t + a_0, \\ \quad t \leq s, \quad \xi_{i-1} \leq s \leq \xi_i, \quad i = 1, 2, \ldots, m-1, \\ b_3t^3 + b_2t^2 + b_1t + b_0, \\ \quad t \geq s, \quad \xi_{i-1} \leq s \leq \xi_i, \quad i = 1, 2, \ldots, m-1, \end{cases} \tag{15}$$

where a_i, b_i, $i = 0, 1, 2, 3$ are unknown coefficients. Considering the properties of Green's function and boundary condition (12), we have

$$\begin{aligned}
a_3 s^3 + a_2 s^2 + a_1 s + a_0 &= b_3 s^3 + b_2 s^2 + b_1 s + b_0,\\
3a_3 s^2 + 2a_2 s + a_1 &= 3b_3 s^2 + 2b_2 s + b_1,\\
6a_3 s + 2a_2 &= 6b_3 s + 2b_2,\\
6a_3 - 6b_3 &= 1,\\
a_3 &= 0,\\
a_2 &= 0,\\
a_1 &= 0,\\
b_3 + b_2 + b_1 + b_0 &= \sum_{k=0}^{i=1} \beta_k \left(a_3 \xi_k^3 + a_2 \xi_k^2 + a_1 \xi_k + a_0\right)\\
&\quad + \sum_{k=i}^{m-2} \beta_k \left(b_3 \xi_k^3 + b_2 \xi_k^2 + b_1 \xi_k + b_0\right).
\end{aligned} \tag{16}$$

A straightforward calculation shows that

$$\begin{aligned}
a_3 &= a_2 = a_1 = 0,\\
a_0 &= -\frac{1}{6}s^3 + \left(\frac{1}{6} - \frac{1}{2}s + \frac{1}{2}s^2 - \frac{1}{6}\sum_{k=0}^{i-1}\beta_k s^3 + \sum_{k=i}^{m-2}\beta_k \xi_k \left(-\frac{1}{6}\xi_k^2 + \frac{1}{2}\xi_k s - \frac{1}{2}s^2\right)\right)\\
&\quad \cdot \left(1 - \sum_{k=0}^{m-1}\beta_k\right)^{-1},\\
b_3 &= -\frac{1}{6}, \qquad b_2 = \frac{s}{2}, \qquad b_1 = -\frac{s^2}{2},\\
b_0 &= \left(\frac{1}{6} - \frac{1}{2}s + \frac{1}{2}s^2 - \frac{1}{6}\sum_{k=0}^{i-1}\beta_k s^3 + \sum_{k=i}^{m-2}\beta_k \xi_k \left(-\frac{1}{6}\xi_k^2 + \frac{1}{2}\xi_k s - \frac{1}{2}s^2\right)\right)\\
&\quad \cdot \left(1 - \sum_{k=0}^{m-1}\beta_k\right)^{-1}.
\end{aligned} \tag{17}$$

These give the explicit expression of Green's function. Then, we have

$$u(t) = \int_0^1 G(t,s)\, y(s)\, ds. \tag{18}$$

□

Lemma 6. *One can see that $G(t,s) \geq 0$, $t, s \in [0,1]$.*

Proof. For $\xi_{i-1} \leq s \leq \xi_i$, $i = 1, 2, \ldots, m-1$,

$$\frac{\partial G(t,s)}{\partial t} = \begin{cases} 0, & t \leq s,\ \xi_{i-1} \leq s \leq \xi_i,\\ -\frac{1}{2}(t-s)^2, & t \geq s,\ \xi_{i-1} \leq s \leq \xi_i. \end{cases} \tag{19}$$

Then, $\partial G(t,s)/\partial t \leq 0$, $0 \leq t, s \leq 1$, which induces that $G(t,s)$ is decreasing on t. By a simple computation, we see

$$\begin{aligned}
G(1,s) &= -\frac{1}{6} + \frac{1}{2}s - \frac{1}{2}s^2\\
&\quad + \left(\frac{1}{6} - \frac{1}{2}s + \frac{1}{2}s^2 - \frac{1}{6}\sum_{k=0}^{i-1}\beta_k s^3 + \sum_{k=i}^{m-2}\beta_k \xi_k \left(-\frac{1}{6}\xi_k^2 + \frac{1}{2}\xi_k s - \frac{1}{2}s^2\right)\right)\\
&\quad \cdot \left(1 - \sum_{k=0}^{m-1}\beta_k\right)^{-1}\\
&= \left(\frac{1}{6}\sum_{k=0}^{m-1}\beta_k (1-s)^3 + \frac{1}{6}\sum_{k=i}^{m-2}\frac{1}{2}\beta_k (s-\xi_k)^3\right)\\
&\quad \cdot \left(1 - \sum_{k=0}^{m-1}\beta_k\right)^{-1}\\
&\geq 0.
\end{aligned} \tag{20}$$

This ensures that $G(t,s) \geq 0$, $t, s \in [0,1]$. □

Lemma 7. *If $y(t) \geq 0$, $t \in [0,1]$ and $u(t)$ is the solution of problem (11), (12), then*

$$\min_{0 \leq t \leq 1} |u(t)| \geq \gamma \max_{0 \leq t \leq 1} |u(t)|, \tag{21}$$

where $\gamma = \sum_{i=1}^{m-2} \beta_i (1 - \xi_i) / (1 - \sum_{i=1}^{m-2} \beta_i \xi_i)$.

Proof. Since $u^{(4)}(t) \leq 0, t \in [0,1]$, then $u'''(t)$ is decreasing on $[0,1]$. Considering $u'''(0) = 0$, we have $u'''(t) \leq 0$, $t \in [0,1]$. Thus, $u''(t)$ is decreasing on $[0,1]$. Considering this together with the boundary condition $u''(0) = 0$, we conclude that $u''(t) \leq 0$. Then, $u(t)$ is concave on $[0,1]$. Taking into account that $u'(0) = 0$, we get that

$$\max_{0 \leq t \leq 1} u(t) = u(0), \qquad \min_{0 \leq t \leq 1} u(t) = u(1). \tag{22}$$

From the concavity of $u(t)$, we have

$$\xi_i (u(1) - u(0)) \leq u(\xi_i) - u(0). \tag{23}$$

Multiplying both sides with β_i and considering the boundary condition, we have

$$\begin{aligned}\min_{0\le t\le 1}|u(t)| &= u(1)\\ &\ge \frac{\left(\sum_{i=1}^{m-2}\beta_i(1-\xi_i)\right)}{\left(1-\sum_{i=1}^{m-2}\beta_i\xi_i\right)u(0)}\\ &= \gamma\max_{0\le t\le 1}|u(t)|.\end{aligned} \tag{24}$$

Problem (1), (2) has a solution $u = u(t)$ if and only if u solves the operator equation

$$u(t) = \int_0^1 G(t,s)\,a(s)\,f(u(s))\,ds := Au(t), \quad 0\le t\le 1. \tag{25}$$

Denote the cone

$$K = \left\{u\in E : u\ge 0,\ \min_{0\le t\le 1}u(t)\ge \gamma\|u\|\right\}. \tag{26}$$

Since K is a cone, by Lemma 7 we have that $A(K)\subset K$. For the convenience, we denote

$$\begin{aligned}M &= \max_{0\le t\le 1}\int_0^1 G(t,s)\,a(s)\,ds,\\ m &= \min_{0\le t\le 1}\int_0^1 G(t,s)\,a(s)\,ds.\end{aligned} \tag{27}$$

□

Theorem 8. *Problem (1), (2) has at least one positive solution if*

(1) $f_0 = 0$, $f_\infty = \infty$ *or*
(2) $f_0 = \infty$, $f_\infty = 0$.

Proof. Firstly, we consider case (1). By $f_0 = 0$, we choose $\varepsilon_1 > 0$ satisfying $\varepsilon_1 M\le 1$. Then, there exists $H_1 > 0$ such that

$$f(u)\le \varepsilon_1 u, \quad u\le H_1. \tag{28}$$

Define $\Omega_1 = \{u\in E : \|u\| < H_1\}$; then, for $u\in K\cap\partial\Omega_1$, we have

$$\begin{aligned}\|Au\| &= \max_{0\le t\le 1}\int_0^1 G(t,s)\,a(s)\,f(u(s))\,ds\\ &\le \int_0^1 G(t,s)\,a(s)\,ds\times\varepsilon_1\times\|u\|\le\|u\|.\end{aligned} \tag{29}$$

On the other hand, by $f_\infty = \infty$, choose ε_2 satisfying $\varepsilon_2 m\gamma\ge 1$. Then, there exists $\overline{H}_2 > 0$ such that

$$f(u)\ge \varepsilon_2 u, \quad \text{for } u\ge \overline{H}_2. \tag{30}$$

Let $H_2 = \max\{2H_1, (1/\gamma)\overline{H}_2\}$ and let $\Omega_2 = \{u\in E : \|u\| < H_2\}$. Then, for $u\in K\cap\partial\Omega_2$, we see $u\ge\gamma(1/\gamma)\overline{H}_2 = \overline{H}_2$. Then,

$$\begin{aligned}\|Au\| &= \max_{0\le t\le 1}\int_0^1 G(t,s)\,a(s)\,f(u(s))\,ds\\ &\ge \varepsilon_2 mu\ge\varepsilon_2 m\gamma\|u\|\ge\|u\|.\end{aligned} \tag{31}$$

Then, by Lemma 3, A has a fixed point u, which implies problem (1), (2) has at least one positive solution $K\cap(\Omega_2\setminus\overline{\Omega}_1)$.

Next, the case $f_0 = \infty$, $f_\infty = 0$ is considered. For $f_0 = \infty$, there exists $H_3 > 0$ such that

$$f(u)\ge\varepsilon_3 u, \quad \text{for } 0 < u\le H_3, \tag{32}$$

where $\varepsilon_3 > 0$ satisfies $\varepsilon_3 m\gamma\ge 1$. Denote $\Omega_3 = \{u\in E : \|u\| < H_3\}$; then, for $u\in K\cap\partial\Omega_3$, we have

$$\begin{aligned}\|Au\| &= \max_{0\le t\le 1}\int_0^1 G(t,s)\,a(s)\,f(u(s))\,ds\\ &\ge \varepsilon_3 mu\ge\varepsilon_3 m\gamma\|u\|\ge\|u\|.\end{aligned} \tag{33}$$

Considering $f_\infty = 0$, there exists $\overline{H}_4 > 0$ such that

$$f(u)\le\varepsilon_4 u, \quad \text{for } u\ge\overline{H}_4, \tag{34}$$

where $\varepsilon_4 > 0$ satisfies $\varepsilon_4 M\le 1$. We distinguish two cases to discuss.

Case 1. Suppose that f is bounded; then, there exists $N > 0$ satisfying $f(u) < N$. Taking $H_4 = \max\{2H_3, NM_1\}$ and $\Omega_4 = \{u\in E : \|u\| < H_4\}$, then for $u\in K\cap\partial\Omega_4$, we have

$$\begin{aligned}\|Au\| &= \max_{0\le t\le 1}|Au(t)|\\ &= \max_{0\le t\le 1}\int_0^1 G(t,s)\,a(s)\,f(u(s))\,ds\\ &\le NM\le H_4\le\|u\|.\end{aligned} \tag{35}$$

Case 2. Suppose that f is unbounded. Since $f\in C([0,+\infty),[0,+\infty))$, then there certainly exists $H_4 > \max\{2H_3, \overline{H_4}\}$ such that

$$f(u)\le f(H_4), \quad 0\le u\le H_4. \tag{36}$$

For $u\in K\cap\partial\Omega_4$, where $\Omega_4 = \{u\in E : \|u\| < H_4\}$, we have

$$\begin{aligned}\|Au\| &= \max_{0\le t\le 1}\int_0^1 G(t,s)\,a(s)\,f(u(s))\,ds\\ &\le Mf(H_4)\le\varepsilon_4 MH_4\le\|u\|.\end{aligned} \tag{37}$$

So, in either case, we can choose $H_4 > 0$ such that $\|Au\|\le\|u\|$, for $u\in K\cap\partial\Omega_4$. Then, Lemma 3 implies that A has a fixed point. Consequently, problem (1), (2) has at least one positive solution. □

Theorem 9. *Problem (1), (2) has at least one positive solution in one of the following cases:*

(3) $f_0 = a\in(0,+\infty)$, $f_\infty = b\in(0,+\infty)$, *and* $aM < 1$, $bm\gamma > 1$;
(4) $f_0 = a\in(0,+\infty)$, $f_\infty = b\in(0,+\infty)$, *and* $bM < 1$, $am\gamma > 1$;
(5) $f_0 = a\in(0,+\infty)$, $f_\infty = \infty$, *and* $aM < 1$;
(6) $f_0 = a$, $f_\infty = 0$, *and* $am\gamma > 1$;

(7) $f_0 = 0$, $f_\infty = b \in (0, +\infty)$, *and* $bm\gamma > 1$;

(8) $f_0 = \infty$, $f_\infty = b$, *and* $bM < 1$.

Proof. We consider the case $f_0 = a \in (0, +\infty)$, $f_\infty = b \in (0, +\infty)$ firstly. Since $f_0 = a \in (0, +\infty)$, for $\varepsilon_5 > 0$ satisfying $(a + \varepsilon_5)M \le 1$, there exists $H_5 > 0$ such that

$$f(u) \le (a + \varepsilon_5)u, \quad \text{for } u \le H_5. \tag{38}$$

Define $\Omega_5 = \{u \in E : \|u\| < H_5\}$; then, for $x \in K \cap \partial\Omega_5$, we see

$$\begin{aligned} \|Au\| &= \max_{0\le t\le 1} \int_0^1 G(t,s)\, a(s)\, f(u(s))\, ds \\ &\le (a + \varepsilon_5) Mu \le \|u\|. \end{aligned} \tag{39}$$

Considering $f_\infty = b$, there exists $\overline{H}_6 \ge 0$ and ε_6 satisfies $(b - \varepsilon_6)m\gamma \ge 1$ such that

$$f(u) \ge (b - \varepsilon_6)u, \quad \text{for } u \ge \overline{H}_6. \tag{40}$$

Let $H_6 = \max\{2H_5, (1/\gamma)\overline{H}_6\}$ and let $\Omega_6 = \{u \in E : \|u\| < H_6\}$; then, for $u \in K \cap \partial\Omega_6$, we get $u \ge \overline{H}_6$; then,

$$\begin{aligned} \|Au\| &= \max_{0\le t\le 1} \int_0^1 G(t,s)\, a(s)\, f(u(s))\, ds \\ &\ge (b - \varepsilon_6) mu \ge (b - \varepsilon_6) m\gamma \|u\| \ge \|u\|. \end{aligned} \tag{41}$$

Then, by Lemma 3, problem (1), (2) has at least one positive solution. As for other cases (4), (5), (6), (7), (8), the proof is considerably analogous with the case (1), (2), (3) and is omitted here. □

Theorem 10. *Assume following conditions are satisfied:*

(C_1) $f_0 = f_\infty = 0$;

(C_2) *there exist* $r^* > 0$ *such that* $f(u) \ge r^*/m$ *for* $u \in [r^*, r^*/\gamma]$;

then, problem (1), (2) has at least two positive solutions u_1, u_2 *satisfying*

$$0 \le \|u_1\| \le r^* \le \|u_2\|. \tag{42}$$

Proof. Considering $f_0 = 0$, similarly with Theorem 8, we claim there exists $r_1 \in (0, r^*)$ such that

$$i(A, K_{r_1}, K) = 1, \quad \text{where } K_{r_1} = \{u \in E : \|u\| < r_1\}. \tag{43}$$

Since $f_\infty = 0$, similarly with above, there exists $r_2 \in (r^*, +\infty)$ such that

$$i(A, K_{r_2}, K) = 1, \quad \text{where } K_{r_2} = \{u \in E : \|u\| < r_2\}. \tag{44}$$

For $u \in K \cap \partial K_{r^*}$, $K_{r^*} = \{u \in E : \|u\| < r^*\}$,

$$\begin{aligned} \|Au\| &= \max_{0\le t\le 1} \int_0^1 G(t,s)\, a(s)\, f(u(s))\, ds \\ &\ge m \times \frac{r^*}{m} = r^* \ge \|u\|. \end{aligned} \tag{45}$$

Hence, $i(A, K_{r^*}, K) = 0$.

Therefore,

$$i(A, K_{r^*} \setminus K_{r_1}, K) = -1, \qquad i(A, K_{r_2} \setminus K_{r^*}, K) = 1. \tag{46}$$

Thus, there exists at least two positive solutions u_1, u_2 such that

$$0 < \|u_1\| \le r^* \le \|u_2\|. \tag{47}$$

□

Theorem 11. *Assume the following conditions are satisfied:*

(C_3) $f_0 = f_\infty = \infty$;

(C_4) *there exists* $R^* > 0$ *such that* $f(u) \le R^*/M$ *for* $u \in [R^*, R^*/\gamma]$;

then, problem (1), (2) has at least two positive solutions u_1, u_2 *satisfying*

$$0 \le \|u_1\| \le R^* \le \|u_2\|. \tag{48}$$

Proof. Considering $f_0 = \infty$, $f_\infty = \infty$, we claim there exists $R_1 \in (0, R^*)$, $R_2 \in (R^*, +\infty)$ such that

$$\begin{aligned} &i(A, K_{R_1}, K) = 0, \quad \text{where } K_{R_1} = \{u \in E : \|u\| < R_1\}, \\ &i(A, K_{R_2}, K) = 0, \quad \text{where } K_{R_2} = \{u \in E : \|u\| < R_2\}. \end{aligned} \tag{49}$$

For $u \in K \cap \partial K_{R^*}$, $K_{R^*} = \{u \in E : \|u\| < R^*\}$,

$$\begin{aligned} \|Au\| &= \max_{0\le t\le 1} \int_0^1 G(t,s)\, a(s)\, f(u(s))\, ds \\ &\le M \times \frac{R^*}{M} = R^* \le \|u\|. \end{aligned} \tag{50}$$

Hence, $i(A, K_{R^*}, K) = 1$. Therefore,

$$i(A, K_{R^*} \setminus K_{R_1}, K) = 1, \qquad i(A, K_{R_2} \setminus K_{R^*}, K) = -1. \tag{51}$$

Thus, there exists at least two positive solutions u_1, u_2 such that

$$0 < \|u_1\| \le R^* \le \|u_2\|. \tag{52}$$

□

4. Existence Results of Problem (1), (3)

Lemma 12. *Denoting* $\xi_0 = 0$, $\xi_{m-1} = 1$, *and* $\beta_0 = \beta_{m-1} = 0$, *Green's function of problem*

$$-u^{(4)}(t) = 0, \tag{53}$$

$$u'''(1) = 0, \qquad u''(1) = 0, \qquad u'(1) = 0,$$

$$u(0) = \sum_{i=1}^{m-2} \beta_i u(\xi_i) \tag{54}$$

is

$$H(t,s)=\begin{cases}\dfrac{1}{6}t^3-\dfrac{1}{2}st^2+\dfrac{1}{2}s^2t \\ \quad+\left(\sum_{k=0}^{i-1}\beta_k\left(\dfrac{1}{6}\xi_k^3-\dfrac{1}{2}\xi_k^2s+\dfrac{1}{2}\xi_ks^2\right)\right. \\ \quad\left.+\dfrac{1}{6}\sum_{k=i}^{m-2}\beta_ks^3\right)\cdot\left(1-\sum_{k=0}^{m-1}\beta_k\right)^{-1}, \\ \qquad t\le s,\quad \xi_{i-1}\le s\le\xi_i \\ \dfrac{1}{6}s^3+\left(\sum_{k=0}^{i-1}\beta_k\left(\dfrac{1}{6}\xi_k^3-\dfrac{1}{2}\xi_k^2s+\dfrac{1}{2}\xi_ks^2\right)\right. \\ \quad\left.+\dfrac{1}{6}\sum_{k=i}^{m-2}\beta_ks^3\right)\cdot\left(1-\sum_{k=0}^{m-1}\beta_k\right)^{-1}, \\ \qquad t\ge s,\quad \xi_{i-1}\le s\le\xi_i,\end{cases} \tag{55}$$

for $i=1,2,\ldots,m-1$.

Proof. For $\xi_{i-1}\le s\le\xi_i$, $i=1,2,\ldots,m-1$, we suppose

$$H(t,s)=\begin{cases}a_3t^3+a_2t^2+a_1t+a_0, & t\le s,\\ b_3t^3+b_2t^2+b_1t+b_0, & t\ge s.\end{cases} \tag{56}$$

Considering the properties of Green's function together with the boundary condition (54), we have

$$\begin{gathered}a_3s^3+a_2s^2+a_1s+a_0=b_3s^3+b_2s^2+b_1s+b_0,\\ 3a_3s^2+2a_2s+a_1=3b_3s^2+2b_2s+b_1,\\ 6a_3s+2a_2=6b_3s+2b_2,\\ 6a_3-6b_3=1,\qquad b_3=0,\qquad 6b_3+2b_2=0,\\ 3b_3+2b_2+b_1=0,\\ a_0=\sum_{k=0}^{i=1}\beta_k\left(a_3\xi_k^3+a_2\xi_k^2+a_1\xi_k+a_0\right)\\ \quad+\sum_{k=i}^{m-2}\beta_k\left(b_3\xi_k^3+b_2\xi_k^2+b_1\xi_k+b_0\right).\end{gathered} \tag{57}$$

Hence,

$$\begin{gathered}a_3=\frac{1}{6},\qquad a_2=-\frac{1}{2}s,\qquad a_1=\frac{1}{2}s^2,\\ a_0=\left(\sum_{k=0}^{i-1}\beta_k\left(\frac{1}{6}\xi_k^3-\frac{1}{2}\xi_k^2s+\frac{1}{2}\xi_ks^2\right)\right.\\ \quad\left.+\frac{1}{6}\sum_{k=i}^{m-2}\beta_ks^3\right)\cdot\left(1-\sum_{k=0}^{m-1}\beta_k\right)^{-1},\\ b_3=0,\qquad b_2=0,\qquad b_1=0,\\ b_0=\frac{1}{6}s^3+\left(\sum_{k=0}^{i-1}\beta_k\left(\frac{1}{6}\xi_k^3-\frac{1}{2}\xi_k^2s+\frac{1}{2}\xi_ks^2\right)\right.\\ \quad\left.+\frac{1}{6}\sum_{k=i}^{m-2}\beta_ks^3\right)\cdot\left(1-\sum_{k=0}^{m-1}\beta_k\right)^{-1}.\end{gathered} \tag{58}$$

The proof of Lemma 12 is completed. □

Lemma 13. *One can see that* $H(t,s)\ge 0$, $t,s\in[0,1]$.

Proof. For $\xi_{i-1}\le s\le\xi_i$, $i=1,2,\ldots,m-1$,

$$\frac{\partial H(t,s)}{\partial t}=\begin{cases}\dfrac{1}{2}(s-t)^2, & t\le s,\ \xi_{i-1}\le s\le\xi_i,\\ 0, & t\ge s,\ \xi_{i-1}\le s\le\xi_i.\end{cases} \tag{59}$$

Then, $\partial H(t,s)/\partial t\ge 0$, $0\le t,s\le 1$, which implies that $H(t,s)$ is increasing on t. The fact that

$$\begin{aligned}H(0,s)=&\left(\sum_{k=0}^{i-1}\beta_k\left(\frac{1}{6}\xi_k^3-\frac{1}{2}\xi_k^2s+\frac{1}{2}\xi_ks^2\right)\right.\\ &\left.+\frac{1}{6}\sum_{k=i}^{m-2}\beta_ks^3\right)\cdot\left(1-\sum_{k=0}^{m-1}\beta_k\right)^{-1}\ge 0\end{aligned} \tag{60}$$

ensures that $H(t,s)\ge 0$, $t,s\in[0,1]$. □

Lemma 14. *If* $y(t)\ge 0$, $t\in[0,1]$, *and* $u(t)$ *is the solution of problem (53), (54), then*

$$\min_{0\le t\le 1}|u(t)|\ge\gamma_1\max_{0\le t\le 1}|u(t)|, \tag{61}$$

where $\gamma_1=\sum_{i=1}^{m-2}\beta_i\xi_i/(1-\sum_{i=1}^{m-2}\beta_i(1-\xi_i))$.

Proof. It follows from the same methods as Lemma 7 that $u(t)$ is concave on $[0,1]$. Taking into account that $u'(1)=0$, one see that $u(t)$ is increasing on $[0,1]$ and

$$\max_{0\le t\le 1}u(t)=u(1),\qquad \min_{0\le t\le 1}u(t)=u(0). \tag{62}$$

From the concavity of $u(t)$, we have

$$\xi_i(u(1)-u(0))\le u(\xi_i)-u(0). \tag{63}$$

Multiplying both sides with β_i and considering the boundary condition, we have

$$\sum_{i=1}^{m-2}\beta_i\xi_iu(1)\le\left(1-\sum_{i=1}^{m-2}\beta_i(1-\xi_i)\right)u(0). \tag{64}$$

Thus,

$$\min_{0\le t\le 1}|u(t)|\ge\gamma_1\max_{0\le t\le 1}|u(t)|. \tag{65}$$

Problem (1), (3) has a solution $u = u(t)$ if and only if u solves the operator equation

$$u(t) = \int_0^1 H(t,s)\, a(s)\, f(u)\, ds := A_1 u(t), \quad 0 \le t \le 1. \tag{66}$$

Denote the cone

$$K_1 = \left\{ u \in E : u \ge 0,\ \min_{0\le t\le 1} u(t) \ge \gamma_1 \|u\| \right\}. \tag{67}$$

It is obvious that $A_1(K_1) \subset K_1$. We denote

$$M_1 = \max_{0\le t\le 1} \int_0^1 H(t,s)\, a(s)\, ds,$$
$$m_1 = \min_{0\le t\le 1} \int_0^1 H(t,s)\, a(s)\, ds. \tag{68}$$

□

Theorem 15. *Problem (1), (3) has at least one positive solution if*

(1) $f_0 = 0$, $f_\infty = \infty$,
(2) $f_0 = \infty$, $f_\infty = 0$.

Theorem 16. *Problem (1), (3) has at least one positive solution in one of the following cases:*

(3) $f_0 = a \in (0, +\infty)$, $f_\infty = b \in (0, +\infty)$, *and* $aM_1 < 1$, $bm_1\gamma_1 > 1$;
(4) $f_0 = a \in (0, +\infty)$, $f_\infty = b \in (0, +\infty)$, *and* $bM_1 < 1$, $am_1\gamma_1 > 1$;
(5) $f_0 = a \in (0, +\infty)$, $f_\infty = \infty$, *and* $aM_1 < 1$;
(6) $f_0 = a$, $f_\infty = 0$, *and* $am_1\gamma_1 > 1$;
(7) $f_0 = 0$, $f_\infty = b \in (0, +\infty)$, *and* $bm_1\gamma_1 > 1$;
(8) $f_0 = \infty$, $f_\infty = b$, *and* $bM_1 < 1$.

Theorem 17. *Assume following conditions are satisfied:*

(C_5) $f_0 = f_\infty = 0$;
(C_6) *there exist* $r^* > 0$ *such that* $f(u) \ge r^*/m_1$ *for* $u \in [r^*, r^*/\gamma_1]$;

then, problem (1), (3) has at least two positive solutions u_1, u_2 *satisfying*

$$0 \le \|u_1\| \le r^* \le \|u_2\|. \tag{69}$$

Theorem 18. *Assume following conditions are satisfied:*

(C_7) $f_0 = f_\infty = \infty$;
(C_8) *there exists* $R^* > 0$ *such that* $f(u) \le R^*/M_1$ *for* $u \in [R^*, R^*/\gamma_1]$;

then, problem (1), (3) has at least two positive solutions u_1, u_2 *satisfying*

$$0 \le \|u_1\| \le R^* \le \|u_2\|. \tag{70}$$

5. Examples

In this section, we given some examples to illustrate the main results established in this paper.

Example 1. Consider the fourth-order boundary value problem

$$u^{(4)}(t) + \pi (1-t)^2 u^\alpha(t) = 0, \quad 0 \le t \le 1,$$
$$u'''(0) = u''(0) = u'(0) = 0, \quad u(1) = \frac{1}{2} u\left(\frac{2}{3}\right), \tag{71}$$

where $\alpha \in (0,1) \cup (1, \infty)$. Then, problems (71) have at least one positive solution. In fact, when $\alpha \in (0,1)$, $f_0 = \infty$ and $f_\infty = 0$; when $\alpha \in (1, \infty)$, $f_0 = 0$ and $f_0 = \infty$. From Theorem 8, there exists at least one positive solution of problem (71).

Example 2. Consider the fourth-order boundary value problem

$$u^{(4)} + (1-t)^{1/2} \left(u^{1/2} + u^{5/3}\right) = 0, \quad 0 \le t \le 1,$$
$$u'''(1) = u''(1) = u'(1) = 0, \quad u(0) = \frac{2}{3} u\left(\frac{1}{2}\right). \tag{72}$$

We see that

$$H(t,s) = \begin{cases} \dfrac{1}{6}t^3 - \dfrac{1}{2}st^2 + \dfrac{1}{2}s^2 t + \dfrac{1}{3}s^3, & t \le s,\ 0 \le s \le \dfrac{2}{3}, \\ \dfrac{1}{2}s^3, & t \ge s,\ 0 \le s \le \dfrac{2}{3}, \\ \dfrac{1}{6}t^3 - \dfrac{1}{2}st^2 + \dfrac{1}{2}s^2 t + \dfrac{1}{2}s^2 - \dfrac{1}{4}s + \dfrac{1}{24}, & t \le s,\ \dfrac{2}{3} \le s \le 1, \\ \dfrac{1}{6}s^3 + \dfrac{1}{2}s^2 - \dfrac{1}{4}s + \dfrac{1}{24}, & t \ge s,\ \dfrac{2}{3} \le s \le 1, \end{cases}$$
$$\gamma_1 = \frac{1}{2}, \qquad M = \max_{0\le t\le 1} \int_0^1 H(t,s)\, a(s)\, ds = \frac{37}{1920},$$
$$m = \min_{0\le t\le 1} \int_0^1 H(t,s)\, a(s)\, ds = \frac{31}{3840}. \tag{73}$$

It is easy to check that $f_0 = f_\infty = \infty$. Taking $R = 8$, we see

$$f(u) \le \frac{15360}{37}, \qquad u \in [8, 16]. \tag{74}$$

From Theorem 18, there exist at least two positive solutions of problem (72) such that

$$\|u_1\| \le 8 \le \|u_2\|. \tag{75}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The work is sponsored by the NSFC (11201109), Anhui Provincial Natural Science Foundation (1408085QA07), and the Higher School Natural Science Project of Anhui Province (KJ2014A200).

References

[1] A. R. Aftabizadeh, "Existence and uniqueness theorems for fourth-order boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 116, no. 2, pp. 415–426, 1986.

[2] C. P. Gupta, "Existence and uniqueness theorems for the bending of an elastic beam equation," *Applicable Analysis*, vol. 26, no. 4, pp. 289–304, 1988.

[3] M. A. del Pino and R. F. Manásevich, "Existence for a fourth-order boundary value problem under a two-parameter nonresonance condition," *Proceedings of the American Mathematical Society*, vol. 112, no. 1, pp. 81–86, 1991.

[4] G. Bonanno and B. Di Bella, "A boundary value problem for fourth-order elastic beam equations," *Journal of Mathematical Analysis and Applications*, vol. 343, no. 2, pp. 1166–1176, 2008.

[5] Z. Wei, "A class of fourth order singular boundary value problems," *Applied Mathematics and Computation*, vol. 153, no. 3, pp. 865–884, 2004.

[6] X. Zhang and L. Liu, "A necessary and sufficient condition of positive solutions for nonlinear singular differential systems with four-point boundary conditions," *Applied Mathematics and Computation*, vol. 215, no. 10, pp. 3501–3508, 2010.

[7] H. Feng, D. Ji, and W. Ge, "Existence and uniqueness of solutions for a fourth-order boundary value problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 10, pp. 3561–3566, 2009.

[8] H. Pang and W. Ge, "Existence results for some fourth-order multi-point boundary value problem," *Mathematical and Computer Modelling*, vol. 49, no. 7-8, pp. 1319–1325, 2009.

[9] X. Zhang and L. Liu, "A necessary and sufficient condition for positive solutions for fourth-order multi-point boundary value problems with p-Laplacian," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 10, pp. 3127–3137, 2008.

[10] E. Alves, T. F. Ma, and M. L. Pelicer, "Monotone positive solutions for a fourth order equation with nonlinear boundary conditions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 9, pp. 3834–3841, 2009.

[11] J. R. Graef, C. Qian, and B. Yang, "A three point boundary value problem for nonlinear fourth order differential equations," *Journal of Mathematical Analysis and Applications*, vol. 287, no. 1, pp. 217–233, 2003.

[12] X. Zhang and L. Liu, "Eigenvalue of fourth-order m-point boundary value problem with derivatives," *Computers & Mathematics with Applications*, vol. 56, no. 1, pp. 172–185, 2008.

[13] J. Yang and Z. Wei, "Existence of positive solutions for fourth-order m-point boundary value problems with a one-dimensional p-Laplacian operator," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2985–2996, 2009.

[14] R. Ma, "Existence of positive solutions of a fourth-order boundary value problem," *Applied Mathematics and Computation*, vol. 168, no. 2, pp. 1219–1231, 2005.

[15] R. Ma and J. Xu, "Bifurcation from interval and positive solutions of a nonlinear fourth-order boundary value problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 1, pp. 113–122, 2010.

[16] C. Bai, D. Yang, and H. Zhu, "Existence of solutions for fourth order differential equation with four-point boundary conditions," *Applied Mathematics Letters*, vol. 20, no. 11, pp. 1131–1136, 2007.

[17] Y.-R. Yang, "Triple positive solutions of a class of fourth-order two-point boundary value problems," *Applied Mathematics Letters*, vol. 23, no. 4, pp. 366–370, 2010.

[18] E. R. Kaufmann and N. Kosmatov, "Elastic beam problem with higher order derivatives," *Nonlinear Analysis. Real World Applications*, vol. 8, no. 3, pp. 811–821, 2007.

[19] C. Bai, "Triple positive solutions of three-point boundary value problems for fourth-order differential equations," *Computers & Mathematics with Applications*, vol. 56, no. 5, pp. 1364–1371, 2008.

[20] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Orlando, Fla, USA, 1988.

Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 954925, 9 pages
http://dx.doi.org/10.1155/2015/954925

Research Article

Boundary Value Problems of Fractional Order Differential Equation with Integral Boundary Conditions and Not Instantaneous Impulses

Peiluan Li[1] and Changjin Xu[2]

[1]*Department of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, China*
[2]*Guizhou Key Laboratory of Economics System Simulation, Guizhou College of Finance and Economics, Guiyang 550004, China*

Correspondence should be addressed to Peiluan Li; lpllpl_lpl@163.com

Received 3 October 2014; Revised 11 February 2015; Accepted 22 February 2015

Academic Editor: Jesús G. Falset

Copyright © 2015 P. Li and C. Xu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate the existence of mild solutions for fractional order differential equations with integral boundary conditions and not instantaneous impulses. By some fixed-point theorems, we establish sufficient conditions for the existence and uniqueness of solutions. Finally, two interesting examples are given to illustrate our theory results.

1. Introduction

Impulsive differential equations are used to describe many practical dynamical systems including evolutionary processes characterized by abrupt changes of the state at certain instants. Such processes are naturally seen in biology, physics, engineering, and so forth. Due to their significance, many authors have established the solvability of impulsive differential equations. Nowadays, the theory of impulsive differential equations has received great attention. Differential equations with instantaneous impulses have been treated in several works (see, e.g., the monographs [1–3], the works on time variable impulses problem [4–7], and the references therein).

However, in almost all the papers concerning impulsive differential equations, the impulses are all instantaneous impulses, and the classical models with instantaneous impulses cannot characterize many practical problems, for example, the dynamics of evolution processes in pharmacotherapy. Let us consider the hemodynamic equilibrium of a person. The introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes. In fact, this situation should be characterized by a new case of impulsive action, which starts at an arbitrary fixed point t_i and stays active on a finite time interval $[t_i, s_i]$. To this end, Hernández and O'Regan [8] initially offered to study a new class of abstract semilinear impulsive differential equations with not instantaneous impulses in a PC-normed Banach space. In [8], the authors discussed the following problem:

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t)), \quad t \in (s_i, t_{i+1}], \ i = 0, \ldots, N, \\ u(t) &= g_i(t, u(t)), \quad t \in (t_i, s_i], \ i = 1, \ldots, N, \\ u(0) &= x_0, \end{aligned} \tag{1}$$

where $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup of bounded linear operators $(T(t))_{t>0}$ defined on a Banach space $(X, \|\cdot\|)$, $0 = t_0 = s_0 < t_1 \le s_1 \le \cdots \le t_N \le s_N \le t_{N+1} = a$ are prefixed numbers, $x_0 \in X$, $g_i \in C((t_i, s_i] \times X, X)$ for all $i = 1, \ldots, N$, and $f : [0, a] \times X \to X$ is a suitable function. Meanwhile, Pierri et al. [9] continued the work in [8] in a PC_α-normed Banach space.

On the one hand, the absorption of drugs has a memory effect; thus, the new class of impulsive conditions introduced by [8] may not explain this phenomenon very well. On the other hand, fractional calculus provides a powerful tool for the description of hereditary properties of various materials and memory processes [10, 11]. Fractional differential equations have recently proved to be strong tools in the modeling

of medical, physics, economics, and technical sciences. For more details on fractional calculus theory, one can see the monographs of Diethelm [12], Kilbas et al. [13], Lakshmikantham et al. [14], Miller and Ross [15], Podlubny [16], and Tarasov [17]. Fractional differential equations involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attention (see [10, 11, 18–22]).

The theory of boundary value problems (BVPs) with integral boundary conditions for differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. For BVPs with integral boundary conditions and comments on their importance, we refer the readers to the papers by Gallardo [23], Karakostas and Tsamatos [24], Lomtatidze and Malaguti [25], and the references therein. For more information about the general theory of integral equations and their relation with BVPs, we refer to the books of Corduneanu [26] and Agarwal and O'Regan [27]. Moreover, BVPs with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint, and nonlocal BVPs as special cases. The existence and multiplicity of positive solutions for such problems have received a great deal of attention. To identify a few, we refer the readers to [28–31] and references therein.

In [32], the authors consider the following problem:

$$\begin{aligned} {}^{c}D_{t}^{q}u(t) &= f\left(t, u(t), \int_{0}^{t} k(t,s,u(s))\,ds\right), \\ t &\in J' := J \setminus \{t_1, t_2, \ldots, t_m\}, \quad J := [0,T], \\ u(t_k^+) &= u(t_k^-) + u_k, \quad u_k \in R,\ k = 0,1,\ldots,m, \\ u(0) &= \int_{0}^{1} g(s)\,u(s)\,ds, \end{aligned} \tag{2}$$

where ${}^{c}D_{t}^{q}$ is the Caputo fractional derivative of order $q \in (0,1)$ with the lower limit zero, $f : J \times X \times X \to X$ is a given function, $k : \Delta \times X \to X$, $g \in L^1([0,1], R^+)$, $g(t) \in [0,1)$, $\Delta = \{(t,s) : 0 \le s \le t \le 1\}$, and t_k satisfy $0 = t_0 < t_1 < \cdots < t_{m+1} = T$, $u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$ and $u(t_k^-) = \lim_{h \to 0^-} u(t_k + h)$ represent the right and left limits of $u(t)$ at $t = t_k$. Obviously, the impulses in (2) are instantaneous. Motivated by the work in [8, 9, 32], in this paper, we consider the following impulsive fractional differential equations with integral boundary conditions and not instantaneous impulses:

$$\begin{aligned} {}^{c}D_{t}^{q}u(t) &= f\left(t, u(t), \int_{0}^{t} k(t,s,u(s))\,ds\right), \\ t &\in (s_i, t_{i+1}], \quad i = 0,\ldots,N, \\ u(t) &= g_i(t, u(t)), \quad t \in (t_i, s_i],\ i = 1,\ldots,N, \\ u(0) &= \int_{0}^{T} w(s)\,u(s)\,ds, \end{aligned} \tag{3}$$

where ${}^{c}D_{t}^{q}$ is the Caputo fractional derivative of order $q \in (0,1)$ with the lower limit zero, $0 = t_0 = s_0 < t_1 \le s_1 \le \cdots \le t_N \le s_N \le t_{N+1} = T$ are prefixed numbers, $g_i \in C((t_i, s_i] \times R, R)$, for $i = 1,\ldots,N$, $J = [0,T]$, $f : J \times R \times R \to R$, $k : \Delta \times R \to R$, $w \in L^1([0,T], R^+)$, and $w \in [0, 1/T)$, $\Delta = \{(t,s) : 0 \le s \le t \le T\}$.

The rest of this paper is organized as follows. In Section 2, some lemmas which are essential to prove our main results are stated. In Section 3, we give the main results. In Section 4, two interesting examples are given to illustrate our theory results.

2. Preliminaries

At first, we present the necessary definitions for the fractional calculus theory.

Definition 1 (see [13]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a suitable function $y : (0,\infty) \to R$ is given by

$$I_{0_+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} y(s)\,ds, \tag{4}$$

where the right side is pointwise defined on $(0, +\infty)$.

Definition 2 (see [13]). The Caputo fractional derivative of order $\alpha > 0$ of a suitable function $y : (0,\infty) \to R$ is given by

$${}^{c}D^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} y^{(n)}(s)\,ds, \tag{5}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α and the right side is pointwise defined on $(0, +\infty)$.

Lemma 3 (see [13]). *Let $\alpha > 0$; then the fractional differential equation ${}^{c}D^{\alpha}u(t) = 0$ has solutions*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{6}$$

where $c_i \in R$, $i = 0, 1, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 4 (see [13]). *Let $\alpha > 0$, then one has*

$$I_{0_+}^{\alpha}\, {}^{c}D^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{7}$$

where $c_i \in R$, $i = 0, 1, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 5 (Krasnoselskii's fixed point theorem [33]). *Let M be a closed convex and nonempty subset of a Banach space X. Let A and B be two operators such that*

(1) *$Ax + By \in M$ whenever $x, y \in M$;*

(2) *A is compact and continuous;*

(3) *B is a contraction mapping.*

Then there exists $z \in M$ such that $z = Az + Bz$.

In order to study problem (3), we define $X = PC(J, R) = \{x : J \to R; x \in C((t_k, t_{k+1}], R)$, $k = 0, 1, \ldots, N$, and $x(t_k^+)$, $x(t_k^-)$ exist with $x(t_k^-) = x(t_k)$, $k = 1, \ldots, N\}$.

It is easy to check that X is a Banach space with the norm $\|x\|_{PC} = \sup_{t \in J} |x(t)|$.

Let $Bu(t) = \int_0^t k(t,s,u(s))ds$, $\sigma = \int_0^{t_1} w(t)dt$; then $0 \le \sigma \le \int_0^T w(t)dt < 1$.

If $u \in \mathrm{PC}(J,R)$ satisfies problem (3), then for $t \in (s_i, t_{i+1}]$, $i = 1,\ldots,N$, integrating the first equation of (3) from s_i to t by virtue of Definition 1, one can obtain

$$u(t) = u(s_i) + \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} f(s, u(s), Bu(s))\, ds. \tag{8}$$

From the second equation in (3), we know $u(s_i) = g_i(s_i, u(s_i))$. Then, for $t \in (s_i, t_{i+1}]$, we have

$$u(t) = g_i(s_i, u(s_i)) + \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} f(s, u(s), Bu(s))\, ds. \tag{9}$$

For $t \in [0, t_1]$, integrating the first equation in (3) from 0 to t by virtue of Definition 1, one can obtain

$$u(t) = u(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s), Bu(s))\, ds. \tag{10}$$

By the boundary conditions, we have

$$u(t) = \int_0^T w(t)\, u(t)\, dt + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s), Bu(s))\, ds. \tag{11}$$

Multiplying (11) with $w(t)$ and integrating from 0 to t_1, we have

$$\int_0^{t_1} w(t)\, u(t)\, dt = \int_0^{t_1} w(t)\, dt \int_0^T w(s)\, u(s)\, ds + \frac{1}{\Gamma(q)} \int_0^{t_1} w(t) \int_0^t (t-s)^{q-1} f(s, u(s), Bu(s))\, ds\, dt, \tag{12}$$

Multiplying (9) with $w(t)$ and integrating from s_i to t_{i+1}, $i = 1,\ldots,N$, we have

$$\int_{s_i}^{t_{i+1}} w(t)\, u(t)\, dt = g_i(s_i, u(s_i)) \int_{s_i}^{t_{i+1}} w(t)\, dt + \frac{1}{\Gamma(q)} \cdot \int_{s_i}^{t_{i+1}} w(t) \frac{1}{\Gamma(q)} \int_{s_i}^t (t-s)^{q-1} f(s, u(s), Bu(s))\, ds\, dt. \tag{13}$$

Multiplying the second equation of (3) with $w(t)$ and integrating from t_i to s_i, $i = 1,\ldots,N$, we can obtain

$$\int_{t_i}^{s_i} w(t)\, u(t)\, dt = \int_{t_i}^{s_i} w(t)\, g_i(t, u(t))\, dt. \tag{14}$$

Adding (12), (13), and (14), one has

$$\begin{aligned} &\int_0^T w(t)\, u(t)\, dt \\ &= \int_0^{t_1} w(t)\, dt \int_0^T w(s)\, u(s)\, ds + \sum_{i=1}^N g_i(s_i, u(s_i)) \int_{s_i}^{t_{i+1}} w(t)\, dt \\ &\quad + \frac{1}{\Gamma(q)} \sum_{i=0}^N \int_{s_i}^{t_{i+1}} w(t) \int_{s_i}^t (t-s)^{q-1} f(s, u(s), Bu(s))\, ds\, dt \\ &\quad + \sum_{i=1}^N \int_{t_i}^{s_i} w(t)\, g_i(t, u(t))\, dt. \end{aligned} \tag{15}$$

Hence

$$\begin{aligned} &\int_0^T w(t)\, u(t)\, dt \\ &= \frac{1}{1-\sigma} \cdot \left\{ \sum_{i=1}^N g_i(s_i, u(s_i)) \int_{s_i}^{t_{i+1}} w(t)\, dt \right. \\ &\quad + \frac{1}{\Gamma(q)} \sum_{i=0}^N \int_{s_i}^{t_{i+1}} w(t) \int_{s_i}^t (t-s)^{q-1} f(s, u(s), Bu(s))\, ds\, dt \\ &\quad \left. + \sum_{i=1}^N \int_{t_i}^{s_i} w(t)\, g_i(t, u(t))\, dt \right\}. \end{aligned} \tag{16}$$

So, for $t \in [0, t_1]$, we have

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s), Bu(s))\, ds \\ &\quad + \frac{1}{1-\sigma} \cdot \left\{ \sum_{i=1}^N g_i(s_i, u(s_i)) \int_{s_i}^{t_{i+1}} w(t)\, dt + \frac{1}{\Gamma(q)} \right. \\ &\quad \cdot \sum_{i=0}^N \int_{s_i}^{t_{i+1}} w(t) \int_{s_i}^t (t-s)^{q-1} f(s, u(s), Bu(s))\, ds\, dt \\ &\quad \left. + \sum_{i=1}^N \int_{t_i}^{s_i} w(t)\, g_i(t, u(t))\, dt \right\}. \end{aligned} \tag{17}$$

Then similar to Definition 2.1 in [9], we can define the mild solution for (3).

Definition 6. A function $u \in \mathrm{PC}(J,R)$ is a mild solution of problems (3) if

$$u(0) = \int_0^T w(s)\,u(s)\,ds,$$
$$u(t) = g_i(t,u(t)), \quad \forall t \in (t_i,s_i],\ i = 1,\ldots,N,$$
$$\begin{aligned} u(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds + \frac{1}{1-\sigma} \\ &\cdot \left\{ \sum_{i=1}^{N} g_i(s_i,u(s_i)) \int_{s_i}^{t_{i+1}} w(t)\,dt + \frac{1}{\Gamma(q)} \right. \\ &\cdot \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} w(t) \int_{s_i}^{t} (t-s)^{q-1} f(s,u(s),Bu(s))\,ds\,dt \\ &\left. + \sum_{i=1}^{N} \int_{t_i}^{s_i} w(t)\,g_i(t,u(t))\,dt \right\} \end{aligned} \tag{18}$$

for all $t \in [0,t_1]$ and

$$\begin{aligned} u(t) &= g_i(s_i,u(s_i)) + \frac{1}{\Gamma(q)}\int_{s_i}^{t} (t-s)^{q-1} f(s,u(s),Bu(s))\,ds \end{aligned} \tag{19}$$

for all $t \in (s_i,t_{i+1}]$, $i = 1,\ldots,N$.

3. Main Results

This section deals with the existence of mild solutions for (3). Before stating and proving the main results, we make the following hypotheses.

(H_1) $f : J \times R \times R \to R$ is jointly continuous. There exists a function $e \in C(J,R)$ such that

$$\begin{aligned} &\left|f(t,u_1(t),v_1(t)) - f(t,u_2(t),v_2(t))\right| \\ &\quad \le e(t)\left(\left|u_1(t)-u_2(t)\right| + \left|v_1(t)-v_2(t)\right|\right), \end{aligned} \tag{20}$$

for all $u_1,v_1,u_2,v_2 \in \mathrm{PC}(J,R)$, for all $t \in J$.

(H_2) $k : \Delta \times R \to R$ is continuous and there exists a function $d \in C[J,R^+]$ such that

$$\begin{aligned} &|k(t,s,u(s)) - k(t,s,v(s))| \le d(t)\,(|u(t)-v(t)|), \\ &\qquad \forall u,v \in \mathrm{PC}(J,R), \quad (t,s) \in \Delta. \end{aligned} \tag{21}$$

(H_3) $g_i \in C((t_i,s_i] \times R, R)$ and there exist $l_i \in C[J,R]$, $i = 1,\ldots,N$ such that

$$\begin{aligned} &\left|g_i(t,x(t)) - g_i(t,y(t))\right| \le l_i(t)\left|x(t)-y(t)\right|, \\ &\qquad \forall x,y \in \mathrm{PC}(J,R), \quad \forall t \in J. \end{aligned} \tag{22}$$

Let

$$\begin{aligned} L &= \max_{1\le i\le N} \sup_{t\in J} \left|l_i(t)\right|, \qquad M = \sup_{t\in J} |e(t)|, \\ D &= \sup_{t\in J} |d(t)|. \end{aligned} \tag{23}$$

Now we are in the position to establish the main results. Our first theorem is based on contraction mapping principle.

Theorem 7. *Let (H_1)–(H_3) hold and $n < 1$; then the problem (3) has a unique mild solution, where*

$$n = \frac{T^q M(1+TD)(2-\sigma)}{\Gamma(q+1)(1-\sigma)} + \frac{2L}{1-\sigma}. \tag{24}$$

Proof. Let $A : \mathrm{PC}(J,R) \to \mathrm{PC}(J,R)$ be the map defined by

$$Au(t) = g_i(t,u(t)), \quad \text{for } t \in (t_i,s_i],\ i = 1,\ldots,N,$$
$$\begin{aligned} Au(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,u(s),Bu(s))\,ds + \frac{1}{1-\sigma} \\ &\cdot \left\{ \sum_{i=1}^{N} g_i(s_i,u(s_i)) \int_{s_i}^{t_{i+1}} w(t)\,dt + \frac{1}{\Gamma(q)} \right. \\ &\cdot \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} w(t) \int_{s_i}^{t} (t-s)^{q-1} f(s,u(s),Bu(s))\,ds\,dt \\ &\left. + \sum_{i=1}^{N} \int_{t_i}^{s_i} w(t)\,g_i(t,u(t))\,dt \right\} \end{aligned} \tag{25}$$

for $t \in [0,t_1]$ and

$$\begin{aligned} Au(t) &= g_i(s_i,u(s_i)) \\ &+ \frac{1}{\Gamma(q)}\int_{s_i}^{t} (t-s)^{q-1} f(s,u(s),Bu(s))\,ds \end{aligned} \tag{26}$$

for $t \in (s_i,t_{i+1}]$, $i = 1,\ldots,N$. Clearly, A is well defined.

Next we show that A is contraction on $\mathrm{PC}(J,R)$.

Fix $x,y \in \mathrm{PC}(J,R)$; we consider three cases.

Case 1. If $t \in (s_i,t_{i+1}]$, $i = 1,\ldots,N$, by the assumptions (H_1)–(H_3) and the property $\Gamma(q+1) = q\Gamma(q)$, we have

$$\begin{aligned} &\left|Ax(t) - Ay(t)\right| \\ &\le \left|g_i(s_i,x(s_i)) - g_i(s_i,y(s_i))\right| \\ &+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} \\ &\qquad \cdot \left|f(s,x(s),Bx(s))\,ds - f(s,y(s),By(s))\right| ds \end{aligned}$$

$$\begin{aligned}
&\le L\left\|x-y\right\|_{\mathrm{PC}}+\frac{M}{\Gamma(q)}\left[-\frac{(t-s)^q}{q}\right]\Big|_0^t\\
&\quad\cdot\left(\left\|x-y\right\|_{\mathrm{PC}}+\left\|Bx-By\right\|_{\mathrm{PC}}\right)\\
&=L\left\|x-y\right\|_{\mathrm{PC}}+\frac{M}{\Gamma(q)}\frac{t^q}{q}\left(\left\|x-y\right\|_{\mathrm{PC}}+\left\|Bx-By\right\|_{\mathrm{PC}}\right)\\
&\le L\left\|x-y\right\|_{\mathrm{PC}}+\frac{T^qM}{\Gamma(q+1)}\left(\left\|x-y\right\|_{\mathrm{PC}}+\left\|Bx-By\right\|_{\mathrm{PC}}\right)\\
&\le\left[L+\frac{T^qM(1+TD)}{\Gamma(q+1)}\right]\left\|x-y\right\|_{\mathrm{PC}}.
\end{aligned}\tag{27}$$

Case 2. If $t\in[0,t_1]$, by (H_1), (H_2), one can obtain

$$\begin{aligned}
&\left|Ax(t)-Ay(t)\right|\\
&\le\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\\
&\qquad\cdot\left|f(s,x(s),Bx(s))-f(s,y(s),By(s))\right|ds\\
&\quad+\frac{1}{1-\sigma}\left\{\sum_{i=1}^N\left|g_i(s_i,x(s_i))-g_i(s_i,y(s_i))\right|\int_{s_i}^{t_{i+1}}w(t)\,dt\right.\\
&\qquad+\frac{1}{\Gamma(q)}\sum_{i=0}^N\int_{s_i}^{t_{i+1}}w(t)\int_{s_i}^t(t-s)^{q-1}\\
&\qquad\qquad\cdot\left|f(s,x(s),Bx(s))\right.\\
&\qquad\qquad\left.-f(s,y(s),By(s))\right|dt\\
&\qquad\left.+\sum_{i=1}^N\int_{t_i}^{s_i}w(t)\left|g_i(t,x(t))-g_i(t,y(t))\right|dt\right\}.
\end{aligned}\tag{28}$$

Then, by a similar argument, we can get

$$\begin{aligned}
&\left|Ax(t)-Ay(t)\right|\\
&\le\left(\frac{T^qM(1+TD)(2-\sigma)}{\Gamma(q+1)(1-\sigma)}+\frac{2L}{1-\sigma}\right)\left\|x-y\right\|_{\mathrm{PC}}.
\end{aligned}\tag{29}$$

Case 3. If $t\in(t_i,s_i]$, $i=1,\dots,N$, from the assumption (H_3), we get

$$\left|Ax(t)-Ay(t)\right|\le L\left\|x-y\right\|_{\mathrm{PC}}.\tag{30}$$

Therefore, $\|Ax-Ay\|_{\mathrm{PC}}\le n\|x-y\|_{\mathrm{PC}}$, for all $x,y\in\mathrm{PC}(J,R)$, which implies that A is a contraction mapping. Then, there exists a unique mild solution of (3).

In order to get the second main result, we give assumption (H_4).

(H_4) The function $f:J\times R\times R\to R$ is jointly continuous and strongly measurable on J. There exist $m_f\in C(J,R^+)$ and a nondecreasing function $h_f\in C(R^+,R^+)$ such that

$$\left|f(t,x,y)\right|\le m_f(t)\,h_f\left(\left|x+y\right|\right)\quad\forall t\in J,\ x,y\in R.\tag{31}$$

Our second result is based on Krasnoselskii's fixed point theorem. □

Theorem 8. *Assume that (H_2)–(H_4) hold; if $2L/(1-\sigma)<1$ and there exists a constant $r>0$ such that*

$$\begin{aligned}
&r(1-\sigma-2L)\\
&\quad\ge\frac{T^q(2-\sigma)}{\Gamma(q+1)}\left\|m_f\right\|_{PC}h_f\left[r(1+TD)+TK\right]+2a,
\end{aligned}\tag{32}$$

where $a=\max_{i=1,\dots,N}|g_i(t,0)|$, then the problem (3) has at least a mild solution.

Proof. Let $Au(t)$ be the map introduced in the proof of Theorem 7. We consider the decomposition $Au(t)=A^1u(t)+A^2u(t)$, where

$$A^1u(t)=\sum_{i=1}^NA_i^1u(t),\qquad A^2u(t)=\sum_{i=1}^NA_i^2u(t),$$

$$A_i^1u(t)=\begin{cases}
\dfrac{1}{\Gamma(q)}\displaystyle\int_{s_i}^t(t-s)^{q-1}f(s,u(s),Bu(s))\,ds,\\
\qquad\text{if } t\in(s_i,t_{i+1}],\quad i\ge1,\\
\dfrac{1}{\Gamma(q)}\displaystyle\int_0^t(t-s)^{q-1}f(s,u(s),Bu(s))\,ds\\
\quad+\dfrac{1}{(1-\sigma)\Gamma(q)}\\
\quad\cdot\displaystyle\sum_{i=0}^N\int_{s_i}^{t_{i+1}}w(t)\int_{s_i}^t(t-s)^{q-1}\\
\qquad\qquad\cdot f(s,u(s),Bu(s))\,ds\,dt,\\
\qquad\text{if } t\in[0,t_1],\\
0,\quad\text{else},
\end{cases}$$

$$A_i^2u(t)=\begin{cases}
g_i(t,u(t)),\qquad\text{if } t\in(t_i,s_i],\ i\ge1,\\
g_i(s_i,u(s_i)),\qquad\text{if } t\in(s_i,t_{i+1}],\ i\ge1,\\
\dfrac{1}{1-\sigma}\left\{\displaystyle\sum_{i=1}^Ng_i(s_i,u(s_i))\int_{s_i}^{t_{i+1}}w(t)\,dt\right.\\
\qquad\left.+\displaystyle\sum_{i=1}^N\int_{t_i}^{s_i}w(t)\,g_i(t,u(t))\,dt\right\}\\
\qquad\text{if } t\in[0,t_1],\\
0,\qquad\text{else}.
\end{cases}\tag{33}$$

Let $B_r=\{x\in\mathrm{PC}(J,R):\|x\|_{\mathrm{PC}}\le r\}$. We divide our proof into three steps.

Step 1. First we show that $A^1x + A^2y \in B_r$ whenever $x, y \in B_r$.

From (H_2), we know that $k : \Delta \times X \to X$ is continuous, and then $k(t,s,0)$ is bounded, for $(t,s) \in \Delta = \{(t,s) : 0 \le s \le t \le T\}$. Let $K := \max\{k(t,s,0) : (t,s) \in \Delta\}$.

Let $x \in B_r$; if $t \in (s_i, t_{i+1}]$, we have

$$\begin{aligned}
&\left|A_i^1 x(t)\right| \\
&\le \frac{1}{\Gamma(q)} \int_{s_i}^{t} (t-s)^{q-1} \left| f(s, x(s), Bx(s)) \right| ds \\
&\le \frac{1}{\Gamma(q)} \left(\int_{s_i}^{t} (t-s)^{q-1} m_f(s)\, ds \right) h_f\left(\|x + Bx\|_{PC}\right) \\
&\le \frac{1}{\Gamma(q)} \left(\int_{s_i}^{t} (t-s)^{q-1} ds \right) \|m_f\|_{PC}\, h_f\left(\|x + Bx\|_{PC}\right) \\
&\le \frac{1}{\Gamma(q)} \left(\int_{0}^{t} (t-s)^{q-1} ds \right) \|m_f\|_{PC}\, h_f\left(\|x + Bx\|_{PC}\right) \\
&= \frac{1}{\Gamma(q)} \left[-\frac{(t-s)^q}{q} \right]_0^t \|m_f\|_{PC}\, h_f\left(\|x + Bx\|_{PC}\right) \\
&= \frac{1}{\Gamma(q)} \frac{t^q}{q} \|m_f\|_{PC}\, h_f\left(\|x + Bx\|_{PC}\right) \\
&\le \frac{T^q}{q\Gamma(q)} \|m_f\|_{PC}\, h_f\left(\|x\|_{PC} + \|Bx\|_{PC}\right).
\end{aligned} \tag{34}$$

By the definition of B, the integral mean value theorem, (H_2), and the property $\Gamma(q+1) = q\Gamma(q)$, we have

$$\begin{aligned}
&\left|A_i^1 x(t)\right| \\
&\le \frac{T^q}{\Gamma(q+1)} \|m_f\|_{PC} \\
&\quad \cdot h_f\left[\|x\|_{PC} + T\left(\left|k(t,\xi,x) - k(t,\xi,0)\right| + \left|k(t,\xi,0)\right|\right)\right] \\
&\le \frac{T^q}{\Gamma(q+1)} \|m_f\|_{PC} \\
&\quad \cdot h_f\left(\|x\|_{PC} + DT\|x\|_{PC} + T\left|k(t,\xi,0)\right|\right) \\
&\le \frac{T^q}{\Gamma(q+1)} \|m_f\|_{PC}\, h_f\left[r(1+TD) + TK\right],
\end{aligned} \tag{35}$$

where $0 \le \xi \le t \le T$.

By a similar argument, let $x \in B_r$, if $t \in [0, t_1]$; we have

$$\begin{aligned}
&\left|A_i^1 x(t)\right| \\
&\le \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} f(s, x(s), Bx(s))\, ds \\
&\quad + \frac{1}{(1-\sigma)\Gamma(q)} \\
&\quad \cdot \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} w(t) \int_{s_i}^{t} (t-s)^{q-1} f(s, x(s), Bx(s))\, ds\, dt \\
&\le \frac{T^q}{\Gamma(q+1)} \|m_f\|_{PC}\, h_f\left[r(1+TD) + TK\right] \\
&\quad + \frac{1}{(1-\sigma)\Gamma(q)} \\
&\quad \cdot \int_0^T w(t) \frac{T^q}{q} \|m_f\|_{PC}\, h_f\left[r(1+TD) + TK\right] dt \\
&= \frac{T^q}{\Gamma(q+1)} \left(1 + \frac{\int_0^T w(t)\, dt}{1-\sigma} \right) \|m_f\|_{PC} \\
&\quad \cdot h_f\left[r(1+TD) + TK\right].
\end{aligned} \tag{36}$$

From the condition $w \in L^1([0,T], R^+)$, $w \in [0, 1/T)$, we can get

$$\left|A_i^1 x(t)\right| \le \frac{T^q(2-\sigma)}{\Gamma(q+1)(1-\sigma)} \|m_f\|_{PC}\, h_f\left[r(1+TD) + TK\right]. \tag{37}$$

For the other cases, from the definition of $A_i^1 u$, one can get $|A_i^1 x(t)| = 0$.

From the proof above, let $x \in B_r$; then for all $t \in [0,T]$, we have

$$\left|A_i^1 x(t)\right| \le \frac{T^q(2-\sigma)}{\Gamma(q+1)(1-\sigma)} \|m_f\|_{PC}\, h_f\left[r(1+TD) + TK\right]. \tag{38}$$

Let $y \in B_r$; if $t \in (t_i, s_i]$, $i \ge 1$, we can obtain

$$\left|A_i^2 y(t)\right| \le \left|g_i(t, y(t)) - g_i(t,0)\right| + \left|g_i(t,0)\right| \le Lr + a. \tag{39}$$

Let $y \in B_r$; if $t \in [0, t_1]$, we can get

$$\begin{aligned}
&\left|A_i^2 y(t)\right| \\
&\le \frac{1}{1-\sigma} \left\{ \sum_{i=1}^{N} \Bigg(\left(\left|g_i(s_i, y(s_i)) - g_i(s_i,0)\right| \right. \right. \\
&\qquad \left. + \left|g_i(s_i,0)\right|\right) \int_{s_i}^{t_{i+1}} w(t)\, dt \Bigg) \\
&\qquad + \sum_{i=1}^{N} \int_{t_i}^{s_i} w(t) \left(\left|g_i(t, y(t)) - g_i(t,0)\right| \right. \\
&\qquad \left. \left. + \left|g_i(t,0)\right|\right) dt \right\} \\
&\le \frac{2(Lr+a)}{1-\sigma}.
\end{aligned} \tag{40}$$

Proceeding as above, we obtain that $|A_i^2 y(t)| \le Lr + a$, $\forall y \in B_r$, for $t \in (s_i, t_{i+1}]$, $i \ge 1$.

Then, for all $x, y \in B_r$, we have that

$$\begin{aligned}\|Ax + By\|_{PC} &\leq \frac{T^q(2-\sigma)}{\Gamma(q+1)(1-\sigma)}\|m_f\|_{PC} h_f[r(1+TD)+TK] \\ &\quad + \frac{2(Lr+a)}{1-\sigma} \leq r.\end{aligned} \tag{41}$$

Step 2. We show that $A^2 = \sum_{i=1}^{N} A_i^2$ is a contraction mapping.

From the definition of $A^2u(t)$, $A_i^2u(t)$, and (H_3), we can easily get

$$\left|A_i^2x(t) - A_i^2y(t)\right| \leq \frac{2L}{1-\sigma}\|x-y\|_{PC}, \quad \forall x, y \in B_r, \ \forall t \in J, \tag{42}$$

which implies that A^2 is a contraction mapping.

Step 3. Next we will prove that A^1 is compact and continuous.

We also divide the proof into 3 steps.

(I) We show that A^1 is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC(J, R)$. Then for all $t \in J$, by the definition of $A^1u(t)$, $A_i^1u(t)$, we have

$$\begin{aligned}&\left|A^1x_n(t) - A^1x(t)\right| \\ &\leq \frac{T^q(2-\sigma)}{\Gamma(q+1)(1-\sigma)} \cdot \left|f(t, x_n(t), Bx_n(t)) - f(t, x(t), Bx(t))\right| \\ &\leq \frac{T^q(2-\sigma)}{\Gamma(q+1)(1-\sigma)} \cdot \left(\left|f(t, x_n(t), Bx_n(t)) - f(t, x(t), Bx_n(t))\right| \right. \\ &\qquad \left. + \left|f(t, x(t), Bx_n(t)) - f(t, x(t), Bx(t))\right|\right).\end{aligned} \tag{43}$$

From (H_2), (H_4), we can get the continuity of f and B. Then one has

$$\left\|A^1x_n - A^1x\right\|_{PC} \longrightarrow 0, \quad \text{as } x_n \longrightarrow x \ (n \longrightarrow \infty), \tag{44}$$

which shows that the operator A^1 is continuous.

(II) We show that A^1 maps bounded sets into bounded sets in $PC(J, R)$.

Indeed, it is enough to show that, for any $R > 0$, there exists a $R' > 0$ such that, for each $x \in B_R = \{u \in PC(J, R) : \|u\|_{PC} \leq R\}$, we have $\|A^1x\|_{PC} \leq R'$.

For all $t \in J$, from the definition of $A^1u(t)$, $A_i^1u(t)$, and (H_2), (H_4), one can obtain

$$\left|A^1x(t)\right| \leq \frac{T^q(2-\sigma)}{\Gamma(q+1)(1-\sigma)}\|m_f\|_{PC} h_f[R(1+TD)+TK] := R'. \tag{45}$$

Then we conclude that A^1 maps bounded sets into bounded sets in $PC(J, R)$.

(III) At last, we prove that A^1 maps bounded sets into equicontinuous sets in $PC(J, R)$.

For interval $t \in (s_i, t_{i+1}]$, $s_i \leq l_1 < l_2 \leq t_{i+1}$, $i = 1, \ldots, N$, $\forall x(t) \in B_r$, by definition of $A^1u(t)$ and (H_3), we have

$$\begin{aligned}&\left|\left(A^1x\right)(l_2) - \left(A^1x\right)(l_1)\right| \\ &= \left|\frac{1}{\Gamma(q)}\int_{s_i}^{l_2}(l_2-s)^{q-1} f(s, x(s), Bx(s))\,ds \right. \\ &\qquad \left. - \frac{1}{\Gamma(q)}\int_{s_i}^{l_1}(l_1-s)^{q-1} f(s, x(s), Bx(s))\,ds\right| \\ &\leq \frac{1}{\Gamma(q)}\int_{l_1}^{l_2}(l_2-s)^{q-1}\left|f(s, x(s), Bx(s))\right|ds \\ &\quad + \frac{1}{\Gamma(q)}\int_{s_i}^{l_1}\left|f(s, x(s), Bx(s))\right| \\ &\qquad \cdot \left[(l_1-s)^{q-1} - (l_2-s)^{q-1}\right]ds \\ &\leq \frac{\|m_f\|_{PC} h_f[r(1+TD)+TK]}{\Gamma(q)} \\ &\quad \cdot \left\{\int_{l_1}^{l_2}(l_2-s)^{q-1}\,ds + \int_{s_i}^{l_1}\left[(l_1-s)^{q-1} - (l_2-s)^{q-1}\right]ds\right\} \\ &\leq \frac{\|m_f\|_{PC} h_f[r(1+TD)+TK]}{\Gamma(q+1)} \\ &\quad \cdot \left[(l_2-l_1)^q + \left|(l_1-s_i)^q - (l_2-s_i)^q\right|\right],\end{aligned} \tag{46}$$

which is independent of x. As $l_1 \to l_2$, the right-hand side of the above inequality tends to zero. Therefore A^1 is equicontinuous on interval $(s_i, t_{i+1}]$, $i \geq 1$.

Proceeding as above, we can also prove that A^1 is equicontinuous for the time interval $[0, t_1]$. From the definition of A_i^1, it is easy to see that A^1 is equicontinuous for the other cases.

By Arzela-Ascoli Theorem, A^1 is continuous and compact. □

As a consequence of Lemma 5, we deduce that the operator A has at least a fixed point on B_r which means that problem (3) has at least a mild solution.

4. Examples

Consider the following impulsive system of fractional differential equations.

Example 1. Consider

$${}^cD_t^{1/2}u(t) = \frac{1}{16}\left(\frac{u(t)\sin t^2}{1+e^{t^2}} + \int_0^t \frac{\sin\left(t^2+\sqrt{s}\right)}{2}u(s)\,ds\right), \quad t \in (s_i, t_{i+1}], \quad i = 1, \ldots, N,$$

$$u(t) = \frac{u(t)}{18e^t(1+|u(t)|)}, \quad t \in (t_i, s_i], \ i = 1, \ldots, N,$$
$$u(0) = \int_0^1 \frac{u(t)}{3} dt, \tag{47}$$

where $0 = t_0 = s_0 < t_1 = 1/2 \le s_1 \le \cdots \le t_N \le s_N \le t_{N+1} = 1$ are pre-fixed numbers, $J = [0, 1]$, $q = 1/2$, and $w(t) = 1/3 < 1$.

We prove that Example 1 satisfies all the assumptions of Theorem 7.

In Example 1, set

$$f(t, u, v) = \frac{u \sin t^2}{16\left(1 + e^{t^2}\right)} + \frac{v}{16},$$
$$Bu(t) = \int_0^t \frac{\sin\left(t^2 + \sqrt{s}\right)}{2} u(s)\, ds, \tag{48}$$
$$k(t, s, u) = \frac{\sin\left(t^2 + \sqrt{s}\right)}{2} u.$$

It is easy to see that f is jointly continuous. We can also check that (H_1) and (H_2) are satisfied with $e(t) = 1/16$ and $d(t) = 1/2$.

For $t \in [0, 1]$, $u \in R$, and $g_i(t, u) = u/18e^t(1 + |u|)$, then we know

$$\left|g_i(t, u) - g_i(t, v)\right| \le \frac{1}{18} |u - v|, \tag{49}$$

with $l_i(t) = 1/36$, so (H_3) is also satisfied.

From 1, we can get $L = 1/18$, $T = 1$, $M = 1/16$, $D = 1/2$, $q = 1/2$, $\Gamma(3/2) = \sqrt{\pi}/2$, $\sigma = 1/6$, and then

$$n = \frac{T^q M (1 + TD)(2 - \sigma)}{\Gamma(q+1)(1-\sigma)} + \frac{2L}{1-\sigma} = \frac{33}{80\sqrt{\pi}} + \frac{2}{15} < 1. \tag{50}$$

So all the conditions of Theorem 7 are satisfied. As a consequence of Theorem 7, Example 1 has a unique mild solution.

Example 2. Consider

$$^cD_t^{1/2} u(t) = \frac{1}{15\left(1 + e^{\sqrt{\sin t}}\right)} \left(u(t) + \int_0^t \frac{\sin\left(t^2 + \sqrt{s}\right)}{2} u(s)\, ds\right), \quad t \in (s_i, t_{i+1}], \quad i = 1, \ldots, N,$$
$$u(t) = \frac{u(t)}{18(1 + |u(t)|)}, \quad t \in (t_i, s_i], \ i = 1, \ldots, N,$$
$$u(0) = \int_0^1 \frac{u(t)}{2} dt, \tag{51}$$

where $0 = t_0 = s_0 < t_1 = 1/3 \le s_1 \le t_2 \le \cdots \le t_N \le s_N \le t_{N+1} = 1$ are prefixed numbers, $J = [0, 1]$, $q = 1/2$, and $w(t) = 1/2 < 1$.

It is easy to see that $g_i(t, 0) = 0$ is bounded. Set

$$f(t, u, v) = \frac{u + v}{15\left(1 + e^{\sqrt{\sin t}}\right)},$$
$$Bu(t) = \int_0^t \frac{\sin\left(t^2 + \sqrt{s}\right)}{2} u(s)\, ds, \tag{52}$$
$$k(t, s, u) = \frac{\sin\left(t^2 + \sqrt{s}\right)}{2} u.$$

Then we have

$$\left|f(t, u, Bu)\right| \le \frac{1}{15} |u + Bu|, \tag{53}$$

with $m_f = 1/15 \in C(J, R^+)$, $h_f(|u + Bu|) = |u + Bu| \in C(R^+, R^+)$ being nondecreasing. So (H_4) is satisfied. Similarly to the proof of Example 1, we know that (H_2) and (H_3) are satisfied.

From (51), we can obtain $T = 1$, $\sigma = 1/6$, $D = 1/2$, $L = 1/18 < 1$, $a = 0$, $K = 0$, $\Gamma(3/2) = \sqrt{\pi}/2$, $m_f = 1/15$, $q = 1/2$, and $2L/(1 - \sigma) = 2/15 < 1$, and then the inequality (32) becomes $13r/18 \ge (11/30\sqrt{\pi})r$. Hence, inequality (32) holds for all $r > 0$.

Thus, all the assumptions in Theorem 8 are satisfied, and our results can be applied to Example 2. So Example 2 has at least one mild solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by National Natural Science Foundation of China (nos. 11001274, 11101126, and 11261010), China Postdoctoral Science Foundation (no. 20110491249), Key Scientific and Technological Research Project of Department of Education of Henan Province (no. 12B110006), Youth Science Foundation of Henan University of Science and Technology (no. 2012QN010), and Innovative Natural Science Foundation of Henan University of Science and Technology (no. 2013ZCX020). The authors thank the referees for their careful reading of the paper and insightful comments, which help to improve the quality of the paper. The authors would also like to acknowledge the valuable comments and suggestions from the editors, which vastly contribute to the perfection of the paper.

References

[1] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, vol. 14 of *World Scientific Series on Nonlinear Science, Series A: Monographs and Treatises*, World Scientific Publishing, Singapore, 1995.

[2] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive DifferentialEquations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific, Singapore, 1989.

[3] M. Benchohra, J. Henderson, and S. Ntouyas, *Impulsive Differential Equations and Inclusions*, vol. 2 of *Contemporary Mathematics and Its Applications*, Hindawi Publishing Corporation, New York, NY, USA, 2006.

[4] M. Frigon and D. O'Regan, "Existence results for first-order impulsive differential equations," *Journal of Mathematical Analysis and Applications*, vol. 193, no. 1, pp. 96–113, 1995.

[5] S. M. Afonso, E. M. Bonotto, M. Federson, and S. Schwabik, "Discontinuous local semiflows for Kurzweil equations leading to LaSalle's invariance principle for differential systems with impulses at variable times," *Journal of Differential Equations*, vol. 250, no. 7, pp. 2969–3001, 2011.

[6] S. M. Afonso, E. M. Bonotto, M. Federson, and L. P. Gimenes, "Boundedness of solutions of retarded functional differential equations with variable impulses via generalized ordinary differential equations," *Mathematische Nachrichten*, vol. 285, no. 5-6, pp. 545–561, 2012.

[7] S. M. Afonso, E. M. Bonotto, M. Federson, and L. P. Gimenes, "Stability of functional differential equations with variable impulsive perturbations via generalized ordinary differential equations," *Bulletin des Sciences Mathématiques*, vol. 137, no. 2, pp. 189–214, 2013.

[8] E. Hernández and D. O'Regan, "On a new class of abstract impulsive differential equations," *Proceedings of the American Mathematical Society*, vol. 141, no. 5, pp. 1641–1649, 2013.

[9] M. Pierri, D. O'Regan, and V. Rolnik, "Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses," *Applied Mathematics and Computation*, vol. 219, no. 12, pp. 6743–6749, 2013.

[10] Y. Zhou and F. Jiao, "Nonlocal Cauchy problem for fractional evolution equations," *Nonlinear Analysis. Real World Applications*, vol. 11, no. 5, pp. 4465–4475, 2010.

[11] Y. Zhou, F. Jiao, and J. Li, "Existence and uniqueness for fractional neutral differential equations with infinite delay," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 3249–3256, 2009.

[12] K. Diethelm, *The Analysis of Fractional Differential Equations*, vol. 2004 of *Lecture Notes in Mathematics*, Springer, 2010.

[13] A. A. Kilbas, M. H. Srivastava, and J. J. Trujillo, "Theory and applications of fractional differential equations," in *North-Holland Mathematics Studies*, vol. 204, Elsevier Science B.V., Amsterdam, The Netherlands, 2006.

[14] V. Lakshmikantham, S. Leela, and J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, Cambridge, UK, 2009.

[15] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 1993.

[16] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[17] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Nonlinear Physical Science, Springer, Heidelberg, Germany, 2011.

[18] R. P. Agarwal, M. Benchohra, and S. Hamani, "A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions," *Acta Applicandae Mathematicae*, vol. 109, no. 3, pp. 973–1033, 2010.

[19] M. Benchohra, J. Henderson, S. K. Ntouyas, and A. Ouahab, "Existence results for fractional order functional differential equations with infinite delay," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 2, pp. 1340–1350, 2008.

[20] J. Wang and Y. Zhou, "A class of fractional evolution equations and optimal controls," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 262–272, 2011.

[21] J. Wang, Y. Zhou, and W. Wei, "A class of fractional delay nonlinear integrodifferential controlled systems in Banach spaces," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 10, pp. 4049–4059, 2011.

[22] S. Zhang, "Existence of positive solution for some class of nonlinear fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 1, pp. 136–148, 2003.

[23] J. M. Gallardo, "Second-order differential operators with integral boundary conditions and generation of analytic semigroups," *The Rocky Mountain Journal of Mathematics*, vol. 30, no. 4, pp. 1265–1291, 2000.

[24] G. L. Karakostas and P. C. Tsamatos, "Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary value problems," *Electronic Journal of Differential Equations*, vol. 30, pp. 1–17, 2002.

[25] A. Lomtatidze and L. Malaguti, "On a nonlocal boundary value problems for second order nonlinear singular differential equations," *Georgian Mathematical Journal*, vol. 7, no. 1, pp. 133–154, 2000.

[26] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, Cambridge, UK, 1991.

[27] R. P. Agarwal and D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic, Dordrecht, The Netherlands, 2001.

[28] B. Ahmad and A. Alsaedi, "Existence of approximate solutions of the forced Duffing equation with discontinuous type integral boundary conditions," *Nonlinear Analysis. Real World Applications*, vol. 10, no. 1, pp. 358–367, 2009.

[29] A. Boucherif, "Second-order boundary value problems with integral boundary conditions," *Nonlinear Analysis*, vol. 70, no. 1, pp. 364–371, 2009.

[30] Z. Yang, "Existence of nontrivial solutions for a nonlinear Sturm-Liouville problem with integral boundary conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 1, pp. 216–225, 2008.

[31] X. Zhang, M. Feng, and W. Ge, "Existence results for nonlinear boundary-value problems with integral boundary conditions in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 10, pp. 3310–3321, 2008.

[32] A. Anguraj, P. Karthikeyan, M. Rivero, and J. J. Trujillo, "On new existence results for fractional integro-differential equations with impulsive and integral conditions," *Computers & Mathematics with Applications*, vol. 66, no. 12, pp. 2587–2594, 2014.

[33] M. A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, New York, NY, USA, 1964.

Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 350840, 5 pages
http://dx.doi.org/10.1155/2015/350840

Research Article

Common Fixed Point of Two R-Weakly Commuting Mappings in b-Metric Spaces

Poom Kumam,[1,2] Wutiphol Sintunavarat,[3] Shaban Sedghi,[4] and Nabi Shobkolaei[5]

[1] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand*
[2] *Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand*
[3] *Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathumthani 12121, Thailand*
[4] *Department of Mathematics, Islamic Azad University, Qaemshahr Branch, Qaemshahr, Iran*
[5] *Department of Mathematics, Islamic Azad University, Babol Branch, Babol, Iran*

Correspondence should be addressed to Wutiphol Sintunavarat; wutiphol@mathstat.sci.tu.ac.th

Received 21 June 2014; Accepted 9 December 2014

Academic Editor: Calogero Vetro

Copyright © 2015 Poom Kumam et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We prove some common fixed point results for two mappings satisfying generalized contractive condition in b-metric space. Note that b-metric of main results in this work are not necessarily continuous. So our results extend and improve several previous works. We also present one example that shows the applicability and usefulness of our results.

1. Introduction

In 1998, Czerwik [1] introduced the concept of b-metric space. Since then, several papers deal with fixed point theory for single-valued and multivalued operators in b-metric spaces (see also [1–19]). For example, Pacurar [16] proved results on sequences of almost contractions and fixed points in b-metric spaces. Also, Hussain and Shah [11] obtained results on KKM mappings in cone b-metric spaces. Furthermore, Khamsi ([12, 13]) also showed that each cone metric space has a b-metric structure.

The aim of this paper is to present some common fixed point results for two mappings under generalized contractive condition in b-metric space, where the b-metric is not necessarily continuous. Because many of the authors in their works have used the b-metric spaces in which the b-metric is continuous, the techniques used in this paper can be used for many of the results on the context of b-metric space. From this point of view the results obtained in this paper generalize and extend several ones obtained earlier in a lot of papers concerning b-metric space.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$, and $\mathbb{R}$ the sets of positive integers, nonnegative real numbers, and real numbers, respectively.

Consistent with [1] and [18, page 264], the following definition and results will be needed in the sequel.

Definition 1 (see [1]). Let X be a nonempty set and let $b \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+$ is a b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

(b1) $d(x, y) = 0$ if and only if $x = y$;
(b2) $d(x, y) = d(y, x)$;
(b3) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space with coefficient b.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric when $b = 1$.

We present an example which shows that a b-metric on X need not be a metric on X (see also [18, page 264]).

Example 2. Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. We show that ρ is a b-metric with $b = 2^{p-1}$.

Obviously conditions (b1) and (b2) in Definition 1 are satisfied. Now we show that condition (b3) holds for ρ.

It is easy to see that if $1 < p < \infty$, then the convexity of the function $f(x) = x^p$, where $x \geq 0$, implies

$$\left(\frac{a+c}{2}\right)^p \leq \frac{1}{2}\left(a^p + c^p\right), \tag{1}$$

and hence

$$(a+c)^p \leq 2^{p-1}\left(a^p + c^p\right). \tag{2}$$

Therefore, for each $x, y, z \in X$, we obtain that

$$\begin{aligned}\rho(x, y) &= (d(x, y))^p \\ &\leq (d(x, z) + d(z, y))^p \\ &\leq 2^{p-1}\left((d(x, z))^p + (d(z, y))^p\right) \\ &= 2^{p-1}\left(\rho(x, z) + \rho(z, y)\right).\end{aligned} \tag{3}$$

So condition (b3) in Definition 1 holds and then ρ is a b-metric coefficient $s = 2^{p-1}$.

It should be noted that in Example 2, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space (see Example 3).

Example 3. For example, if $X = \mathbb{R}$ and the usual Euclidean metric $d : X \times X \to \mathbb{R}_+$ is defined by

$$d(x, y) = |x - y| \tag{4}$$

for all $x, y \in X$, then $\rho(x, y) = |x - y|^2$ is a b-metric on $\mathbb{R}$ with $b = 2$ but is not a metric on $\mathbb{R}$, because the triangle inequality does not hold.

Before stating and proving our results, we present some definition and proposition in b-metric space. We recall first the notions of convergence, closedness, and completeness in a b-metric space.

Definition 4 (see [6]). Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called

(i) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to +\infty$, and in this case, we write $\lim_{n\to\infty} x_n = x$,

(ii) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to +\infty$.

Proposition 5 (Remark 2.1 in [6]). *In a b-metric space (X, d) the following assertions hold:*

(i) *a convergent sequence has a unique limit;*

(ii) *each convergent sequence is Cauchy;*

(iii) *in general, a b-metric is not continuous.*

Definition 6 (see [6]). The b-metric space (X, d) is complete if every Cauchy sequence in X converges.

It should be noted that in general a b-metric function d for $b > 1$ is not jointly continuous in all the two of its variables.

Since in general a b-metric is not continuous, we need the following simple lemma about the b-convergent sequences.

Lemma 7 (see [20]). *Let (X, d) be a b-metric space with coefficient $b \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively; then one has*

$$\begin{aligned}\frac{1}{b^2}d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \\ \leq b^2 d(x, y).\end{aligned} \tag{5}$$

In particular, if $x = y$, then one has $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$ one has

$$\frac{1}{b}d(x, z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq bd(x, z). \tag{6}$$

3. Common Fixed Point Results

Definition 8. Let f and g be mappings from a b-metric space (X, d) into itself. The mappings f and g are said to be weakly commuting if

$$d(fgx, gfx) \leq d(fx, gx) \tag{7}$$

for each x in X.

Definition 9. Let f and g be mappings from a b-metric space (X, d) into itself. The mappings f and g are said to be R-weakly commuting if there exists some positive real number R such that

$$d(fgx, gfx) \leq Rd(fx, gx) \tag{8}$$

for each x in X.

Remark 10. Weak commutativity implies R-weak commutativity in b-metric space. However, R-weak commutativity implies weak commutativity only when $R \leq 1$.

Example 11. Let $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}_+$ defined as follows:

$$d(x, y) = (x - y)^2 \tag{9}$$

for all $x, y \in X$. Then (X, d) is a b-metric space. Define $fx = 2x - 1$ and $gx = x^2$. Then

$$d(fgx, gfx) = 4(x-1)^4 = 4d(fx, gx) > d(fx, gx). \tag{10}$$

Therefore, for $R = 4$, f and g are R-weakly commuting. But f and g are not weakly commuting.

Theorem 12. *Let (X, d) be a complete b-metric space and let f and g be R-weakly commuting self-mappings on X satisfying the following conditions:*

(a) $f(X) \subseteq g(X)$;

(b) f *or* g *is continuous;*

(c) $d(fx, fy) \leq \gamma((1/b^4)d(gx, gy))$ *for all* $x, y \in X$, *where* $\gamma : [0, \infty) \to [0, \infty)$ *is a continuous and nondecreasing function such that* $\gamma(a) < a$ *for each* $a > 0$ *and* $\gamma(0) = 0$.

Then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. By (a), choose a point x_1 in X such that $fx_0 = gx_1$. In general choose x_{n+1} such that $fx_n = gx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Now we observe that for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leq \gamma\left(\frac{1}{b^4} d(gx_n, gx_{n+1})\right) \\ &= \gamma\left(\frac{1}{b^4} d(fx_{n-1}, fx_n)\right) \\ &\leq \frac{1}{b^4} d(fx_{n-1}, fx_n) \\ &\leq d(fx_{n-1}, fx_n). \end{aligned} \tag{11}$$

This implies that

$$\{c_n\} := \{d(fx_n, fx_{n+1})\} \tag{12}$$

is nonincreasing sequence in $[0, \infty)$. Therefore, it tends to a limit $a \geq 0$. Next, we claim that $a = 0$. Suppose that $a > 0$. Making $n \to \infty$ in the inequality (11), we get

$$a \leq \gamma\left(\frac{1}{b^4} a\right) < \frac{1}{b^4} a \leq a, \tag{13}$$

which is a contradiction. Hence $a = 0$; that is,

$$\lim_{n \to \infty} c_n = \lim_{n \to \infty} d(fx_n, fx_{n+1}) = 0. \tag{14}$$

Now, we prove that $\{fx_n\}$ is a Cauchy sequence in $f(X)$. Suppose that $\{fx_n\}$ is not a Cauchy sequence in $f(X)$. For convenience, let $y_n = fx_n$ for $n = 1, 2, 3, \ldots$. Then there is an $\varepsilon > 0$ such that, for each integer k, there exist integers $m(k)$ and $n(k)$ with $m(k) > n(k) \geq k$ such that

$$d_k := d(y_{n(k)}, y_{m(k)}) \geq \varepsilon \quad \text{for } k = 1, 2, \ldots. \tag{15}$$

We may assume that

$$d(y_{n(k)}, y_{m(k)-1}) < \varepsilon, \tag{16}$$

by choosing $m(k)$ as the smallest number exceeding $n(k)$ for which (15) holds. Using (11), we have

$$\begin{aligned} \varepsilon \leq d_k &\leq b\left[d(y_{n(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)})\right] \\ &< b\varepsilon + bc_{m(k)-1} \leq b\varepsilon + bc_k. \end{aligned} \tag{17}$$

Hence, $\varepsilon \leq \lim_{k \to \infty} d_k \leq b\varepsilon$. Also notice

$$\begin{aligned} d_k &= d(y_{n(k)}, y_{m(k)}) \\ &\leq bd(y_{n(k)}, y_{n(k)+1}) \\ &\quad + b^2\left[d(y_{n(k)+1}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{m(k)})\right] \\ &\leq bc_k + b^2\gamma\left(\frac{1}{b^4} d(y_{n(k)}, y_{m(k)})\right) + b^2 c_k \\ &= bc_k + b^2\gamma\left(\frac{1}{b^4} d_k\right) + b^2 c_k \end{aligned} \tag{18}$$

for each $k \in \mathbb{N}$. Thus, as $k \to \infty$ in the above inequality we have

$$\varepsilon \leq \lim_{k \to \infty} d_k \leq b \lim_{k \to \infty} c_k + b^2\gamma\left(\frac{1}{b^4} \lim_{k \to \infty} d_k\right) + b^2 \lim_{k \to \infty} c_k, \tag{19}$$

and thus

$$\varepsilon \leq b^2\gamma\left(\frac{1}{b^3}\varepsilon\right) < \frac{1}{b}\varepsilon, \tag{20}$$

which is a contradiction. Thus, $\{fx_n\}$ is Cauchy sequence in X and by the completeness of X, $\{fx_n\}$ converges to z in X. Also $\{gx_n\}$ converges to z in X. Let us suppose that the mapping f is continuous. Then $\lim_{n \to \infty} ffx_n = fz$ and $\lim_{n \to \infty} fgx_n = fz$. Further we have that since f and g are R-weakly commuting

$$d(fgx_n, gfx_n) \leq Rd(fx_n, gx_n) \tag{21}$$

for all $n \in \mathbb{N}$. Taking the upper limit as $n \to \infty$ in the above inequality, we get

$$\begin{aligned} \frac{1}{b^2} d\left(fz, \limsup_{n \to \infty} gfx_n\right) &\leq \limsup_{n \to \infty} d(fgx_n, gfx_n) \\ &\leq R \limsup_{n \to \infty} d(fx_n, gx_n) \\ &\leq Rb^2 d(z, z) = 0. \end{aligned} \tag{22}$$

Similarly,

$$\frac{1}{b^2} d\left(fz, \liminf_{n \to \infty} gfx_n\right) = 0, \tag{23}$$

and hence we get $\lim_{n \to \infty} gfx_n = fz$. We now prove that $z = fz$. Suppose that $z \neq fz$ and then $d(z, fz) > 0$. By (c), we have

$$\begin{aligned} \frac{1}{b} d(fz, fx_m) &\leq \limsup_{n \to \infty} d(ffx_n, fx_m) \\ &\leq \gamma\left(\limsup_{n \to \infty} \frac{1}{b^4} d(gfx_n, gx_m)\right) \\ &\leq \gamma\left(\frac{1}{b^3} d(fz, gx_m)\right) \end{aligned} \tag{24}$$

for each $m \in \mathbb{N}$. On taking the upper limit as $m \to \infty$ in the above inequality we get

$$\begin{aligned}\frac{1}{b^2}d(fz,z) &\le \limsup_{m\to\infty}\frac{1}{b}d(fz,fx_m)\\ &\le \gamma\left(\limsup_{m\to\infty}\frac{1}{b^3}d(fz,gx_m)\right)\\ &\le \gamma\left(\frac{1}{b^2}d(fz,z)\right) < \frac{1}{b^2}d(fz,z),\end{aligned} \tag{25}$$

which is a contradiction. Therefore, $z = fz$. Since $f(X) \subseteq g(X)$ we can find z_1 in X such that $z = fz = gz_1$. Now, we have

$$d(ffx_n, fz_1) \le \gamma\left(\frac{1}{b^4}d(gfx_n, gz_1)\right) \tag{26}$$

for all $n \in \mathbb{N}$. Taking limit sup as $n \to \infty$ we get

$$\begin{aligned}\frac{1}{b}d(fz,fz_1) &\le \limsup_{n\to\infty} d(ffx_n, fz)\\ &\le \gamma\left(\limsup_{n\to\infty}\frac{1}{b^4}d(gfx_n, gz_1)\right)\\ &\le \gamma\left(\frac{1}{b^3}d(fz,gz_1)\right)\\ &= 0\end{aligned} \tag{27}$$

since $\gamma(0) = 0$, which implies that $fz = fz_1$; that is, $z = fz = fz_1 = gz_1$. Also,

$$d(fz,gz) = d(fgz_1, gfz_1) \le Rd(fz_1, gz_1) = 0 \tag{28}$$

which again implies that $fz = gz$. Thus z is a common fixed point of f and g.

Now to prove uniqueness let if possible $z' \ne z$ be another common fixed point of f and g. Then $d(z,z') > 0$ and so

$$\begin{aligned}d(z,z') = d(fz,fz') &\le \gamma\left(\frac{1}{b^4}d(gz,gz')\right)\\ &= \gamma\left(\frac{1}{b^4}d(z,z')\right) < \frac{1}{b^4}d(z,z') \le d(z,z'),\end{aligned} \tag{29}$$

which is a contradiction. Therefore, $z = z'$, that is, z is a unique common fixed point of f and g. This completes the proof. □

Now we give an example to support Theorem 12.

Example 13. Let $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}_+$ defined by

$$d(x,y) = (x-y)^2 \tag{30}$$

for all $x, y \in X$. Then (X,d) is a b-metric space for $b = 2$. Define $fx = 1$ and $gx = 2x - 1$ on X. It is evident that $f(X) \subseteq g(X)$ and f is continuous. Now we observe that

$$d(fx,fy) \le \gamma\left(\frac{1}{16}d(gx,gy)\right) \tag{31}$$

for all x, y in X and $\gamma : (0,\infty) \to (0,\infty)$ defined by $\gamma(t) = kt$ for $0 < k < 1$. Moreover, it is easy to see that f and g are R-weakly commuting. Thus all the conditions of Theorem 12 are satisfied and 1 is a common fixed point of f and g.

Corollary 14. *Let (X,d) be a complete b-metric space and let f be a self-mapping on X satisfying the following condition:*

$$d(fx,fy) \le \gamma\left(\frac{1}{b^4}d(x,y)\right) \tag{32}$$

for all $x, y \in X$, where $\gamma : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\gamma(a) < a$ for each $a > 0$ and $\gamma(0) = 0$. Then f has a unique fixed point.

Proof. If we take g as identity mapping on X, then Theorem 12 follows that f has a unique fixed point. □

Corollary 15. *Let (X,d) be a complete metric space and let f and g be R-weakly commuting self-mappings of X satisfying the following conditions:*

(a) *$f(X) \subseteq g(X)$;*

(b) *f or g is continuous;*

(c) *$d(fx,fy) \le \gamma(d(gx,gy))$ for all $x, y \in X$, where $\gamma : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\gamma(a) < a$ for each $a > 0$ and $\gamma(0) = 0$.*

Then f and g have a unique common fixed point.

Proof. If we take $b = 1$, then Theorem 12 follows that f and g have a unique common fixed point. □

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the paper.

Acknowledgment

Poom Kumam and Wutiphol Sintunavarat would like to thank the Thailand Research Fund and Thammasat University under Grant no. TRG5780013 for financial support during the preparation of this paper.

References

[1] S. Czerwik, "Nonlinear set-valued contraction mappings in b-metric spaces," *Atti del Seminario Matematico e Fisico dell'Università di Modena*, vol. 46, no. 2, pp. 263–276, 1998.

[2] M. Akkouchi, "Common fixed point theorems for two selfmappings of a b-metric space under an implicit relation," *Hacettepe Journal of Mathematics and Statistics*, vol. 40, no. 6, pp. 805–810, 2011.

[3] H. Aydi, M.-F. Bota, E. Karapinar, and S. Mitrović, "A fixed point theorem for set-valued quasi-contractions in b-metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 88, 2012.

[4] M. Boriceanu, "Strict fixed point theorems for multivalued operators in b-metric spaces," *International Journal of Modern Mathematics*, vol. 4, no. 3, pp. 285–301, 2009.

[5] M. Boriceanu, "Fixed point theory for multivalued generalized contraction on a set with two b-metrics," *Studia Universitatis Babeş-Bolyai—Series Mathematica*, vol. 54, no. 3, pp. 1–14, 2009.

[6] M. Boriceanu, M. Bota, and A. Petrusel, "Multivalued fractals in b-metric spaces," *Central European Journal of Mathematics*, vol. 8, no. 2, pp. 367–377, 2010.

[7] M. Bota, A. Molnar, and C. Varga, "On Ekeland's variational principle in b-metric spaces," *Fixed Point Theory*, vol. 12, no. 2, pp. 21–28, 2011.

[8] S. Czerwik, "Contraction mappings in *b*-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[9] S. Czerwik, K. Dlutek, and S. L. Singh, "Round-off stability of iteration procedures for operators in *b*-metric spaces," *Journal of Natural & Physical Sciences*, vol. 11, pp. 87–94, 1997.

[10] N. Hussain, D. Đorić, Z. Kadelburg, and S. Radenović, "Suzuki-type fixed point results in metric type spaces," *Fixed Point Theory and Applications*, vol. 2012, article 126, 12 pages, 2012.

[11] N. Hussain and M. H. Shah, "KKM mappings in cone b-metric spaces," *Computers & Mathematics with Applications*, vol. 62, no. 4, pp. 1677–1684, 2011.

[12] M. A. Khamsi, "Remarks on cone metric spaces and fixed point theorems of contractive mappings," *Fixed Point Theory and Applications*, Article ID 315398, 7 pages, 2010.

[13] M. A. Khamsi and N. Hussain, "KKM mappings in metric type spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 9, pp. 3123–3129, 2010.

[14] P. Kumam and W. Sintunavarat, "The existence of fixed point theorems for partial *q*-set-valued quasi-contractions in *b*-metric spaces and related results," *Fixed Point Theory and Applications*, vol. 2014, article 226, 2014.

[15] M. O. Olatinwo, "Some results on multi-valued weakly Jungck mappings in *b*-metric space," *Central European Journal of Mathematics*, vol. 6, no. 4, pp. 610–621, 2008.

[16] M. Pacurar, "Sequences of almost contractions and fixed points in b- metric spaces," *Analele Universitatii de Vest, Timisoara Seria Matematica Informatica XLVIII*, vol. 3, pp. 125–137, 2010.

[17] S. Phiangsungnoen, W. Sintunavarat, and P. Kumam, "Fixed point results, generalized Ulam-Hyers stability and well-posedness via α-admissible mappings in b-metric spaces," *Fixed Point Theory and Applications*, vol. 2014, article 188, 2014.

[18] S. L. Singh and B. Prasad, "Some coincidence theorems and stability of iterative procedures," *Computers & Mathematics with Applications*, vol. 55, no. 11, pp. 2512–2520, 2008.

[19] W. Sintunavarat, S. Plubtieng, and P. Katchang, "Fixed point result and applications on a *b*-metric space endowed with an arbitrary binary relation," *Fixed Point Theory and Applications*, vol. 2013, article 296, 2013.

[20] A. Aghajani, M. Abbas, and J. R. Roshan, "Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces," *Mathematica Slovaca*, vol. 64, no. 4, pp. 941–960, 2014.

Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 413850, 8 pages
http://dx.doi.org/10.1155/2015/413850

Research Article

Some Generalized Difference Sequence Spaces Defined by a Sequence of Moduli in n-Normed Spaces

Abdullah Alotaibi,[1] Kuldip Raj,[2] and S. A. Mohiuddine[1]

[1] *Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[2] *School of Mathematics, Shri Mata Vaishno Devi University, Katra, Jammu and Kashmir 182320, India*

Correspondence should be addressed to S. A. Mohiuddine; mohiuddine@gmail.com

Received 12 December 2014; Revised 17 February 2015; Accepted 25 February 2015

Academic Editor: Mohamed-Aziz Taoudi

Copyright © 2015 Abdullah Alotaibi et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We introduce some new generalized difference sequence spaces by means of ideal convergence, infinite matrix, and a sequence of modulus functions over n-normed spaces. We also make an effort to study several properties relevant to topological, algebraic, and inclusion relations between these spaces.

1. Introduction and Preliminaries

The concept of 2-normed spaces was initially developed by Gähler [1] in the middle of 1960s, while that of n-normed spaces one can see in Misiak [2]. Since then, many others have studied this concept and obtained various results; see Gunawan [3, 4] and Gunawan and Mashadi [5]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of reals of dimension d, where $d \geq n \geq 2$. A real valued function $\|\cdot, \ldots, \cdot\|$ on X^n satisfying the following four conditions:

(1) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in X,

(2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation,

(3) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in \mathbb{R}$,

(4) $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$,

is called a n-norm on X and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called a n-normed space over the field $\mathbb{R}$. For more details about n-normed spaces, see [6, 7] and references therein.

The notion of difference sequence spaces was introduced by Kızmaz [8], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [9] by introducing the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$, and $c_0(\Delta^m)$. Later the concept has been studied by Bektaş et al. [10] and Et and Esi [11]. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [12] who studied the spaces $l_\infty(\Delta_\nu)$, $c(\Delta_\nu)$, and $c_0(\Delta_\nu)$. Recently, Esi et al. [13] and Tripathy et al. [14] have introduced a new type of generalized difference operators and unified those as follows.

Let m and ν be nonnegative integers; then for Z, a given sequence space, we have

$$Z(\Delta_\nu^m) = \{x = (x_k) \in w : (\Delta_\nu^m x_k) \in Z\}, \tag{1}$$

for $Z = c$, c_0, and l_∞ where $\Delta_\nu^m x = (\Delta_\nu^m x_k) = (\Delta_\nu^{m-1} x_k - \Delta_\nu^{m-1} x_{k+\nu})$ and $\Delta_\nu^0 x_k = x_k$, for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta_\nu^m x_k = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} x_{k+\nu\nu}. \tag{2}$$

Taking $\nu = 1$, we get the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$, and $c_0(\Delta^m)$ studied by Et and Çolak [9]. Taking $\nu = m = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ introduced and studied by Kızmaz [8]. For more details about difference sequence spaces, see [15–19].

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,

(2) $f(x + y) \leq f(x) + f(y)$, for all $x, y \geq 0$,

(3) f is increasing,

(4) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = x/(x+1)$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus function $f(x)$ is unbounded. Subsequently, modulus function has been discussed in [20] and references therein.

In [21], Mursaleen introduced the idea of λ-statistical convergence by extending the concept of $[V,\lambda]$ summability. Further, Savaş and Das [22] unified the two approaches and gave new concepts of I-statistical convergence, I-S_λ-convergence, and I-$[V,\lambda]$ convergence. Later some pioneer works have been extended in this direction by numerous authors such as Belen and Mohiuddine [23], Das et al. [24], Gürdal and Sarí [25], and references therein. Quite recently, many authors including Et et al. [26] and Maddox [27] have constructed some sequence spaces by using modulus function and difference sequences and investigate their properties.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,k \in \mathbb{N}$. Then we say that A defines a matrix mapping from X into Y if, for every sequence $x = (x_k)_{k=0}^{\infty} \in X$, the sequence $Ax = \{A_n(x)\}_{n=0}^{\infty}$, the A-transform of x, is in Y, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k \quad (n \in \mathbb{N}). \tag{3}$$

By (X,Y), we denote the class of all matrices A such that $A : X \to Y$. Thus, $A \in (X,Y)$ if and only if the series on the right-hand side of (3) converges for each $n \in \mathbb{N}$ and every $x \in X$.

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) : Ax \in X\}. \tag{4}$$

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors (see [28]).

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers such that $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$, $\lambda_n \to \infty$ as $n \to \infty$, and $I_n = [n - \lambda_n + 1, n]$.

Let $\mathbb{N}$ be a nonempty set. Then a family of sets $I \subseteq 2^{\mathbb{N}}$ (power set of $\mathbb{N}$) is said to be an ideal if I is additive; that is, $A,B \in I \Rightarrow A \cup B \in I$ and $A \in I, B \subseteq A \Rightarrow B \in I$. A nonempty family of sets $\pounds(I) \subseteq 2^{\mathbb{N}}$ is said to be filtered on $\mathbb{N}$ if and only if $\Phi \notin \pounds(I)$; for $A,B \in \pounds(I)$ we have $A \cap B \in \pounds(I)$ and for each $A \in \pounds(I)$ and $A \subseteq B$ implying $B \in \pounds(I)$.

An ideal $I \subseteq 2^{\mathbb{N}}$ is called nontrivial if $I \neq 2^{\mathbb{N}}$. A nontrivial ideal $I \subseteq 2^{\mathbb{N}}$ is called admissible if $\{\{x\} : x \in \mathbb{N}\} \subseteq I$. A nontrivial ideal is maximal if there cannot exist any nontrivial ideal $J \neq I$ containing I as a subset. For each ideal I, there exist a filter $\pounds(I)$ corresponding to I; that is, $\pounds(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} \setminus K$.

Definition 1. A sequence $x = (x_k)$ in a n-normed space $(X, \|\cdot,\dots,\cdot\|)$ is said to be statistically convergent to some $L \in X$ if, for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in \mathbb{N} : \|x_k - L, z_1, \dots, z_{n-1}\| \ge \varepsilon\}$ has its natural density zero.

Definition 2 (see [26]). A sequence $x = (x_k)$ is said to be λ_X^m-statistically convergent to the number L, if, for every $\varepsilon > 0$, $\lim_{n\to\infty}(1/\lambda_n)|\{k \in I_n : \|\Delta^m x_k - L\| \ge \varepsilon\}| = 0$. In this case one writes $S_\lambda(\Delta^m, X) - \lim_{k\to\infty} x_k = L$.

Definition 3 (see [22]). A sequence $x = (x_k)$ is said to be I-$[V,\lambda]$ summable to L, if, for any $\delta > 0$, $\{n \in \mathbb{N} : (1/\lambda_n)\sum_{k\in I_n}\|x_k - L\| \ge \delta\} \in I$, where $I_n = [n - \lambda_n + 1, n]$.

Definition 4 (see [22]). A sequence $x = (x_k)$ is said to be I-λ-statistically convergent or I-S_λ convergent to L if, for every $\varepsilon > 0$ and $\delta > 0$, $\{n \in \mathbb{N} : (1/\lambda_n)|\{k \in I_n : \|x_k - L\| \ge \delta\}| \in I$. In this case one writes $x_k \to L(I\text{-}S_\lambda)$ or $I\text{-}S_\lambda - \lim_{k\to\infty} x_k = L$.

The following well-known lemma is required for establishing some important results in this paper.

Lemma 5. *Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $0 < \delta < 1$. Then for each $x > \delta$ one has $f_k(x) \le 2f_k(1)x/\delta$.*

Throughout the paper, $\mathbb{N}$ will denote the set of all positive integers. By S_X^n we denote the space of all sequences defined over n-normed space $(X, \|\cdot,\dots,\cdot\|)$.

Let $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal, $\mathcal{F} = (f_k)$ a sequence modulus functions, $p = (p_k)$ a bounded sequence of positive (strictly) real numbers, $u = (u_k)$ a sequence of positive real numbers, $A = (a_{nk})$ an infinite matrix, and $(X, \|\cdot,\dots,\cdot\|)$ a n-normed space. Then, for every $\delta > 0$ and $z_1, \dots, z_{n-1} \in X$, we define the following sequence spaces:

$$\begin{aligned}
&V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot,\dots,\cdot\|]_0 \\
&= \left\{ x = (x_k) \in S_X^n : \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k\left(\left\|u_k\Delta_v^m x_k, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_k} \ge \delta \right\} \in I \right\}, \\
&V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot,\dots,\cdot\|] \\
&= \left\{ x = (x_k) \in S_X^n : \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k\left(\left\|u_k\Delta_v^m x_k - L, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_k} \ge \delta \right\} \right. \\
&\qquad \left. \in I \right\},
\end{aligned} \tag{5}$$

for some $L > 0$, and

$$V_{\mathscr{F}}^{I}\left[\Delta_{\nu}^{m}, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|\right]_{\infty} = \left\{ x = (x_k) \in S_X^n : \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f_k \left(\left\| u_k \Delta_{\nu}^{m} x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \geq K \right\} \in I \right\}, \tag{6}$$

for some $K > 0$.

If we take $\mathscr{F}(x) = f_k(x) = x$, $u = (u_k) = 1$ for all $k \in \mathbb{N}$, $\nu = 1$, and $A = (a_{nk}) = I$ (identity matrix) and n-normed space is replaced by 2-normed space then we get the sequence spaces defined by Kumar et al. [29].

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = H$, and let $D = \max\{1, 2^{H-1}\}$. Then, for the factorable sequences (a_k) and (b_k) in the complex plane, we have

$$\left|a_k + b_k\right|^{p_k} \leq D\left(\left|a_k\right|^{p_k} + \left|b_k\right|^{p_k}\right). \tag{7}$$

2. Main Results

The main purpose of this section is to study some topological properties and some inclusion relations between the sequence spaces which we have defined above.

Theorem 6. *Suppose $\mathscr{F} = (f_k)$ is a sequence of modulus functions, $p = (p_k)$ is a bounded sequence of strictly positive real numbers, $u = (u_k)$ is a sequence of positive real numbers, and $A = (a_{nk})$ is an infinite matrix. Then the spaces $V_{\mathscr{F}}^{I}[\Delta_{\nu}^{m}, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]_0$, $V_{\mathscr{F}}^{I}[\Delta_{\nu}^{m}, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$, and $V_{\mathscr{F}}^{I}[\Delta_{\nu}^{m}, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]_{\infty}$ are linear spaces over the real field $\mathbb{R}$.*

Proof. The proof of the theorem is easy so we omit it. □

For the next result we shall define the following sequence space:

$$V_{\mathscr{F}}\left[\Delta_{\nu}^{m}, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|\right]_{\infty} = \left\{ x = (x_k) \in S_X^n : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f_k \left(\left\| u_k \Delta_{\nu}^{m} x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} < \infty \right\}. \tag{8}$$

Theorem 7. *Let $\mathscr{F} = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ a bounded sequence of strictly positive real numbers. Then the space $V_{\mathscr{F}}[\Delta_{\nu}^{m}, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]_0$ is a paranormed space with the paranorm:*

$$g(x) = \sup_n \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f_k \left(\left\| u_k \Delta_{\nu}^{m} x_k, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{1/G}, \tag{9}$$

where $G = \max(1, H)$ and $0 < p_k \leq \sup_k p_k = H$.

Proof. Clearly $g(x) = g(-x)$ for $x \in V_{\mathscr{F}}^{I}[\Delta_{\nu}^{m}, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]_0$. It is trivial that $u_k \Delta_{\nu}^{m} x_k = 0$ for $x = 0$. Since $f_k(0) = 0$, we get $g(x) = 0$ for $x = 0$. Since $p_k/G \leq 1$, using Minkowski's inequality, we have

$$\begin{aligned} &\left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f_k \left(\left\| u_k \Delta_{\nu}^{m} \left(x_k + y_k \right), z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{1/G} \\ &\leq \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f_k \left(\left\| u_k \Delta_{\nu}^{m} x_k, z_1, \ldots, z_{n-1} \right\| + \left\| u_k \Delta_{\nu}^{m} y_k, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{1/G} \\ &\leq \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f_k \left(\left\| u_k \Delta_{\nu}^{m} x_k, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{1/G} \\ &\quad + \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f_k \left(\left\| u_k \Delta_{\nu}^{m} y_k, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{1/G}. \end{aligned} \tag{10}$$

Hence $g(x + y) \leq g(x) + g(y)$. Finally, to check the continuity of scalar multiplication, let α be any complex number; therefore by definition

$$\begin{aligned} g(\alpha x) &= \sup_n \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f_k \left(\left\| u_k \Delta_{\nu}^{m} \alpha x_k, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{1/G} \\ &\leq C_{\alpha}^{H/G} g(x), \end{aligned} \tag{11}$$

where C_α is a positive integer such that $\alpha \le C_\alpha$. Let $\alpha \to 0$ for any fixed x with $g(x) \neq 0$. By definition, for $|\alpha| < 1$, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k\left(\left\|u_k\Delta_\nu^m \alpha x_k, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_k} < \epsilon, \quad \text{for } n > N(\epsilon). \tag{12}$$

Also, for $1 \le n \le N$, taking α small enough, since f_k is coninuous, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k\left(\left\|u_k\Delta_\nu^m \alpha x_k, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_k} < \epsilon. \tag{13}$$

Now (12) and (13) imply that $g(\alpha x) \to 0$ as $\alpha \to 0$. This completes the proof. □

Theorem 8. *Let $u = (u_k)$ be a sequence of positive real numbers. Then for $m \ge 1$ the following inclusions:*

(i) $V_{\mathscr{F}}^I[\Delta_\nu^{m-1}, \lambda, A, u, \|\cdot, \ldots, \cdot\|]_0 \subset V_{\mathscr{F}}^I[\Delta_\nu^m, \lambda, A, u, \|\cdot, \ldots, \cdot\|]_0$,

(ii) $V_{\mathscr{F}}^I[\Delta_\nu^{m-1}, \lambda, A, u, \|\cdot, \ldots, \cdot\|]_\infty \subset V_{\mathscr{F}}^I[\Delta_\nu^m, \lambda, A, u, \|\cdot, \ldots, \cdot\|]_\infty$,

are strict.

Proof. We will prove the result for $V_{\mathscr{F}}^I[\Delta_\nu^{m-1}, \lambda, A, u, \|\cdot, \ldots, \cdot\|]_0$ only. The others can be proved similarly.

Suppose $x \in V_{\mathscr{F}}^I[\Delta_\nu^{m-1}, \lambda, A, u, \|\cdot, \ldots, \cdot\|]_0$; by definition, for every $\delta > 0$ and $z_1, \ldots, z_{n-1} \in X$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k\left(\left\|u_k\Delta_\nu^{m-1}x_k, z_1, \ldots, z_{n-1}\right\|\right)\right] \ge \delta\right\} \in I. \tag{14}$$

By the property of modulus function, we have

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k\left(\left\|u_k\Delta_\nu^m x_k, z_1, \ldots, z_{n-1}\right\|\right)\right] \\
&\le \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k\left(\left\|u_k\Delta_\nu^{m-1}x_k, z_1, \ldots, z_{n-1}\right\|\right)\right. \\
&\qquad \left. + f_k\left(\left\|u_k\Delta_\nu^{m-1}x_{k+1}, z_1, \ldots, z_{n-1}\right\|\right)\right] \\
&\le D\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k\left(\left\|u_k\Delta_\nu^{m-1}x_k, z_1, \ldots, z_{n-1}\right\|\right)\right] \\
&\quad + D\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k\left(\left\|u_k\Delta_\nu^{m-1}x_{k+1}, z_1, \ldots, z_{n-1}\right\|\right)\right] \quad \text{by (7)}.
\end{aligned} \tag{15}$$

Now, for given $\delta > 0$, we have

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k\left(\left\|u_k\Delta_\nu^m x_k, z_1, \ldots, z_{n-1}\right\|\right)\right] \ge \delta\right\} \\
&\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k\left(\left\|u_k\Delta_\nu^{m-1}x_k, z_1, \ldots, z_{n-1}\right\|\right)\right] \ge \frac{\delta}{2D}\right\} \\
&\cup \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k\left(\left\|u_k\Delta_\nu^{m-1}x_{k+1}, z_1, \ldots, z_{n-1}\right\|\right)\right] \ge \frac{\delta}{2D}\right\},
\end{aligned} \tag{16}$$

for each $z_1, \ldots, z_{n-1} \in X$. Since $x \in V_{\mathscr{F}}^I[\Delta_\nu^{m-1}, \lambda, A, u, \|\cdot, \ldots, \cdot\|]_0$, it follows that the sets on the right-hand side in the above containment belong to I. Hence $x \in V_{\mathscr{F}}^I[\Delta_\nu^{m-1}, \lambda, A, u, \|\cdot, \ldots, \cdot\|]_0$. To show that the inclusion is strict, we give the following example.

We take $f_k(x) = x$, $\lambda_n = n$, and $A = (a_{nk}) = I$, for all $n, k \in \mathbb{N}$, $\nu = 1$, and consider a sequence $x = (x_k) = k^{m-1}$; then $x \in V_{\mathscr{F}}^I[\Delta_\nu^m, \lambda, A, u, \|\cdot, \ldots, \cdot\|]_0$ but does not belong to $V_{\mathscr{F}}^I[\Delta_\nu^{m-1}, \lambda, A, u, \|\cdot, \ldots, \cdot\|]_0$ for $u = (u_k) = 1, k \in \mathbb{N}$. This shows that the inclusion is strict. □

Theorem 9. *Let $\mathscr{F}' = (f_k')$ and $\mathscr{F}'' = (f_k'')$ be sequences of modulus functions. If $\limsup_{t\to\infty}(f_k'(t)/f_k''(t)) = P > 0$, then $V_{\mathscr{F}'}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|] \subset V_{\mathscr{F}''}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$.*

Proof. Let $\limsup_{t\to\infty}(f_k'(t)/f_k''(t)) = P$; then there exists a constant $M > 0$ such that $f_k'(t) \ge Mf_k''(t)$ for all $t \ge 0$. Therefore, for each $z_1, \ldots, z_{n-1} \in X$, we have

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k'\left(\left\|u_k\Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \\
&\ge (M)^H \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k''\left(\left\|u_k\Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_k}.
\end{aligned} \tag{17}$$

Then, for every $\delta > 0$ and $z_1, \ldots, z_{n-1} \in X$, we have the following relationship:

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k''\left(\left\|u_k\Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \ge \delta\right\}$$

$$\subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f_k' \left(\left\| u_k \Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \geq \delta (M)^H \right\}. \tag{18}$$

Since $x \in V_{\mathcal{F}'}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$, it follows that the set on left-side of the above containment belongs to I, which gives $x \in V_{\mathcal{F}''}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$. □

Theorem 10. *Let $\mathcal{F} = (f_k)$, $\mathcal{F}' = (f_k')$, and $\mathcal{F}'' = (f_k'')$ be sequences of modulus functions. Then*

(i) $V_{\mathcal{F}'}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|] \subset V_{\mathcal{F}\circ\mathcal{F}'}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$,

(ii) $V_{\mathcal{F}'}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|] \cup V_{\mathcal{F}''}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|] \subset V_{\mathcal{F}'+\mathcal{F}''}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$.

Proof. (i) Let $x = (x_k) \in V_{\mathcal{F}'}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$, for every $\varepsilon > 0$ and for some $L > 0$ such that

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f_k' \left(\left\| u_k \Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I, \tag{19}$$

for each $z_1, \ldots, z_{n-1} \in X$. For given $\varepsilon > 0$, we choose $\delta \in (0, 1)$ such that $f_k(t) < \varepsilon$ for all $0 < t < \delta$. On the other hand, we have

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[\left(f_k \circ f_k' \right) \left(\left\| u_k \Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \\ &= \frac{1}{\lambda_n} \sum_{k \in I_n, \left[f_k' \left(\left\| u_k \Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} < \delta} a_{nk} \left[\left(f_k \circ f_k' \right) \cdot \left(\left\| u_k \Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \\ &\quad + \frac{1}{\lambda_n} \sum_{k \in I_n, \left[f_k' \left(\left\| u_k \Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \geq \delta} a_{nk} \cdot \left[\left(f_k \circ f_k' \right) \cdot \left(\left\| u_k \Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \\ &\leq (\varepsilon)^H + \max \left(1, \left(2 \frac{f_k(1)}{\delta} \right)^H \right) \frac{1}{\lambda_n} \cdot \sum_{k \in I_n} a_{nk} \left[f_k' \left(\left\| u_k \Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k}, \end{aligned} \tag{20}$$

by Lemma 5.

By using (19), we obtain $x \in V_{\mathcal{F}\circ\mathcal{F}'}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$.

(ii) The result of this part is proved by using the following inequality:

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[\left(f_k' + f_k'' \right) \left(\left\| u_k \Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \\ &\leq \frac{D}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f_k' \left(\left\| u_k \Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \\ &\quad + \frac{D}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f_k'' \left(\left\| u_k \Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k}, \end{aligned} \tag{21}$$

where $\sup_k p_k = H$ and $D = \max(1, 2^{H-1})$. □

Theorem 11. *Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ a bounded sequence of strictly positive real numbers. Then $V^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|] \subseteq V_{\mathcal{F}}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$.*

Proof. This can be proved by using the same techniques as in Theorem 10. □

Theorem 12. *Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions. If $\limsup_{t \to \infty} (f_k(t)/t) = M > 0$, then $V_{\mathcal{F}}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|] \subseteq V^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$.*

Proof. Suppose $x = (x_k) \in V_{\mathcal{F}}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$ and $\limsup_{t \to \infty} (f_k(t)/t) = M > 0$, then there exists a constant $K > 0$ such that $f_k(t) \geq Kt$ for all $t \geq 0$. Thus, we have

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f_k \left(\left\| u_k \Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \\ &\geq (K)^H \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[\left(\left\| u_k \Delta_\nu^m x_k - L, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k}, \end{aligned} \tag{22}$$

for each $z_1, \ldots, z_{n-1} \in X$. This completes the proof. □

Theorem 13. *If $0 < p_k \leq q_k$ and (q_k/p_k) be bounded. Then $V_{\mathcal{F}}^I[\Delta_\nu^m, \lambda, A, u, q, \|\cdot, \ldots, \cdot\|] \subset V_{\mathcal{F}}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$.*

Proof. It is easy to prove so we omit the detail. □

3. Statistical Convergence

The notion of statistical convergence introduced by Fast [30] in 1951 and later developed by Fridy [31], Šalát [32] and many

others. Furthermore, Kostyrko et al. [33] presented a very interesting generalization of statistical convergence called as I-convergence. Some recent developments in this regard can be found in [34–37] and many others.

In this section, we define a new class of generalized statistical convergent sequences with the help of an ideal, modulus functions and infinite matrix. We also made an effort to establish a strong connection between this convergence and the sequence space $V_{\mathscr{F}}^{I}[\Delta_{\nu}^{m}, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$.

Definition 14. Let $I \subseteq P(\mathbb{N})$ be a non-trivial ideal and $\lambda = (\lambda_n)$ be a non-decreasing sequence. A sequence $x = (x_k) \in X$ is said to be $S_{\lambda}^{\Delta_{\nu}^{m}}(I, u, A, \|\cdot, \ldots, \cdot\|)$-convergent to a number L provided that for every $\varepsilon > 0$, $\delta > 0$ and $z_1, \ldots, z_{n-1} \in X$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\left|\left\{k \in I_n : a_{nk}\left(\left\|u_k \Delta_{\nu}^{m} x_k - L, z_1, \ldots, z_{n-1}\right\|\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in I. \tag{23}$$

In this case, we write $S_{\lambda}^{\Delta_{\nu}^{m}}(I, u, A) - \lim_{k\to\infty}\|u_k \Delta_{\nu}^{m} x_k, z_1, \ldots, z_{n-1}\| = \|L/\rho, z_1, \ldots, z_{n-1}\|$. Let $S_{\lambda}^{\Delta_{\nu}^{m}}(I, A, u, \|\cdot, \ldots, \cdot\|)$ denotes the set of all $S_{\lambda}^{\Delta_{\nu}^{m}}(I, u, A, \|\cdot, \ldots, \cdot\|)$-convergent sequences in X.

Theorem 15. *Let $\mathscr{F} = (f_k)$ be a sequence of modulus functions and $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$. Then $V_{\mathscr{F}}^{I}[\Delta_{\nu}^{m}, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|] \subset S_{\lambda}^{\Delta_{\nu}^{m}}[I, A, u, \|\cdot, \ldots, \cdot\|]$.*

Proof. Suppose $x = (x_k) \in V_{\mathscr{F}}^{I}[\Delta_{\nu}^{m}, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$ and $\varepsilon > 0$ be given. Then for each $z_1, \ldots, z_{n-1} \in X$, we obtain

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k\left(\left\|u_k \Delta_{\nu}^{m} x_k - L, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \\
&= \frac{1}{\lambda_n}\sum_{k\in I_n, \|u_k \Delta_{\nu}^{m} x_k - L, z_1, \ldots, z_{n-1}\| \geq \varepsilon} a_{nk}\left[f_k\left(\left\|u_k \Delta_{\nu}^{m} x_k - L, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \\
&\quad + \frac{1}{\lambda_n}\sum_{k\in I_n, \|u_k \Delta_{\nu}^{m} x_k - L, z_1, \ldots, z_{n-1}\| < \varepsilon} a_{nk}\left[f_k\left(\left\|u_k \Delta_{\nu}^{m} x_k - L, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \\
&\geq \frac{1}{\lambda_n}\sum_{k\in I_n, \|u_k \Delta_{\nu}^{m} x_k - L, z_1, \ldots, z_{n-1}\| \geq \varepsilon} a_{nk}\left[f_k\left(\left\|u_k \Delta_{\nu}^{m} x_k - L, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \\
&\geq \frac{1}{\lambda_n}\sum_{k\in I_n}\left[f_k(\varepsilon)\right]^{p_k} \\
&\geq \sum_{k\in I_n}\min\left(\left[f_k(\varepsilon)\right]^{h}, \left[f_k(\varepsilon)\right]^{H}\right) \\
&\geq K\frac{1}{\lambda_n}\left|\left\{k \in I_n : a_{nk}\left(\left\|u_k \Delta_{\nu}^{m} x_k - L, z_1, \ldots, z_{n-1}\right\|\right) \geq \varepsilon\right\}\right|,
\end{aligned} \tag{24}$$

where $K = \min([f_k(\varepsilon)]^h, [f_k(\varepsilon)]^H)$. Then for every $\delta > 0$ and $z_1, \ldots, z_{n-1} \in X$, we have

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\left|\left\{k \in I_n : a_{nk}\left(\left\|u_k \Delta_{\nu}^{m} x_k - L, z_1, \ldots, z_{n-1}\right\|\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \\
&\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\left[f_k\left(\left\|u_k \Delta_{\nu}^{m} x_k - L, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \geq K\delta\right\}.
\end{aligned} \tag{25}$$

Since $x_k \to L(V_{\mathscr{F}}^{I}[\Delta_{\nu}^{m}, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|])$ so that $S_{\lambda}^{\Delta_{\nu}^{m}}(I, u, A) - \lim_{k\to\infty}\|u_k \Delta_{\nu}^{m} x_k, z_1, \ldots, z_{n-1}\| = \|L, z_1, \ldots, z_{n-1}\|$. □

Theorem 16. *Let $\mathscr{F} = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$ then $S_{\lambda}^{\Delta_{\nu}^{m}}[I, A, u, \|\cdot, \ldots, \cdot\|] \subset V_{\mathscr{F}}^{I}[\Delta_{\nu}^{m}, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$.*

Proof. By using ([11, Theorem 3.5]) we can prove easily. □

Theorem 17. *Let $\mathscr{F} = (f_k)$ be a bounded sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$. Then $S_{\lambda}^{\Delta_{\nu}^{m}}[I, A, u, \|\cdot, \ldots, \cdot\|] = V_{\mathscr{F}}^{I}[\Delta_{\nu}^{m}, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$ if and only if $\mathscr{F} = (f_k)$ is a bounded.*

Proof. This part can be obtained by combining Theorems 15 and 16. Conversely suppose $\mathscr{F} = (f_k)$ be unbounded defined by $f_k(k) = k$ for all $k \in \mathbb{N}$. We take a fixed set $B \in I$, where I is an admissible ideal and define $x = (x_k)$ as follows

$$x_k = \begin{cases} k^{m+1}, & \text{for } n - \left\lceil\sqrt{\lambda_n}\right\rceil + 1 \leq k \leq n,\ n \notin B \\ k^{m+1}, & \text{for } n - \left\lceil\sqrt{\lambda_n}\right\rceil + 1 \leq k \leq n,\ n \in B \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

For given $\varepsilon > 0$ and for each $z_1, \ldots, z_{n-1} \in X$ we have,

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{k \in I_n : a_{nk}\left(\left\|u_k \Delta_{\nu}^{m} x_k - 0, z_1, \ldots, z_{n-1}\right\|\right) \geq \varepsilon\right\}\right| \\
&< \frac{\left\lceil\sqrt{\lambda_n}\right\rceil}{\lambda_n} \longrightarrow 0,
\end{aligned} \tag{27}$$

for $n \notin B$. Hence for $\delta > 0$, there exists a positive integer n_0 such that $(1/\lambda_n)|\{k \in I_n : a_{nk}\|u_k\Delta_\nu^m x_k - 0, z_1, \ldots, z_{n-1}\| \geq \varepsilon\}| < \delta$ for $n \notin B$ and $n \geq n_0$. Now we have $(1/\lambda_n)|\{k \in I_n : a_{nk}\|u_k\Delta_\nu^m x_k - 0, z_1, \ldots, z_{n-1}\| \geq \varepsilon\}| \geq \delta \subset \{B \cup (1, 2, \ldots, n_0 - 1)\}$. Since I be an admissible ideal it follows that $S_\lambda^{\Delta_\nu^m}(I, u, A) - \lim_{k\to\infty}\|u_k x_k, z_1, \ldots, z_{n-1}\| \to 0$ for each $z_1, \ldots, z_{n-1} \in X$.

On the other hand, if we take $p = (p_k) = 1$, for all $k \in \mathbb{N}$ then $x_k \notin V_{\mathscr{F}}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$. This contradicts the fact $S_\lambda^{\Delta_\nu^m}[I, A, u, \|\cdot, \ldots, \cdot\|] = V_{\mathscr{F}}^I[\Delta_\nu^m, \lambda, A, u, p, \|\cdot, \ldots, \cdot\|]$, so our supposition is wrong. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. (69-130-35-RG). The authors, therefore, acknowledge with thanks DSR technical and financial support.

References

[1] S. Gähler, "Lineare 2-normierte Räume," *Mathematische Nachrichten*, vol. 28, no. 1-2, pp. 1–43, 1964.

[2] A. Misiak, "n-inner product spaces," *Mathematische Nachrichten*, vol. 140, pp. 299–319, 1989.

[3] H. Gunawan, "On n-inner product, n-norms, and the Cauchy-Schwartz inequality," *Scientiae Mathematicae Japonicae*, vol. 5, pp. 47–54, 2001.

[4] H. Gunawan, "The space of p-summable sequences and its natural n-norm," *Bulletin of the Australian Mathematical Society*, vol. 64, no. 1, pp. 137–147, 2001.

[5] H. Gunawan and M. Mashadi, "On n-normed spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 10, pp. 631–639, 2001.

[6] K. Raj, S. K. Sharma, and A. K. Sharma, "Some difference sequence spaces in n-normed spaces defined by Musielak-Orlicz function," *Armenian Journal of Mathematics*, vol. 3, pp. 127–141, 2010.

[7] A. Şahiner, M. Gürdal, S. Saltan, and H. Gunawan, "Ideal convergence in 2–normed spaces," *Taiwanese Journal of Mathematics*, vol. 11, no. 5, pp. 1477–1484, 2007.

[8] H. Kızmaz, "On certain sequence spaces," *Canadian Mathematical Bulletin*, vol. 24, no. 2, pp. 169–176, 1981.

[9] M. Et and R. Çolak, "On some generalized difference sequence spaces," *Soochow Journal of Mathematics*, vol. 21, no. 4, pp. 377–386, 1995.

[10] Ç. A. Bektaş, M. Et, and R. Çolak, "Generalized difference sequence spaces and their dual spaces," *Journal of Mathematical Analysis and Applications*, vol. 292, no. 2, pp. 423–432, 2004.

[11] M. Et and A. Esi, "On Köthe-Toeplitz duals of generalized difference sequence spaces," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 23, no. 1, pp. 25–32, 2000.

[12] B. C. Tripathy and A. Esi, "A new type of difference sequence spaces," *The International Journal of Science & Technology*, vol. 1, pp. 11–14, 2006.

[13] A. Esi, B. C. Tripathy, and B. Sarma, "On some new type generalized difference sequence spaces," *Mathematica Slovaca*, vol. 57, no. 5, pp. 475–482, 2007.

[14] B. C. Tripathy, A. Esi, and B. Tripathy, "On a new type of generalized difference Cesàro sequence spaces," *Soochow Journal of Mathematics*, vol. 31, no. 3, pp. 333–340, 2005.

[15] T. Bilgin, "Some new difference sequences spaces defined by an Orlicz function," *Filomat*, no. 17, pp. 1–8, 2003.

[16] M. Et, "Strongly almost summable difference sequences of order m defined by a modulus," *Studia Scientiarum Mathematicarum Hungarica*, vol. 40, no. 4, pp. 463–476, 2003.

[17] S. A. Mohiuddine, K. Raj, and A. Alotaibi, "On some classes of double difference sequences of interval numbers," *Abstract and Applied Analysis*, vol. 2014, Article ID 516956, 8 pages, 2014.

[18] S. A. Mohiuddine, K. Raj, and A. Alotaibi, "Some paranormed double difference sequence spaces for Orlicz functions and bounded-regular matrices," *Abstract and Applied Analysis*, vol. 2014, Article ID 419064, 10 pages, 2014.

[19] S. A. Mohiuddine, K. Raj, and A. Alotaibi, "Generalized spaces of double sequences for Orlicz functions and bounded-regular matrices over n-normed spaces," *Journal of Inequalities and Applications*, vol. 2014, article 332, 2014.

[20] Y. Altin and M. Et, "Generalized difference sequence spaces defined by a modulus function in a locally convex space," *Soochow Journal of Mathematics*, vol. 31, no. 2, pp. 233–243, 2005.

[21] M. Mursaleen, "λ-statistical convergence," *Mathematica Slovaca*, vol. 50, pp. 111–115, 2000.

[22] E. Savaş and P. Das, "A generalized statistical convergence via ideals," *Applied Mathematics Letters*, vol. 24, no. 6, pp. 826–830, 2011.

[23] C. Belen and S. A. Mohiuddine, "Generalized weighted statistical convergence and application," *Applied Mathematics and Computation*, vol. 219, no. 18, pp. 9821–9826, 2013.

[24] P. Das, E. Savaş, and S. K. Ghosal, "On generalizations of certain summability methods using ideals," *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1509–1514, 2011.

[25] M. Gürdal and H. Sarí, "Extremal A-statistical limit points via ideals," *Journal of the Egyptian Mathematical Society*, vol. 22, no. 1, pp. 55–58, 2014.

[26] M. Et, Y. Altin, and H. Altinok, "On some generalized difference sequence spaces defined by a modulus function," *Filomat*, no. 17, pp. 23–33, 2003.

[27] I. J. Maddox, "Sequence spaces defined by a modulus," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 100, no. 1, pp. 161–166, 1986.

[28] E. Savaş and M. Mursaleen, "Matrix transformations in some sequence spaces," *Istanbul Üniversitesi Fen Fakültesi Matematik Dergisi*, vol. 52, pp. 1–5, 1993.

[29] S. Kumar, V. Kumar, and S. S. Bhatia, "Generalized sequence spaces in 2-normed spaces defined by ideal and a modulus function," *Analele Ştiinţifice ale Universităţii*, 2014.

[30] H. Fast, "Sur la convergence statistique," *Colloquium Mathematicae*, vol. 2, no. 3-4, pp. 241–244, 1951.

[31] J. A. Fridy, "On statistical convergence," *Analysis*, vol. 5, no. 4, pp. 301–313, 1985.

[32] T. Šalát, "On statistically convergent sequences of real numbers," *Mathematica Slovaca*, vol. 30, no. 2, pp. 139–150, 1980.

[33] P. Kostyrko, T. Salat, and W. Wilczynski, "I- convergence," *Real Analysis Exchange*, vol. 26, pp. 669–686, 2000.

[34] M. Et, A. Alotaibi, and S. A. Mohiuddine, "On $(\triangle^m, I)$-statistical convergence of order α," *The Scientific World Journal*, vol. 2014, Article ID 535419, 5 pages, 2014.

[35] F. Gezer and S. Karakuş, "I and I^* convergent function sequences," *Mathematical Communications*, vol. 10, pp. 71–80, 2005.

[36] B. Hazarika and S. A. Mohiuddine, "Ideal convergence of random variables," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 148249, 7 pages, 2013.

[37] M. Mursaleen and S. A. Mohiuddine, "On ideal convergence in probabilistic normed spaces," *Mathematica Slovaca*, vol. 62, no. 1, pp. 49–62, 2012.

Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 905834, 11 pages
http://dx.doi.org/10.1155/2015/905834

Research Article

On a New Faster Implicit Fixed Point Iterative Scheme in Convex Metric Spaces

Renu Chugh,[1] Preety Malik,[1] and Vivek Kumar[2]

[1] *Department of Mathematics, Maharshi Dayanand University, Rohtak 124001, India*
[2] *Department of Mathematics, KLP College, Rewari 123401, India*

Correspondence should be addressed to Vivek Kumar; ratheevivek15@yahoo.com

Received 3 August 2014; Revised 28 September 2014; Accepted 28 September 2014

Academic Editor: Nawab Hussain

Copyright © 2015 Renu Chugh et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The purpose of this paper is to consider a new implicit iteration and study its strong convergence, stability, and data dependence. It is proved through numerical examples that newly introduced iteration has better convergence rate than well known implicit Mann iteration as well as implicit Ishikawa iteration and implicit iterations converge faster as compared to corresponding explicit iterations. Applications of implicit iterations to RNN (Recurrent Neural Networks) analysis are also presented.

1. Introduction

In recent years, numerous papers have been published on explicit iterations in various spaces [1–6], but there are very few works on implicit iterations (regarding convergence rate and data dependence) [7–13]. Implicit iterations have an advantage over explicit iterations for nonlinear problems as they provide better approximation of fixed points and are widely used in many applications when explicit iterations are inefficient. Approximation of fixed points in computer oriented programs by using implicit iterations can reduce the computational cost of the fixed point problems. The study of stability of iterations enjoys a celebrated place in applied sciences and engineering due to chaotic behavior of functions in discrete dynamics and other numerical computations. Data dependence of fixed points is a related and new issue which has been studied by many authors; see [4, 14] and references therein. In computational mathematics, it is of theoretical and practical importance to compare the convergence rate of iterations and to find out, if possible, which one of them converges more rapidly to the fixed point. Recent works in this direction are [1, 3, 4, 15–17]. In concrete, a fixed point iteration is valuable from a numerical point of view and is useful for applications if it satisfies the following requirements:

(a) it converges to fixed point of an operator;

(b) it is T-stable;

(c) it is faster as compared to other iterations existing in the literature;

(d) it shows data dependence results.

Motivated by the fact that three-step iterations give better approximation than one-step and two-step iterations [18], we define a new and more general three-step implicit iteration (IN) which satisfies the above requirements.

Let K be a nonempty convex subset of a convex metric space X and let $T : K \to K$ be a given mapping. For the real sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$ in $[0, 1]$, Noor iteration [19] in convex metric spaces can be written as

$$\begin{aligned} x_n &= W\left(x_{n-1}, Ty_{n-1}, \alpha_n\right), \\ y_{n-1} &= W\left(x_{n-1}, Tz_{n-1}, \beta_n\right), \\ z_{n-1} &= W\left(x_{n-1}, Tx_{n-1}, \gamma_n\right), \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{N}$$

Putting $\gamma_n = 1$ in (N), we get well known Ishikawa iteration [20, 21] in convex metric spaces:

$$\begin{aligned} x_n &= W\left(x_{n-1}, Ty_{n-1}, \alpha_n\right), \\ y_{n-1} &= W\left(x_{n-1}, Tx_{n-1}, \beta_n\right), \quad n = 0, 1, 2 \ldots. \end{aligned} \tag{I}$$

Putting $\gamma_n = \beta_n = 1$ in (N), we get well known Mann iteration [21, 22] in convex metric spaces:

$$x_n = W\left(x_{n-1}, Tx_{n-1}, \alpha_n\right), \quad n = 0, 1, 2 \ldots. \tag{M}$$

For $x_0 \in K$, we define the following iteration, namely, implicit Noor iteration in convex metric spaces:

$$\begin{aligned} x_n &= W\left(y_{n-1}, Tx_n, \alpha_n\right), \\ y_{n-1} &= W\left(z_{n-1}, Ty_{n-1}, \beta_n\right), \\ z_{n-1} &= W\left(x_{n-1}, Tz_{n-1}, \gamma_n\right), \end{aligned} \tag{IN}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$.

Equivalence form of iteration (IN) in linear space can be written as

$$\begin{aligned} x_n &= \alpha_n y_{n-1} + \left(1 - \alpha_n\right) Tx_n, \\ y_{n-1} &= \beta_n z_{n-1} + \left(1 - \beta_n\right) Ty_{n-1}, \\ z_{n-1} &= \gamma_n x_{n-1} + \left(1 - \gamma_n\right) Tz_{n-1}. \end{aligned} \tag{1}$$

Putting $\gamma_n = 1$ in (IN), we get well known implicit Ishikawa iteration [23]:

$$\begin{aligned} x_n &= W\left(y_{n-1}, Tx_n, \alpha_n\right), \\ y_{n-1} &= W\left(x_{n-1}, Ty_{n-1}, \beta_n\right). \end{aligned} \tag{II}$$

Putting $\gamma_n = \beta_n = 1$ in (IN), we get well known implicit Mann iteration [2, 6, 11, 24]:

$$x_n = W\left(x_{n-1}, Tx_n, \alpha_n\right) = \alpha_n x_{n-1} + \left(1 - \alpha_n\right) Tx_n. \tag{IM}$$

Zamfirescu operators [25] are the most general contractive like operators which have been studied by several authors, satisfying the following condition: for each pair of points x, y in X at least one of the following is true:

$$\begin{aligned} &\text{(i)} \ \ d\left(Tx, Ty\right) \le p d\left(x, y\right) \\ &\text{(ii)} \ \ d\left(Tx, Ty\right) \le q\left[d\left(x, Tx\right) + d\left(y, Ty\right)\right] \\ &\text{(iii)} \ \ d\left(Tx, Ty\right) \le r\left[d\left(x, Ty\right) + d\left(y, Tx\right)\right], \end{aligned} \tag{2}$$

where p, q, and r are nonnegative constants satisfying $0 \le p \le 1$, $0 \le q$, and $r \le 1/2$.

Z-operators are equivalent to the following contractive condition:

$$\begin{aligned} d\left(Tx, Ty\right) \le c \max \Bigg\{ & d\left(x, y\right), \frac{\left\{d\left(x, Tx\right) + d\left(y, Ty\right)\right\}}{2}, \\ & \frac{\left\{d\left(x, Ty\right) + d\left(y, Tx\right)\right\}}{2} \Bigg\} \\ & \forall x, y \in X, 0 < c < 1. \end{aligned} \tag{3}$$

The contractive condition (3) implies

$$d\left(Tx, Ty\right) \le 2ad\left(x, Tx\right) + ad\left(x, y\right), \quad \forall x, y \in X, \tag{4}$$

where $a = \max\{c, c/(2 - c)\}$ [see [15]].

In [5], Rhoades used the following more general contractive condition than (4): there exists $c \in [0, 1)$ such that

$$\begin{aligned} d\left(Tx, Ty\right) \le c \max \Bigg\{ & d\left(x, y\right), \frac{\left\{d\left(x, Tx\right) + d\left(y, Ty\right)\right\}}{2}, \\ & d\left(x, Ty\right), d\left(y, Tx\right) \Bigg\} \quad \forall x, y \in X. \end{aligned} \tag{5}$$

In [26], Osilike used a more general contractive definition than those of Rhoades: there exist $a \in [0, 1)$, $L \ge 0$ such that

$$d\left(Tx, Ty\right) \le Ld\left(x, Tx\right) + ad\left(x, y\right) \quad \forall x, y \in X. \tag{6}$$

We use the contractive condition due to Imoru and Olatinwo [27], which is more general than (6): there exist $a \in [0, 1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$, such that

$$\begin{aligned} d\left(Tx, Ty\right) \le \phi\left(d\left(x, Tx\right)\right) + ad\left(x, y\right), \\ a \in [0, 1), \forall x, y \in X. \end{aligned} \tag{7}$$

Also, we use the following definitions and lemmas to achieve our main results.

Definition 1 (see [23]). A map $W : X^2 \times [0, 1] \to X$ is a convex structure on X if

$$d\left(u, W(x, y, \lambda)\right) \le \lambda d\left(u, x\right) + (1 - \lambda) d\left(u, y\right) \tag{8}$$

for all $x, y, u \in X$ and $\lambda \in [0, 1]$. A metric space (X, d) together with a convex structure W is known as convex metric space and denoted by (X, d, W). A nonempty subset C of a convex metric space is convex if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$.

All normed spaces and their subsets are the examples of convex metric spaces. But there are many examples of convex metric spaces which are not embedded in any normed space (see [23, 28]). After that several authors extended this concept in many ways; one such convex structure is hyperbolic space which was introduced by Kohlenbach [29] as follows.

Definition 2 (see [29]). A hyperbolic space (X, d, W) is a metric space (X, d) together with a convexity mapping $W : X^2 \times [0, 1] \to X$ satisfying

(W1) $d(z, W(x, y, \lambda)) \le (1 - \lambda)d(z, x) + \lambda d(z, y)$,

(W2) $d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2| d(x, y)$,

(W3) $W(x, y, \lambda) = W(y, x, 1 - \lambda)$,

(W4) $d(W(x, z, \lambda), W(y, w, \lambda)) \le (1 - \lambda)d(x, y) + \lambda d(z, w)$ for all $x, y, z, w \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$.

Evidently every hyperbolic space is a convex metric space but the converse may not true. For example, if we take $X = R$, $W(x, y, \lambda) = \lambda x + (1 - \lambda) y$ and define $d(x, y) = |x - y|/(1 + |x - y|)$ for $x, y \in R$, then (X, d, W) is a convex metric space but not a hyperbolic space.

The stability of explicit as well as implicit iterations has extensively been studied by various authors [4, 7, 21, 27, 30–32] due to its increasing importance in computational mathematics, especially due to revolution in computer programming. The concept of T-stability in convex metric space setting was given by Olatinwo [21].

Definition 3 (see [21]). Let (X, d, W) be a convex metric space and let $T : X \to X$ a self-mapping.

Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iterative scheme involving T which is defined by

$$x_{n+1} = f_{T,\alpha_n}^{x_n}, \quad n = 0, 1, 2 \ldots, \tag{9}$$

where $x_0 \in X$ is the initial approximation and $f_{T,\alpha_n}^{x_n}$ is some function having convex structure such that $\alpha_n \in [0, 1]$. Suppose that $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set $\varepsilon_n = d(y_{n+1}, f_{T,\alpha_n}^{y_n})$. Then, the iteration (9) is said to be T-stable with respect to T if and only if $\lim_{n\to\infty}\varepsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$.

Lemma 4 (see [4, 15]). *If δ is a real number such that $0 \le \delta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty}\epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \le \delta u_n + \epsilon_n, \quad n = 0, 1, 2, \ldots, \tag{10}$$

one has $\lim_{n\to\infty} u_n = 0$.

Definition 5 (see [15]). Suppose $\{a_n\}$ and $\{b_n\}$ are two real convergent sequences with limits a and b, respectively. Then $\{a_n\}$ is said to converge faster than $\{b_n\}$ if

$$\lim_{n\to\infty}\left|\frac{a_n - a}{b_n - b}\right| = 0. \tag{11}$$

Definition 6 (see [15]). Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iterations that converge to the same fixed point p on a normed space X such that the error estimates

$$\begin{aligned} \|u_n - p\| &\le a_n, \\ \|v_n - p\| &\le b_n \end{aligned} \tag{12}$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers (converging to zero). If $\{a_n\}$ converge faster than $\{b_n\}$, then one says that $\{u_n\}$ converge faster to p than $\{v_n\}$.

Definition 7 (see [14]). Let T, T_1 be two operators on X. One says T_1 is approximate operator of T if, for all $x \in X$ and for a fixed $\varepsilon > 0$, one has $d(Tx, T_1x) \le \varepsilon$.

Lemma 8 (see [4, 14]). *Let $\{a_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which there exists $n_0 \in N$ such that, for all $n \ge n_0$, one has the following inequality:*

$$a_{n+1} \le (1 - r_n)\, a_n + r_n t_n, \tag{13}$$

where $r_n \in (0, 1)$, for all $n \in N$, $\sum_{n=1}^{\infty} r_n = \infty$, and $t_n \ge 0$ $\forall n \in N$.

Then, $0 \le \lim_{n\to\infty} \sup a_n \le \lim_{n\to\infty} \sup t_n$.

Having introduced the implicit Noor iteration (IN), we use it to prove the results concerning convergence, stability, and convergence rate for contractive condition (7) in convex metric spaces. Also, data dependence result of the same iteration is proved in hyperbolic spaces. Moreover, applications of implicit iterations in RNN analysis will be discussed in the last section.

2. Convergence and Stability Results of New Implicit Iteration in Convex Metric Spaces

Theorem 9. *Let K be a nonempty closed convex subset of a convex metric space X and let T be a quasi-contractive operator satisfying* (7) *with $F(T) \ne \phi$. Then, for $x_0 \in C$, the sequence $\{x_n\}$ defined by* (IN), with $\sum(1 - \alpha_n) = \infty$, converges to the fixed point of T.

Proof. Using (IN) and (7), we have, for $p \in F(T)$,

$$\begin{aligned} d(x_n, p) &= d\left(W\left(y_{n-1}, Tx_n, \alpha_n\right), p\right) \le \alpha_n d\left(y_{n-1}, p\right) \\ &\quad + (1 - \alpha_n)\, d\left(Tx_n, p\right) \le \alpha_n d\left(y_{n-1}, p\right) \\ &\quad + (1 - \alpha_n)\, a d\left(x_n, p\right), \end{aligned} \tag{14}$$

which further implies

$$\begin{aligned} \left\{1 - (1 - \alpha_n)\, a\right\} d\left(x_n, p\right) &\le \alpha_n d\left(y_{n-1}, p\right) \\ d\left(x_n, p\right) &\le \frac{\alpha_n}{1 - (1 - \alpha_n)\, a} d\left(y_{n-1}, p\right). \end{aligned} \tag{15}$$

Again from (IN), we have the following estimates:

$$\begin{aligned} d\left(y_{n-1}, p\right) &= d\left(W\left(z_{n-1}, Ty_{n-1}, \beta_n\right), p\right) \le \beta_n d\left(z_{n-1}, p\right) \\ &\quad + (1 - \beta_n)\, d\left(Ty_{n-1}, p\right) \le \beta_n d\left(z_{n-1}, p\right) \\ &\quad + (1 - \beta_n)\, a d\left(y_{n-1}, p\right), \end{aligned} \tag{16}$$

which gives

$$\begin{aligned} d\left(y_{n-1}, p\right) &\le \frac{\beta_n}{1 - (1 - \beta_n)\, a} d\left(z_{n-1}, p\right), \\ d\left(z_{n-1}, p\right) &= d\left(W\left(x_{n-1}, Tz_{n-1}, \gamma_n\right), p\right) \le \gamma_n d\left(x_{n-1}, p\right) \\ &\quad + (1 - \gamma_n)\, d\left(Tz_{n-1}, p\right) \le \gamma_n d\left(x_{n-1}, p\right) \\ &\quad + (1 - \gamma_n)\, a d\left(z_{n-1}, p\right), \end{aligned} \tag{17}$$

implying that

$$d\left(z_{n-1}, p\right) \le \frac{\gamma_n}{1 - (1 - \gamma_n)\, a} d\left(x_{n-1}, p\right). \tag{18}$$

Using (14)–(18), we arrive at

$$\begin{aligned} d\left(x_n, p\right) \le &\left(\frac{\alpha_n}{1 - (1 - \alpha_n)\, a}\right)\left(\frac{\beta_n}{1 - (1 - \beta_n)\, a}\right) \\ &\times \left(\frac{\gamma_n}{1 - (1 - \gamma_n)\, a}\right) d\left(x_{n-1}, p\right). \end{aligned} \tag{19}$$

If we take $\alpha_n/(1-(1-\alpha_n)a) = A_n/B_n$, then

$$1-\frac{A_n}{B_n} = 1-\frac{\alpha_n}{1-(1-\alpha_n)a} = \frac{1-[(1-\alpha_n)a+\alpha_n]}{1-(1-\alpha_n)a} \geq 1-[(1-\alpha_n)a+\alpha_n], \tag{20}$$

and hence

$$\frac{A_n}{B_n} \leq (1-\alpha_n)a+\alpha_n = 1-(1-a)(1-\alpha_n). \tag{21}$$

Similarly, with ease we can show that

$$\frac{\beta_n}{1-(1-\beta_n)a} \leq \beta_n+(1-\beta_n)a = 1-(1-\beta_n)(1-a) \leq 1, \tag{22}$$

$$\frac{\gamma_n}{1-(1-\gamma_n)a} \leq \gamma_n+(1-\gamma_n)a = 1-(1-\gamma_n)(1-a) \leq 1. \tag{23}$$

Using (21)–(23), (19) becomes

$$\begin{aligned} d(x_n,p) &\leq [1-(1-\alpha_n)(1-a)]\,d(x_{n-1},p) \\ &\leq \prod_{i=1}^{n}[1-(1-\alpha_i)(1-a)]\,d(x_0,p) \\ &\leq e^{-\sum_{i=1}^{n}(1-\alpha_i)(1-a)}d(x_0,p). \end{aligned} \tag{24}$$

But $\sum_{i=1}^{n}(1-\alpha_i) = \infty$; hence (24) yields $\lim_{n\to\infty}d(x_n,p) = 0$. Therefore $\{x_n\}$ converges to p. □

Theorem 10. *Let K be a nonempty closed convex subset of a convex metric space X and let T be a quasi-contractive operator satisfying* (7) *with $F(T) \neq \phi$. Then, for $x_0 \in C$, the sequence $\{x_n\}$ defined by* (IN) with $\alpha_n \leq \alpha < 1$, $\sum(1-\alpha_n) = \infty$, is T-stable.

Proof. Suppose that $\{p_n\}_{n=0}^{\infty} \subset K$ is an arbitrary sequence, $\varepsilon_n = d(p_n, W(q_{n-1}, Tp_n, \alpha_n))$, where $q_{n-1} = W(r_{n-1}, Tq_{n-1}, \beta_n)$, $r_{n-1} = W(p_{n-1}, Tr_{n-1}, \gamma_n)$ and let $\lim_{n\to\infty}\varepsilon_n = 0$.

Then, using (7) we have

$$\begin{aligned} d(p_n,p) &\leq d(p_n, W(q_{n-1}, Tp_n, \alpha_n)) \\ &\quad + d(W(q_{n-1}, Tp_n, \alpha_n), p) \\ &\leq \varepsilon_n + \alpha_n d(q_{n-1},p) + (1-\alpha_n)\,d(Tp_n,p) \\ &\leq \varepsilon_n + \alpha_n d(q_{n-1},p) + (1-\alpha_n)\,\varphi d(Tp,p) \\ &\quad + (1-\alpha_n)\,ad(p_n,p), \end{aligned} \tag{25}$$

which implies

$$\{1-(1-\alpha_n)a\}\,d(p_n,p) \leq \varepsilon_n + \alpha_n d(q_{n-1},p), \tag{26}$$

and therefore

$$d(p_n,p) \leq \frac{\alpha_n}{1-(1-\alpha_n)a}d(q_{n-1},p) + \frac{\varepsilon_n}{1-(1-\alpha_n)a}. \tag{27}$$

Using (21), (27) becomes

$$d(p_n,p) \leq [1-(1-\alpha_n)(1-a)]\,d(q_{n-1},p) + \frac{\varepsilon_n}{1-(1-\alpha_n)a}. \tag{28}$$

Now, using (22) and (23), we have the following estimates:

$$\begin{aligned} d(q_{n-1},p) &\leq \frac{\beta_n}{1-(1-\beta_n)a}d(r_{n-1},p) \leq d(r_{n-1},p), \\ d(r_{n-1},p) &\leq \frac{\gamma_n}{1-(1-\gamma_n)a}d(p_{n-1},p) \leq d(p_{n-1},p). \end{aligned} \tag{29}$$

Using $\alpha_n \leq \alpha < 1$and $a \in [0,1)$, we have $1-(1-\alpha_n)(1-a) < 1$. Hence using Lemma 4, together with estimates (29), (28) yields $\lim_{n\to\infty}p_n = p$.

Conversely, if we let $\lim_{n\to\infty}p_n = p$, then using contractive condition (7), it is easy to see that $\lim_{n\to\infty}\varepsilon_n = 0$.

Therefore, the iteration (IN) is T-stable. □

Remark 11. As contractive condition (7) is more general than those of (2)–(6), the convergence and stability results for implicit iteration (IN) using contractive conditions (2)–(6) can be obtained as special cases.

Remark 12. As implicit Mann iteration (IM) and implicit Ishikawa iteration (II) are special cases of implicit iteration (IN), results similar to Theorems 9 and 10 hold for these iterations as well.

3. Convergence Rate of Implicit Iterations

Theorem 13. *Let K be a nonempty closed convex subset of a convex metric space X and let T be a quasi-contractive operator satisfying* (7) *with $F(T) \neq \phi$. Then, for $x_0 \in C$, the sequence $\{x_n\}$ defined by* (IN) with $\sum(1-\alpha_n) = \infty$, $\alpha_n \leq \alpha < 1$, converges faster than implicit Mann (IM) and implicit Ishikawa (II) iterations to the fixed point of T. Moreover, implicit iterations converge faster than the corresponding explicit iterations.

Proof. For implicit Mann iteration (IM), we have

$$\begin{aligned} d(x_n,p) &= d(W(x_{n-1}, Tx_n, \alpha_n), p) \leq \alpha_n d(x_{n-1},p) \\ &\quad + (1-\alpha_n)\,d(Tx_n,p) \leq \alpha_n d(x_{n-1},p) \\ &\quad + (1-\alpha_n)\,ad(x_n,p), \end{aligned} \tag{30}$$

which further yield

$$[1-(1-\alpha_n)a]\,d(x_n,p) \leq \alpha_n d(x_{n-1},p), \tag{31}$$

and so

$$d(x_n,p) \leq \frac{\alpha_n}{1-(1-\alpha_n)a}d(x_{n-1},p). \tag{32}$$

Similarly, for implicit Ishikawa (II) and implicit Noor (IN) iterations, we have the estimates (33) and (34), respectively, as follows:

$$d(x_n, p) \le \left[\frac{\alpha_n}{1-(1-\alpha_n)a}\right]\left[\frac{\beta_n}{1-(1-\beta_n)a}\right] d(x_{n-1}, p), \tag{33}$$

$$d(x_n, p) \le \left[\frac{\alpha_n}{1-(1-\alpha_n)a}\right]\left[\frac{\beta_n}{1-(1-\beta_n)a}\right] \times \left[\frac{\gamma_n}{1-(1-\gamma_n)a}\right] d(x_{n-1}, p). \tag{34}$$

Also, for explicit Mann iteration (M), we have

$$\begin{aligned} d(x_n, p) &= d(W(x_{n-1}, Tx_{n-1}, \alpha_n), p) \le \alpha_n d(x_{n-1}, p) \\ &\quad + (1-\alpha_n) d(Tx_{n-1}, p) \le \alpha_n d(x_{n-1}, p) \\ &\quad + (1-\alpha_n) a d(x_{n-1}, p) \\ &\le [\alpha_n + (1-\alpha_n) a] d(x_{n-1}, p). \end{aligned} \tag{35}$$

For explicit Ishikawa iteration (I), we have

$$\begin{aligned} d(x_n, p) &= d(W(x_{n-1}, Ty_{n-1}, \alpha_n), p) \le \alpha_n d(x_{n-1}, p) \\ &\quad + (1-\alpha_n) d(Ty_{n-1}, p) \le \alpha_n d(x_{n-1}, p) \\ &\quad + (1-\alpha_n) a d(y_{n-1}, p), \end{aligned} \tag{36}$$

$$\begin{aligned} d(y_{n-1}, p) &= d(W(x_{n-1}, Tx_{n-1}, \beta_n), p) \\ &\le (\beta_n + (1-\beta_n) a) d(x_{n-1}, p). \end{aligned} \tag{37}$$

Using (37), (36) becomes

$$d(x_n, p) \le [\alpha_n + (1-\alpha_n) a(\beta_n + (1-\beta_n) a)] d(x_{n-1}, p). \tag{38}$$

Similarly, for explicit Noor iteration (N), we have

$$d(x_n, p) \le \{\alpha_n + (1-\alpha_n) a\beta_n + (1-\alpha_n)(1-\beta_n) a^2 \times [\gamma_n + (1-\gamma_n) a]\} d(x_{n-1}, p). \tag{39}$$

Now, using (21) and (22), we obtain

$$\begin{aligned} &\left[\frac{\alpha_n}{1-(1-\alpha_n)a}\right]\left[\frac{\beta_n}{1-(1-\beta_n)a}\right] \\ &\le [\alpha_n + (1-\alpha_n) a][\beta_n + (1-\beta_n) a] \\ &= [\alpha_n + (1-\alpha_n) a][1 - (1-\beta_n)(1-a)] \\ &\le \alpha_n + (1-\alpha_n) a\beta_n + (1-\alpha_n)(1-\beta_n) a^2. \end{aligned} \tag{40}$$

Similarly using (21)–(23), we get

$$\begin{aligned} &\left[\frac{\alpha_n}{1-(1-\alpha_n)a}\right]\left[\frac{\beta_n}{1-(1-\beta_n)a}\right]\left[\frac{\gamma_n}{1-(1-\gamma_n)a}\right] \\ &\le [\alpha_n + (1-\alpha_n) a][\beta_n + (1-\beta_n) a][\gamma_n + (1-\gamma_n) a] \\ &\le \alpha_n + (1-\alpha_n) a\beta_n + (1-\alpha_n)(1-\beta_n) a^2 \\ &\quad \times [\gamma_n + (1-\gamma_n) a]. \end{aligned} \tag{41}$$

Keeping in mind Berinde's Definitions 5 and 6, inequalities (21), (32), and (35) yield that implicit Mann iteration (IM) converges faster than explicit Mann iteration (M), inequalities (33), (38), and (40) yield that implicit Ishikawa iteration (II) converges faster than explicit Ishikawa iteration (I), and inequalities (34), (39), and (41) yield that implicit Noor iteration (IN) converges faster than explicit Noor iteration (N).

Moreover, again using Berinde's Definitions 5 and 6 with

$$\begin{aligned} &\left[\frac{\alpha_n}{1-(1-\alpha_n)a}\right]\left[\frac{\beta_n}{1-(1-\beta_n)a}\right] \le \left[\frac{\alpha_n}{1-(1-\alpha_n)a}\right], \\ &\left[\frac{\alpha_n}{1-(1-\alpha_n)a}\right]\left[\frac{\beta_n}{1-(1-\beta_n)a}\right]\left[\frac{\gamma_n}{1-(1-\gamma_n)a}\right] \\ &\le \left[\frac{\alpha_n}{1-(1-\alpha_n)a}\right]\left[\frac{\beta_n}{1-(1-\beta_n)a}\right], \end{aligned} \tag{42}$$

and inequalities (32)–(34), we conclude that decreasing order of convergence speed of implicit iterations is as follows: implicit Noor, implicit Ishikawa, and implicit Mann iteration. □

Example 14. Let $K = [0, 1]$, $T(x) = x/4$, $x_0 \ne 0$, $\alpha_n = \beta_n = \gamma_n = 1 - (4/\sqrt{n})$, $n \ge 25$, and for $n = 1, 2, \ldots, 24$, $\alpha_n = \beta_n = \gamma_n = 0$; then for implicit Mann iteration (IM), we have

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) Tx_n = \left(1 - \frac{4}{\sqrt{n}}\right) x_{n-1} + \frac{4}{\sqrt{n}} \frac{x_n}{4}, \tag{43}$$

which further implies

$$x_n \left[1 - \frac{1}{\sqrt{n}}\right] = \left(1 - \frac{4}{\sqrt{n}}\right) x_{n-1}, \tag{44}$$

and so

$$x_n = \frac{\sqrt{n}-4}{\sqrt{n}-1} x_{n-1} = \prod_{i=25}^{n} \left(\frac{\sqrt{i}-4}{\sqrt{i}-1}\right) x_0. \tag{45}$$

Also, for implicit Ishikawa iteration (II), we have

$$x_n = \frac{\sqrt{n}-4}{\sqrt{n}-1} y_{n-1}, \tag{46}$$

with

$$y_{n-1} = \frac{\sqrt{n}-4}{\sqrt{n}-1}x_{n-1}. \tag{47}$$

Using (47), (46) becomes

$$x_n = \left(\frac{\sqrt{n}-4}{\sqrt{n}-1}\right)^2 x_{n-1} = \prod_{i=25}^{n}\left(\frac{\sqrt{i}-4}{\sqrt{i}-1}\right)^2 x_0. \tag{48}$$

Similarly, for implicit Noor iteration (IN), we have

$$x_n = \prod_{i=25}^{n}\left(\frac{\sqrt{i}-4}{\sqrt{i}-1}\right)^3 x_0. \tag{49}$$

Using (48) and (49), we have

$$\frac{x_n\,(\mathrm{IN})}{x_n\,(\mathrm{II})} = \prod_{i=25}^{n}\left(\frac{\sqrt{i}-4}{\sqrt{i}-1}\right) = \prod_{i=25}^{n}\left(1-\frac{3}{\sqrt{i}-1}\right) \tag{50}$$

with

$$\begin{aligned} 0 &\le \lim_{n\to\infty}\prod_{i=25}^{n}\left(1-\frac{3}{\sqrt{i}-1}\right) \le \lim_{n\to\infty}\prod_{i=25}^{n}\left(1-\frac{1}{i}\right) \\ &= \lim_{n\to\infty}\frac{24}{25}\times\frac{25}{26}\cdots\frac{n-1}{n} = \lim_{n\to\infty}\frac{24}{n} = 0. \end{aligned} \tag{51}$$

Hence $\lim_{n\to\infty}|(x_n(\mathrm{IN})-0)/(x_n(\mathrm{II})-0)| = 0$. Therefore using Definition 5, implicit Noor iteration (IN) converges faster than the implicit Ishikawa iteration (II) to the fixed point $p = 0$.

Similarly, using (45) and (48), we have

$$\lim_{n\to\infty}\left|\frac{x_n\,(\mathrm{II})-0}{x_n\,(\mathrm{IM})-0}\right| = \lim_{n\to\infty}\prod_{i=25}^{n}\left(\frac{\sqrt{i}-4}{\sqrt{i}-1}\right) = 0. \tag{52}$$

That is, implicit Ishikawa iteration (II) converges faster than the implicit Mann iteration (IM) to the fixed point $p = 0$.

Now, we compare implicit iterations with their corresponding explicit iterations.

For explicit Mann iteration (M), we have

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)Tx_{n-1} = \left(1-\frac{4}{\sqrt{n}}\right)x_{n-1} \\ &\quad + \frac{4}{\sqrt{n}}\frac{x_{n-1}}{4} = \left(1-\frac{3}{\sqrt{n}}\right)x_{n-1} \\ &= \prod_{i=25}^{n}\left(1-\frac{3}{\sqrt{i}}\right)x_0. \end{aligned} \tag{53}$$

Similarly, for explicit Ishikawa and explicit Noor iterations, we have following estimates (54) and (55), respectively:

$$x_{n-1} = \left(1-\frac{3}{\sqrt{n}}-\frac{3}{n}\right)x_{n-1} = \prod_{i=25}^{n}\left(1-\frac{3}{\sqrt{i}}-\frac{3}{i}\right)x_0, \tag{54}$$

$$\begin{aligned} x_{n-1} &= \left(1-\frac{3}{\sqrt{n}}-\frac{3}{n}-\frac{3}{n^{3/2}}\right)x_{n-1} \\ &= \prod_{i=25}^{n}\left(1-\frac{3}{\sqrt{i}}-\frac{3}{i}-\frac{3}{i^{3/2}}\right)x_0. \end{aligned} \tag{55}$$

Using (45) and (53), we arrive at

$$\frac{x_n\,(\mathrm{IM})}{x_n\,(\mathrm{M})} = \prod_{i=25}^{n}\left(\frac{\sqrt{i}-4}{\sqrt{i}-1}\right)\left(\frac{\sqrt{i}}{\sqrt{i}-3}\right) = \prod_{i=25}^{n}\left(\frac{i-4\sqrt{i}}{i-4\sqrt{i}+3}\right), \tag{56}$$

with

$$\begin{aligned} 0 &\le \lim_{n\to\infty}\prod_{i=25}^{n}\left(\frac{i-4\sqrt{i}}{i-4\sqrt{i}+3}\right) \le \lim_{n\to\infty}\prod_{i=25}^{n}\left(1-\frac{3}{i-4\sqrt{i}+3}\right) \\ &\le \lim_{n\to\infty}\prod_{i=25}^{n}\left(1-\frac{1}{i}\right) = \lim_{n\to\infty}\frac{24}{25}\times\frac{25}{26}\cdots\frac{n-1}{n} \\ &= \lim_{n\to\infty}\frac{24}{n} = 0. \end{aligned} \tag{57}$$

Hence $\lim_{n\to\infty}|(x_n(\mathrm{IM})-0)/(x_n(\mathrm{M})-0)| = 0$. Therefore using Definition 5, implicit Mann iteration (IM) converges faster than the corresponding explicit Mann iteration (M) to the fixed point $p = 0$.

Similarly, using (48) and (54), we have

$$\begin{aligned} \frac{x_n\,(\mathrm{II})}{x_n\,(\mathrm{I})} &= \prod_{i=25}^{n}\left(\frac{\sqrt{i}-4}{\sqrt{i}-1}\right)^2\left(\frac{i}{i-3\sqrt{i}-3}\right) \\ &= \prod_{i=25}^{n}\left(\frac{i^2-8i^{3/2}+16i}{i^2-5i^{3/2}+4i+3\sqrt{i}-3}\right), \end{aligned} \tag{58}$$

with

$$\begin{aligned} 0 &\le \lim_{n\to\infty}\prod_{i=25}^{n}\left(\frac{i^2-8i^{3/2}+16i}{i^2-5i^{3/2}+4i+3\sqrt{i}-3}\right) \\ &\le \lim_{n\to\infty}\prod_{i=25}^{n}\left(1-\frac{\left(3i^{3/2}-12i+3\sqrt{i}-3\right)}{i^2-5i^{3/2}+4i+3\sqrt{i}-3}\right) \\ &\le \lim_{n\to\infty}\prod_{i=25}^{n}\left(1-\frac{1}{i}\right) = \lim_{n\to\infty}\frac{24}{25}\times\frac{25}{26}\cdots\frac{n-1}{n} \\ &= \lim_{n\to\infty}\frac{24}{n} = 0, \end{aligned} \tag{59}$$

which implies that $\lim_{n\to\infty}|(x_n(\mathrm{II})-0)/(x_n(\mathrm{I})-0)| = 0$. That is, implicit Ishikawa iteration (II) converges faster than the corresponding explicit Ishikawa iteration (I) to the fixed point $p = 0$.

Again, similarly using (49) and (55), we have

$$\begin{aligned} &\frac{x_n\,(\mathrm{IN})}{x_n\,(\mathrm{N})} \\ &= \prod_{i=25}^{n}\left(\frac{\sqrt{i}-4}{\sqrt{i}-1}\right)^3\left(\frac{i^{3/2}}{i^{3/2}-3i-3\sqrt{i}-3}\right) \\ &= \prod_{i=25}^{n}\left(\frac{i^3-12i^{5/2}+48i^2-64i^{3/2}}{i^3-6i^{5/2}+9i^2-4i^{3/2}+3i-6\sqrt{i}+3}\right), \end{aligned} \tag{60}$$

Table 1: Convergence rate comparison of various iterations.

Number of iterations (n)	Mann iteration (M)	Ishikawa iteration (I)	Noor iteration (N)	Implicit Mann iteration (IM)	Implicit Ishikawa iteration (II)	Implicit Noor iteration (IN)
25	0.4	0.28	0.256	0.25	0.0625	0.015625
26	0.164661	0.082955	0.070051	0.0670294	0.004493	0.000301
27	0.0695938	0.025844	0.020326	0.0191074	0.000365	$6.97593e-06$
28	0.0301378	0.008423	0.006213	0.00575025	$3.306536e-05$	$1.90134e-07$
29	0.0133485	0.002859	0.00199	0.00181636	$3.299167e-06$	$5.992477e-09$
30	0.00603721	0.001007	0.000665	0.000599294	$3.591534e-07$	$2.152385e-10$
31	0.00278426	0.000367	0.000231	0.000205692	$4.230913e-08$	$8.702644e-12$
32	0.00130768	0.000138	$8.287781e-05$	$7.31828e-005$	$5.355715e-09$	$3.91946e-13$
33	0.000624769	$5.337885e-05$	$3.075038e-05$	$2.69091e-005$	$7.240995e-10$	$1.948486e-14$
34	0.000303328	$2.120576e-05$	$1.175085e-05$	$1.01987e-005$	$1.040128e-10$	$1.060791e-15$
35	0.000149513	$8.63484e-06$	$4.614614e-06$	$3.97501e-006$	$1.580069e-11$	$6.280786e-17$
36	$7.47563e-005$	$3.59785e-06$	$1.858664e-06$	$1.59e-006$	$2.52811e-12$	$4.019703e-18$
—	—	—	—	—	—	—
—	—	—	—	—	—	—

with

$$\begin{aligned}
0 &\le \lim_{n\to\infty}\prod_{i=25}^{n}\left(\frac{i^3-12i^{5/2}+48i^2-64i^{3/2}}{i^3-6i^{5/2}+9i^2-4i^{3/2}+3i-6\sqrt{i}+3}\right)\\
&\le \lim_{n\to\infty}\prod_{i=25}^{n}\left(1-\frac{\left(6i^{5/2}-39i^2+60i^{3/2}+3i-6\sqrt{i}+3\right)}{i^3-6i^{5/2}+9i^2-4i^{3/2}+3i-6\sqrt{i}+3}\right)\\
&\le \lim_{n\to\infty}\prod_{i=25}^{n}\left(1-\frac{1}{i}\right)=\lim_{n\to\infty}\frac{24}{25}\times\frac{25}{26}\cdots\frac{n-1}{n}\\
&= \lim_{n\to\infty}\frac{24}{n}=0,
\end{aligned} \tag{61}$$

which implies that $\lim_{n\to\infty}|(x_n(\mathrm{IN})-0)/(x_n(\mathrm{N})-0)|=0$. That is, implicit Noor iteration (IN) converges faster than the explicit Noor iteration (N) to the fixed point $p=0$.

Using computer programs in C++, the convergence speed of various iterations is compared and observations are listed in Table 1 by taking initial approximation $x_0=1$, $T(x)=x/4$ and $\alpha_n=\beta_n=\gamma_n=1-(4/\sqrt{n})$, $n\ge 25$. The table reveals that implicit Noor iteration has better convergence rate as compared to implicit Mann iteration as well as implicit Ishikawa iteration and implicit iterations converge faster than corresponding explicit iterations.

4. Data Dependence of Implicit Iterations in Hyperbolic Spaces

Theorem 15. *Let $T: K\to K$ be a mapping satisfying* (7). *Let T_1 be an approximate operator of T as in Definition 7, and let $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ be two implicit iterations associated to T, T_1 and defined by* (IN)

$$\begin{aligned}
u_n &= W\left(v_{n-1}, T_1u_n, \alpha_n\right),\\
v_{n-1} &= W\left(w_{n-1}, T_1v_{n-1}, \beta_n\right),\\
w_{n-1} &= W\left(u_{n-1}, T_1w_{n-1}, \gamma_n\right),
\end{aligned} \tag{62}$$

respectively, where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying $\sum_{n=0}^{\infty}(1-\alpha_n)=\infty$, $\alpha_n\le\beta_n$, $\alpha_n\le\gamma_n$. Let $p=Tp$ and $q=T_1q$, then, for $\varepsilon>0$, we have the following estimate:

$$d\left(p,q\right)\le\frac{\varepsilon}{(1-a)^2}. \tag{63}$$

Proof. Using Definition 2, iteration (IN), and iteration (62), we have the following estimates:

$$\begin{aligned}
d\left(x_n,u_n\right) &= d\left(W\left(y_{n-1},Tx_n,\alpha_n\right),W\left(v_{n-1},T_1u_n,\alpha_n\right)\right)\\
&\le \alpha_n d\left(y_{n-1},v_{n-1}\right)+\left(1-\alpha_n\right)d\left(Tx_n,T_1u_n\right)\\
&\le \alpha_n d\left(y_{n-1},v_{n-1}\right)+\left(1-\alpha_n\right)\\
&\quad\times\left\{d\left(Tx_n,T_1x_n\right)+d\left(T_1x_n,T_1u_n\right)\right\}\\
&\le \alpha_n d\left(y_{n-1},v_{n-1}\right)+\left(1-\alpha_n\right)\\
&\quad\times\left\{\varepsilon+\varphi d\left(x_n,T_1x_n\right)+ad\left(x_n,u_n\right)\right\},
\end{aligned} \tag{64}$$

which further gives

$$\begin{aligned}\{1-(1-\alpha_n)a\}\,d(x_n,u_n)&\le \alpha_n d(y_{n-1},v_{n-1})\\&\quad+(1-\alpha_n)\{\varepsilon+\varphi(d(x_n,T_1x_n))\},\\ d(x_n,u_n)&\le \frac{\alpha_n}{1-(1-\alpha_n)a}d(y_{n-1},v_{n-1})\\&\quad+\frac{(1-\alpha_n)}{1-(1-\alpha_n)a}\{\varepsilon+\varphi(d(x_n,T_1x_n))\},\end{aligned}\tag{65}$$

with

$$\begin{aligned}&d(y_{n-1},v_{n-1})\\&=d(W(z_{n-1},Ty_{n-1},\beta_n),W(w_{n-1},T_1v_{n-1},\beta_n))\\&\le \beta_n d(z_{n-1},w_{n-1})+(1-\beta_n)d(Ty_{n-1},T_1v_{n-1}),\end{aligned}\tag{66}$$

$$\begin{aligned}d(Ty_{n-1},T_1v_{n-1})&\le d(Ty_{n-1},T_1y_{n-1})\\&\quad+d(T_1y_{n-1},T_1v_{n-1})\\&\le \varepsilon+\varphi d(y_{n-1},T_1y_{n-1})\\&\quad+ad(y_{n-1},v_{n-1}).\end{aligned}\tag{67}$$

Using (66) and (67), we have

$$\begin{aligned}d(y_{n-1},v_{n-1})&\le \frac{\beta_n}{1-(1-\beta_n)a}d(z_{n-1},w_{n-1})\\&\quad+\frac{(1-\beta_n)}{1-(1-\beta_n)a}\\&\quad\times\{\varepsilon+\varphi(d(y_{n-1},T_1y_{n-1}))\},\end{aligned}\tag{68}$$

with

$$\begin{aligned}d(z_{n-1},w_{n-1})&\le \gamma_n d(x_{n-1},u_{n-1})\\&\quad+(1-\gamma_n)d(Tz_{n-1},T_1w_{n-1}),\end{aligned}\tag{69}$$

$$\begin{aligned}d(Tz_{n-1},T_1w_{n-1})&\le d(Tz_{n-1},T_1z_{n-1})\\&\quad+d(T_1z_{n-1},T_1w_{n-1})\\&\le \varepsilon+\varphi d(z_{n-1},T_1z_{n-1})\\&\quad+ad(z_{n-1},w_{n-1}).\end{aligned}\tag{70}$$

Using (69) and (70) and simplifying, we get the following estimate:

$$\begin{aligned}d(z_{n-1},w_{n-1})&\le \frac{\gamma_n}{1-(1-\gamma_n)a}d(x_{n-1},u_{n-1})\\&\quad+\frac{(1-\gamma_n)}{1-(1-\gamma_n)a}\{\varepsilon+\varphi(d(z_{n-1},T_1z_{n-1}))\}.\end{aligned}\tag{71}$$

Using estimates (65), (68), and (71), we arrive at

$$\begin{aligned}d(x_n,u_n)&\le\left(\frac{\alpha_n}{1-(1-\alpha_n)a}\right)\left(\frac{\beta_n}{1-(1-\beta_n)a}\right)\\&\quad\times\left(\frac{\gamma_n}{1-(1-\gamma_n)a}\right)d(x_{n-1},u_{n-1})\\&\quad+\left(\frac{\alpha_n}{1-(1-\alpha_n)a}\right)\left(\frac{\beta_n}{1-(1-\beta_n)a}\right)\\&\quad\times\left(\frac{1-\gamma_n}{1-(1-\gamma_n)a}\right)\{\varepsilon+\varphi(d(z_{n-1},T_1z_{n-1}))\}\\&\quad+\left(\frac{\alpha_n}{1-(1-\alpha_n)a}\right)\left(\frac{1-\beta_n}{1-(1-\beta_n)a}\right)\\&\quad\times\{\varepsilon+\varphi(d(y_{n-1},T_1y_{n-1}))\}\\&\quad+\left(\frac{\alpha_n}{1-(1-\alpha_n)a}\right)\left(\frac{1-\beta_n}{1-(1-\beta_n)a}\right)\\&\quad\times\{\varepsilon+\varphi(d(x_n,T_1x_n))\}.\end{aligned}\tag{72}$$

Keeping in mind inequalities (21)–(23), (72) reduces to

$$\begin{aligned}d(x_n,u_n)&\le[1-(1-\alpha_n)(1-a)]\,d(x_{n-1},u_{n-1})\\&\quad+\left(\frac{1-\gamma_n}{1-(1-\gamma_n)a}\right)\{\varepsilon+\varphi(d(z_{n-1},T_1z_{n-1}))\}\\&\quad+\left(\frac{1-\beta_n}{1-(1-\beta_n)a}\right)\{\varepsilon+\varphi(d(y_{n-1},T_1y_{n-1}))\}\\&\quad+\left(\frac{1-\alpha_n}{1-(1-\alpha_n)a}\right)\{\varepsilon+\varphi(d(x_n,T_1x_n))\}\\&\le[1-(1-\alpha_n)(1-a)]\,d(x_{n-1},u_{n-1})\\&\quad+\left(\frac{1-\alpha_n}{1-a}\right)\left(\frac{1-a}{1-a}\right)\{\varepsilon+\varphi(d(z_{n-1},T_1z_{n-1}))\}\\&\quad+\left(\frac{1-\alpha_n}{1-a}\right)\left(\frac{1-a}{1-a}\right)\{\varepsilon+\varphi(d(y_{n-1},T_1y_{n-1}))\}\\&\quad+\left(\frac{1-\alpha_n}{1-a}\right)\left(\frac{1-a}{1-a}\right)\{\varepsilon+\varphi(d(x_n,T_1x_n))\}.\end{aligned}\tag{73}$$

Putting $a_n = d(x_n,u_n)$, $r_n = (1-\alpha_n)(1-a)$, and $t_n = \{\varphi d(x_n,T_1x_n)+\varphi d(y_{n-1},T_1y_{n-1})+\varphi d(z_{n-1},T_1z_{n-1})+3\varepsilon\}/(1-a)^2$, the above inequality becomes

$$a_n\le(1-r_n)a_{n-1}+r_nt_n.\tag{74}$$

Now from Theorem 9, we have $\lim_{n\to\infty}d(x_n,p)=0$, $\lim_{n\to\infty}d(u_n,p)=0$ and since φ is continuous, hence $\lim_{n\to\infty}\varphi d(x_n,Tx_n)=\lim_{n\to\infty}\varphi d(y_n,Ty_n)=\lim_{n\to\infty}\varphi d(z_n,Tz_n)=0$.

Table 2: Convergence of implicit iterations to the fixed point of decreasing function.

Number of iterations (n)	Implicit Noor iteration (IN)	Implicit Ishikawa iteration (II)	Implicit Mann iteration (IM)
1	0.38215	0.384209	0.409165
2	0.381968	0.382091	0.388393
3	**0.381966**	0.381972	0.38334
4	**0.381966**	**0.381966**	0.382236
5	**0.381966**	**0.381966**	0.382015
6	**0.381966**	**0.381966**	0.381974
7	**0.381966**	**0.381966**	0.381967
8	**0.381966**	**0.381966**	**0.381966**
9	**0.381966**	**0.381966**	**0.381966**
10	**0.381966**	**0.381966**	**0.381966**

Table 3: Convergence of implicit iterations to the fixed point of increasing function.

Number of iterations (n)	Implicit Noor iteration (IN)	Implicit Ishikawa iteration (II)	Implicit Mann iteration (IM)
0	7	7	7
1	6.194906	6.336163	6.579796
2	6.019386	6.072164	6.268634
3	6.001241	6.011546	6.107454
4	6.000057	6.001482	6.038496
5	6.000002	6.000159	6.012624
6	**6**	6.000015	6.003844
7	**6**	6.000001	6.001098
8	**6**	**6**	6.000297
9	**6**	**6**	6.000076
10	**6**	**6**	6.000019
11	**6**	**6**	6.000004
12	**6**	**6**	6.000001
13	**6**	**6**	**6**
14	**6**	**6**	**6**
15	**6**	**6**	**6**
16	**6**	**6**	**6**

Therefore, using Lemma 8, (74) yields

$$d(p,q) \leq \frac{3\varepsilon}{(1-a)^2}. \tag{75}$$

□

Remark 16. Putting $\gamma_n = \beta_n = 1$ in (IN) and (62), data dependence result of implicit Mann (IM) iteration can be proved easily on the same lines as in Theorem 15.

Remark 17. Putting $\beta_n = 1$ in (IN) and (62), data dependence result of implicit Ishikawa (II) iteration can be proved easily on the same lines as in Theorem 15.

5. Applications

5.1. Implicit Iterations in RNN (Recurrent Neural Networks) Analysis. Neutral networks are a class of nonlinear functions approximations and stable state is achieved in recurrent autoassociative neural networks using iterations. Here we analyze the convergence speed of implicit iterations in recurrent network and many important results will be studied for decreasing and increasing functions. The achieved results possess multifaceted real line applications and in particular can be helpful to design the inner product kernel of support vector machine with faster convergence rate. (For details about RNN and SVM please refer to [33].)

5.1.1. Decreasing Function $(1-x)^2$. The function $f : [0,1] \to [0,1]$ defined by $f(x) = (1-x)^2$ is a decreasing function with fixed point $p = 0.381966$. By taking initial approximation $x_0 = 0.7$, $1-\alpha_n = 1-\beta_n = 1-\gamma_n = 1/\sqrt{n+4}$, the comparison of convergence of implicit iterations to the fixed point is listed in Table 2.

5.1.2. Increasing Function $(x/2)+3$. The function $f : [6,8] \to [6,8]$ defined by $f(x) = (x/2) + 3$ is an increasing function with fixed point $p = 6$. By taking initial approximation $x_0 = 7$, $1-\alpha_n = 1-\beta_n = 1-\gamma_n = 1/\sqrt{5n+1}$, the comparison of convergence of implicit iterations to the fixed point is listed in Table 3.

From computational study of convergence of implicit iterations to the fixed point of decreasing as well as increasing functions, the following observations are made.

Decreasing Function. For $x_0 = 0.7$, $\alpha_n = \beta_n = \gamma_n = 1/\sqrt{n+4}$, implicit Noor iteration converges in 3 iterations, implicit Ishikawa iteration converges in 4 iterations, and implicit Mann iteration converges in 8 iterations.

For $x_0 = 0.7$, $\alpha_n = \beta_n = \gamma_n = 1/\sqrt{n+8}$, implicit Noor, implicit Ishikawa, and implicit Mann iterations converge in 2, 3, and 7 iterations, respectively.

Increasing Function. For $x_0 = 7$, $\alpha_n = \beta_n = \gamma_n = 1/\sqrt{5n+1}$, implicit Noor iteration converges in 7 iterations, implicit Ishikawa iteration converges in 9 iterations, and implicit Mann iteration converges in 14 iterations.

For $x_0 = 7$, $\alpha_n = \beta_n = \gamma_n = 1/\sqrt{5n+3}$, implicit Noor, implicit Ishikawa, and implicit Mann iterations converge in 6, 8, and 13 iterations, respectively.

6. Conclusions

(i) The speed of implicit iterations depends on parameters α_n, β_n, and γ_n.

(ii) In every case the newly introduced three-step implicit Noor iteration has better convergence rate as compared to other iterations and hence has a good potential for further applications in various disciplines of science.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] R. Chugh and V. Kumar, "Convergence of SP iterative scheme with mixed errors for accretive Lipschitzian and strongly accretive Lipschitzian operators in Banach spaces," *International Journal of Computer Mathematics*, vol. 90, no. 9, pp. 1865–1880, 2013.

[2] L. Ćirić, J. S. Ume, and M. S. Khan, "On the convergence of the Ishikawa iterates to a common fixed point of two mappings," *Archivum Mathematicum*, vol. 39, no. 2, pp. 123–127, 2003.

[3] N. Hussain, R. Chugh, V. Kumar, and A. Rafiq, "On the rate of convergence of Kirk-type iterative schemes," *Journal of Applied Mathematics*, vol. 2012, Article ID 526503, 22 pages, 2012.

[4] A. R. Khan, V. Kumar, and N. Hussain, "Analytical and numerical treatment of Jungck-type iterative schemes," *Applied Mathematics and Computation*, vol. 231, pp. 521–535, 2014.

[5] B. E. Rhoades, "Fixed point theorems and stability results for fixed point iteration procedures," *Indian Journal of Pure and Applied Mathematics*, vol. 24, no. 11, pp. 691–703, 1993.

[6] D. Ariza-Ruiz, "Convergence and stability of some iterative processes for a class of quasinonexpansive type mappings," *Journal of Nonlinear Science and its Applications*, vol. 5, no. 2, pp. 93–103, 2012.

[7] P. Ky and T. Quoc, "Stability and convergence of implicit iteration processes," *Vietnam Journal of Mathematics*, vol. 32, no. 4, pp. 467–473, 2004.

[8] C. E. Chidume and N. Shahzad, "Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 62, no. 6, pp. 1149–1156, 2005.

[9] L. Ćirić and N. T. Nikolić, "Convergence of the Ishikawa iterates for multi-valued mappings in convex metric spaces," *Georgian Mathematical Journal*, vol. 15, no. 1, pp. 39–43, 2008.

[10] L. Ćirić, A. Rafiq, N. Cakić, and J. S. Ume, "Implicit Mann fixed point iterations for pseudo-contractive mappings," *Applied Mathematics Letters*, vol. 22, no. 4, pp. 581–584, 2009.

[11] L. Ćirić, A. Rafiq, S. Radenović, M. Rajović, and J. S. Ume, "On Mann implicit iterations for strongly accretive and strongly pseudo-contractive mappings," *Applied Mathematics and Computation*, vol. 198, no. 1, pp. 128–137, 2008.

[12] A. R. Khan, H. Fukhar-ud-din, and M. A. A. Khan, "An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces," *Fixed Point Theory and Applications*, vol. 2012, article 54, 12 pages, 2012.

[13] N. Shahzad and H. Zegeye, "On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 3-4, pp. 838–844, 2009.

[14] F. Gursoy, V. Karakaya, and B. E. Rhoades, "Data dependence results of new multi-step and S-iterative schemes for contractive-like operators," *Fixed Point Theory and Applications*, vol. 2013, article 76, pp. 1–12, 2013.

[15] V. Berinde, "Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators," *Fixed Point Theory and Applications*, no. 2, pp. 94–105, 2004.

[16] L. Ciric, B. S. Lee, and A. Rafiq, "Faster Noor iterations," *Indian Journal of Mathematics*, vol. 52, no. 3, pp. 429–436, 2010.

[17] V. Kumar, A. Latif, A. Rafiq, and N. Hussain, "*S*-iteration process for quasi-contractive mappings," *Journal of Inequalities and Applications*, vol. 2013, article 206, 2013.

[18] R. Glowinski and P. Le Tallec, *Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics*, SIAM, Philadelphia, Pa, USA, 1989.

[19] M. A. Noor, "New approximation schemes for general variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 217–229, 2000.

[20] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, no. 1, pp. 147–150, 1974.

[21] M. O. Olatinwo, "Stability results for some fixed point iterative processes in convex metric spaces," *International Journal of Engineering*, vol. 9, pp. 103–106, 2011.

[22] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.

[23] W. Takahashi, "A convexity in metric space and nonexpansive mappings," *Kodai Mathematical Seminar Reports*, vol. 22, pp. 142–149, 1970.

[24] Z. Xue and F. Zhang, "The convergence of implicit Mann and Ishikawa iterations for weak generalized φ-hemicontractive mappings in real Banach spaces," *Journal of Inequalities and Applications*, vol. 2013, article 231, pp. 1–11, 2013.

[25] T. Zamfirescu, "Fixed point theorems in metric spaces," *Archiv der Mathematik*, vol. 23, pp. 292–298, 1972.

[26] M. O. Osilike, "Stability results for fixed point iteration procedures," *Journal of the Nigerian Mathematical Society*, vol. 14, no. 15, pp. 17–29, 1995.

[27] C. O. Imoru and M. O. Olatinwo, "On the stability of Picard and Mann iteration processes," *Carpathian Journal of Mathematics*, vol. 19, no. 2, pp. 155–160, 2003.

[28] M. Moosaei, "Fixed point theorems in convex metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 164, 6 pages, 2012.

[29] U. Kohlenbach, "Some logical metatheorems with applications in functional analysis," *Transactions of the American Mathematical Society*, vol. 357, no. 1, pp. 89–128, 2005.

[30] V. Berinde, "Stability of Picard iteration for contractive mappings satisfying an implicit relation," *Carpathian Journal of Mathematics*, vol. 27, no. 1, pp. 13–23, 2011.

[31] A. M. Ostrowski, "The round-off stability of iterations," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 47, pp. 77–81, 1967.

[32] I. Timiş, "Stability of Jungck-type iterative procedure for some contractive type mappings via implicit relations," *Miskolc Mathematical Notes*, vol. 13, no. 2, pp. 555–567, 2012.

[33] M. F. Barnsley, *Fractals Everywhere*, Academic Press Professional, Boston, Mass, USA, 2nd edition, 1993.

Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2014, Article ID 914398, 7 pages
http://dx.doi.org/10.1155/2014/914398

Research Article

$(\alpha - \psi)$-Contractive Mappings on Generalized Quasimetric Spaces

Erdal Karapınar[1,2] and Hosein Lakzian[3]

[1] *Department of Mathematics, Atilim University, Incek, 06836 Ankara, Turkey*
[2] *Nonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University, Jeddah, Saudi Arabia*
[3] *Department of Mathematics, Payame Noor University, Tehran 19395-4697, Iran*

Correspondence should be addressed to Hosein Lakzian; h_lakzian@pnu.ac.ir

Received 26 June 2014; Revised 30 July 2014; Accepted 30 July 2014; Published 18 August 2014

Academic Editor: Nawab Hussain

Copyright © 2014 E. Karapınar and H. Lakzian. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider the existence of a fixed point of $(\alpha - \psi)$-contractive mappings in the context of generalized quasimetric spaces without Hausdorff assumption. The obtained results extend several results on the topic in the literature.

1. Introduction and Preliminaries

In the last decade, quasimetric spaces have been one of the interesting topics for the researchers in the field of fixed point theory due to two reasons. The first reason is that the assumptions of quasimetric are weaker than the more general metric. Consequently, the obtained fixed point results in this space are more general and hence the corresponding results in metric space are covered. The second reason is the fact that fixed point problems in G-metric space (introduced by Mustafa and Sims [1]) can be reduced to related fixed point problems in the context of quasimetric space (see, e.g., [2, 3]). Very recently, Lin et al. [4] introduced the notion of generalized quasimetric spaces and investigated the existence of a certain operator on such spaces. In this paper [4], the authors assumed that the generalized quasimetric space is Hausdorff to get a fixed point.

In this paper, we examine the existence of $(\alpha$-$\psi)$-contractive mappings in the context of generalized quasimetric space without the Hausdorffness assumption. Consequently, our results extend, improve, and generalize several results in the literature.

In what follows we recall the basic definitions and results on the topics for the sake of completeness. Throughout the paper, the symbols $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{N}_0$ denote the real numbers, the natural numbers, and the positive integers, respectively.

Let X be a nonempty set and let $d : X \times X \to [0, \infty)$. Then d is called a distance function if, for every $x, y, z \in X$, it satisfies

(d_1) $d(x, x) = 0$;
(d_2) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$;
(d_3) $d(x, y) = d(y, x)$;
(d_4) $d(x, z) \leq d(x, y) + d(y, z)$.

Notice that if d satisfies the conditions (d_2), (d_3), and (d_4), then d is called a dislocated metric on X. If d satisfies the conditions (d_1), (d_2), and (d_4), then d is called a quasimetric on X. On the other hand, if d satisfies the conditions (d_1)–(d_4), then d is called a metric on X.

One of the very natural generalizations of the notion of a metric was introduced by Branciari [5] in 2000 by replacing the triangle inequality assumption of a metric with a weaker condition, quadrilateral inequality.

Definition 1 (see [5]). Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ be a mapping such that, for all $x, y \in X$ and for all distinct points $u, v \in X$ each of them different from x and y, one has

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality).

Then (X, d) is called a generalized metric space (or shortly g.m.s).

We present an example to show that not every generalized metric on a set X is a metric on X.

Example 2 (see, e.g., [4]). Let $X = \{t, 2t, 3t, 4t, 5t\}$ with $t > 0$ be a constant, and we define $d : X \times X \to [0, \infty)$ by

(1) $d(x, x) = 0$, for all $x \in X$;

(2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(3) $d(t, 2t) = 3s$;

(4) $d(t, 3t) = d(2t, 3t) = s$;

(5) $d(t, 4t) = d(2t, 4t) = d(3t, 4t) = 2s$;

(6) $d(t, 5t) = d(2t, 5t) = d(3t, 5t) = d(4t, 5t) = (3/2)s$,

where $s > 0$ is a constant. Then (X, d) is a generalized metric space, but it is not a metric space, because

$$d(t, 2t) = 3s > d(t, 3t) + d(3t, 2t) = 2s. \tag{1}$$

Despite the analogy between the definitions of metric and generalized metric their topological properties differ from each other. For example, for a generalized metric space (X, d), we have the following.

($P1$), a generalized metric, need not be continuous;

($P2$), a convergent sequence in generalized metric space, need not be Cauchy;

($P3$), an open ball $(B_r(x) = \{y \in X : d(x, y) < r\}, r > 0)$, need not be open set;

($P4$), a generalized metric space, need not be Hausdorff, and hence the uniqueness of limits cannot be guaranteed.

Example 3 (see [6], Example 1.1). Let $X = A \cup B$, where $A = \{0, 2\}$ and $A = \{1/n : n \in \mathbb{N}\}$. Define $d : X \times X \to [0, \infty)$ in the following way:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y, \ [\{x, y\} \subset A \text{ or } \{x, y\} \subset B], \\ y, & \text{if } x \in A, y \in B. \end{cases} \tag{2}$$

Notice that $d(a, b) = d(b, a) = b$ whenever $a \in A$ and $b \in B$. Furthermore, (X, d) is a complete generalized metric space. Clearly, we have ($P1$)–($P4$). Indeed, the sequence $\{1/n : n \in \mathbb{N}\}$ converges to both 0 and 2. There is no $r > 0$ such that $B_r(0) \cap B_r(2) = \emptyset$ and hence it is not Hausdorff. It is clear that the ball $B_{2/3}(1/3) = \{0, 1/3, 2\}$ since there is no $r > 0$ such that $B_r(0) \subset B_{2/3}(1/3)$; that is, open balls may not be an open set. The function d is not continuous since $\lim_{n \to \infty} d(1/n, 1/2) \neq d(0, 1/2)$ although $\lim_{n \to \infty} 1/n = 0$. For more details, see, e.g., [6, 7].

Regarding the weakness of the topology of generalized metric space, mentioned above, the authors add some additional conditions to get the analog of existing fixed point results in the literature; see, e.g., [8–15]. Very recently, Suzuki [16] underlined the importance of generalized metric space by emphasizing that generalized metric space and metric space have no compatible topology.

The following is the definition of the notion of generalized quasimetric space defined by Lin et al. [4]

Definition 4. Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ be a mapping such that, for all $x, y \in X$ and for all distinct points $u, v \in X$ each of them different from x and y, one has

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Then (X, d) is called a generalized quasimetric space (or shortly g.q.m.s).

It is evident that any generalized metric space is a generalized quasimetric space, but the converse is not true in general. We give an example to show that not every generalized quasimetric on a set X is a generalized metric on X.

Example 5 (see [4]). Let $X = \{t, 2t, 3t, 4t, 5t\}$ with $t > 0$ be a constant, and we define $d : X \times X \to [0, \infty)$ by

(1) $d(x, x) = 0$, for all $x \in X$;

(2) $d(t, 2t) = d(2t, t) = 3s$;

(3) $d(t, 3t) = d(2t, 3t) = d(3t, t) = d(3t, 2t) = s$;

(4) $d(t, 4t) = d(2t, 4t) = d(3t, 4t) = d(4t, t) = d(4t, 2t) = d(4t, 3t) = 2s$;

(5) $d(t, 5t) = d(2t, 5t) = d(3t, 5t) = d(4t, 5t) = (3/2)s$;

(6) $d(5t, t) = d(5t, 2t) = d(5t, 3t) = d(5t, 4t) = (5/4)s$,

where $s > 0$ is a constant. Then (X, d) is a generalized quasimetric space, but it is not a generalized metric space, because

$$d(t, 5t) = \frac{3}{2}s \neq d(5t, t) = \frac{5}{4}s. \tag{3}$$

We next give the definitions of convergence and completeness on generalized quasimetric spaces.

Definition 6 (see [4]). Let (X, d) be a g.q.m.s; let $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is g.q.m.s convergent to x if and only if

$$\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0. \tag{4}$$

Definition 7 (see [4]). Let (X, d) be a g.q.m.s and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is left-Cauchy if and only if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n \geq m > k$. We say that $\{x_n\}$ is right-Cauchy if and only if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n > k$. We say that $\{x_n\}$ is Cauchy if and only if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > k$.

Remark 8. A sequence $\{x_n\}$ in a g.q.m.s is Cauchy if and only if it is left-Cauchy and right-Cauchy.

Definition 9 (see [4]). Let (X, d) be a g.q.m.s. We say that

(1) (X, d) is left-complete if and only if each left-Cauchy sequence in X is convergent;

(2) (X, d) is right-complete if and only if each right-Cauchy sequence in X is convergent;

(3) (X, d) is complete if and only if each Cauchy sequence in X is convergent.

Notice that, in the literature in several reports for fixed point results in generalized metric space, an additional but superfluous condition, "Hausdorffness," was assumed. Recently, Jleli and Samet [17], Kirk and Shahzad [18], Karapınar [19], Kadeburg, and Radenović [7], and Aydi et al. [20] reported new some fixed point results by removing the assumption of Hausdorffness in the context of generalized metric spaces. The following crucial lemma is inspired from [7, 17].

Lemma 10. *Let (X, d) be a generalized quasimetric space and let $\{x_n\}$ be a Cauchy sequence in X such that $x_m \neq x_n$ whenever $m \neq n$. Then the sequence $\{x_n\}$ can converge to at most one point.*

Proof. Given $\varepsilon > 0$, since $\{x_n\}$ is a Cauchy sequence, there exists $k_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon, \quad \forall m, n > k_0. \tag{5}$$

We use the method of *Reductio ad absurdum*. Suppose, on the contrary, that there exist two distinct points x and y in X such that the sequence $\{x_n\}$ converges to x and y; that is,

$$\begin{aligned} \lim_{n\to\infty} d(x_n, x) &= \lim_{n\to\infty} d(x, x_n) = 0, \\ \lim_{n\to\infty} d(x_n, y) &= \lim_{n\to\infty} d(y, x_n) = 0. \end{aligned} \tag{6}$$

By assumption for any $n \in \mathbb{N}$, $x_n \neq x_m$, and since $x \neq y$, there exists $k_1 \in \mathbb{N}$ such that $x_n \neq x$ and $x_n \neq y$ for any $n > k_1 \geq k_0$. Due to quadrilateral inequality, we have

$$d(x, y) \leq d(x, x_n) + d(x_n, x_m) + d(x_m, y). \tag{7}$$

Letting $n, m \to \infty$, we can obtain that $d(x, y) = 0$ by regarding (5) and (6). Hence, we get $x = y$ which is a contradiction. □

2. Main Results

In this section, we state and prove the main result of this paper. We start by introducing the following family of functions.

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(Ψ_1) ψ is nondecreasing;

(Ψ_2) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

These functions are known in the literature as (c)-comparison functions. It is easily proved that if ψ is a (c)-comparison function, then $\psi(t) < t$ for any $t > 0$. For more details about such function, we refer the reader to [21, 22]. In this study, we discuss the notion of α-admissible mappings; see, e.g., [23–27]. The following definition was introduced in [23].

Definition 11. Let $f : X \to X$ be a self-mapping of a set X and $\alpha : X \times X \to \mathbb{R}^+$. Then f is called a α-admissible if

$$x, y \in X, \qquad \alpha(x, y) \geq 1 \Longrightarrow \alpha(fx, fy) \geq 1. \tag{8}$$

In what follows we define the $(\alpha-\psi)$-contractive mapping in the setting of generalized quasimetric space.

Definition 12. Let (X, d) be a g.q.m.s. and let $f : X \to X$ be a given mapping. We say that f is an $(\alpha - \psi)$-contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)\, d(fx, fy) \leq \psi(d(x, y)), \quad \forall x, y \in X. \tag{9}$$

Now, we state the following fixed point theorem.

Theorem 13. *Let (X, d) be a complete g.q.m.s, and let $f : X \to X$ be an $(\alpha - \psi)$-contractive mapping. Suppose that*

(i) *f is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$, $\alpha(fx_0, x_0) \geq 1$, $\alpha(x_0, f^2x_0) \geq 1$, and $\alpha(f^2x_0, x_0) \geq 1$;*

(iii) *f is continuous.*

Then f has a periodic point.

Proof. Due to statement (ii) of theorem, there exists $x_0 \in X$ which is an arbitrary point such that $\alpha(x_0, fx_0) \geq 1$ and $\alpha(x_0, fx_0) \geq 1$. We will construct a sequence $\{x_n\}$ in X by $x_{n+1} = fx_n = f^{n+1}x_0$ for all $n \geq 0$. If we have $x_{n_0} = x_{n_0+1}$ for some n_0, then $u = x_{n_0}$ is a fixed point of f. Hence, for the rest of the proof, we presume that

$$x_n \neq x_{n+1} \quad \forall n. \tag{10}$$

Since f is α-admissible, we have

$$\begin{aligned} \alpha(x_0, x_1) = \alpha(x_0, fx_0) \geq 1 \\ \Longrightarrow \alpha(fx_0, fx_1) = \alpha(x_1, x_2) \geq 1. \end{aligned} \tag{11}$$

Utilizing the expression above, we obtain that

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \forall n = 0, 1, \ldots. \tag{12}$$

By repeating the same steps with starting with the assumption $\alpha(x_1, x_0) = \alpha(fx_0, x_0) \geq 1$, we conclude that

$$\alpha(x_{n+1}, x_n) \geq 1, \quad \forall n = 0, 1, \ldots. \tag{13}$$

In a similar way, we derive that

$$\alpha(x_0, x_2) = \alpha\left(x_0, f^2 x_0\right) \geq 1 \Longrightarrow \alpha(fx_0, fx_2) = \alpha(x_1, x_3) \geq 1. \tag{14}$$

Recursively, we get that

$$\alpha(x_n, x_{n+2}) \geq 1, \quad \forall n = 0, 1, \ldots. \tag{15}$$

Analogously, we can easily derive that

$$\alpha(x_{n+2}, x_n) \geq 1, \quad \forall n = 0, 1, \ldots. \tag{16}$$

Step 1. We will show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$. Regarding (8) and (12), we deduce that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq \alpha(x_{n-1}, x_n)\, d(fx_{n-1}, fx_n) \\ &\leq \psi(d(x_{n-1}, x_n)), \end{aligned} \tag{17}$$

for all $n \geq 1$.

Iteratively, we find that

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)) \quad \forall n \geq 1. \tag{18}$$

Similarly,

$$d(x_n, x_{n+2}) \leq \psi^n(d(x_0, x_2)) \quad \forall n \geq 1. \tag{19}$$

By the properties of ψ we can conclude that $\lim_{n\to\infty} \psi^n(d(x_0, x_1)) = 0$; that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{20}$$

Similarly, $\lim_{n\to\infty} \psi^n(d(x_0, x_2)) = 0$, that is;

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{21}$$

Step 2. We will prove that $\{x_n\}$ is a right-Cauchy sequence; that is,

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0 \quad \forall k \in \mathbb{N}. \tag{22}$$

The cases $k = 1$ and $k = 2$ are proved, respectively, by (20) and (21). Now, take $k \geq 3$ arbitrary. It is sufficient to examine two cases.

Case (I). Suppose that $k = 2m+1$, where $m \geq 1$. Then, by using Step 1 and the quadrilateral inequality together with (18), we find

$$\begin{aligned} d(x_n, x_{n+k}) &= d(x_n, x_{n+2m+1}) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\ &\quad + \cdots + d(x_{n+2m}, x_{n+2m+1}) \\ &\leq \sum_{p=n}^{n+2m} \psi^p(d(x_0, x_1)) \\ &\leq \sum_{p=n}^{+\infty} \psi^p(d(x_0, x_1)) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{23}$$

Case (II). Suppose that $k = 2m$, where $m \geq 2$. Again, by applying the quadrilateral inequality and Step 1 together with (18) and (19), we find

$$\begin{aligned} d(x_n, x_{n+k}) &= d(x_n, x_{n+2m}) \\ &\leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) \\ &\quad + \cdots + d(x_{n+2m-1}, x_{n+2m}) \\ &\leq d(x_n, x_{n+2}) + \sum_{p=n+2}^{n+2m-1} \psi^p(d(x_0, x_1)) \\ &\leq d(x_n, x_{n+2}) \\ &\quad + \sum_{p=n}^{+\infty} \psi^p(d(x_0, x_1)) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{24}$$

By combining the expressions (23) and (24), we have

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0 \quad \forall k \geq 3. \tag{25}$$

We conclude that $\{x_n\}$ is a right-Cauchy sequence in (X, d).

In the same way $\{x_n\}$ is a left-Cauchy sequence in (X, d). So it is a Cauchy sequence. Since X is a complete g.q.m.s, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = \lim_{n\to\infty} d(u, x_n) = 0. \tag{26}$$

Also, we can easily see that $x_n \neq x_m$ for whenever $m \neq n$. Indeed, if $x_n = x_m$, for some $m, n \in \mathbb{N}$ with $n < m$, then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_m, x_{m+1}) \\ &\leq \psi^{m-n} d(x_n, x_{n+1}) < d(x_n, x_{n+1}), \end{aligned} \tag{27}$$

which is a contradiction. Analogously, we derive the same conclusion for the case $n > m$. Therefore, we conclude that the sequence $\{x_n\}$ cannot have two limits due to Lemma 10.

Step 3. We claim that f has a periodic point in X. Suppose, on the contrary, that f has no periodic point. Since f is continuous, from Step 2, we have $u = fu$, $(p = 1)$ which contradicts the assumption that f has no periodic point. Therefore, there exists $u \in X$ such that $u = f^p(u)$ for some $p \in \mathbb{N}$. So f has a periodic point in X. □

Now, we state the following fixed point theorem.

Theorem 14. *Let (X, d) be a complete g.q.m.s, and let $f : X \to X$ be an $(\alpha - \psi)$-contractive mapping. Suppose that*

(i) *f is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$, $\alpha(fx_0, x_0) \geq 1$, $\alpha(x_0, f^2x_0) \geq 1$, and $\alpha(f^2x_0, x_0) \geq 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then f has a periodic point.

Proof. Following the proof of Theorem 13, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n$ for all $n \geq 0$ converges for some $u \in X$. It is sufficient to show that f admits a periodic point. Suppose, on the contrary, that f has no periodic point. Notice that $x_n \neq u$ and $x_n \neq fu$ for sufficiently large n. By the quadrilateral inequality, for this n, we have

$$d(u, fu) \leq d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, fu). \quad (28)$$

On account of the fact that $\psi(t) < t$, for all $t > 0$, and regarding the assumption (iii), we get that

$$\begin{aligned} d(u, fu) &\leq d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, fu) \\ &\leq d(u, x_n) + d(x_n, x_{n+1}) + d(fx_n, fu) \\ &\leq d(u, x_n) + d(x_n, x_{n+1}) + \alpha(x_n, u)\, d(fx_n, fu) \\ &\leq d(u, x_n) + d(x_n, x_{n+1}) + \psi(d(x_n, u)) \\ &< d(u, x_n) + d(x_n, x_{n+1}) + d(x_n, u). \end{aligned} \quad (29)$$

Letting $n \to \infty$ in the above equality, from (20) we find that

$$\lim_{n\to\infty} d(x_n, u) = 0 \text{ and so } \lim_{n\to\infty} d(x_{n+1}, fu) = 0. \quad (30)$$

Again from (20), (26), and (30), we can obtain $d(u, fu) = 0$ and hence u is a periodic point of f. □

In what follows we give an example to illustrate Theorem 13.

Example 15. In Example 5 define the mapping $f : X \to X$ as

$$fx \overset{\text{def}}{=} \begin{cases} 5t & \text{if } x \neq 4t \text{ or } t, \\ t & \text{if } x = 4t \text{ or } t, \end{cases} \quad (31)$$

First, we can see easily that the classic Branciari contraction [5] cannot be applied in this case since

$$d(f3t, ft) = \frac{5}{4}s > s = d(3t, t). \quad (32)$$

Now we define the mapping α from $X \times X \to [0, \infty)$ by

$$\begin{aligned} \alpha(x, x) &= \alpha(t, 4t) = \alpha(2t, 3t) = \alpha(2t, 5t) \\ &= \alpha(3t, 5t) = 1; \quad \text{otherwise } \alpha(x, y) = 0, \end{aligned} \quad (33)$$

for all $x, y \in X$. For $\psi(r) = r/2$, where $r \geq 0$, we have

$$\alpha(x, y)\, d(fx, fy) \leq \frac{1}{2} d(x, y). \quad (34)$$

Obviously f is α-admissible and also for $x_0 = t$ we have

$$\alpha(t, ft) = \alpha(ft, t) = \alpha\left(t, f^2 t\right) = \alpha\left(f^2 t, t\right) = \alpha(t, t) = 1. \quad (35)$$

Finally f is continuous. Therefore f satisfies in Theorem 13 and we can see that f has two fixed points t and $3t$.

2.1. Existence Theorem. Let (X, d) be a g.q.m.s and $\text{Per}(f_X)$ denote the set of periodic points of $f : X \to X$.

Property (E). For all $a \in \text{Per}(f_X)$, we have $\alpha(f^p a, f^{p+1} a) \geq 1$ and $\alpha(f^{p+1} a, f^p a) \geq 1$, for any $p \geq 1$.

Theorem 16. *Adding Property (E) to the hypotheses of Theorem 13 (res. Theorem 14), one obtains existence of a fixed point of f.*

Proof. Suppose that a is a periodic point of f; that is, $f^p a = a$. If $p = 1$, then a is a fixed point of f; that is, $f^p a = fa = a$. Assume that $p > 1$. We will show that $z = f^{p-1} a$ is a fixed point of f.

Suppose, on the contrary, that $f^{p-1} a \neq f^p a$. Then $d(f^{p-1} a, f^p a) > 0$, $d(f^p a, f^{p-1} a) > 0$, $\psi(d(f^{p-1} a, f^p a)) > 0$, and $\psi(d(f^p a, f^{p-1} a)) > 0$.

By Property (E), we have $\alpha(a, f^p a) \geq 1$ and $\alpha(f^p a, a) \geq 1$. Due to (9), we can obtain

$$\begin{aligned} d(a, fa) &= d\left(f^p a, f^{p+1} a\right) \\ &\leq \alpha\left(f^p a, f^{p+1} a\right) d\left(f^p a, f^{p+1} a\right) \\ &\leq \psi\left(d\left(f^{p-1} a, f^p a\right)\right). \end{aligned} \quad (36)$$

Due to property $\psi(t) < t$, we get that

$$d(a, fa) < d\left(f^{p-1} a, f^p a\right). \quad (37)$$

Again by (9), we have

$$\begin{aligned} d\left(f^{p-1} a, f^p a\right) &\leq \alpha\left(f^{p-1} a, f^p a\right) d\left(f^{p-1} a, f^p a\right) \\ &\leq \psi\left(d\left(f^{p-2} a, f^{p-1} a\right)\right) \\ &\vdots \\ &\leq \psi^p(d(a, fa)) \\ &< d(a, fa). \end{aligned} \quad (38)$$

Consequently, we get the following contradiction: $d(a, fa) < d(a, fa)$. Hence, the assumption that z is not a fixed point of f is not true and thus $z = f^{p-1} a$ is a fixed point of f. □

To assure the uniqueness of the fixed point, we will consider the following properties.

Property (U). For all $x, y \in \text{Fix}(f)$, we have $\alpha(x, y) \geq 1$.

Theorem 17. *Adding property (U) to the hypotheses of Theorem 16, one obtains uniqueness of the fixed point of f.*

Proof. Suppose that u and v are two distinct fixed points of f. By property (U), $\alpha(Tu, Tv) = \alpha(u, v) \geq 1$.

Thus by α-admissibility of f and the above relation, we can obtain

$$\begin{aligned} d(u,v) &\leq \alpha(u,v)\, d(u,v) \\ &= \alpha(fu,fv)\, d(fu,fv) \\ &\leq \psi d(u,v) < d(u,v), \end{aligned} \tag{39}$$

which is a contradiction. Therefore $u = v$. □

3. Consequences

Definition 18. Let (X,d) be a g.q.m.s, and let $f : X \to X$ be a function satisfying

$$d(fx,fy) \leq \psi(d(x,y)), \tag{40}$$

for all $x, y \in X$. Then f is said to be a ψ-contractive mapping.

Theorem 19. *Let (X,d) be a complete g.q.m.s, and let f be a continuous ψ-contractive mapping. Then f is a unique point u in X.*

Proof. It is sufficient to take $\alpha(x,y) = 1$ for all $x, y \in X$. □

Theorem 20. *Let (X,d) be a complete g.q.m.s, and let f be a continuous mapping. Suppose that there exists $k \in [0,1)$ such that*

$$d(fx,fy) \leq kd(x,y), \quad \forall x, y \in X. \tag{41}$$

Then f is a unique point u in X.

Proof. Take $\psi(t) = kt$, where $k \in [0,1)$. It is clear that all conditions of Theorem 19 are satisfied. □

4. Conclusion

It is evident that almost all fixed point theorems, in the context of generalized metric spaces, can be represented in the setting of generalized quasimetric spaces. Thus, all fixed point results obtained in Section 2 infer the analog of the fixed point theorems in the context of generalized metric spaces. Consequently, several results in the literature (see, e.g., [5, 8, 9, 19, 28]) can be derived from our main results.

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

Acknowledgment

The authors thank the anonymous referees for their remarkable comments, suggestion, and ideas that helped to improve this paper.

References

[1] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.

[2] M. Jleli and B. Samet, "Remarks on G-metric spaces and fixed point theorems," *Fixed Point Theory and Applications*, vol. 2012, article 210, 2012.

[3] N. Bilgili, E. Karapınar, and B. Samet, "Generalized α-ψ contractive mappings in quasi-metric spaces and related fixed-point theorems," *Journal of Inequalities and Applications*, vol. 2014, article 36, 2014.

[4] I. Lin, C. Chen, and E. Karapınar, "Periodic points of weaker meir-keeler contractive mappings on generalized quasimetric spaces," *Abstract and Applied Analysis*, vol. 2014, Article ID 490450, 6 pages, 2014.

[5] A. Branciari, "A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces," *Publicationes Mathematicae Debrecen*, vol. 57, no. 1-2, pp. 31–37, 2000.

[6] I. R. Sarma, J. M. Rao, and S. S. Rao, "Contractions over generalized metric spaces," *Journal of Nonlinear Science and Its Applications*, vol. 2, no. 3, pp. 180–182, 2009.

[7] Z. Kadeburg and S. Radenović, "On generalized metric spaces: a survey," *TWMS Journal of Pure and Applied Math*, vol. 5, no. 1, pp. 3–13, 2014.

[8] A. Azam and M. Arshad, "Kannan fixed point theorem on generalized metric spaces," *Journal of Nonlinear Sciences and Its Applications*, vol. 1, no. 1, pp. 45–48, 2008.

[9] P. Das, "A fixed point theorem on a class of generalized metric spaces," *Korean Journal of Mathematical Sciences*, vol. 9, pp. 29–33, 2002.

[10] M. Erhan, E. Karapınar, and T. Sekulic, "Fixed points of (ψ,ϕ) contractions on rectangular metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 138, 2012.

[11] D. Mihet, "On Kannan fixed point principle in generalized metric spaces," *Journal of Nonlinear Science and its Applications: TJNSA*, vol. 2, no. 2, pp. 92–96, 2009.

[12] B. Samet, "A fixed point theorem in a generalized metric space for mappings satisfying a contractive condition of integral type," *International Journal of Mathematical Analysis*, vol. 3, no. 25–28, pp. 1265–1271, 2009.

[13] B. Samet, "Disscussion on: a fixed point theorem of Banach-Caccioppli type on a class of generalized metric spaces," *Publicationes Mathematicae*, vol. 76, no. 4, pp. 493–494, 2010.

[14] H. Lakzian and B. Samet, "Fixed points for (ψ,φ)-weakly contractive mappings in generalized metric spaces," *Applied Mathematics Letters*, vol. 25, no. 5, pp. 902–906, 2012.

[15] H. Aydi, E. Karapınar, and H. Lakzian, "Fixed point results on a class of generalized metric spaces," *Mathematical Sciences*, vol. 6, article 46, 2012.

[16] T. Suzuki, "Generalized metric spaces do not have the compatible topology," *Abstract and Applied Analysis*, vol. 2014, Article ID 458098, 2014.

[17] M. Jleli and B. Samet, "The Kannan's fixed point theorem in a cone rectangular metric space," *Journal of Nonlinear Science and Its Applications*, vol. 2, no. 3, pp. 161–167, 2009.

[18] W. A. Kirk and N. Shahzad, "Generalized metrics and Caristi's theorem," *Fixed Point Theory and Applications*, vol. 2013, article 129, 2013.

[19] E. Karapınar, "Discussion on α-ψ contractions on generalized metric spaces," *Abstract and Applied Analysis*, vol. 2014, Article ID 962784, 7 pages, 2014.

[20] H. Aydi, E. Karapınar, and B. Samet, "Fixed points for generalized (alpha,psi)-contractions on generalized metric spaces," *Journal of Inequalities and Applications*, vol. 2014, article 229, 2014.

[21] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, Romania, 2001.

[22] R. M. Bianchini and M. Grandolfi, "Trasformazioni di tipo contrattivo generalizzato in uno spazio metrico," *Atti della Accademia Nazionale dei Lincei: Rendiconti: Classe di Scienze Fisiche, Matematiche e Naturali*, vol. 45, pp. 212–216, 1968.

[23] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for $\alpha - -\psi$-contractive type mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.

[24] N. Hussain, E. Karapınar, P. Salimi, and P. Vetro, "Fixed point results for G^m-Meir-Keeler contractive and G-(α,ψ)-Meir-Keeler contractive mappings," *Fixed Point Theory and Applications*, vol. 2013, article 34, 2013.

[25] N. Hussain, E. Karapınar, P. Salimi, and F. Akbar, "α-admissible mappings and related fixed point theorems," *Journal of Inequalities and Applications*, vol. 2013, article 114, 11 pages, 2013.

[26] P. Salimi and E. Karapınar, "Suzuki-Edelstein type contractions via auxiliary functions," *Mathematical Problems in Engineering*, vol. 2013, Article ID 648528, 8 pages, 2013.

[27] L. B. Ćirić, S. M. Alsulami, P. Salimi, and P. Vetro, "PPF dependent fixed point results for triangular α_c-admissible mappings," *The Scientific World Journal*, vol. 2014, Article ID 673647, 10 pages, 2014.

[28] M. A. Alghamdi, C.-M. Chen, and E. Karapınar, "A generalized weaker (alpha-phi-varphi)-contractive mappings and related fixed point results in complete generalized metric spaces," *Abstract and Applied Analysis*, vol. 2014, Article ID 985080, 10 pages, 2014.