Mathematical Problems in Engineering

Theory, Methods, and Applications

Editor-in-Chief: Jose Manoel Balthazar

Special Issue
Time-Dependent Billiards

Guest Editors: Edson Denis Leonel and Alexander Loskutov

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Volume 2009

Time-Dependent Billiards

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Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 848619, 4 pages
doi:10.1155/2009/848619

Editorial

Time-Dependent Billiards

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Received 24 August 2009; Accepted 24 August 2009

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1. Introduction

Dynamical systems of a billiard type are a fundamental notion relevant for the understanding of numerous phenomena observed in statistical mechanics, Hamiltonian dynamics, nonlinear physics, and others. By means of billiard models, the principal ideas as Boltzmann ergodic hypothesis, related to the foundation of statistical physics, and deterministic diffusion gained more deep insight. However, more sophisticated and realistic results may be obtained if billiards are considered with time-dependent boundaries. Such kind of billiard systems represent a natural generalization of mathematical billiards and more adequately reflects the observed physical phenomena. This Focus Issue presents recent progress in this area with contributions on new ways and specific model studies.

2. Context

In the last four decades of the twentieth century, mathematical billiards and related fields became one of the most active research areas in statistical mechanics and theory of dynamical systems. However, the history of billiard problems started in 1927 with a remarkable paper by Birkhoff [1], who considered a problem concerning the free motion of a point particle (billiard ball) in some bounded manifold. A more complete consideration of the problems related to the dynamics of mass points in a bounded region was given by Krylov [2]. However, the contemporary investigations of billiards were initiated by Sinai [3] and some time later by Bunimovich [4] (see also a classical paper on the Lorenz gas [5]), who made the rigorous analysis of the billiard dynamics. A great progress of billiard's dynamics was achieved in a solution, in a certain sense, of the known Boltzmann ergodic hypothesis.

A billiard dynamical system is generated by the free motion of a point mass particle (billiard ball) in some region with a piecewise-smooth boundary and the condition of the elastic collision from this boundary. If the boundary in the collision point is smooth, then the billiard ball reflects from it in such a way that the velocity tangent component remains constant, and the normal component changes its sign. If the ball hits a corner of the billiard table, then its dynamics is not determined.

Nowadays, the main focus in the study of billiard problems is moving from the investigations of billiards with the fixed boundary to time-dependent billiards. Indeed, the Lorentz gas has been proposed for the description of the motion of electrons between heavy ions in the lattice of metals. In the reality, however, ions should weakly oscillate near their equilibrium state. Moreover, some important problems of mathematical physics can be described by nonstationary billiard models (see [6]).

For the most part, investigations of classical time-dependent billiards concerned two main questions: descriptions of their statistical properties and the study of trajectories for which the particle velocity grows indefinitely. This problem is related to the unbounded increase of energy in periodically forced Hamiltonian systems and known as Fermi acceleration [7]. In his paper, Fermi supposed that in a typical environment, the probability of a head-on collision is greater than the probability of a head-tail collision, so particles would, on average, be accelerated. Later many approaches regarding the description of such a phenomenon were introduced for both the continuous and discrete time models (see [6, 8–11] and references cited therein).

It is obvious that in this context dynamical properties of billiards play a principal role: if it is chaotic, then the boundary perturbation may lead to the particle acceleration. In papers [12, 13], on the basis of these ideas, the following conjecture, known in literature as Loskutov-Ryabov-Akinshin- (LRA-) conjecture (see, e.g., [14–16]) has been advanced: *Fermi acceleration will be observed in time-dependent billiards if in the autonomous case these billiards possess chaotic properties*. This conjecture has been confirmed for Bunimovich stadium [17] and for annular billiards [18]. Lately, using the theory of dynamical systems, it was proved that the Fermi acceleration in some nonautonomous billiard-like systems should be observed [19, 20]. Recently this phenomenon has been detected for driven elliptical billiard [9]. Finally, applying thermodynamic methods, it was shown [21] that Fermi acceleration is inherent in the Lorentz gas of a quite general configuration.

The next physical generalization of billiard ideas is to consider the corresponding dynamical systems with inelastic collisions. These models admits more realistic analysis of some natural phenomena. For example, they allow us to study the presence and lack of Fermi acceleration in certain billiard-like models.

3. This Focus Issue

The contributions to this focus issue can be grouped into three main parts:

(i) Hamiltonian dynamics and related fields;

(ii) dynamics of time-dependent billiard-like models;

(iii) applications.

The first part on *Hamiltonian systems* addresses problems such as integrability and nonintegrability, scaling results for the phase transition, properties of chaotic regimes in the

region below the first invariant spanning curve, chaotic properties and structure of the phase space, and stability of periodic time-dependent systems.

The second part contains in-depth studies of specific models of *time-dependent billiard-like models* and nonautonomous systems. Here, the following questions are considered: in what cases Fermi acceleration can be observed?, how one can describe the scaling properties of billiard-like systems?, does adding the inelasticity lead to suppression of Fermi acceleration?

The third part on *Applications* contains contributions on several generalizations related to accelerator models, analysis of certain mechanical billiard-like systems, and some quantum abstractions.

We do hope that the presented papers will be interested in a wide audience of readers.

Acknowledgments

Alexander Loskutov gratefully acknowledges FAPESP and Departamento de Estatística, Matemática Aplicada e Computação (Brazil) for the summer grant during his stay in the Universidade Estadual Paulista, Rio Claro, Brazil. E. D. Leonel acknowledges the financial support from CNPq, FAPESP, CAPES, FUNDUNESP and Pró Reitoria de Pesquisa (PROPe, UNESP) and Pró Reitoria de Pós-Graduação (PROPG, UNESP). They are grateful to Professors Makoto Yoshida (UNESP, Brazil), Jafferson Silva (UFMG, Brazil), and Leonid Bunimovich (Georgia Institute of Tecnhology, USA), and all the contributors of the past International Meeting "Billiards'09" (Águas de Lindóia, Brazil), as well as the authors for their papers sent to the present issue.

Alexander Loskutov
Edson D. Leonel

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Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 367921, 22 pages
doi:10.1155/2009/367921

Research Article

Phase Transition in Dynamical Systems: Defining Classes of Universality for Two-Dimensional Hamiltonian Mappings via Critical Exponents

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Received 8 April 2009; Accepted 20 July 2009

Recommended by Alexander Loskutov

A phase transition from integrability to nonintegrability in two-dimensional Hamiltonian mappings is described and characterized in terms of scaling arguments. The mappings considered produce a mixed structure in the phase space in the sense that, depending on the combination of the control parameters and initial conditions, KAM islands which are surrounded by chaotic seas that are limited by invariant tori are observed. Some dynamical properties for the largest component of the chaotic sea are obtained and described in terms of the control parameters. The average value and the deviation of the average value for chaotic components of a dynamical variable are described in terms of scaling laws, therefore critical exponents characterizing a scaling function that describes a phase transition are obtained and then classes of universality are characterized. The three models considered are: The Fermi-Ulam accelerator model, a periodically corrugate waveguide, and variant of the standard nontwist map.

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1. Introduction

Dynamical systems described by two-dimensional nonlinear area preserving mappings have been considered for many years as prototype for the study of low-dimensional physical systems. Applications of the description of dynamical systems by mappings are observed in the study of magnetic field lines in toroidal plasma devices with reversed shear (like tokamaks), channel flows, waveguide, Fermi acceleration and many other [1–10].

In this paper we consider a special class of two-dimensional area preserving mappings exhibiting a phase transition from integrable to nonintegrable. For certain values of the control parameter, the phase space of the system has only periodic and quasiperiodic orbits, thus the phase space is filled by straight lines. As this control parameter varies, the phase

space exhibits a mixed form containing both Kolmogorov-Arnold-Moser (KAM) islands surrounded by a chaotic sea (conservative chaos) and a set of invariant spanning curves limiting the size of the chaotic sea. This form of the phase space is generic for nondegenerate Hamiltonian systems [11] and can be observed in many different problems including time dependent potential [12–16], tokamaks [17–20] and also in billiards [21–23].

The main approach considered in this paper is the description of the behavior of some average dynamical variables as a function of the control parameter, like the first momenta and the deviation around the average variable. This characterization is made along the chaotic sea near a transition from integrable to nonintegrable. This formalism furnishes critical exponents that can be used to define classes of universality.

This paper is organized as follows. In Section 2, we present and discuss some details needed to construct a generic two-dimensional area preserving mapping. Then application for a specific mapping is made including a discussion of the stability of fixed points and properties of the phase transition from integrable to nonintegrable. In Section 3 the discussion is devoted to a periodically corrugate waveguide. Some results for the phase space are also discussed in connection with the Standard Mapping. The scaling results for the phase transition are obtained and discussed. In Section 4, we extend the applicability of the theory for the one-dimensional Fermi accelerator model confirming that the rather two distinct models discussed in Sections 3 and 4 belong to the same class of universality and experience the same transition . Finally, the conclusion and final remarks are drawn in Section 5.

2. A Generic Two-Dimensional Area-Preserving Map

In this section we present and discuss some dynamical properties for a two-dimension mapping. We assume that there is a two-dimensional integrable system that is slightly perturbed. Then, the Hamiltonian function that describes the system is written as

$$H(I_1, I_2, \theta_1, \theta_2) = H_0(I_1, I_2) + \varepsilon H_1(I_1, I_2, \theta_1, \theta_2), \tag{2.1}$$

where the variables I_i and θ_i with $i = 1, 2$ correspond, respectively, to the action and angle. Considering the Poincaré section given by the plane $I_1 \times \theta_1$ assuming that θ_2 is constant (mod 2π), one can write a modified twist mapping as

$$T : \begin{cases} I_{n+1} = I_n + \varepsilon h(\theta_n, I_{n+1}), \\ \theta_{n+1} = \left[\theta_n + K(I_{n+1}) + \varepsilon p(\theta_n, I_{n+1})\right] \quad \text{mod } (2\pi), \end{cases} \tag{2.2}$$

where both h, K, and p are, in principle, nonlinear functions of their variables and the index n correspond to the nth iteration of the mapping. The variables θ and I corresponds indeed to I_1 and θ_1.

Since the mapping (2.2) must be area preserving (it was obtained from an Hamiltonian) then the functions $h(\theta_n, I_{n+1})$ and $p(\theta_n, I_{n+1})$ must obey some intrinsic relations.

The relations are obtained considering that the determinant of the Jacobian matrix is the unity. Then we have that the coefficients are written as

$$\begin{aligned}
\frac{\partial I_{n+1}}{\partial I_n} &= \frac{1}{1-\varepsilon(\partial h(\theta_n, I_{n+1})/\partial I_{n+1})}, \\
\frac{\partial I_{n+1}}{\partial \theta_n} &= \varepsilon\frac{\partial h(\theta_n, I_{n+1})}{\partial \theta_n} + \varepsilon\frac{\partial h(\theta_n, I_{n+1})}{\partial I_{n+1}}\frac{\partial I_{n+1}}{\partial \theta_n}, \\
\frac{\partial \theta_{n+1}}{\partial I_n} &= \left[\frac{\partial K(I_{n+1})}{\partial I_{n+1}} + \varepsilon\frac{\partial p(\theta_n, I_{n+1})}{\partial I_{n+1}}\right]\frac{\partial I_{n+1}}{\partial I_n}, \\
\frac{\partial \theta_{n+1}}{\partial \theta_n} &= 1 + \varepsilon\frac{\partial p(\theta_n, I_{n+1})}{\partial \theta_n} + \left[\frac{\partial K(I_{n+1})}{\partial I_{n+1}} + \varepsilon\frac{\partial p(\theta_n, I_{n+1})}{\partial I_{n+1}}\right]\frac{\partial I_{n+1}}{\partial \theta_n}.
\end{aligned} \tag{2.3}$$

Given the above coefficients, the determinant of the Jacobian matrix is given by

$$\operatorname{Det} J = \frac{\left[1+\varepsilon\left(\partial p(\theta_n, I_{n+1})/\partial \theta_n\right)\right]}{\left[1-\varepsilon(\partial h(\theta_n, I_{n+1})/\partial I_{n+1})\right]}. \tag{2.4}$$

It is easy then to conclude that area preservation will be observed only if the condition

$$\frac{\partial p(\theta_n, I_{n+1})}{\partial \theta_n} + \frac{\partial h(\theta_n, I_{n+1})}{\partial I_{n+1}} = 0 \tag{2.5}$$

is matched. For many mappings, the function $p(\theta_n, I_{n+1}) = 0$. Therefore, if we keep the function h as $h(\theta_n) = \sin(\theta_n)$, just to illustrate applicability of the formalism, the following mappings have already been studied:

(i) $K(I_{n+1}) = I_{n+1}$, then the Taylor-Chirikov's map is recovered;

(ii) $K(I_{n+1}) = 2/I_{n+1}$, then the Fermi-Ulam accelerator model is obtained;

(iii) $K(I_{n+1}) = \zeta I_{n+1}$, with ζ constant, then the bouncer model is found;

(iv) for the case of

$$K(I_{n+1}) = \begin{cases} 4\zeta^2\left(I_{n+1} - \sqrt{I_{n+1}^2 - \dfrac{1}{\zeta^2}}\right), & \text{if } I_{n+1} > \dfrac{1}{\zeta}, \\ 4\zeta^2 I_{n+1}, & \text{if } I_{n+1} \le \dfrac{1}{\zeta}. \end{cases} \tag{2.6}$$

where ζ is a constant, then we recovered the so-called Hybrid-Fermi-Ulam-bouncer model;

(v) considering $K(I_{n+1}) = I_{n+1} + \zeta I_{n+1}^2$, the logistic twist mapping is obtained.

Figure 1: Phase space generated by mapping (2.7) for the control parameters $a = 2$ and: (a) $b = 10^{-3}$ and (b) $b = 10^{-2}$.

In this section, we shall consider the following expression for the two-dimensional area-preserving mapping:

$$T : \begin{cases} y_{n+1} = \left| y_n - b \sin(2\pi x_n) \right|, \\ x_{n+1} = \left[x_n + \dfrac{a}{\sqrt{y_{n+1}}} \right] \mod (1), \end{cases} \tag{2.7}$$

where the variables I and θ were replaced, by sake of simplicity, by y and x. The control parameters of the model are a and b. The modulus function is requested since the argument of the denominator of the second equation above is not defined for nonpositive values. It is easy to see that when the control parameter $b = 0$, then the system is integrable. For $b \neq 0$, the phase space of the system exhibits chaotic components and periodic orbits, as we can see in Figure 1. The determinant of the Jacobian matrix of the mapping (2.7) is the unity, so area preservation is observed.

The fixed points of the mapping (2.7) are obtained by matching the following conditions: $y_{n+1} = y_n = y$ and $x_{n+1} = x_n = x + m$ with $m = 1, 2, 3, \ldots$. After solving these two conditions, the two sets of fixed points are given by: fixed point 1 $(x, y) = (0, a^2/m^2)$; fixed point 2 $(x, y) = (1/2, a^2/m^2)$. The fixed point 1 is always hyperbolic while fixed point 2 is elliptic for

$$a > \sqrt[4]{\frac{b^2 \pi^2 m^6}{16}}. \tag{2.8}$$

From now on, we shall consider as fixed the control parameter $a = 2$. Results for $a \neq 0$ will be presented elsewhere.

We can see in Figure 1 that the size of the chaotic sea varies as the control parameter b varies. As b rises, the position of the lowest invariant spanning curve also rises. It is then expected that the average value of y, say $\overline{y}$ changes as a function of the control parameter. To

Figure 2: Behavior of $\overline{y}$ as function of n for the following control parameters: (a) $b = 10^{-3}$, $b = 3 \times 10^{-3}$ and $b = 7 \times 10^{-3}$; (b) same plot of (a) after a transformation $n \to nb^2$.

obtain the average value of y, we proceed as follows. First we obtain the average value along an orbit

$$y(n,b) = \frac{1}{n}\sum_{i=1}^{n} y_i, \tag{2.9}$$

and then we evolve the average over an ensemble of M different initial conditions along the chaotic sea. The average value is then written as

$$\overline{y}(n,b) = \frac{1}{M}\sum_{j=1}^{M} y_j, \tag{2.10}$$

where y_j is a sample of the ensemble. It is shown in Figure 2 the behavior of $\overline{y} \times n$ for three different control parameters. We can see that $\overline{y}$ starts growing and then, after reaching a characteristic crossover iteration number, it bends towards a regime of convergence. We see also that the three curves start growing at different positions. Therefore, a transformation $n \to nb^2$ coalesces all curves at small iterations, as we can see in Figure 2(b).

The behavior shown in Figure 2 allow us to suppose the following.

(i) For low iterations and after a short initial transient, that is, $1 \ll n \ll n_x$, the average y is described as $\overline{y} \propto (nb^2)^{\beta}$, where β is a critical exponent.

(ii) For large enough n, say $n \gg n_x$, then $\overline{y}_{\text{sat}} \propto b^{\alpha}$, where α is a critical exponent.

(iii) The number of iterations which characterizes the changeover from growth to the saturation is given $n_x \propto b^z$ where z is also a critical exponent.

Plotting the behavior of $\overline{y}_{\text{sat}} \times b$ and $n_x \times b$ we can obtain the critical exponents α and z. The corresponding plots of $\overline{y}_{\text{sat}} \times b$ and $n_x \times b$ are shown in Figure 3. After applying a power law

Figure 3: Plot of: (a) $\overline{y}_{\text{sat}} \times b$ and (b) $n_x \times b$. A power law fitting furnishes that $\alpha = 0.676(3)$ and $z = -0.686(2)$.

Figure 4: Collapse of different curves of $\overline{y}$, after a suitable rescale in the axis, into a single and universal plot.

fitting we obtain that $\alpha = 0.676(3)$ and $z = -0.686(2)$. The critical exponent β was obtained for several different time series. Its average value is around $\beta \approx 0.5$.

Rescaling properly, we can see that all curves coalesce together, as they are shown in Figure 4. Such collapse is a clear evidence of the criticality for the chaotic sea near a transition from integrability to nonintegrability. It must be emphasized that the values obtained for

the critical exponents are useful to define classes of universality. As we will see in the two next sections, for the periodically corrugate waveguide and for the one-dimensional Fermi accelerator model, the exponents obtained here are different from the critical exponents obtained there, thus confirming that, despite the similarity of the two-dimensional mappings, the classes of universality of such models are different.

3. Periodically Corrugate Waveguide

Let us now discuss in this section some properties of a periodically corrugated waveguide. This section is a review of papers [24, 25]. The interest in describing the problem of *guiding a light ray* inside a periodically corrugated boundary has increased along last years, mainly because the topic is applicable in so many different fields of science. Applications involve, for example, ray chaos in underwater acoustics [26–28], quantum transport in ballistic cavities [29, 30], transport through a finite $GaAs/Al_xGa_{1-x}As$ hetero-structure [31], quantized ballistic conductance in a periodically modulated quantum channel [32], comparisons of classical versus quantum behavior in periodic mesoscopic systems [33, 34], scattering of a quantum particle in a rippled waveguide [35] and anomalous wave transmittance in the stop band of a corrugated parallel-plane waveguide [36].

There are many different ways to describe problems involving waveguides. One of them consists in considering the well-known billiard approach (A billiard problem consists of a system in which a point-like particle moves freely inside a bounded region and suffers specular reflections with the boundaries.). Generally, the dynamics is described using the formalism of discrete maps. Therefore, depending on the combinations of the control parameters as well as on the initial conditions, the phase space for such mappings might be included in three different classes namely: (i) regular, (ii) ergodic, and (iii) mixed. Roughly speaking, the integrability of the regular cases is generally related to the angular momentum preservation and the static circular billiard is a typical example. On the other hand, for the completely ergodic billiards, only chaotic and therefore unstable periodic orbits are present in the dynamics. The well-known Bunimovich stadium [37] and the Sinai billiard [38] (sometimes also-called as Lorenz gas) are typical examples of case (ii). On these two kinds of systems, the time evolution of a single initial condition, for the appropriated combinations of control parameters, is enough to fill up ergodically the entire phase space. Finally, there are many different billiards that present mixed phase space structure [39–43], which have control parameters with different physical significance. Depending on the combination of both initial conditions and control parameters, the phase space presents a very rich structure which contains invariant spanning curves (sometimes also-called as invariant tori), KAM islands and chaotic seas.

In this section, scaling arguments are used to describe the behavior of the deviation of the average reflection angle within the chaotic sea, for the problem of a *classical* light ray inside a periodically corrugated waveguide suffering specular reflections. The model consists of a classical light ray which is specularly reflected between a flat plane at $y = 0$ and a corrugated surface given by $y = y_0 + d\cos(kx)$, where y_0 is the average distance between the flat plane and the corrugated surface, d is the amplitude of the corrugation and k is the wave number. The dynamical variables, which are used in the description of the problem are the angle θ of the trajectory measured from the positive horizontal axis and the corresponding value of the coordinate x at the instant of the reflection. It must be emphasized that the mapping is iterated when the ray light *hits* the flat plane, thus multiple reflections with

Figure 5: (a) Reflection from the corrugated surface of a light ray coming from the flat surface at $y = 0$. (b) Details of the trajectory before and after a collision with the corrugated surface.

the periodically corrugated surface are neglected. (This is a standard approximation which is very useful to speed up the numerical simulations since no transcendental equations must be solve, as they have to be in the corresponding complete version of the problem).

3.1. The Model and the Mapping

Let us discuss now all the details needed to describe the model and obtain the equations that describe the dynamics of the problem. The model thus consists in obtaining a mapping $T(\theta_n, x_n) = (\theta_{n+1}, x_{n+1})$, given the initial conditions (θ_n, x_n) as shown in Figure 5(a). The mapping is obtained from purely geometrical considerations of Figure 5(b). It is easy to see that

$$x_n^* - x_n = \frac{\left(y_0 + d\cos(kx_n^*)\right)}{\theta_n}. \tag{3.1}$$

and in a similar way

$$x_{n+1} - x_n^* = \frac{\left(y_0 + d\cos(kx_n^*)\right)}{\theta_{n+1}}. \tag{3.2}$$

We stress that the term x_n^* gives the exact location of the collision on the corrugated surface. On the other hand, the angle θ_n is written as

$$\theta_{n+1} = \theta_n - 2\psi_n, \tag{3.3}$$

where $\tan(\psi_n(x)) = dy(x)/dx = -dk\sin(kx_n^*)$ gives the slope of the corrugated surface at the point $x = x_n^*$. The condition of specular reflection (A specular reflection is that in which the angle of incidence is equals to the angle of reflection. See details in Figure 5(b)) was used in the derivation of the (3.3). The Equations (3.2) and (3.3) correspond to the exact form of the mapping. However, I will discuss in this paper only a simplified version. To do this, the following approximations are assumed.

(1) The relative corrugation is assumed to be small, so that $d/y_0 \ll 1$, yielding consequently $y_0 + d\cos(kx_n^*) \cong y_0$.

(2) In the limit of small corrugation, it is also assumed that $\tan(\psi_n) \cong \psi_n$.

Figure 6: Phase space for the mapping (3.5) with the control parameter $\delta = 10^{-3}$.

Let us now argue on the approximation of small relative corrugation. It is easy to see that, for $d = 0$, the system is integrable, since the two boundaries are parallel plates and the phase space shows only straight lines. However, if $d \neq 0$ the phase space birth as mixed therefore containing both chaos, invariant spanning curves and KAM islands. Thus, there is an abrupt transition from integrability to nonintegrability when the control parameter goes from $d = 0$ to $d \neq 0$. This is the transition we shall address in this section. We stress also that the formalism can be extendable to many other systems and even to other different kinds of transitions. Before write the equations of the mapping, we can see that there are an excessive number of control parameters, namely d, k, y_0. The dynamics however does not depend on all of them. It is therefore convenient to define the following dimensionless and more convenient variables $\delta = d/y_0$, $\gamma_n = \theta_n/k$ and finally $X_n = kx_n/y_0$. Thus the only relevant control parameter is δ. With these new variables, the two-dimensional nonlinear mapping is given by

$$T : \begin{cases} X_{n+1} = X_n + \left[\dfrac{1}{\gamma_n} + \dfrac{1}{\gamma_{n+1}}\right] \quad \mathrm{mod}\ (2\pi), \\ \gamma_{n+1} = \gamma_n + 2\delta \sin\left(X_n + \dfrac{1}{\gamma_n}\right). \end{cases} \tag{3.4}$$

This mapping preserves the phase space measure since that the determinant of the Jacobian matrix is the unity.

It is shown in Figure 6, the phase space generated by the iteration of mapping (3.4). We can see that there is a complex structure which contains KAM islands surrounded by a large chaotic sea that is confined between two invariant spanning curves. The chaotic sea was characterized using the well-known Lyapunov exponent. Such exponent shows great

applicability as a practical tool that can quantify the average expansion or contraction rate for a small volume of initial conditions. As discussed in [44], the Lyapunov exponents are defined as

$$\lambda_j = \lim_{n\to\infty} \frac{1}{n} \ln\left|\Lambda_j\right|, \quad j = 1, 2, \tag{3.5}$$

where Λ_j are the eigenvalues of $M = \prod_{i=1}^{n} J_i(\gamma_i, X_i)$, and J_i is the Jacobian matrix evaluated over the orbit (γ_i, X_i). It is important to stress that a direct implementation of a computational algorithm to evaluate (3.5) has a severe limitation to obtain M. Even in the limit of very short n, the components of the matrix M can assume very different orders of magnitude for chaotic orbits and periodic attractors, yielding impracticably the implementation of the algorithm. In order to avoid such problem, we note that J can be written as $J = \Theta T$ where Θ is an orthogonal matrix and T is a right triangular matrix. Thus we rewrite M as $M = J_n J_{n-1} \ldots J_2 \Theta_1 \Theta_1^{-1} J_1$, where $T_1 = \Theta_1^{-1} J_1$. A product of $J_2\Theta_1$ defines a new J_2'. In a next step, it is easy to show that $M = J_n J_{n-1} \cdots J_3 \Theta_2 \Theta_2^{-1} J_2' T_1$. The same procedure can be used to obtain $T_2 = \Theta_2^{-1} J_2'$ and so on. Using this procedure, the problem is reduced to evaluate the diagonal elements of $T_i : T_{11}^i, T_{22}^i$. Finally, the Lyapunov exponents are now given by

$$\lambda_j = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln\left|T_{jj}^i\right|, \quad j = 1, 2. \tag{3.6}$$

If at least one of the λ_j is positive then the orbit is classified as chaotic. Therefore we define λ as the larger value of the λ_j. It is shown in Figure 7 the behavior of the positive Lyapunov exponent for an ensemble of 10 different initial conditions on the chaotic sea for the control parameter $\delta = 10^{-3}$. The initial conditions were chosen such that $X_0 = \delta$ and 10 different initial values of γ_0 were uniformly distributed along the interval $\gamma_0 \in [0, 2\pi]$. Each initial condition was iterated up to 5×10^8 times. The average value obtained was $\overline{\lambda} = 1.624(7)$ where the error 0.007 corresponds to the standard deviation of the 10 samples.

Let us now discuss one of the consequences of the variation of the control parameter δ on the location of the invariant spanning curves. Since the two first invariant spanning curves define the limit for the chaotic sea, the variation of δ consequently change the edges of the chaotic sea. Before do that, I must first give a brief review in the well-known *Standard Map* (see also [45] for specific details). It is defined as

$$T : \begin{cases} I_{n+1} = I_n + K \sin(\Theta_n), \\ \Theta_{n+1} = \Theta_n + I_{n+1}, \end{cases} \tag{3.7}$$

where K is a control parameter. This mapping can also shows mixed phase space for $K \neq 0$. Moreover, one of the most important properties is that it shows a transition from local to global chaos (see [45, 46] for specific details) when the control parameter matches the condition $K > 0.971\ldots$. After this transition, no invariant spanning curves are present and consequently the chaotic sea spreads over the phase space. The connection of this result with the phase space of the mapping that describes the behavior of the light ray in a periodically corrugate waveguide suppose that, near the invariant spanning curves that limit the chaotic

Figure 7: The positive Lyapunov exponent obtained via triangularization algorithm for the control parameter $\delta = 10^{-3}$.

sea, the dynamics could be locally described by using the standard mapping. Thus we can suppose that near the invariant spanning curve, the reflection angle can be written as

$$\gamma_{n+1} \cong \gamma^* + \Delta\gamma_{n+1}, \tag{3.8}$$

where γ^* is a typical value of the reflection angle along the invariant spanning curve, and $\Delta\gamma_{n+1}$ is a small perturbation of the angle. After defining $Z_n = X_n + 1/\gamma_n$, the first equation of the mapping (3.4) is written as

$$Z_{n+1} = Z_n + \frac{2}{\gamma_{n+1}}. \tag{3.9}$$

Using (3.8), we can rewrite (3.9) as

$$Z_{n+1} = Z_n + \frac{2}{\gamma^*}\left[1 + \frac{\Delta\gamma_{n+1}}{\gamma^*}\right]^{-1}. \tag{3.10}$$

Expanding (3.10) in Taylor series, we obtain that

$$Z_{n+1} = Z_n + \frac{2}{\gamma^*}\left[1 - \frac{\Delta\gamma_{n+1}}{\gamma^*} + O\left(\frac{\Delta\gamma_{n+1}}{\gamma^*}\right)^2\right]. \tag{3.11}$$

Considering the first representative term in the series, we obtain that

$$Z_{n+1} = Z_n + \frac{2}{\gamma^*}\left[1 - \frac{\Delta\gamma_{n+1}}{\gamma^*}\right]. \tag{3.12}$$

On the other hand, the second equation of mapping (3.4) may be rewritten as

$$\gamma^* + \Delta\gamma_{n+1} = \gamma^* + \Delta\gamma_n + 2\delta\sin(Z_n). \tag{3.13}$$

Multiplying both sides of (3.13) by $-2/\gamma^{*2}$ and adding again in both sides the term $2/\gamma^*$, defining the term

$$I_{n+1} = -\frac{2\Delta\gamma_{n+1}}{\gamma^{*2}} + \frac{2}{\gamma^*}, \tag{3.14}$$

and calling $\phi_n = Z_n + \pi$, we obtain the corresponding mapping

$$T : \begin{cases} I_{n+1} = I_n + \dfrac{4\delta}{\gamma^{*2}}\sin(\phi_n), \\ \phi_{n+1} = \phi_n + I_{n+1}. \end{cases} \tag{3.15}$$

An immediate comparison of (3.7) and (3.15) allow us to conclude that there is an effective control parameter given by

$$K_{\text{eff}} = \frac{4\delta}{\gamma^{*2}}. \tag{3.16}$$

Since the transition from local to global chaos occurs at $K_{\text{eff}} \cong 0.971\ldots$, we obtain that the two invariant spanning curves that limit the chaotic sea are given by

$$\gamma^* \cong \pm 2\sqrt{\frac{\delta}{0.971\ldots}}. \tag{3.17}$$

Therefore we can conclude that the *size* of the chaotic sea is proportional to $\sqrt{\delta}$ [25].

3.2. Scaling Approach

This section is devoted to discuss a scaling property which is present in the chaotic sea. As it was discussed in Section 2, an initial condition in the region of the chaotic sea wanders in a chaotic way along the accessible region, however it is always confined between two invariant spanning curves. The location of the first positive and negative invariant spanning curves depend on the value of the control parameter δ. As a consequence, the "amplitude"

Figure 8: (a) Plot of ω as function of the iteration number n for three different control parameters. (b) After a suitable changing of variables, all three curves start growing together.

of a chaotic time series is dependent on the control parameter too. We will now explore the behavior of the deviation of the average value for the angle γ, an observable defined as

$$\omega(n,\delta) = \frac{1}{M}\sum_{i=1}^{M}\sqrt{\overline{\gamma}_i^2(n,\delta) - \overline{\gamma}_i^{\,2}(n,\delta)}, \tag{3.18}$$

where it is used as

$$\overline{\gamma}(n,\delta) = \frac{1}{n}\sum_{i=0}^{n}\gamma_i. \tag{3.19}$$

The (3.19) gives the correspondent average reflection angle averaged over the orbit, while the M in (3.18) corresponds to an ensemble of different initial conditions. Therefore, (3.18) was iterated using an ensemble of $M = 5\times10^3$ different initial conditions. The variable γ_0 was kept fixed as $\gamma_0 = 10^{-2}\delta$ while the 5000 values of X_0 were uniformly distributed along $X_0 \in [0, 2\pi)$. It is shown in Figure 8(a) the behavior of three different values of ω for different control parameters. It is easy to see that ω grows for short iterations and then suddenly it bends towards a regime of saturation for large enough iterations. The changeover from growth to the saturation regime is marked by a characteristic crossover iteration number denoted as n_x. It is worth stressing that different values for the control parameters generate different curves for ω for short iteration n. This might be an indicative that n is not a "good" scaling variable. Thus, a trick transformation $n \rightarrow n\delta^2$ coalesces all the curves for short iteration number, as can be seen in Figure 8(b). Based on the behavior of Figure 8, the following three scaling hypotheses are proposed.

(i) For a short iteration number, say $n \ll n_x$, ω behaves according to the power law

$$\omega\left(n\delta^2, \delta\right) \propto \left(n\delta^2\right)^{\beta}, \tag{3.20}$$

where β is a critical exponent.

(ii) For large enough iteration number, say $n \gg n_x$, ω approaches a regime of saturation marked by a constant "plateau" given by

$$\omega_{\text{sat}} \propto \delta^{\alpha}, \tag{3.21}$$

where the exponent α is a critical exponent;

(iii) The number of iterations that characterizes the crossover, that is, the iterations that marks the change from growth to the saturation is written as

$$n_x \propto \delta^{z}. \tag{3.22}$$

The exponent z is defined as the *dynamical* exponent.

This scaling formalism is commonly used in the description of phase transitions in critical phenomena. It is also very useful in studies of problems of surface sciences (see, e.g., [47]). Considering these three initial suppositions, ω can now be formally described in terms of a scaling function of the type

$$\omega\left(n\delta^2, \delta\right) = l\omega\left(l^a n\delta^2, l^b\delta\right), \tag{3.23}$$

where l is a scaling factor, a and b are the so-called scaling exponents. Moreover, the exponents a and b must be necessarily related to the critical exponents α, β and z. Since l is a scaling factor, it could be chosen that $l = (n\delta^2)^{(-1/a)}$ and then (3.23) is rewritten as

$$\omega\left(n\delta^2\right) = \left(n\delta^2\right)^{(-1/a)}\omega_1\left(\left[n\delta^2\right]^{(-b/a)}\delta\right), \tag{3.24}$$

where the function $\omega_1 = \omega(1, [n\delta^2]^{-b/a}\delta)$ is assumed to be constant on the limit of $n \ll n_x$. A comparison of (3.24) and (3.20) allows us to conclude that $\beta = -1/a$. Choosing now that $l = \delta^{-1/b}$, (3.23) is given by

$$\omega\left(n\delta^2, \delta\right) = \delta^{-1/b}\omega_2\left(\delta^{-(a/b)}n\delta^2\right), \tag{3.25}$$

where the function $\omega_2 = \omega(\delta^{-(a/b)}n\delta^2, 1)$ is assumed to be constant for $n \gg n_x$. Comparing now (3.25) and (3.21), it is easy to see that $\alpha = -1/b$. Given the two different expressions of the scaling factor l, we can obtain the relation between the critical exponents as

$$z = \frac{\alpha}{\beta} - 2. \tag{3.26}$$

Figure 9: Plot of (a) ω_{sat} and (b) crossover iteration number n_x as function of the control parameter δ.

Note however that the scaling exponents are all determined if the critical exponents α and β were numerically obtained. Thus, it is shown in Figure 9 the behavior of (a) $\omega_{\text{sat}} \times \delta$ and (b) $n_x \times \delta$. The point which is important to stress is that the saturation values were obtained via extrapolation since that even after almost $10^3 n_x$, the saturation value for ω has not yet been reached. A power law on Figures 9(a) and 9(b) gives that $\alpha = 0.509(1)$, $\beta = 0.4997(8)$ and $z = -0.979(4)$. Since the dynamical exponent z can also be obtained of (3.26), from evaluation of the numerical values of α and β, it was found that $z = -0.9814(3)$. Such result is indeed in good agreement with the numerical result obtained in Figure 9(b).

Given that the values of the critical exponents are now obtained, the scaling hypotheses can be verified. In this sense, it is shown in Figure 10 the collapse for three different ω curves generated from different values of the control parameters onto a single and universal plot. The control parameters used in Figure 10(a) were $\delta = 3\times10^{-3}$, $\delta = 3\times10^{-4}$ and $\delta = 3\times10^{-5}$. The collapse obtained in Figure 10(b) reinforces that the scaling suppositions are indeed correct and therefore confirms a criticality near $\delta = 0$.

4. One-Dimensional Fermi Accelerator Model

In this section we will use the formalism described in Section 3 to characterize the same transition in a rather different model, the well-known one-dimensional Fermi accelerator model (FAM). This section is a short review of [48]. The physical motivation of this problem is far away different of the corrugated waveguide, however and as we will see, the two different model belong to the same class of universality. The origin of the FAM back to earlier 1949 when Enrico Fermi [49] proposed a model in order to study the cosmic ray acceleration. After that, his original model was modified and studied using many different approaches [50–54]. A version which we address now, as it is shown in Figure 11, consists in a classical particle which is confined between two rigid walls and suffers elastic collisions with them. One of the walls is fixed at $x = l$ and the other one is periodically time varying $x_w(t) = \epsilon' \cos(wt)$. The dynamics is described in terms of a two-dimensional nonlinear mapping for the variables v_n and t_n, where the index n corresponds to the nth collision with the moving wall. Before obtain the equations of the mapping, let us argue on a simplification commonly used. We will

Figure 10: (a) Different curves of ω for three control parameters and (b) their collapse onto a single and universal plot.

suppose that both walls are fixed but that, when the particle suffers a collision with one of the walls, it exchanges momentum as if the wall was moving. This simplification carries the huge advantage of allowing us substantially to speed up the numerical simulations compared with the complete model (In the complete model, the movement of the oscillating wall is taken into account. Therefore, the instant of the collisions is obtained via a solution of a transcendental equation.) It is usefully applicable because the main dynamical properties of the system are preserved under such conditions. Incorporating this simplification in the model and using dimensionless variables $V_n = v_n/(wl)$, $\epsilon = \epsilon'/l$ and measuring the time in term of $\phi_n = wt_n$, the mapping is written as

$$T : \begin{cases} V_{n+1} = \left| V_n - 2\epsilon \sin(\phi_{n+1}) \right|, \\ \phi_{n+1} = \phi_n + \dfrac{2}{V_n} \quad \mod 2\pi. \end{cases} \tag{4.1}$$

The term $2/V_n$ specifies the length of time during which the particle travels between collisions, while $-2\epsilon \sin(\phi_{n+1})$ gives the corresponding fraction of velocity gained or lost in the collision. The modulus function is introduced to avoid the particle leaving the region between the walls. We stress that the approximation of using the simplified FAM is valid in the limit of small ϵ. In such a limit, the same transition observed for the periodically corrugated waveguide is observed in the FAM [48].

The phase space generated for iteration of the mapping (4.1) is shown in Figure 12. It is easy to see that there is a large chaotic sea surrounding KAM islands. The chaotic sea is also limited by an invariant spanning curve. This model exhibits also scaling properties near the phase transition from integrable to nonintegrable like those of the periodically corrugate waveguide. However, the observable ω is now obtained as

$$\omega(n,\epsilon) \equiv \frac{1}{M} \sum_{j=1}^{M} \left[\sqrt{\overline{V^2}_j(n,\epsilon) - \overline{V}_j^2(n,\epsilon)} \right], \tag{4.2}$$

Figure 11: Illustrative picture of the one-dimensional Fermi accelerator model.

Figure 12: Phase space for the mapping (4.1) for the control parameter $\epsilon = 10^{-2}$.

where the average velocity is defined as

$$\overline{V}(n,\epsilon) = \frac{1}{n}\sum_{i=1}^{n} V_i, \tag{4.3}$$

and M is an ensemble of different initial conditions.

It is shown in Figure 13(a), the behavior of ω for two different control parameters, while Figure 13(b) shows the curves of ω after a suitable transformation $n \to n\epsilon^2$.

Figure 13: (a) Behavior of ω as a function of the iteration number n. (b) Behavior of ω as a function of $n\epsilon^2$. Both curves were derived from an ensemble average of 5×10^4 different initial conditions starting with $V_0 \approx 0$.

Figure 14: (a) Plot of ω_{sat} against the control parameter ϵ. (b) The crossover iteration number n_x as a function of ϵ.

The exponent α is obtained in the asymptotic limit of large iteration number and it is independent of V_0. Figure 14(a) illustrates an attempt to characterize this exponent using the extrapolated saturation of ω. Extrapolation is also required because, even after $10^3 n_x$ iterations, ω has still not quite reached saturation. From a power law fit, we obtain $\alpha = 0.512(3) \approx 1/2$. After averaging over different values of the control parameter ϵ in the range $\epsilon \in [10^{-4}, 10^{-1}]$, we then obtain $\beta = 0.496(6) \approx 1/2$. It is found that the scaling relation for the exponent z is $z = \alpha/\beta - 2$. Considering the previous values of both α and β, it is found $z = -1$. The exponent z can be also obtained numerically. Figure 14(b) shows the behavior of the crossover iteration number n_x as function of the control parameter ϵ. The power law fit gives us that $z = -1.01(2)$, in good accord with the scaling result. The scaling for $V_0 \approx 0$

Figure 15: (a) Evolution of ω for different values of the control parameter ϵ. (b) Collapse of the curves from (a) onto a universal curve. Both (a) and (b) were obtained using $V_0 \approx 0$.

is demonstrated in Figure 15, where the three different curves for ω in Figure 15(a) are very well collapsed onto the universal curve seen in Figure 15(b).

Since the critical exponents are the same of those obtained for the corrugated waveguide, we can conclude that the one-dimensional Fermi accelerator model belongs to the same class of universality of the corrugated waveguide and experiences the same transition from integrability to nonintegrability when the control parameter ϵ goes from $\epsilon = 0$ to $\epsilon \neq 0$. However the critical exponents obtained for a variant of a standard nontwist map are different from those obtained in Sections 3 and 4. Despite the similarities of the model presented in Section 2 and those models discussed in Sections 3 and 4, it belongs to a different class of universality from the corrugate waveguide and Fermi accelerator model since the critical exponents are rather different.

5. Conclusion

In summary, we have studied a phase transition from integrability to nonintegrability in three different two-dimensional mappings. Critical exponents for the behavior of the chaotic time series were obtained and used to define classes of universality. Even though the mappings are similar in their forms, the models discussed in Sections 3 and 4 belong to the same class of universality but then, they do not belong to the same class of universality of a variant of a standard nontwist map since the critical exponents are rather different. Indeed the term that seem to define the universality class is the expression of $K(I_{n+1})$ (see (2.2)) and the exponent of the variable I_{n+1} in that equation.

Acknowledgments

Financial support from CNPq, FAPESP and FUNDUNESP, Brazilian agencies are gratefully acknowledged. The present paper contains totally new results, those shown in Section 2 and also bring a short review of the articles [24], [25], and [48].

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Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 213857, 13 pages
doi:10.1155/2009/213857

Research Article

Scaling Properties of a Hybrid Fermi-Ulam-Bouncer Model

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Received 28 January 2008; Revised 25 July 2008; Accepted 29 September 2008

Recommended by Francesco Pellicano

Some dynamical properties for a one-dimensional hybrid Fermi-Ulam-bouncer model are studied under the framework of scaling description. The model is described by using a two-dimensional nonlinear area preserving mapping. Our results show that the chaotic regime below the lowest energy invariant spanning curve is scaling invariant and the obtained critical exponents are used to find a universal plot for the second momenta of the average velocity.

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1. Introduction

The investigation of nonlinear dynamical systems has awaken special interest along last decades since they can explain and/or predict some phenomena until then incomprehensible. A special class of systems that present nonlinear phenomena and that can be described via recursive equations is the so-called classical nonlinear billiard problems [1, 2]. Generally, a billiard problem consists of a system in which a point-like particle moves freely inside a bounded region and suffers specular reflections with the boundaries. This system has been extensively studied during several years considering both its classical and quantized formulation.

It is well known in the literature that the phase space of billiard problems highly depends on the shape of the boundary. The dynamics of the particle might generate phase spaces of different kinds that can be settled in three different classes of universality including (i) integrable, (ii) ergodic, and (iii) mixed. A typical example of case (i) is the circle billiard, as the integrability of such a case resembles the angular momentum conservation. Two examples

of case (ii) are the Bunimovich stadium [3] and the Sinai billiard [4]. In case (iii), the most important property present in the phase space is that chaotic seas, generally surrounding Kolmogorov-Arnold-Moser (KAM) islands, are confined by invariant spanning curves [2] sometimes also called as primary KAM tori. Particularly, such curves cross the phase plane from one side to the other one thus separating different portions of the phase space. The mixed structure in the phase portrait is such that regions of regular motion and regions with chaotic behavior could coexist together. This property is generic for nondegenerate Hamiltonian systems [5]. Some examples of systems with this particular structure are the one-dimensional Fermi-Ulam model [6–12], the bouncer model [13–19], time-dependent potentials well [20, 21], and many other different problems considering different degrees of freedom.

In this paper, we will consider a 1D model that is described using the formalism of discrete mappings, the so-called hybrid Fermi-Ulam-bouncer model [22, 23]. We are seeking to understand and describe a scaling property present in the chaotic sea of such a model. The model was originally proposed to merge together two different nonlinear problems commonly studied apart from each other, that is, the Fermi-Ulam model and the bouncer model. The Fermi-Ulam model (FUM) consists of a classical particle, in the total absence of any external field, which is confined to bounce between two rigid walls, where one of them is fixed and the other one is periodically time varying. The returning mechanism of the particle for a next collision with the moving wall is due to a reflection with the fixed wall. On the other hand, the bouncer model consists of a classical particle falling in a constant gravitational field and that hits a periodically oscillating wall. The returning mechanism of the bouncer model is related to the gravitational field only. Despite the very similar models, the different returning mechanisms of the two models cause profound consequences on the dynamics of the particle [24]. Particularly, depending on the combination of the control parameters and initial conditions, the average velocity of the particle reaches a constant value for sufficient long time in the FUM, while it diverges on the bouncer model. Such a divergency is basically related to the phenomenon of Fermi acceleration. Therefore, Fermi acceleration (FA) is a phenomenon in which a classical particle acquires unbounded energy from collisions with a massive moving wall [25, 26]. Applications of FA have acquired a broad interest in different fields of physics including plasma physics [27], astrophysics [28, 29], atomic physics [30], optics [31, 32], and even in the well-known time-dependent billiard problems [33, 34]. The interesting posed question that should be answered is whether the FA results from the nonlinear dynamics itself considering the complete absence of any imposed random motion. An one-dimensional model exhibiting such a phenomenon, which is modelled by a nonlinear mapping, is the bouncer model. For two-dimensional time-dependent billiards (billiards with moving boundaries), the answer for this question is not unique and it depends on the kind of phase space for the corresponding static version of the problem. Therefore, as conjectured by Loskutov et al. [35] (such a conjecture is known in the literature as the LRA conjecture), the regular dynamics for a fixed boundary implies a bound to the energy gained by the bouncing particle, but the chaotic dynamics of a billiard with a fixed boundary is a sufficient condition for FA in the system when a boundary perturbation is introduced. Such a conjecture was confirmed in different billiards [36–38]. It was, therefore, observed recently FA in a time varying elliptical billiard [39]. The oval billiard, however, seems not to exhibit FA for the breathing case [40].

Thus, the hybrid Fermi-Ulam-bouncer model consists of a classical particle which is confined in and bouncing elastically between two rigid walls in the presence of a constant gravitational field. Thus, properties that are individually observed in the FUM and bouncer

model come together and coalesce in the hybrid Fermi-Ulam-bouncer model [22]. As we will show, the phase space of this model is of mixed kind. Our main goal is to characterize some properties on the chaotic regime for the region below the first invariant spanning curve, particularly the behavior of the average velocity and the deviation of the average velocity.

This paper is organized as follows. In Section 2, we give all the details needed for the construction of the nonlinear mapping. We illustrate some results for the phase space of the model considering both the simplified and the complete versions of the Fermi-Ulam-bouncer model. The scaling hypotheses are present and numerical results are discussed in Section 3. Finally, in Section 4, we draw our conclusions.

2. The Model and the Mapping

We discuss in this section all the details needed for the mapping construction. We also present the phase space and obtain the positive Lyapunov exponent for the low-energy chaotic sea. The one-dimensional hybrid Fermi-Ulam-bouncer model thus consists of a classical particle confined to bounce elastically between two rigid walls. One of the walls is assumed to be fixed at the position $y = l$, while the other one moves periodically in time according to the equation $y_w(t) = \varepsilon\cos(wt)$. Here, ε and w denote, respectively, the amplitude of oscillation and the frequency of the moving wall. Additionally, the particle is suffering the action of a constant gravitational field g'. As it is so usual in the literature, the dynamics of the particle is described in terms of a two-dimensional nonlinear area preserving map T that gives the velocity of the particle v_n and the corresponding time t_n at the instant of the nth impact with the moving wall, that is, $T(v_n, t_n) = (v_{n+1}, t_{n+1})$. Before we write the equations of the mapping, it is convenient to define dimensionless and much more appropriated variables to describe the dynamics of the particle. We define $\epsilon = \varepsilon/l$, $g = g'/(w^2 l)$, and $V_n = v_n/(wl)$. Finally, we measure the time in terms of $\phi_n = wt_n$. Incorporating this new set of variables in the dynamics, the map is written as

$$T : \begin{cases} V_{n+1} = V_n^* + g\phi_c - 2\epsilon\sin\left(\phi_{n+1}\right), \\ \phi_{n+1} = \left(\phi_n + \Delta T_n\right)\mathrm{mod}(2\pi), \end{cases} \tag{2.1}$$

where the expressions for both V_n^* and ΔT_n depend on the kind of collision occurs. There are three different possible situations, namely (i) multiple collisions, (ii) collisions without reflection in the upper wall, and (iii) collisions with reflection in the upper wall. Considering the first case, where the particle suffers an impact before leaving the collision zone, which is given by the interval $y \in [-\epsilon, \epsilon]$, the expressions are $V_n^* = -V_n$ and $\Delta T_n = \phi_c$, where ϕ_c is obtained as the smallest solution of $G(\phi_c) = 0$ with $\phi_c \in (0, 2\pi]$. Such solution is equivalent to the position of the particle being the same as the position of the moving wall. Thus, the transcendental equation $G(\phi_c)$ is given by

$$G(\phi_c) = \epsilon\cos\left(\phi_n + \phi_c\right) - \epsilon\cos\left(\phi_n\right) - V_n\phi_c + \frac{g\phi_c^2}{2}. \tag{2.2}$$

If $G(\phi_c)$ does not have a solution in the interval $\phi_c \in (0, 2\pi]$, the particle leaves the collision zone without suffering a successive hit. Considering the case (ii), the needed condition for the particle not colliding with the upper wall after leaving the collision zone is

$V_n \leq \sqrt{2g[1-\epsilon\cos(\phi_n)]}$. In this case, $V_n^* = \sqrt{V_n^2 + 2g\epsilon[\cos(\phi_n)-1]}$ and $\Delta T_n = \phi_u + \phi_d + \phi_c$, where the auxiliary terms assume the following expressions:

$$\phi_u = \frac{V_n}{g},$$
$$\phi_d = \frac{\sqrt{V_n^2 + 2g\epsilon[\cos(\phi_n)-1]}}{g}. \tag{2.3}$$

The term ϕ_c is obtained by solution of $F(\phi_c) = 0$. Again, considering the condition that matches the same position of the moving wall and of the particle. Thus, $F(\phi_c)$ is given by

$$F(\phi_c) = \epsilon\cos(\phi_n + \phi_u + \phi_d + \phi_c) - \epsilon - V_n^*\phi_c + \frac{g\phi_c^2}{2}. \tag{2.4}$$

Finally, for the case (iii), where $V_n > \sqrt{2g[1-\epsilon\cos(\phi_n)]}$, the corresponding expressions for V_n^* and ΔT_n are the same as those of the case (ii). However, the auxiliary terms, ϕ_u and ϕ_d, assume the following expressions:

$$\phi_u = \frac{V_n - \sqrt{V_n^2 - 2g[1-\epsilon\cos(\phi_n)]}}{g},$$
$$\phi_d = \frac{\sqrt{V_n^2 + 2g\epsilon[\epsilon\cos(\phi_n)-1]} - \sqrt{V_n^2 - 2g[1-\epsilon\cos(\phi_n)]}}{g}. \tag{2.5}$$

The value of ϕ_c is obtained from (2.4) using, however, the new expressions for ϕ_u and ϕ_d.

Figure 1(a) shows the phase space for the complete hybrid Fermi-Ulam-bouncer model generated from (2.1). The control parameters used in Figure 1 are $\epsilon = 10^{-3}$ and $g = 5.628 \times 10^{-5}$. It is easy to see the presence of KAM islands surrounded by a chaotic sea, that is, limited by an invariant spanning curve. The presence of the invariant spanning curves in the phase space implies that the particle cannot acquire unlimited energy. To illustrate the behavior of the average energy, we define

$$E_j = \frac{1}{n+1}\sum_{i=0}^{n}\overline{V^2}_{i,j}, \tag{2.6}$$

and finally obtain the average energy as

$$\overline{E}(n,\epsilon) = \frac{1}{M}\sum_{j=1}^{M}E_j, \tag{2.7}$$

where M corresponds to an ensemble of different initial conditions.

Figure 1: (a) Phase space for the complete hybrid Fermi-Ulam-bouncer model. (b) Behavior of the average energy, $\overline{E}$, as function of n. The control parameters used in (a) and (b) were $\epsilon = 10^{-3}$ and $g = 5.628 \times 10^{-5}$.

Figure 1(b) shows the behavior of the average energy, $\overline{E}$, as a function of n. We can see that the energy grows for small n and then, after a changeover, it reaches a regime of saturation for large n. The initial velocity was considered fixed as $V_0 = 2.1\epsilon$, while we have considered an ensemble of $M = 5000$ different initial phases ϕ_0 uniformly distributed in the interval $[0, 2\pi)$. Each initial condition was evolved up to 10^7 iterations. Since the instant of each impact of the particle with the moving wall can only be obtained numerically, the computational time for evaluation of (2.7) is very large. As an attempt to reduce the time consuming in the numerical simulations, we will discuss in Section 2.1 a simplified version of the hybrid Fermi-Ulam-bouncer model.

2.1. A Simplified Hybrid Fermi-Ulam-Bouncer Model

In this section, we discuss a simplification used in the model. For the complete model, the instant of the impact of the particle with the moving wall is obtained via a solution of a transcendental equation, which yields the simulations to be long-time consuming. However, instead of considering solving transcendental equations, we will use a simplification in the model which is commonly used in the literature [2]. We will suppose that both walls are fixed (one of them is fixed at $y = 0$ and the other one is fixed at $y = l$), but that, when the particle suffers a collision with one of then (say, with the one at $y = 0$), the particle exchanges momentum and energy as if the wall was moving. This simplification carries a great advantage of allowing us to speed up our numerical simulations when compared to the complete version of the model. Such a simplification also retains the nonlinearity of the problem. Considering the same dimensionless variables defined for the complete model, the map for the simplified hybrid Fermi-Ulam-bouncer model can be written as

$$T : \begin{cases} V_{n+1} = \left| V_n - 2\epsilon \sin\left(\phi_{n+1}\right) \right|, \\ \phi_{n+1} = \left(\phi_n + \Delta T_n\right) \mathrm{mod}(2\pi). \end{cases} \tag{2.8}$$

The modulus function used in the equation of the velocity on the mapping (2.8) was introduced to preserve the particle into the region between the walls. The expressions for ΔT_n depend on the following conditions:

(1) collision without reflection in the upper wall. Such a condition is verified if $V_n \leq \sqrt{2g(1-\epsilon\cos(\phi_n))}$. In this case, the particle rises and decelerates due to the gravitational field only until reaches instantaneously the rest. Then the particle is accelerated downward until it collides with the wall at $y = 0$. Thus, ΔT_n is given by

$$\Delta T_n = \frac{2V_n}{g}. \tag{2.9}$$

(2) Collision with reflection in the upper wall. For such a collision, the condition that must be observed is $V_n > \sqrt{2g(1-\epsilon\cos(\phi_n))}$. In this case, the particle goes in the upward direction and hits the upper wall. It is then reflected downward and is also accelerated by the gravitational field. The expression for ΔT_n is written as

$$\Delta T_n = \frac{V_n - \sqrt{V_n^2 - 2g}}{g}. \tag{2.10}$$

The mapping (2.8) preserves the phase space measure since that $\det(J) = \pm 1$.

The phase space for the mapping (2.8) is shown in Figure 2(a) for the same control parameters used in Figure 1. We have also evaluated numerically the positive Lyapunov exponent for the chaotic sea of Figure 2(a). It is well known that the Lyapunov exponent has great applicability as a practical tool that can quantify the average expansion or contraction rate for a small volume of initial conditions. As discussed in [41], the Lyapunov exponents are defined as

$$\lambda_j = \lim_{n\to\infty} \frac{1}{n} \ln\left|\Lambda_j\right|, \quad j = 1, 2, \tag{2.11}$$

where Λ_j are the eigenvalues of $M = \prod_{i=1}^{n} J_i(V_i, \phi_i)$ and J_i is the Jacobian matrix evaluated over the orbit (V_i, ϕ_i). However, a direct implementation of a computational algorithm to evaluate (2.11) has a severe limitation to obtain M. Even in the limit of short n, the components of M can assume very different orders of magnitude for chaotic orbits and periodic attractors, yielding impracticable the implementation of the algorithm. In order to avoid such problem, we note that J can be written as $J = \Theta T$, where Θ is an orthogonal matrix and T is a right triangular matrix. Thus, we rewrite M as $M = J_n J_{n-1}, \ldots, J_2 \Theta_1 \Theta_1^{-1} J_1$, where $T_1 = \Theta_1^{-1} J_1$. A product of $J_2\Theta_1$ defines a new J_2'. In a next step, it is easy to show that $M = J_n J_{n-1}, \ldots, J_3 \Theta_2 \Theta_2^{-1} J_2' T_1$. The same procedure can be used to obtain $T_2 = \Theta_2^{-1} J_2'$, and so on. Using this procedure, the problem is reduced to evaluate the diagonal elements of T_i: T_{11}^i, T_{22}^i. Finally, the Lyapunov exponents are now given by

$$\lambda_j = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln\left|T_{jj}^i\right|, \quad j = 1, 2. \tag{2.12}$$

Figure 2: (a) Phase space generated from iteration of mapping (2.8) for the control parameters $\epsilon = 10^{-3}$ and $g = 5.628 \times 10^{-5}$. (b) Behavior of the positive Lyapunov exponent of the chaotic sea for the same control parameters used in (a). The average value obtained was $\overline{\lambda} = 1.55 \pm 0.01$, where the error 0.01 corresponds to the standard deviation of the 10 samples. (c) Behavior of the average energy, $\overline{E}$, as function of n.

If at least one of the λ_j is positive, then the orbit is classified as chaotic. Figure 2(b) shows the behavior of the positive Lyapunov exponent plotted against the number of collisions n with the wall located at $y = 0$ for 10 different initial conditions on the chaotic sea. The control parameters used in the construction of Figure 2 were $\epsilon = 10^{-3}$ and $g = 5.628 \times 10^{-5}$. The average of the positive Lyapunov exponent for the ensemble of the 10-time series gives $\overline{\lambda} = 1.55 \pm 0.01$, where the value 0.01 corresponds to the standard deviation of the ten samples. Finally, in Figure 2(c), we present the behavior of the average energy of the particle for the simplified model.

3. Scaling Analysis

The main goal of this section is to describe a scaling present in the low-energy regime. We discuss in full detail the investigation for the simplified and then, at the end of section, we present the corresponding results for the complete version. Thus, we now discuss the procedures used to obtain the average velocity on the chaotic low-energy region. The average velocity (average along an orbit) is defined as

$$\overline{V}(n, \epsilon) = \frac{1}{n+1} \sum_{i=0}^{n} V_i. \tag{3.1}$$

Since we obtain the average velocity, it is also easy to obtain the deviation of the average velocity, sometimes also called as the second momenta of $\overline{V}$. It is defined as

$$\omega(n,\epsilon) = \frac{1}{M}\sum_{i=1}^{M}\sqrt{\overline{V_i^2}(n,\epsilon) - \overline{V_i}^2(n,\epsilon)}, \tag{3.2}$$

where the sum on M refers to an average over an ensemble of M different initial conditions. In order to iterate (3.2) for the simplified version of the model, we evolved our simulations considering an ensemble of $M = 5 \times 10^3$ different initial conditions. Figure 3 shows the behavior of the deviation around the average velocity for the simplified model as function of n for three different control parameters, as labelled in the figures. It is easy to see in Figure 3 two different kinds of behaviors. For short n, the deviation of the average velocity grows according to a power law and suddenly it bends toward a regime of saturation for long enough values of n. The changeover from growth to the saturation is marked by a cross-over iteration number n_x. It must be emphasized that different values of the parameter ϵ generate different behaviors for short n. However, applying the transformation $n \to n\epsilon^2$ coalesces all the curves at short n, as shown in Figure 3(b). We can see that different values of ϵ yield each curve to saturate at distinct values, thus we can suppose that

(1) when $n \ll n_x$, the deviation of the average velocity grows according to

$$\omega(n\epsilon^2,\epsilon) \propto (n\epsilon^2)^{\beta}, \tag{3.3}$$

where the exponent β is a critical exponent;

(2) as the iteration number increases, $n \gg n_x$, the deviation of the average velocity approaches a regime of saturation, that is described as

$$\omega_{\text{sat}}(\epsilon) \propto \epsilon^{\alpha}, \tag{3.4}$$

where the exponent α is also a critical exponent;

(3) the cross-over iteration number that marks the change from growth to the saturation is written as

$$n_x \propto \epsilon^{z}, \tag{3.5}$$

where z is a dynamic exponent.

After considering these three initial suppositions, we are now able to describe the deviation of the average velocity in terms of a scaling function of the type

$$\omega(n\epsilon^2,\epsilon) = l\omega(l^a n\epsilon^2, l^b\epsilon), \tag{3.6}$$

where l is the scaling factor, a and b are scaling exponents. If we chose properly the scaling factor l, we can relate the scaling exponents a and b with the critical exponents α, β, and z.

Figure 3: (a) Behavior of the deviation of the average velocity for different values of the control parameter ϵ for the simplified model. (b) Their initial collapse after the transformation $n\epsilon^2$ for the simplified model. The control parameter g was fixed as $g = 5.628 \times 10^{-5}$.

Figure 4: (a) Plot of ω_{sat} as function of the control parameter ϵ for the simplified model. (b) Behavior of the cross-over number n_x against ϵ for the simplified model.

We begin considering that $l = (n\epsilon^2)^{-1/a}$. Thus, (3.6) is rewritten as

$$\omega(n\epsilon^2, \epsilon) = (n\epsilon^2)^{-1/a}\omega_1((n\epsilon^2)^{-b/a}\epsilon), \tag{3.7}$$

where the function $\omega_1((n\epsilon^2)^{-b/a}\epsilon) = \omega(1, (n\epsilon^2)^{-b/a}\epsilon)$ is assumed to be constant for $n \ll n_x$. Comparing (3.3) and (3.7), we obtain $\beta = -1/a$.

Choosing now $l^b\epsilon = 1$, we have $l = \epsilon^{-1/b}$ and (3.6) is given by

$$\omega(n\epsilon^2, \epsilon) = \epsilon^{-1/b}\omega(\epsilon^{-a/b}n\epsilon^2), \tag{3.8}$$

Figure 5: (a) Different curves of the ω for four different control parameters for the simplified model. (b) Their collapse onto a single and universal plots.

where the function ω_2 is defined as $\omega_2(\epsilon^{-a/b}n\epsilon^2) = \omega(\epsilon^{-a/b}n\epsilon^2, 1)$. It is also assumed as constant for $n \gg n_x$. An immediate comparison of (3.4) and (3.6) gives us $\alpha = -1/b$. Given the two different expressions of the scaling factor l, we obtain a relation for the dynamic exponent z, that is, given by

$$z = \frac{\alpha}{\beta} - 2. \tag{3.9}$$

Note that the scaling exponents are determined if the critical exponents β and α were numerically obtained. The exponent β is obtained from a power law fitting for the deviation of the average velocity curves for the parameter $\epsilon \in [10^{-4}, 10^{-2}]$ for short iteration number. Thus, the average of these values gives that $\beta = 0.502(4) \cong 1/2$. Figure 4 shows the behaviors of (a) $\omega_{\text{sat}} \times \epsilon$ and (b) $n_x \times \epsilon$. Applying power law fittings on the figure, we obtain $\alpha = 0.5003(6) \cong 0.5$ and $z = -0.96(1) \cong -1$. We can also obtain the exponent z considering (3.9) and the previous values of both α and β; we found that $z = -1.003(6) \cong -1$. Such result indeed agrees with our numerical result. In order to confirm the initial hypotheses, and since the values of the scaling exponents α, β, and z are now known, we will collapse all the curves onto a single and universal plots, as demonstrated in Figure 5. With this good collapse of all the curves of the deviation of the average velocity and considering that the critical exponents are $\alpha \cong 0.5$, $\beta \cong 0.5$ and $z \cong -1$, we can conclude that the hybrid Fermi-Ulam-bouncer model belongs to the same class of universality of a periodically corrugated waveguide [42]. It also belongs to the same class of universality of the Fermi-Ulam [43, 44] itself and that the presence of a gravitational field does not seem to create a new universality class, at least for the control parameters considered in the present study.

Let us now discuss our numerical results for the complete version of the model. Once the equations of the mapping now be solved numerically, we have considered an ensemble of less different initial conditions. Such a consideration is mainly to produce a simulation not so longer. However, it is still relevant to characterize statistical properties of the model. For the complete version of the model, we have considered an ensemble of $M = 10^3$ different initial conditions. The behavior of ω for the complete version is rather similar to that observed for

Figure 6: (a) Different curves of the ω for three different control parameters for the complete model. (b) Their collapse onto a single and universal plots.

the simplified version of the model. After an extensive simulation, we obtain that the critical exponents are $\beta = 0.50(2) \cong 0.5$, $\alpha = 0.501(3)$, and $z = -0.95(3)$. Evaluating (3.9), we found that $z = -1.00(3)$. These critical exponents also allow us to produce a good collapse of all the curves of ω onto a single and universal plots, as shown in Figure 6. Thus, we can conclude that the scaling properties are unaffected by considering the simplified or complete versions of the model.

4. Final Remarks

As a final remark of the present paper, we have studied a simplified and the complete version of the hybrid Fermi-Ulam-bouncer model considering elastic collisions with the walls. We show that the average energy as well as the deviation around the average velocity for chaotic orbits for both the complete and simplified versions of the model exhibit scaling properties with the same critical exponents. Moreover, we have shown that there is an analytical relation between the critical exponents α, β, and z. Our scaling hypotheses are confirmed by a good collapse of all the curves of ω onto a single and universal plot, therefore, confirming that this model also belongs to the same class of universality of the Fermi-Ulam model [43], for the range of control parameters studied, and the periodically corrugated waveguide [42].

Acknowledgment

D. F. M. Oliveira and R. A. Bizão are grateful to CNPq; E. D. Leonel thanks FAPESP, FUNDUNESP, and CNPq for financial support.

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Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 298906, 40 pages
doi:10.1155/2009/298906

Research Article

Motion Switching and Chaos of a Particle in a Generalized Fermi-Acceleration Oscillator

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Received 23 September 2008; Revised 21 December 2008; Accepted 26 January 2009

Recommended by Edson Denis Leonel

Dynamic behaviors of a particle (or a bouncing ball) in a generalized Fermi-acceleration oscillator are investigated. The motion switching of a particle in the Fermi-oscillator causes the complexity and unpredictability of motion. Thus, the mechanism of motion switching of a particle in such a generalized Fermi-oscillator is studied through the theory of discontinuous dynamical systems, and the corresponding analytical conditions for the motion switching are developed. From solutions of linear systems in subdomains, four generic mappings are introduced, and mapping structures for periodic motions can be constructed. Thus, periodic motions in the Fermi-acceleration oscillator are predicted analytically, and the corresponding local stability and bifurcations are also discussed. From the analytical prediction, parameter maps of periodic and chaotic motions are achieved for a global view of motion behaviors in the Fermi-acceleration oscillator. Numerical simulations are carried out for illustrations of periodic and chaotic motions in such an oscillator. In existing results, motion switching in the Fermi-acceleration oscillator is not considered. The motion switching for many motion states of the Fermi-acceleration oscillator is presented for the first time. This methodology will provide a useful way to determine dynamical behaviors in the Fermi-acceleration oscillator.

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1. Introduction

The Fermi-acceleration oscillator was first presented [1] by Fermi in 1949, which was used to explain the very high energy of cosmic ray. Since then, such an oscillator has been extensively investigated to interpret many physical and mechanical phenomena. In 1961, Ulam [2] pointed out the statistical properties of a particle in the Fermi-oscillator. In 1964, Zaslavskii and Chirikov [3] gave a comprehensive study of the Fermi-acceleration mechanism in the one-dimensional case. In 1972, Lieberman and Lichtenberg [4] discussed the stochastic and adiabatic behaviors of particles accelerated by periodic forcing, but such analysis was

based on the model presented by Zaslavskii and and Chirikov [3]. The corresponding stability of periodic motion was discussed. Such results can be found from Lichtenberg and Lieberman [5]. In 1978, Pustyl'nikov [6] discussed the reducibility of non-autonomous systems to normal forms in the neighborhood of an equilibrium point, and in 1995 he gave a detailed description about the Fermi-acceleration mechanism [7]. In 1986, Jose and Cordery [8] studied a quantum Fermi-accelerator consisting of a particle moving between a fixed wall and a periodic oscillator. In 1987, Celaschi and Zimmerman [9] gave an experimental investigation to observe the period-doubling route to chaos for a 1D system with two parameters. In 1988, Kowalik et al. [10] completed an experimental investigation on chaotic behaviors of a ball in the bouncing ball system, and used the Zaslavski's mapping to analyze such chaotic behaviors. In 1989, Luna-Acosta [11] investigated the dynamics of the Fermi-accelerator subject to a viscous friction. In 1996, Warr et al. [12] determined experimentally a probability distribution function for a single-particle vibrating in one dimension. In 1998, Saif et al. [13] studied the dynamics of both classical and quantum Fermi-acceleration oscillators and determined the dynamical localization of position and momentum for a modulation amplitude. Lopac and Dananić [14] investigated chaotic dynamics and energy conservation in a gravitationally driven Fermi-accelerator. In 2004, Bouchet et al. [15] presented a simple stochastic system to generate anomalous diffusion of both position and velocity for the Fermi-accelerator. Leonel et al. [16] completed a scaling analysis of chaotic motion in the Fermi-Ulam-simplified accelerator, and Leonel et al. [17] investigated the effect of a time-dependent perturbation on a Fermi-accelerator model using discrete dynamical systems. In 2006, Leonel and McClintock [18] discussed the influence of dissipation on a simplified Fermi-Ulam accelerator model. In 2007, Leonel and de Carvalho [19] presented the Fermi-accelerator model with inelastic collisions through a two-dimensional nonlinear area-contracting map. In 2007, Karlis et al. [20] investigated the Fermi-acceleration of an ensemble of noninteracted particles in two stochastic two-moving walls in the Fermi-Ulam model. Leonel and Silva [21] discussed the dynamical properties of a prototype for the Fermi-acceleration modeled by two nonlinear terms. Kamphorst et al. [22] numerically studied the energy change of a particle bouncing in a time-varying billiard. In 2008, Leonel and Livorati [23] presented the average velocity behavior on a dissipative Fermi-acceleration model with a scaling approach. The scaling characteristics of a breathing circular billiard were investigated by Ladeira and da Silva [24]. All the aforesaid investigations were based on the one motion state with impact. The motion switching was not considered. In this paper, the motion switching from a motion state into another one will be investigated.

The similar studies of impacting systems have also been carried out in mechanical engineering because the impact is an important phenomenon in mechanical engineering. For instance, in 1982, Holmes [25] investigated the dynamics of repeated impacts of a ball with a sinusoidal vibrating table. It was assumed that the mass of the ball is much smaller than the mass of the table, and the ball and table always impact at the same position. In 1983, Bapat et al. [26] investigated the asymptotically stable periodic motions of a ball in an impact-pair. Shaw and Holmes [27] studied harmonic, subharmonic, and chaotic motions of a single-degree-of-freedom nonlinear oscillator and analyzed the corresponding bifurcations. In 1987, Whiston [28] studied the steady-state, vibroimpacting responses of one-dimensional, harmonically excited, linear oscillator. In 1988, C. N. Bapat and C. Bapat [29] used Fourier series and perturbation method to determine the stability regions of two equispaced impact motion of an impact-pair under a prescribed periodic displacement. In 1991, Nordmark [30] studied the singularity caused by grazing impact in a single degree of freedom impact oscillator. In 1992, Foale and Bishop [31] investigated a periodically forced linear oscillator

with instantaneous impacts at one or two stops. In 1994, Budd and Dux [32] investigated the chattering behavior of a periodically forced, single-degree-of-freedom impact oscillator with a restitution law for each impact. Foale [33] tried to determine bifurcations in a periodically driven impact oscillator analytically, and the impact wall is fixed. In 1995, Bapat [34] studied the motion of an inclined impact damper with friction and collision on both sides. In order to understand dynamical behaviors of a bouncing ball on a vibrating table, in 1996, Luo and Han [35] presented a reasonable model and analytically predicted the periodic motions. In 2002, Luo [36] investigated the stability and bifurcation of unsymmetrical periodic motions in a horizontal impact oscillator with a periodic excitation. In 2004, Giusepponi et al. [37] discussed the randomness of motion of the bouncing ball on a sinusoidally vibrating table. Luo [38] discussed the stability, saddle-node and period-doubling bifurcations for the LR model motion in a horizontal impact oscillator.

In 2005, Luo [39] presented the mapping dynamics of periodic motions in a nonsmooth piecewise system. To understand the complexity in discontinuous dynamical systems, Luo [40, 41] developed a theory of the non-smooth dynamical systems on connectable and accessible subdomains. Luo and Chen [42] applied such a theory to investigate the grazing bifurcation and periodic motions in an idealized gear-transmission system with impacts. In 2006, Luo and Gegg [43] used such a theory to develop the force criteria of stick and nonstick motion in harmonically forced, friction-induced oscillators. In 2007, Luo [44] discussed switching bifurcations of a flow to the separation boundary. Luo and Rapp [45] studied the switching dynamics of a flow from one domain into another adjacent domain in a periodically driven, discontinuous dynamical system. Luo and O'Connor [46, 47] discussed the dynamics mechanism of impact chatters and possible stick motions in a gear transmission system, and the moving boundaries are controlled by other dynamical systems. In the gear model, the two boundaries are movable. However, the Fermi-acceleration oscillator possesses both static and time-varying boundaries in phase space for impacts and motion switching. In existing investigations on the Fermi-acceleration oscillator or impact oscillators, the dynamical systems are not switched except for impacts. Thus, in this paper, the impacting and sticking of two systems on the boundaries will be considered, which will provide a practical model to predict the motions of the Fermi-acceleration oscillator.

This paper will investigate dynamics of a generalized Fermi-acceleration oscillator. The domains and boundaries for such a problem will be introduced because of the discontinuity, and the analytical conditions of stick and grazing motions will be developed at the boundaries. The mapping technique will be used for the analytical prediction of periodic motions of the extended Fermi-acceleration oscillator. The local stability and bifurcation of periodic motions will be discussed using the eigenvalue analysis. Bifurcation scenario and analytical prediction of motions will be presented, and numerical simulations of periodic and chaotic motions will be carried out. In addition, the Poincare mapping sections will be presented for illustrations of chaotic motions, and the parameter maps will be presented as well.

2. Physical Problem Statement

A generalized Fermi-acceleration oscillator consists of a particle moving vertically between a fixed wall and the moving piston in a vibrating oscillator. The piston of mass m_2 is connected with a spring of constant k and a damper with coefficient c, and they are set on a periodically oscillating base, as shown in Figure 1. The mass of particle is m_1 and the

Figure 1: Mechanical model.

restitution coefficients of impact are e_1 and e_2 for the wall and piston, respectively. The gap between the fixed wall and the equilibrium position of piston is h. The displacement of the oscillating base is $D(t) = Q\cos\Omega t$.

If the particle does not stay on the piston, the corresponding motion of such a system is called the *nonstick motion*. For this case, the equation of motion for such a system is given from the Newton's law, that is,

$$\begin{gathered}\ddot{x}^{(1)} = -g,\\ \ddot{x}^{(2)} + 2d^{(2)}\dot{x}^{(2)} + \left(\omega^{(2)}\right)^2 x^{(2)} = \left(\omega^{(2)}\right)^2 Q\cos\Omega t - 2d^{(2)}Q\Omega\sin\Omega t,\end{gathered} \tag{2.1}$$

where $d^{(2)} = c/2m_2$, and $\omega^{(2)} = \sqrt{k/m_2}$. $\ddot{x}^{(i)}$, $\dot{x}^{(i)}$ and $x^{(i)}$ $(i = 1,2)$ are acceleration, velocity, and displacement, respectively. g is the gravitational acceleration. The superscripts $i = 1$ and 2 represent the particle and piston, accordingly.

If the particle stays on the piston and they move together, such a motion is called *the stick motion*. For the stick motion, there is a relation ($x^{(1)} = x^{(2)}$ and $\dot{x}^{(1)} = \dot{x}^{(2)}$) and the corresponding equation of motion becomes

$$\ddot{x}^{(i)} + 2d^{(0)}\dot{x}^{(i)} + \left(\omega^{(0)}\right)^2 x^{(i)} = \left(\omega^{(0)}\right)^2 Q\cos\Omega t - 2d^{(0)}Q\Omega\sin\Omega t, \tag{2.2}$$

where $d^{(0)} = c/2(m_1 + m_2)$, $\omega^{(0)} = \sqrt{k/(m_1 + m_2)}$. The superscripts $i = 1,2$ represent the particle and piston, respectively.

For nonstick motion, the impact relation between the particle and the fixed wall is

$$\begin{gathered}x_+^{(1)} = x_-^{(1)} = h, \qquad \dot{x}_+^{(1)} = -e_1\dot{x}_-^{(1)},\\ x_+^{(2)} = x_-^{(2)}, \qquad \dot{x}_+^{(2)} = \dot{x}_-^{(2)},\end{gathered} \tag{2.3}$$

where (+) and (−) represent the after and before impact. The impact relations between the particle and the vibrating piston are

$$x_+^{(1)} = x_+^{(2)} = x_-^{(1)} = x_-^{(2)},$$
$$\dot{x}_+^{(1)} = \frac{m_1\dot{x}_-^{(1)} + m_2\dot{x}_-^{(2)} - m_2 e_2(\dot{x}_-^{(1)} - \dot{x}_-^{(2)})}{m_1 + m_2}, \tag{2.4}$$
$$\dot{x}_+^{(2)} = \frac{m_1\dot{x}_-^{(1)} + m_2\dot{x}_-^{(2)} + m_1 e_2(\dot{x}_-^{(1)} - \dot{x}_-^{(2)})}{m_1 + m_2}.$$

3. Analytical Conditions

In this section, the domains and boundaries of the extended Fermi-oscillator will be introduced in the absolute and relative coordinates. From such domains and boundaries, the analytical conditions of stick and grazing motions to each switching boundary will be developed from the theory of discontinuous dynamical systems.

3.1. Domains and Boundaries for Absolute Motions

To analyze the motion discontinuity in the Fermi-acceleration oscillator, the origin of the absolute coordinates is set at the equilibrium of piston. The domains for the particle and piston without stick in the absolute coordinates are defined as

$$\begin{aligned} \Omega_1^{(1)} &= \{(x^{(1)}, \dot{x}^{(1)}) \mid x^{(1)} \in (x^{(2)}, h)\}, \\ \Omega_1^{(2)} &= \{(x^{(2)}, \dot{x}^{(2)}) \mid x^{(2)} \in (-\infty, x^{(1)})\}. \end{aligned} \tag{3.1}$$

The corresponding boundaries are defined as

$$\begin{aligned} \partial\Omega_{1(+\infty)}^{(i)} &= \{(x^{(i)}, \dot{x}^{(i)}) \mid \varphi_{1(+\infty)}^{(1)} \equiv x^{(1)} - h = 0,\ \dot{x}^{(1)} \neq 0\}, \\ \partial\Omega_{1(-\infty)}^{(i)} &= \{(x^{(i)}, \dot{x}^{(i)}) \mid \varphi_{1(-\infty)}^{(i)} \equiv x^{(i)} - x^{(\bar{i})} = 0,\ \dot{x}^{(i)} \neq \dot{x}^{(\bar{i})}\}, \end{aligned} \tag{3.2}$$

where $\bar{i} = 1$ and 2 are for $i = 2$ and 1, accordingly. The subscripts "$\pm\infty$" represent the permanent boundaries at $x^{(1)} = h$ and $x^{(i)} = x^{(\bar{i})}$. If a flow in a subdomain of a dynamical system cannot pass over a switching boundary into another subdomain without any transport law, such a boundary is called the permanent boundary (e.g., Luo [48]). For this problem, the impact law, as a transport law, should be applied for such permanent boundaries. In other words, the impact relations in (2.3) and (2.4) will be applied for such permanent boundaries. The subdomains and boundaries for motions of particle and piston in phase space are sketched in Figure 2. The subdomains are filled with diagonal lines. The boundary ($\partial\Omega_{1(+\infty)}^{(1)}$) for the particle is a dashed line with $x^{(1)} = h$ and the boundaries ($\partial\Omega_{1(-\infty)}^{(1)}$ and $\partial\Omega_{1(-\infty)}^{(2)}$) with $x^{(1)} = x^{(2)}$ for the particle and piston are two dashed curves.

Figure 2: Absolute domains and boundaries without stick: (a) particle and (b) piston.

For this system, there is a stick motion of the particle and piston. Thus, the stick motion will appear and vanish under certain conditions. Such onset and disappearance of the sticking motion will generate new boundaries and domains. The domains $\Omega_0^{(i)}$ and $\Omega_1^{(i)}$ $(i = 1, 2)$ for the particle and piston with stick in the absolute coordinates are defined as

$$\begin{aligned}
\Omega_0^{(1)} &= \left\{ \left(x^{(1)}, \dot{x}^{(1)}\right) \mid x^{(1)} \in \left(-\infty, x_{cr}^{(2)}\right), \dot{x}^{(1)} = \dot{x}^{(2)} \right\}, \\
\Omega_0^{(2)} &= \left\{ \left(x^{(2)}, \dot{x}^{(2)}\right) \mid x^{(2)} \in \left(x^{(1)}, h\right), \dot{x}^{(1)} = \dot{x}^{(2)} \right\}, \\
\Omega_1^{(1)} &= \left\{ \left(x^{(1)}, \dot{x}^{(1)}\right) \mid x^{(1)} \in \left(x_{cr}^{(2)}, h\right), \dot{x}^{(1)} \neq \dot{x}^{(2)} \right\}, \\
\Omega_1^{(2)} &= \left\{ \left(x^{(2)}, \dot{x}^{(2)}\right) \mid x^{(2)} \in \left(-\infty, x_{cr}^{(1)}\right), \dot{x}^{(1)} \neq \dot{x}^{(2)} \right\}.
\end{aligned} \tag{3.3}$$

The corresponding boundaries are defined as

$$\begin{aligned}
\partial\Omega_{1(+\infty)}^{(i)} &= \left\{ \left(x^{(i)}, \dot{x}^{(i)}\right) \middle| \varphi_{1(+\infty)}^{(1)} \equiv x^{(1)} - h = 0, \dot{x}^{(1)} \neq 0 \right\}, \\
\partial\Omega_{10}^{(i)} &= \left\{ \left(x^{(i)}, \dot{x}^{(i)}\right) \middle| \varphi_{10}^{(i)} \equiv x^{(i)} - x_{cr}^{(\bar{i})} = 0, \dot{x}^{(i)} = \dot{x}_{cr}^{(\bar{i})} \right\},
\end{aligned} \tag{3.4}$$

where $x_{cr}^{(i)}$ is an instantaneous value for the appearance or vanishing of stick motion with $\dot{x}_{cr}^{(1)} = \dot{x}_{cr}^{(2)}$ and $x_{cr}^{(1)} = x_{cr}^{(2)}$. The instantaneous values are dependent on time, which cannot give specified values. The domains ($\Omega_1^{(i)}$ and $\Omega_0^{(i)}$) describe the domains for the nonstick and stick motions of the particle and piston, respectively. $\partial\Omega_{1(-\infty)}^{(1)}$ is the impact chatter boundary for the particle impacting with the piston, and $\partial\Omega_{10}^{(1)}$ and $\partial\Omega_{01}^{(1)}$ are the stick onset and vanishing boundaries for the particle sticking with the piston. Similarly, $\partial\Omega_{1(-\infty)}^{(2)}$ is the impact chatter boundary for the piston impacting with the particle, and $\partial\Omega_{10}^{(2)}$ and $\partial\Omega_{01}^{(2)}$ are the stick onset and vanishing boundaries for the piston sticking with particle. In Figure 3, the nonstick

Figure 3: Absolute domains and boundaries with stick: (a) particle and (b) piston.

and stick domains ($\Omega_1^{(i)}$ and $\Omega_0^{(i)}$) are presented through the filled and shaded regions. The boundaries ($\partial\Omega_{10}^{(i)}$ and $\partial\Omega_{01}^{(i)}$, $i = 1,2$) for the particle and piston are given by the dashed curves in the absolute frame. The nonstick, permanent boundary ($\partial\Omega_{1(+\infty)}^{(1)}$) for the particle is a dashed line, and such a boundary is also applied for the piston as a permanent boundary.

From the aforedefined domains, the vectors for absolute motions are defined as

$$\mathbf{x}_\lambda^{(i)} = \left(x_\lambda^{(i)}, \dot{x}_\lambda^{(i)}\right)^T, \quad \mathbf{f}_\lambda^{(i)} = \left(\dot{x}_\lambda^{(i)}, F_\lambda^{(i)}\right)^T \quad \text{for } i = 1,2,\ \lambda = 0,1, \tag{3.5}$$

where the subscript ($\lambda = 0,1$) represents the corresponding domain. The equation of absolute motion of the generalized Fermi-oscillator is rewritten in a state vector form of

$$\dot{\mathbf{x}}_\lambda^{(i)} = \mathbf{f}_\lambda^{(i)}\left(\mathbf{x}_\lambda^{(i)}, t\right) \quad \text{for } i = 1,2,\ \lambda = 0,1, \tag{3.6}$$

and one obtains

$$\begin{gathered} F_1^{(1)}\left(x_1^{(1)}, t\right) = -g, \\ F_1^{(2)}\left(x_1^{(2)}, t\right) = -2d\dot{x}_1^{(2)} - \omega^2 x_1^{(2)} + \omega^2 Q\cos\Omega t - 2dQ\Omega\sin\Omega t \end{gathered} \tag{3.7}$$

for the nonstick motion, and

$$F_1^{(i)}\left(x_0^{(i)}, t\right) = -2d^{(0)}\dot{x}_0^{(i)} - \left(\omega^{(0)}\right)^2 x_0^{(i)} + \left(\omega^{(0)}\right)^2 Q\cos\Omega t - 2d^{(0)}Q\Omega\sin\Omega t \tag{3.8}$$

for the stick motion.

Figure 4: Relative domains and boundaries definition: (a) $(z, \dot{z})$-plane for particle, (b) $(\dot{z}, \ddot{z})$-plane for particle, (c) $(z, \dot{z})$-plane for piston, (d) $(\dot{z}, \ddot{z})$-plane for piston.

3.2. Domains and Boundaries for Relative Motions

Because the switching boundary varies with time, it is very difficult to develop the switching conditions. Thus, the relative coordinates for such a Fermi-oscillator are adopted herein. The relative displacement, relative velocity, and relative acceleration between the particle and the piston are $z^{(i)} = x^{(i)} - x^{(\bar{i})}$, $\dot{z}^{(i)} = \dot{x}^{(i)} - \dot{x}^{(\bar{i})}$, and $\ddot{z}^{(i)} = \ddot{x}^{(i)} - \ddot{x}^{(\bar{i})}$, where the index $i = 1, 2$ (with $\bar{i} = 2, 1$) represents the particle and piston, respectively. The stick domain and boundaries in the relative phase space become points, as shown in Figures 4(a) and 4(c). Because the piston and particle have the same displacement and velocity for the entire stick motion, the boundary and domains in the relative phase plane become zero. However, the relative interaction force between the particle and position are different before or after the stick. Thus, the plane of relative velocity and acceleration should be considered, and the stick domains and boundaries in the relative velocity and acceleration (i.e., $(\dot{z}^{(i)}, \ddot{z}^{(i)})$) are presented in

Figures 4(b) and 4(d). The filled domains are for stick motion. The shaded regions are for nonstick motion. The domains $\Omega_0^{(i)}$ and $\Omega_1^{(i)}$ for the relative motions of the particle and piston are defined as

$$\begin{aligned}
\Omega_0^{(i)} &= \left\{\left(z^{(i)}, \dot{z}^{(i)}\right) \mid \dot{z}^{(i)} = 0, z^{(i)} = 0\right\} \quad \text{for } i = 1, 2, \\
\Omega_1^{(1)} &= \left\{\left(z^{(1)}, \dot{z}^{(1)}\right) \mid z^{(1)} \in \left(0, h - x^{(2)}\right)\right\}, \\
\Omega_1^{(2)} &= \left\{\left(z^{(2)}, \dot{z}^{(2)}\right) \mid z^{(2)} \in (-\infty, 0)\right\}.
\end{aligned} \tag{3.9}$$

The boundaries $\partial\Omega_{1(+\infty)}^{(i)}$, $\partial\Omega_{1(-\infty)}^{(i)}$, $\partial\Omega_{10}^{(i)}$, and $\partial\Omega_{01}^{(i)}$ for the relative motions of the particle and piston are defined as

$$\begin{aligned}
\partial\Omega_{1(+\infty)}^{(i)} &= \left\{\left(z^{(i)}, \dot{z}^{(i)}\right) \middle| \varphi_{1(+\infty)}^{(1)} \equiv z^{(1)} - h + x^{(2)} = 0\right\}, \\
\partial\Omega_{1(-\infty)}^{(i)} &= \left\{\left(z^{(i)}, \dot{z}^{(i)}\right) \middle| \varphi_{1(-\infty)}^{(i)} \equiv z^{(i)} = 0\right\}, \\
\partial\Omega_{10}^{(i)} = \partial\Omega_{01}^{(i)} &= \left\{\left(z^{(i)}, \dot{z}^{(i)}\right) \middle| \varphi_{10}^{(i)} \equiv \dot{z}_{cr}^{(i)} = 0, z_{cr}^{(i)} = 0\right\},
\end{aligned} \tag{3.10}$$

where $\partial\Omega_{1(-\infty)}^{(1)}$ is the impact chatter boundary for the particle, and two stick boundaries for the particle are $\partial\Omega_{10}^{(1)}$ and $\partial\Omega_{01}^{(1)}$. Similarly, $\partial\Omega_{1(-\infty)}^{(2)}$ is the impact chatter boundary for the piston, and $\partial\Omega_{10}^{(2)}$ and $\partial\Omega_{01}^{(2)}$ are two stick boundaries for the piston. $\partial\Omega_{1(+\infty)}^{(i)}$ is the impact boundary for the particle and the fixed wall.

The vectors for relative motions are

$$\mathbf{z}_\lambda^{(i)} = \left(z_\lambda^{(i)}, \dot{z}_\lambda^{(i)}\right)^T, \qquad \mathbf{g}_\lambda^{(i)} = \dot{\mathbf{z}}_\lambda^{(i)} = \left(\dot{z}_\lambda^{(i)}, g_\lambda^{(i)}\right)^T, \tag{3.11}$$

where the superscript $(i = 1, 2)$ represents the particle and piston, respectively. The subscript $(\lambda = 0, 1)$ represents the corresponding domains. The equation of relative motion is rewritten in a vector form of

$$\dot{\mathbf{z}}_\lambda^{(i)} = \mathbf{g}_\lambda^{(i)}\left(\mathbf{z}_\lambda^{(i)}, \mathbf{x}_\lambda^{(\bar{i})}, t\right), \qquad \dot{\mathbf{x}}_\lambda^{(\bar{i})} = \mathbf{f}_\lambda^{(\bar{i})}\left(\mathbf{x}_\lambda^{(\bar{i})}, t\right), \tag{3.12}$$

where

$$\begin{aligned}
g_1^{(1)}\left(\mathbf{z}_1^{(1)}, \mathbf{x}_1^{(2)}, t\right) &= -g + 2d^{(2)}\dot{x}_1^{(2)} + \left(\omega^{(2)}\right)^2 x_1^{(2)} - \left(\omega^{(2)}\right)^2 Q\cos\Omega t + 2d^{(2)}Q\Omega\sin\Omega t, \\
g_1^{(2)}\left(\mathbf{z}_1^{(2)}, \mathbf{x}_1^{(1)}, t\right) &= -2d^{(2)}\dot{z}_1^{(2)} - \left(\omega^{(2)}\right)^2 z_1^{(2)} + \left(\omega^{(2)}\right)^2 Q\cos\Omega t - 2d^{(2)}Q\Omega\sin\Omega t \\
&\quad - 2d^{(2)}\dot{x}_1^{(1)} - \left(\omega^{(2)}\right)^2 x_1^{(1)} + g,
\end{aligned} \tag{3.13}$$

for nonstick motion

$$\begin{aligned} \dot{z}_0^{(i)} &= 0, \\ g_0^{(i)}\left(\mathbf{z}_0^{(i)}, \mathbf{x}_0^{(\bar{i})}, t\right) &= 0, \end{aligned} \quad \text{for } i = 1,2 \text{ with } \bar{i} = 2,1, \tag{3.14}$$

for stick motion.

3.3. Analytical Conditions

The analytical conditions of stick and grazing motion will be developed from the theory of non-smooth dynamical systems in Luo [40, 41, 44, 49, 50]. The grazing motion is a motion tangential to the switching boundary in phase space. In the Fermi-acceleration oscillator, once the ball meets with the piston or the wall, there is a motion tangential to the switching boundary in phase space. Such a motion is called the grazing motion. To develop the analytical conditions of the grazing and stick motion to the switching boundaries, the normal vectors of such boundaries should be determined, that is,

$$\begin{aligned} \mathbf{n}_{\partial\Omega_{\alpha\beta}^{(i)}} &= \nabla\varphi_{\alpha\beta}^{(i)} = \left(\frac{\partial\varphi_{\alpha\beta}^{(i)}}{\partial z^{(i)}}, \frac{\partial\varphi_{\alpha\beta}^{(i)}}{\partial \dot{z}^{(i)}}\right)^T \quad \text{for the relative frame,} \\ \mathbf{n}_{\partial\Omega_{\alpha\beta}} &= \nabla\varphi_{\alpha\beta}^{(i)} = \left(\frac{\partial\varphi_{\alpha\beta}^{(i)}}{\partial x^{(i)}}, \frac{\partial\varphi_{\alpha\beta}^{(i)}}{\partial \dot{x}^{(i)}}\right)^T \quad \text{for the absolute frame.} \end{aligned} \tag{3.15}$$

The normal vectors to the stick boundaries $(\mathbf{n}_{\partial\Omega_{10}^{(1)}}, \mathbf{n}_{\partial\Omega_{01}^{(1)}})$ and impact boundary $(\mathbf{n}_{\partial\Omega_{1(+\infty)}^{(1)}}, \mathbf{n}_{\partial\Omega_{1(-\infty)}^{(1)}})$ are

$$\begin{aligned} \mathbf{n}_{\partial\Omega_{10}^{(1)}} = \mathbf{n}_{\partial\Omega_{01}^{(1)}} = (0,1)^T, \quad \mathbf{n}_{\partial\Omega_{1(-\infty)}^{(1)}} = (1,0)^T \quad \text{in the relative frame,} \\ \mathbf{n}_{\partial\Omega_{1(+\infty)}^{(1)}} = (1,0)^T \quad \text{in the absolute frame.} \end{aligned} \tag{3.16}$$

Before the analytical conditions are developed, the zero-order and first-order G-functions in the relative frame will be presented. For the straight-line boundary, the zero-order and first-order G-functions are given from Luo [49, 50], that is,

$$\begin{aligned} G_{\partial\Omega_{01}^{(i)}}^{(0,0)}\left(\mathbf{z}_0^{(i)}, t_{m\pm}\right) &= \mathbf{n}_{\partial\Omega_{01}^{(1)}}^T \cdot \mathbf{g}^{(i)}\left(\mathbf{z}_0^{(i)}, \mathbf{x}_0^{(\bar{i})}, t_{m\pm}\right) = g_0^{(i)}\left(\mathbf{z}_0^{(i)}, \mathbf{x}_0^{(\bar{i})}, t_{m\pm}\right) = \ddot{z}_0^{(i)}\left(t_{m\pm}\right), \\ G_{\partial\Omega_{10}^{(i)}}^{(0,1)}\left(\mathbf{z}_1^{(i)}, t_{m\pm}\right) &= \mathbf{n}_{\partial\Omega_{10}^{(1)}}^T \cdot \mathbf{g}^{(i)}\left(\mathbf{z}_1^{(i)}, \mathbf{x}_1^{(\bar{i})}, t_{m\pm}\right) = g_1^{(i)}\left(\mathbf{z}_1^{(i)}, \mathbf{x}_1^{(\bar{i})}, t_{m\pm}\right) = \ddot{z}_1^{(i)}\left(t_{m\pm}\right), \\ G_{\partial\Omega_{01}^{(i)}}^{(1,0)}\left(\mathbf{z}_0^{(i)}, t_{m\pm}\right) &= \mathbf{n}_{\partial\Omega_{01}^{(1)}}^T \cdot D\mathbf{g}^{(i)}\left(\mathbf{z}_0^{(i)}, \mathbf{x}_0^{(\bar{i})}, t_{m\pm}\right) = \frac{d}{dt} g_0^{(i)}\left(\mathbf{z}_0^{(i)}, \mathbf{x}_0^{(\bar{i})}, t\right)\Big|_{t=t_{m\pm}} = \dddot{z}_0^{(i)}\left(t_{m\pm}\right), \\ G_{\partial\Omega_{10}^{(i)}}^{(1,1)}\left(\mathbf{z}_1^{(i)}, t_{m\pm}\right) &= \mathbf{n}_{\partial\Omega_{10}^{(1)}}^T \cdot D\mathbf{g}^{(i)}\left(\mathbf{z}_1^{(i)}, \mathbf{x}_1^{(\bar{i})}, t_{m\pm}\right) = \frac{d}{dt} g_1^{(i)}\left(\mathbf{z}_1^{(i)}, \mathbf{x}_1^{(\bar{i})}, t\right)\Big|_{t=t_{m\pm}} = \dddot{z}_1^{(i)}\left(t_{m\pm}\right). \end{aligned} \tag{3.17}$$

Notice that the switching time t_m is the motion at the corresponding switching boundary, and $t_{m\pm} = t_m \pm 0$ represents the motion in domains rather than on the boundary. In $G^{(k,\lambda)}_{\partial\mathfrak{D}_1^{(i)}}$, two superscripts λ ($\lambda = 0, 1$) and k ($k = 0, 1$) represent the λ-subdomain and the kth order of G-functions, respectively. Again $i = 1$ and 2 are for the particle and piston. The zero-order G-function ($k = 0$) is the normal component of the vector field in the normal direction of boundary, and the first-order G-function ($k = 1$) is the time-change rate of the normal component of vector fields. The detailed discussion of G-functions can be referred to Luo [48–50]. Similarly, the G-functions for the impact boundaries are given by

$$\begin{aligned}
&G^{(0,1)}_{\partial\Omega^{(1)}_{1(+\infty)}}\left(\mathbf{x}_1^{(1)}, t_{m\pm}\right) = \mathbf{n}^T_{\partial\Omega^{(1)}_{1(+\infty)}} \cdot \mathbf{f}^{(1)}\left(\mathbf{x}_1^{(1)}, t_{m\pm}\right) = f^{(1)}\left(\mathbf{x}_1^{(1)}, t_{m\pm}\right) = \ddot{x}_1^{(1)} = -g,\\
&G^{(1,1)}_{\partial\Omega^{(1)}_{1(+\infty)}}\left(\mathbf{x}_1^{(1)}, t_{m\pm}\right) = \mathbf{n}^T_{\partial\Omega^{(1)}_{1(+\infty)}} \cdot D\mathbf{f}^{(1)}\left(\mathbf{x}_1^{(1)}, t_{m\pm}\right) = \frac{d}{dt} f^{(1)}\left(\mathbf{x}_1^{(1)}, t\right)\Big|_{t=t_{m\pm}} = \dddot{x}_1^{(1)} = 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \partial\Omega^{(1)}_{1(+\infty)} \text{ in the absolute frame,}\\
&G^{(0,1)}_{\partial\Omega^{(i)}_{1(-\infty)}}\left(\mathbf{z}_1^{(i)}, t_{m\pm}\right) = \mathbf{n}^T_{\partial\Omega^{(i)}_{1(-\infty)}} \cdot \mathbf{g}^{(i)}\left(\mathbf{z}_1^{(i)}, \mathbf{x}_1^{(\bar{i})}, t_{m\pm}\right) = g^{(i)}\left(\mathbf{z}_1^{(i)}, \mathbf{x}_1^{(\bar{i})}, t_{m\pm}\right) = \ddot{z}_1^{(i)}\left(t_{m\pm}\right),\\
&G^{(1,1)}_{\partial\Omega^{(i)}_{1(-\infty)}}\left(\mathbf{z}_1^{(i)}, t_{m\pm}\right) = \mathbf{n}^T_{\partial\Omega^{(i)}_{1(-\infty)}} \cdot D\mathbf{g}^{(i)}\left(\mathbf{z}_1^{(i)}, \mathbf{x}_1^{(\bar{i})}, t_{m\pm}\right) = \frac{d}{dt} g^{(i)}\left(\mathbf{z}_1^{(i)}, \mathbf{x}_1^{(\bar{i})}, t\right)\Big|_{t=t_{m\pm}} = \dddot{z}_1^{(i)}\left(t_{m\pm}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \partial\Omega^{(i)}_{1(-\infty)} \text{ in the relative frame.}
\end{aligned} \tag{3.18}$$

Based on the normal vectors and G-functions, the analytical condition for stick motion is obtained for the condition for a passable flow from domain $\Omega_1^{(i)}$ to $\Omega_0^{(i)}$ in Luo [49] or [50], that is,

$$(-1)^i G^{(0,1)}_{\partial\Omega^{(i)}_{10}}\left(\mathbf{z}_1^{(i)}, t_{m-}\right) > 0, \qquad (-1)^i G^{(0,1)}_{\partial\Omega^{(i)}_{10}}\left(\mathbf{z}_0^{(i)}, t_{m+}\right) > 0. \tag{3.19}$$

Therefore, one obtains

$$(-1)^i g_1^{(i)}\left(\mathbf{z}_1^{(i)}, \mathbf{x}_1^{(\bar{i})}, t_{m-}\right) > 0, \qquad (-1)^i g_0^{(i)}\left(\mathbf{z}_0^{(i)}, \mathbf{x}_0^{(\bar{i})}, t_{m+}\right) > 0. \tag{3.20}$$

With the relative force function per unit mass (i.e., relative acceleration $\ddot{z}^{(i)} = \ddot{x}^{(i)} - \ddot{x}^{(\bar{i})}$), equations of (3.20) gives

$$\ddot{x}^{(2)}\left(t_{m\pm}\right) > \ddot{x}^{(1)}\left(t_{m\pm}\right) = -g. \tag{3.21}$$

From (3.21), the stick motion requires that the acceleration of the piston $\ddot{x}^{(2)}(t_{m\pm})$ is greater than $\ddot{x}^{(1)}(t_{m\pm}) = -g$. In a similar fashion, the condition for vanishing of the stick motion at $\partial\Omega_{01}^{(i)}$ in the relative frame is given by

$$\begin{aligned}
G^{(0,0)}_{\partial\Omega^{(i)}_{01}}\left(\mathbf{z}_0^{(i)}, t_{m-}\right) &= 0, \qquad G^{(0,1)}_{\partial\Omega^{(i)}_{01}}\left(\mathbf{z}_1^{(i)}, t_{m+}\right) = 0,\\
(-1)^i G^{(1,0)}_{\partial\Omega^{(i)}_{01}}\left(\mathbf{z}_0^{(i)}, t_{m-}\right) &< 0, \qquad (-1)^i G^{(1,1)}_{\partial\Omega^{(i)}_{01}}\left(\mathbf{z}_1^{(i)}, t_{m+}\right) < 0.
\end{aligned} \tag{3.22}$$

From equations of (3.22), one obtains the relative force relations or $\partial\Omega_{01}^{(i)}$ as

$$\begin{aligned} g_0^{(i)}\left(\mathbf{z}_0^{(i)}, \mathbf{x}_0^{(\bar{i})}, t_{m-}\right) &= 0, \qquad g_1^{(i)}\left(\mathbf{z}_1^{(i)}, \mathbf{x}_1^{(\bar{i})}, t_{m+}\right) = 0, \\ (-1)^i \frac{d}{dt} g_0^{(i)}\left(\mathbf{z}_0^{(i)}, \mathbf{x}_0^{(\bar{i})}, t_{m-}\right) &< 0, \qquad (-1)^i \frac{d}{dt} g_1^{(i)}\left(\mathbf{z}_1^{(i)}, \mathbf{x}_1^{(\bar{i})}, t_{m+}\right) < 0. \end{aligned} \tag{3.23}$$

With the relative acceleration ($\ddot{z}^{(i)} = \ddot{x}^{(i)} - \ddot{x}^{(\bar{i})}$) and relative jerk ($\dddot{z}^{(i)} = \dddot{x}^{(i)} - \dddot{x}^{(\bar{i})}$), equations of (3.22) can be rewritten as

$$\ddot{x}^{(2)}\left(t_{m\pm}\right) = \ddot{x}^{(1)}\left(t_{m\pm}\right) = -g, \quad \dddot{x}_{m\pm}^{(2)} < \dddot{x}_{m\pm}^{(1)} = 0 \quad \text{for } \partial\Omega_{01}^{(i)}. \tag{3.24}$$

The stick motion vanishing at the switching boundary requires that the piston's jerk is less than zero as both accelerations are the gravitational acceleration.

Based on the G-functions of a flow to the impacting boundaries, the conditions of grazing motions are also developed, that is,

$$\begin{aligned} G_{\partial\Omega_{1(+\infty)}^{(1)}}^{(0,1)}\left(\mathbf{x}_1^{(1)}, t_{m\pm}\right) &= 0, \quad G_{\partial\Omega_{1(+\infty)}^{(1)}}^{(1,1)}\left(\mathbf{x}_1^{(1)}, t_{m\pm}\right) < 0 \quad \text{for } \partial\Omega_{1(+\infty)}^{(i)}, \\ (-1)^i G_{\partial\Omega_{1(-\infty)}^{(1)}}^{(0,1)}\left(\mathbf{z}_1^{(i)}, t_{m\pm}\right) &= 0, \quad (-1)^i G_{\partial\Omega_{1(-\infty)}^{(1)}}^{(0,1)}\left(\mathbf{z}_1^{(i)}, t_{m\pm}\right) < 0 \quad \text{for } \partial\Omega_{1(-\infty)}^{(i)}. \end{aligned} \tag{3.25}$$

The grazing conditions for the boundaries of nonstick motions reduce to

$$\begin{aligned} \dot{x}^{(1)} &= 0, \quad \ddot{x}^{(1)} = -g < 0 \quad \text{for } \partial\Omega_{1(+\infty)}^{(1)}, \\ \dot{x}^{(1)} &= \dot{x}^{(2)}, \quad \ddot{x}^{(2)} < \ddot{x}^{(1)} = -g \quad \text{for } \partial\Omega_{1(-\infty)}^{(i)}. \end{aligned} \tag{3.26}$$

From the first equation of (3.26), the ball grazing at the fixed wall requires that the velocity of ball be zero because the acceleration of ball ($\ddot{x}^{(1)} = -g$) is always zero. However, from the second equation of (3.26), for the ball grazing at the impacting boundary with piston, the piston acceleration should be less than the ball acceleration (i.e., $\ddot{x}^{(1)} = -g$). In other words, this condition is also for the piston motion grazing at this impacting boundary with the ball.

In a similar fashion, the grazing conditions for the boundary of the stick motion are

$$\begin{aligned} G_{\partial\Omega_{10}^{(i)}}^{(0,1)}\left(\mathbf{z}_1^{(i)}, t_{m\pm}\right) &= 0, \quad (-1)^i G_{\partial\Omega_{10}^{(i)}}^{(1,1)}\left(\mathbf{z}_1^{(i)}, t_{m\pm}\right) < 0 \quad \text{for } \partial\Omega_{10}^{(i)}, \\ G_{\partial\Omega_{01}^{(i)}}^{(0,0)}\left(\mathbf{z}_0^{(i)}, t_{m\pm}\right) &= 0, \quad (-1)^i G_{\partial\Omega_{01}^{(i)}}^{(1,0)}\left(\mathbf{z}_0^{(i)}, t_{m\pm}\right) > 0 \quad \text{for } \partial\Omega_{01}^{(i)}. \end{aligned} \tag{3.27}$$

Figure 5: (a) Passable motion and (b) grazing motion.

Again, with the relative acceleration ($\ddot{z}^{(i)} = \ddot{x}^{(i)} - \ddot{x}^{(\bar{i})}$) and relative jerk ($\dddot{z}^{(i)} = \dddot{x}^{(i)} - \dddot{x}^{(\bar{i})}$), the grazing conditions are given by the accelerations and jerks of piston and ball, that is,

$$\begin{aligned} \ddot{x}^{(2)} = \ddot{x}^{(1)} = -g, \quad \dddot{x}^{(2)} < \dddot{x}^{(1)} = 0 \quad \text{for } \partial\Omega_{10}^{(i)}, \\ \ddot{x}^{(2)} = \ddot{x}^{(1)} = -g, \quad \dddot{x}^{(2)} > \dddot{x}^{(1)} = 0 \quad \text{for } \partial\Omega_{01}^{(i)}. \end{aligned} \tag{3.28}$$

To geometrically explain the above analytical conditions, consider a flow of the motion approaching the stick boundary $\partial\Omega_{10}^{(1)}$ at $(\mathbf{z}_1^{(1)}, t_{m-})$. In order for such a flow to pass the boundary, the normal vector fields of $G_{\partial\Omega_{10}^{(i)}}^{(0,1)}$ and $G_{\partial\Omega_{01}^{(i)}}^{(0,0)}$ have to point toward the stick domain $\Omega_1^{(i)}$ (i.e., $(-1)^i G_{\partial\Omega_{10}^{(i)}}^{(0,1)} > 0$, $(-1)^i G_{\partial\Omega_{01}^{(i)}}^{(0,0)} > 0$) because the normal vector points to the domain $\Omega_1^{(i)}$. Such conditions are given in (3.19) (or (3.21)) and the flow of motion will pass over the boundary and gets into the stick domain $\Omega_0^{(i)}$. Such a motion is a passable motion to the boundary $\partial\Omega_{10}^{(1)}$, as shown in Figure 5(a). For this case, the stick motion is formed. Similarly, for a flow from a stick domain to the nonstick domain, a flow of motion needs to satisfy the conditions in (3.22) (or (3.23)). Such a passable flow is called the vanishing of stick motion. When a flow of motion approaches the impact chatter boundary $\partial\Omega_{1(-\infty)}^{(1)}$ at $(\mathbf{z}_1^{(1)}, t_{m-})$, if the normal vector field $G_{\partial\Omega_{1(-\infty)}^{(1)}}^{(0,1)}$ be equal to zero and the total change rate of the normal vector field $G_{\partial\Omega_{1(-\infty)}^{(1)}}^{(1,1)}$ is greater than zero in domain $\Omega_1^{(1)}$ (i.e., $G_{\partial\Omega_{1(-\infty)}^{(1)}}^{(0,1)} = 0$ and $G_{\partial\Omega_{1(-\infty)}^{(1)}}^{(1,1)} > 0$), the flow of motion will leave the boundary and stay in domain $\Omega_1^{(1)}$, as sketched in Figure 5(b). Such a motion is called a grazing motion to the boundary $\partial\Omega_{1(-\infty)}^{(1)}$. The corresponding condition for grazing at $\partial\Omega_{1(-\infty)}^{(1)}$ is presented in (3.25). Similarly, for the stick boundary $\partial\Omega_{01}^{(1)}$, the grazing condition in (3.27) can be geometrically explained.

4. Mapping Structures and Motions

In this section, the switching sets and mapping structures will be introduced to symbolically describe motions in the generalized Fermi-acceleration oscillator. The switching sets will be

defined from the switching boundaries. From the switching sets, the generic mappings in domains will be introduced. The mapping structure for periodic and chaotic motions will be constructed from such generic mappings. The stability and bifurcation conditions will also be discussed via the eigenvalue analysis.

4.1. Switching Sets and Generic Mappings

Based on the switching boundaries in (3.2), the switching sets of the generalized Fermi-acceleration oscillator without stick are defined as

$$\begin{aligned} \Sigma_{1(-\infty)} &= \Sigma^{(1)}_{1(-\infty)} \otimes \Sigma^{(2)}_{1(-\infty)} = \left\{ \left(x_i^{(1)}, \dot{x}_i^{(1)}, x_i^{(2)}, \dot{x}_i^{(2)}, t_i \right) \mid x_i^{(1)} = x_{i+}^{(2)}, \dot{x}_i^{(1)} \neq \dot{x}_i^{(2)} \right\}, \\ \Sigma_{1(+\infty)} &= \Sigma^{(1)}_{1(+\infty)} \otimes \Sigma^{(2)}_{1(+\infty)} = \left\{ \left(x_i^{(1)}, \dot{x}_i^{(1)}, x_i^{(2)}, \dot{x}_i^{(2)}, t_i \right) \mid x_i^{(1)} = h, \dot{x}_i^{(1)} \neq 0 \right\}, \end{aligned} \tag{4.1}$$

where the symbol $\otimes$ represents the direct product of switching subsets, and the switching subsets ($\Sigma^{(i)}_{1(-\infty)}$ and $\Sigma^{(i)}_{1(+\infty)}$) on the boundaries ($\partial\Omega^{(i)}_{1(-\infty)}$ and $\partial\Omega^{(i)}_{1(+\infty)}$) are defined as

$$\begin{aligned} \Sigma^{(i)}_{1(-\infty)} &= \left\{ \left(x_k^{(i)}, \dot{x}_k^{(i)}, t_k \right) \mid x_k^{(i)} = x_k^{(\bar{i})}, \dot{x}_k^{(i)} \neq \dot{x}_k^{(\bar{i})} \right\} \subset \partial\Omega^{(i)}_{1(-\infty)}, \\ \Sigma^{(i)}_{1(+\infty)} &= \left\{ \left(x_k^{(i)}, \dot{x}_k^{(i)}, t_k \right) \mid x_k^{(1)} = h, \dot{x}_k^{(1)} \neq 0 \right\} \subset \partial\Omega^{(i)}_{1(+\infty)}. \end{aligned} \tag{4.2}$$

Thus, the generic mappings for motions without stick are defined as

$$P_1 : \Sigma_{1(-\infty)} \longrightarrow \Sigma_{1(+\infty)}, \qquad P_2 : \Sigma_{1(+\infty)} \longrightarrow \Sigma_{1(-\infty)}, \qquad P_3 : \Sigma_{1(-\infty)} \longrightarrow \Sigma_{1(-\infty)}, \tag{4.3}$$

where the mappings P_1 and P_2 are the global mappings and the mapping P_3 is a local mapping. No any mapping exists from $\Sigma_{1(+\infty)}$ to $\Sigma_{1(+\infty)}$ because $\ddot{x}^{(1)} = -g$. The global mapping will map one switching set into another one. However, the local mapping will map a switching set into itself. Since the resultant switching set is the direct product of the two switching subsets, a resultant mapping should have two components, that is,

$$\begin{gathered} P_1 = ({}^{(1)}P_1, {}^{(2)}P_1), \qquad P_2 = ({}^{(1)}P_2, {}^{(2)}P_1), \qquad P_3 = ({}^{(1)}P_3, {}^{(2)}P_1), \\ {}^{(i)}P_1 : \Sigma^{(i)}_{1(-\infty)} \longrightarrow \Sigma^{(i)}_{1(+\infty)}, \qquad {}^{(i)}P_2 : \Sigma^{(i)}_{1(+\infty)} \longrightarrow \Sigma^{(i)}_{1(-\infty)}, \qquad {}^{(i)}P_3 : \Sigma^{(i)}_{1(-\infty)} \longrightarrow \Sigma^{(i)}_{1(-\infty)}. \end{gathered} \tag{4.4}$$

From the above definitions, the switching subsets and submappings are sketched in Figures 6(a) and 6(b) for the particle and piston. The switching points in the switching sets are presented by the circular symbols, and the mappings are depicted by the curves with arrows.

Similarly, from the switching boundaries in (3.2) and (3.4), the switching sets of the generalized Fermi-acceleration oscillator with stick are defined as

$$\begin{aligned} \Sigma_{10} &= \Sigma^{(1)}_{10} \otimes \Sigma^{(2)}_{10} = \left\{ \left(x_k^{(1)}, \dot{x}_k^{(1)}, x_k^{(2)}, \dot{x}_k^{(2)}, t_k \right) \mid x_k^{(1)} = x_k^{(2)}, \dot{x}_k^{(1)} = \dot{x}_k^{(2)} \right\}, \\ \Sigma_{1(+\infty)} &= \Sigma^{(1)}_{1(+\infty)} \otimes \Sigma^{(2)}_{1(+\infty)} = \left\{ \left(x_k^{(1)}, \dot{x}_k^{(1)}, x_k^{(2)}, \dot{x}_k^{(2)}, t_k \right) \mid x_k^{(1)} = h, \dot{x}_k^{(1)} \neq 0 \right\}, \end{aligned} \tag{4.5}$$

Figure 6: Switching sets and generic mappings for nonstick motion (in absolute coordinates): (a) particle and (b) piston.

where the switching set Σ_{10} is defined on the boundary $\partial\Omega_{10}$

$$\begin{aligned} \Sigma^{(i)}_{10} &= \left\{ \left(x^{(i)}_k, \dot{x}^{(i)}_k, t_k \right) \mid x^{(i)}_k = x^{(\bar{i})}_k, \dot{x}^{(i)}_k = \dot{x}^{(\bar{i})}_k \right\} \subset \partial\Omega^{(i)}_{10}, \\ \Sigma^{(i)}_{1(+\infty)} &= \left\{ \left(x^{(i)}_k, \dot{x}^{(i)}_k, t_k \right) \mid x^{(1)}_k = h, \dot{x}^{(1)}_k \neq 0 \right\} \subset \partial\Omega^{(i)}_{1(+\infty)}. \end{aligned} \tag{4.6}$$

Thus, the generic mappings are defined as

$$P_0 : \Sigma_{10} \longrightarrow \Sigma_{10}, \qquad P_1 : \Sigma_{10} \longrightarrow \Sigma_{1(+\infty)}, \qquad P_2 : \Sigma_{1(+\infty)} \longrightarrow \Sigma_{10}, \qquad P_3 : \Sigma_{1(-\infty)} \longrightarrow \Sigma_{10}. \tag{4.7}$$

Similarly, a resultant mapping should have two components, so one obtains

$$\begin{gathered} P_0 = \left({}^{(1)}P_0, {}^{(2)}P_0 \right), \qquad P_1 = \left({}^{(1)}P_1, {}^{(2)}P_1 \right), \qquad P_2 = \left({}^{(1)}P_2, {}^{(2)}P_2 \right), \qquad P_3 = \left({}^{(1)}P_3, {}^{(2)}P_3 \right), \\ {}^{(i)}P_0 : \Sigma^{(i)}_{10} \longrightarrow \Sigma^{(i)}_{10}, \qquad {}^{(i)}P_1 : \Sigma^{(i)}_{1(-\infty)} \longrightarrow \Sigma^{(i)}_{1(+\infty)}, \\ {}^{(i)}P_2 : \Sigma^{(i)}_{1(+\infty)} \longrightarrow \Sigma^{(i)}_{1(-\infty)}, \qquad {}^{(i)}P_3 : \Sigma^{(i)}_{1(-\infty)} \longrightarrow \Sigma^{(i)}_{1(-\infty)}, \end{gathered} \tag{4.8}$$

where P_1 and P_2 are the global mappings and P_0 and P_3 are the local mappings. The mappings in (4.8), are sketched in Figure 7.

From the above definitions, a mapping will map a switching set into another switching set (or itself) through the corresponding dynamical system in a specified domain. In such a domain, the dynamical system given in (2.1) or (2.2) is continuous. The solutions for such a continuous system are listed in the appendix. From the displacement and velocity solutions

Figure 7: Switching sets and generic mappings for stick motion: (a) particle and (b) piston.

of the particle and piston in the appendix, with switching sets, four algebraic governing equations for a generic mapping P_j $(j = 1, 2, 3)$ can be expressed in a form of

$$P_j : \begin{cases} f_1^{(j)}\left(x_k^{(1)}, \dot{x}_k^{(1)}, x_k^{(2)}, \dot{x}_k^{(2)}, t_k, x_{k+1}^{(1)}, \dot{x}_{k+1}^{(1)}, x_{k+1}^{(2)}, \dot{x}_{k+1}^{(2)}, t_{k+1}\right) = 0, \\ f_2^{(j)}\left(x_k^{(1)}, \dot{x}_k^{(1)}, x_k^{(2)}, \dot{x}_k^{(2)}, t_k, x_{k+1}^{(1)}, \dot{x}_{k+1}^{(1)}, x_{k+1}^{(2)}, \dot{x}_{k+1}^{(2)}, t_{k+1}\right) = 0, \\ f_3^{(j)}\left(x_k^{(1)}, \dot{x}_k^{(1)}, x_k^{(2)}, \dot{x}_k^{(2)}, t_k, x_{k+1}^{(1)}, \dot{x}_{k+1}^{(1)}, x_{k+1}^{(2)}, \dot{x}_{k+1}^{(2)}, t_{k+1}\right) = 0, \\ f_4^{(j)}\left(x_k^{(1)}, \dot{x}_k^{(1)}, x_k^{(2)}, \dot{x}_k^{(2)}, t_k, x_{k+1}^{(1)}, \dot{x}_{k+1}^{(1)}, x_{k+1}^{(2)}, \dot{x}_{k+1}^{(2)}, t_{k+1}\right) = 0, \end{cases} \tag{4.9}$$

with

$$\begin{aligned} &x_k^{(1)} = h, \quad x_{k+1}^{(1)} = x_{k+1}^{(2)} \quad \text{for } P_1, \\ &x_k^{(1)} = x_k^{(2)}, \quad x_{k+1}^{(1)} = h \quad \text{for } P_2, \\ &x_k^{(1)} = x_k^{(1)}, \quad x_{k+1}^{(1)} = x_{k+1}^{(2)} \quad \text{for } P_3. \end{aligned} \tag{4.10}$$

The governing equations for the stick mapping P_0 can be expressed by

$$P_0 : \begin{cases} f_1^{(0)}\left(x_k^{(i)}, \dot{x}_k^{(i)}, t_k, x_{k+1}^{(i)}, \dot{x}_{k+1}^{(i)}, t_{k+1}\right) = 0, \\ f_2^{(0)}\left(x_k^{(i)}, \dot{x}_k^{(i)}, t_k, x_{k+1}^{(i)}, \dot{x}_{k+1}^{(i)}, t_{k+1}\right) = 0, \\ f_4^{(0)}\left(x_{k+1}^{(i)}, \dot{x}_{k+1}^{(i)}, t_{k+1}\right) = g_1^{(i)}\left(0, \mathbf{x}_{k+1}^{(i)}, t_{k+1}\right) = 0, \end{cases} \tag{4.11}$$

with

$$\begin{aligned} &x_k^{(i)} = x_k^{(\bar{i})}, \quad \dot{x}_k^{(i)} = \dot{x}_k^{(\bar{i})}, \\ &x_{k+1}^{(i)} = x_{k+1}^{(\bar{i})}, \quad \dot{x}_{k+1}^{(i)} = \dot{x}_{k+1}^{(\bar{i})}, \quad \text{for mapping } P_0. \\ &\ddot{x}_k^{(2)} > -g, \quad \ddot{x}_{k+1}^{(2)} < -g, \end{aligned} \tag{4.12}$$

4.2. Motions with Mapping Structures

The notation for mapping action is introduced as

$$P_{j_k j_{k-1} \cdots j_1} = P_{j_k} \circ P_{j_{k-1}} \circ \cdots \circ P_{j_1}, \tag{4.13}$$

where $j_m \in \{0,1,2,3\}$ with $m = 1,2,\ldots,k$. For a motion with n-time repeats of mapping structure of $P_{j_1 j_2 \cdots j_k}$, the mapping structure is expressed as

$$P_{j_k j_{k-1} \cdots j_1}^{(m)} = \underbrace{\left(P_{j_k} \circ P_{j_{k-1}} \circ \cdots \circ P_{j_1}\right) \circ \cdots \circ \left(P_{j_k} \circ P_{j_{k-1}} \circ \cdots \circ P_{j_1}\right)}_{n} = P_{(j_k j_{k-1} \cdots j_1)^n}. \tag{4.14}$$

Any periodic motion can be described using the above notations. Consider a mapping structure for the generalized Fermi-acceleration oscillator as

$$P = P_{3^n} \circ P_{0^{k_3}} \circ P_{3^m} \circ P_{2^{k_2}} \circ P_{1^{k_1}} = P_{3^n 0^{k_3} 3^m 2^{k_2} 1^{k_1}}, \tag{4.15}$$

where $k_j \in \{0,1\}$ and $m,n \in \{0,\mathbb{N}\}$. From the mapping of (4.15), a periodic motion can be expressed. By period-doubling, the corresponding mapping structures for motion relative to the foregoing mapping structure $P_{3^n 0^{k_3} 3^m 2^{k_2} 1^{k_1}}$ of (4.15) are

$$\begin{aligned} &P = P_{3^n 0^{k_3} 3^m 2^{k_2} 1^{k_1}} \circ P_{3^n 0^{k_3} 3^m 2^{k_2} 1^{k_1}} = P_{(3^n 0^{k_3} 3^m 2^{k_2} 1^{k_1})^2}, \\ &\vdots \\ &P = P_{(3^n 0^{k_3} 3^m 2^{k_2} 1^{k_1})^{2^{l-1}}} \circ P_{(3^n 0^{k_3} 3^m 2^{k_2} 1^{k_1})^{2^{l-1}}} = P_{(3^n 0^{k_3} 3^m 2^{k_2} 1^{k_1})^{2^l}}. \end{aligned} \tag{4.16}$$

For $l \to \infty$, the chaotic motion relative to $P_{3^n 0^{k_3} 3^m 2^{k_2} 1^{k_1}}$ can be obtained.

Once the grazing bifurcation of the periodic motion of $P_{3^n 0^{k_3} 3^m 2^{k_2} 1^{k_1}}$ occurs, the corresponding mapping structure will change dramatically. For instance, the changed mapping structure can be expressed by two parts of mapping structures, that is,

$$\begin{aligned} P &= P_{3^{n_2} 0^{k_{32}} 3^{m_2} 2^{k_{22}} 1^{k_{12}}} \circ P_{3^{n_1} 0^{k_{31}} 3^{m_1} 2^{k_{21}} 1^{k_{11}}} \\ &= P_{(3^{n_2} 0^{k_{32}} 3^{m_2} 2^{k_{22}} 1^{k_{12}})(3^{n_1} 0^{k_{31}} 3^{m_1} 2^{k_{21}} 1^{k_{11}})}, \end{aligned} \tag{4.17}$$

where $k_{js} \in \{0,1\}$ and $m_s, n_s \in \mathbb{N}$ $(s = 1,2)$. For a generalized case, the mapping structure is

$$P = \underbrace{P_{3^{n_l}0^{k_{3l}}3^{m_l}2^{k_{2l}}1^{k_{1l}}} \circ \cdots \circ P_{3^{n_1}0^{k_{31}}3^{m_1}2^{k_{21}}1^{k_{11}}}}_{l\text{-terms}} = P_{\underbrace{(3^{n_l}0^{k_{3l}}3^{m_l}2^{k_{2l}}1^{k_{1l}})\cdots(3^{n_1}0^{k_{31}}3^{m_1}2^{k_{21}}1^{k_{11}})}_{l\text{-terms}}}, \tag{4.18}$$

where $k_{js} \in \{0,1\}$ and $m_s, n_s \in \mathbb{N}$ $(s = 1,2,\ldots,l)$. Such mapping structure can be used to express all possible motions in the generalized Fermi-acceleration oscillator. In addition, the grazing bifurcation of chaotic motions will cause the strange attractor fragmentation (see Luo [41]).

Without stick, there are two types of motions: (i) impact only at the boundary $\partial\Omega_{1(-\infty)}$ and (ii) impacts at both of the boundaries $\partial\Omega_{1(-\infty)}$ and $\partial\Omega_{1(+\infty)}$. For the first case, the motion interacts with the boundary $\partial\Omega_{1(-\infty)}$ ($k_{1s} = 0$, $k_{2s} = 0$, and $k_{3s} = 0$, $s \in \mathbb{N}$) only. This motion can be described by the generic mapping P_3. The motion pattern relative to P_{3^m}, caused by the period-doubling bifurcation, is given by

$$\begin{aligned} &P = P_{3^m}, \\ &P = P_{3^m} \circ P_{3^m} = P_{3^{2m}}, \\ &\vdots \\ &P = P_{3^{2^{l-1}m}} \circ P_{3^{2^{l-1}m}} = P_{3^{2^l m}}. \end{aligned} \tag{4.19}$$

The motion pattern relative to P_{3^m} can be generated by the grazing bifurcation, and the corresponding mapping structures are

$$\begin{aligned} &P = P_{3^m}, \\ &P = P_{3^{m_1}} \circ P_3 \circ P_{3^{m_2}} = P_{3^{m+1}}, \\ &\vdots \\ &P = P_{3^m} \circ P_{3^n} = P_{3^{m+n}}. \end{aligned} \tag{4.20}$$

The second case is that the motion interacts with boundaries $\partial\Omega_{1(-\infty)}$ and $\partial\Omega_{1(+\infty)}$ ($k_{3s} = 0$, $s \in \mathbb{N}$, $m, n \in \{0, \mathbb{N}\}$). Consider a mapping structure of $P_{3^m 2^{k_2} 1^{k_1}}$. The motion pattern generated by the period-doubling is expressed by

$$\begin{aligned} &P = P_{3^m 2^{k_2} 1^{k_1}} \circ P_{3^m 2^{k_2} 1^{k_1}} = P_{(3^m 2^{k_2} 1^{k_1})^2}, \ldots, \\ &P = P_{(3^m 2^{k_2} 1^{k_1})^{2^{l-1}}} \circ P_{(3^m 2^{k_2} 1^{k_1})^{2^{l-1}}} = P_{(3^m 2^{k_2} 1^{k_1})^{2^l}}. \end{aligned} \tag{4.21}$$

Similarly, the motion pattern generated by the grazing bifurcation can be discussed. To illustrate such mapping structures, the relation between the mappings and motions in phase plane are sketched in Figure 8. The mappings and motions for the piston can be similarly presented. A motion pertaining to mapping $P_{3^m 21}$ is sketched in Figure 8(a). In such a motion, no any stick motions exists. However, a motion with stick is associated with mapping $P_{3^n 0 3^m 21}$ in Figure 8(b). The mappings are depicted through thick curves, and the impacts are represented by the thin arrows. The stick motion is in the shaded area with mapping P_0. In a similar fashion, any motion with a specific mapping structure can be illustrated.

Figure 8: Mapping structure for motions: (a) P_{13^m2} and (b) $P_{13^n03^m2}$.

Figure 9: The mapping structure of a simple periodic motion of P_{21}.

4.3. Periodic Motions and Local Stability

Consider a map P_j $(j = 0, 1, 2, 3)$ mapping from the initial state $(x_k^{(1)}, \dot{x}_k^{(1)}, x_k^{(2)}, \dot{x}_k^{(2)}, t_k)$ to the final state $(x_{k+1}^{(1)}, \dot{x}_{k+1}^{(1)}, x_{k+1}^{(2)}, \dot{x}_{k+1}^{(2)}, t_{k+1})$:

$$P_j : \left(x_k^{(1)}, \dot{x}_k^{(1)}, x_k^{(2)}, \dot{x}_k^{(2)}, t_k\right) \longrightarrow \left(x_{k+1}^{(1)}, \dot{x}_{k+1}^{(1)}, x_{k+1}^{(2)}, \dot{x}_{k+1}^{(2)}, t_{k+1}\right). \tag{4.22}$$

From (4.9), the mapping in (4.22) gives four algebraic equations. With boundary constraints, the five variables for the final state can be determined. Without loss of generality, substitution of the boundary constraint into the four algebraic governing equations for P_j $(j = 0, 1, 2, 3)$ yields

$$\mathbf{f}^{(j)}\left(\mathbf{X}_k, \mathbf{Y}_{k+1}\right) = 0, \tag{4.23}$$

where $\mathbf{X}_k = (X_{k1}, X_{k2}, X_{k3}, X_{k4})^T$ with $X_{kr} \in \{x_k^{(1)}, \dot{x}_k^{(1)}, x_k^{(2)}, \dot{x}_k^{(2)}, t_k\}$ and $\mathbf{Y}_k = (Y_{k1}, Y_{k2}, Y_{k3}, Y_{k4})^T$ with $Y_{kr} \in \{x_k^{(1)}, \dot{x}_k^{(1)}, x_k^{(2)}, \dot{x}_k^{(2)}, t_k\}$ and $\mathbf{f}^{(j)} = (f_1^{(j)}, f_2^{(j)}, f_3^{(j)}, f_4^{(j)})^T$. Taking the total derivative of (4.23) with respect to $\mathbf{X}_k$ leads to

$$\begin{gathered}\left[\frac{\partial \mathbf{f}^{(j)}}{\partial \mathbf{X}_k}\right] + \left[\frac{\partial \mathbf{f}^{(j)}}{\partial \mathbf{Y}_{k+1}}\right]\left[\frac{\partial \mathbf{Y}_{k+1}}{\partial \mathbf{X}_k}\right] = 0 \quad \text{or} \\ \frac{\partial f_I^{(j)}}{\partial X_{kJ}} + \sum_{R=1}^{4} \frac{\partial f_I^{(j)}}{\partial Y_{(k+1)R}} \frac{\partial Y_{(k+1)R}}{\partial X_{kJ}} = 0 \quad \text{for } I, J = 1, 2, 3, 4.\end{gathered} \tag{4.24}$$

From the equation of (4.24), one obtains

$$DP_j = \left[\frac{\partial \mathbf{Y}_{k+1}}{\partial \mathbf{X}_k}\right] = -\left[\frac{\partial \mathbf{f}^{(j)}}{\partial \mathbf{Y}_{k+1}}\right]^{-1}\left[\frac{\partial \mathbf{f}^{(j)}}{\partial \mathbf{X}_k}\right], \tag{4.25}$$

where

$$\begin{gathered}\left[\frac{\partial \mathbf{f}^{(j)}}{\partial \mathbf{Y}_{k+1}}\right] = \left[\frac{\partial f_I^{(j)}}{\partial Y_{(k+1)J}}\right]_{4\times 4} \quad \text{for } I, J = 1, 2, 3, 4; \\ \left[\frac{\partial \mathbf{f}^{(j)}}{\partial \mathbf{X}_k}\right] = \left[\frac{\partial f_I^{(j)}}{\partial X_{kJ}}\right]_{4\times 4} \quad \text{for } I, J = 1, 2, 3, 4; \\ DP_j = \left[\frac{\partial \mathbf{Y}_{k+1}}{\partial \mathbf{X}_k}\right] = \left[\frac{\partial Y_{(k+1)I}}{\partial X_{kJ}}\right]_{4\times 4} \quad \text{for } I, J = 1, 2, 3, 4.\end{gathered} \tag{4.26}$$

Consider a simple periodic motion of the mapping structure of $P_{21} = P_2 \circ P_1$ as an example, sketched in Figure 9. With the mapping structure $P_{21} = P_2 \circ P_1$, one obtains

$$\begin{gathered}P_1 : \left(t_k, x_k^{(2)}, \dot{x}_k^{(1)}, \dot{x}_k^{(2)}\right) \longrightarrow \left(t_{k+1}, x_{k+1}^{(2)}, \dot{x}_{k+1}^{(1)}, \dot{x}_{k+1}^{(2)}\right), \\ P_2 : \left(t_{k+1}, x_{k+1}^{(2)}, \dot{x}_{k+1}^{(1)}, \dot{x}_{k+1}^{(2)}\right) \longrightarrow \left(t_{k+2}, x_{k+2}^{(2)}, \dot{x}_{k+2}^{(1)}, \dot{x}_{k+2}^{(2)}\right).\end{gathered} \tag{4.27}$$

Let $X_{k1} = x_k^{(2)} = x_k^{(1)}$, $X_{k2} = \dot{x}_k^{(1)}$, $X_{k3} = \dot{x}_k^{(2)}$, $X_{k4} = t_k$ because the impact occurs at $x_k^{(1)} = x_k^{(2)}$. However, $Y_{k1} = x_k^{(2)}$, $Y_{k2} = \dot{x}_k^{(1)}$, $Y_{k3} = \dot{x}_k^{(2)}$, $Y_{k4} = t_k$ because the impact occurs at $x_k^{(1)} = h$. The corresponding governing equations for such a periodic motion are

$$\begin{gathered}\mathbf{f}^{(1)}\left(\mathbf{X}_k, \mathbf{Y}_{k+1}\right) = 0 \quad \text{for } P_1, \\ \mathbf{f}^{(2)}\left(\mathbf{Y}_{k+1}, \mathbf{X}_{k+2}\right) = 0 \quad \text{for } P_2.\end{gathered} \tag{4.28}$$

For a period-N motion, the periodicity requires

$$\mathbf{X}_k^* = \mathbf{X}_{k+2}^*, \tag{4.29}$$

or

$$\begin{aligned} x_{k+2}^{*(i)} &= x_k^{*(i)}, \quad \dot{x}_{k+2}^{*(i)} = \dot{x}_k^{*(i)}, \\ t_{k+2}^* &= t_k^* + \frac{2N\pi}{\Omega}, \quad N = 1, 2, \ldots, \end{aligned} \quad \text{for } i = 1, 2. \tag{4.30}$$

Solving (4.28) and (4.29) produces the switching sets of the periodic motion. The Jacobian matrix for each mapping is developed from (4.28) for $j = 1, 2$

$$DP_j = \left[\frac{\partial\left(t_{k+j}, x_{k+j}^{(2)}, \dot{x}_{k+j}^{(1)}, \dot{x}_{k+j}^{(2)}\right)}{\partial\left(t_{k+j-1}, x_{k+j-1}^{(2)}, \dot{x}_{k+j-1}^{(1)}, \dot{x}_{k+j-1}^{(2)}\right)} \right]_{4\times 4}. \tag{4.31}$$

The total Jacobian matrix of the periodic motion can be computed by

$$DP = DP_2 \cdot DP_1. \tag{4.32}$$

The Jacobian matrix of a periodic motion with a generalized mapping structure can be developed in a similar fashion. For a stable periodic motion, the magnitudes of all the eigenvalues should be inside a unit circle, that is,

$$|\lambda_i| < 1, \quad (i = 1, 2, 3, 4). \tag{4.33}$$

If (4.33) cannot be satisfied, then the periodic motion is unstable. Consider a pair of complex eigenvalues ($\lambda_{1,2}$) and the others ($\lambda_{3,4}$). If $|\lambda_{1,2}| = 1$ and $|\lambda_{3,4}| < 1$, the Neirmark bifurcation of the periodic motion takes place. Consider two real eigenvalues ($\lambda_{1,2}$) and the others ($\lambda_{3,4}$). If $\lambda_1 = -1$ and $|\lambda_{2,3,4}| < 1$, the period doubling bifurcation of the periodic motion occurs. If $\lambda_1 = 1$ and $|\lambda_{2,3,4}| < 1$, the saddle-node bifurcation of the periodic motion occurs. However, the eigenvalue analysis cannot be used to predict stick motion and grazing bifurcation. Both of them should be determined through the normal vector fields, and the stick motion is determined by (3.19) and the grazing bifurcation is determined by (3.25).

5. Predictions and Simulations

In this section, bifurcation scenario for the generalized Fermi-acceleration oscillator will be presented first, and analytical predictions of periodic motions will be completed through the mapping structures. Periodic and chaotic motions will be presented for illustration of the analytical conditions. Poincare mapping sections will be given for illustrations of chaotic motions in the Fermi-acceleration oscillator. The parameter maps for certain parameters will also be presented for an overview of dynamical behaviors of particle and piston in the generalized Fermi-acceleration oscillator.

5.1. Bifurcation Scenario

To obtain a bifurcation scenario in the generalized Fermi-acceleration oscillator, the closed-form solutions of particle and piston in the appendix are used for numerical simulations. To achieve a motion with stick, the stick conditions in (3.21) (or $\ddot{x}_k^{(2)} > -g$) with $x_k^{(1)} = x_k^{(2)}$ and $\dot{x}_k^{(1)} = \dot{x}_k^{(2)}$ should be embedded in the computer program. Before stick, the particle and piston are separated. One should set a tolerance of the velocity difference (i.e., $|\dot{x}_k^{(1)} - \dot{x}_k^{(2)}| < \varepsilon$). In computation, the tolerance $\varepsilon < 10^{-4}$ is used for the stick motion. In other words, when $|\dot{x}_k^{(1)} - \dot{x}_k^{(2)}| < 10^{-4}$, it was assumed $\dot{x}_k^{(1)} \approx \dot{x}_k^{(2)}$. If $\ddot{x}_k^{(2)} > -g$, the stick motion of particle and piston is formed, and the two separated dynamical systems in (2.1) will be switched into one dynamical system in (2.2) for the stick motion. Further, the solution of the stick motion in the appendix will be employed for numerical simulations. For the vanishing of the stick motion, the condition in (3.24) (i.e., $\ddot{x}^{(2)}(t_{m\pm}) = \ddot{x}^{(1)}(t_{m\pm}) = -g$ and $\dddot{x}_{m\pm}^{(2)} < \dddot{x}_{m\pm}^{(1)} = 0$) will also be embedded in the computer program. Once the stick motion disappears, the system in (2.2) will become two separated dynamical systems for particle and piston in (2.1). Similarly, the grazing conditions in (3.26) and (3.28) at the corresponding boundaries were embedded in the computer program. The switching points for the particle and piston with switching times are recorded for bifurcation scenarios.

Consider a set of parameters ($m_1 = 0.01$, $m_2 = 1$, $k = 10$, $c = 6$, $e_1 = 0.9$, $e_2 = 0.7$, $h = 1$, $Q = 0.5$). Switching points on the boundaries versus excitation frequency in a bifurcation scenario are shown in Figure 10. The acronyms PD, SN, and GB denote the period-doubling, saddle node, and grazing bifurcations, respectively. In Figure 10(a), the switching displacement of particle versus excitation frequency is presented. The corresponding bifurcations are marked by dashed vertical lines and labeled by the corresponding acronyms (i.e., PD, SN, and GB). The switching displacements for particle impacts for $x^{(1)} = h$ lie on a straight line. However, the switching displacement for the particle impacting with the piston will be changed. The region for the stick motion is labeled. It is observed that the stick motion occurs at the low excitation frequency. Many windows of periodic motions are also observed. The first obvious window of periodic motions is relative to mapping P_{21} for $\Omega \in (21.2, 35.1)$ with one excitation period ($1T$). However, the second obvious window of periodic motions is relative to mapping P_{21} for $\Omega \in (59.3, 70.2)$ with two excitation periods ($2T$). In Figure 10(b), shown are the switching velocities of particle before impact. So the upper branch of the switching velocity of particle is relative to impacts at the fixed wall, and the lower branch of the switching velocity shows the impacts between the particle and piston. In Figure 10(c), the switching displacements of piston are plotted. The switching displacements of impacts and stick between the piston and particle are the same, which are presented by the lower branch of switching displacement. The upper branch of the switching displacement of piston is to record the displacement of piston at impacts between the particle and the fixed wall. Before and after such impacts, the corresponding switching velocity of the piston will not be changed. Therefore, the switching velocities of piston before impacts are plotted in Figure 10(d). In addition, switching times for switching points are very important. Thus, switching phases pertaining to the switching times are plotted in Figure 10(e). From the bifurcation scenario, the motion of P_{21} lies in the range of $\Omega \in (21.2, 35.1)$ and $\Omega \in (59.3, 70.2)$. The motion of $P_{(21)^2}$ lies in the range of $\Omega \in (19.4, 21.2)$ and $\Omega \in (56.8, 59.3)$. The motion of $P_{(21)^4}$ is in the range of $\Omega \in (19.1, 19.4)$ and $\Omega \in (56.4, 56.8)$. In the region of $\Omega \in (0, 19.1)$, the zone of complex motions exists, and in the region of $\Omega \in (35.1, 56.4)$ and $\Omega \in (70.2, 80)$,the

Figure 10: Bifurcation scenario varying with driving frequency: (a) switching displacement and (b) switching velocity of the particle; (c) switching displacement and (d) switching velocity of the piston; (e) Switching phase ($m_1 = 0.01,\ m_2 = 1,\ k = 10,\ c = 6,\ e_1 = 0.9,\ e_2 = 0.7,\ h = 1,\ Q = 0.5$).

zone of chaos exists. In the zone of complex motion, the majority of motions are observable periodic motions with complicated mapping structure. In the zone of chaos, the majority of motions is chaotic. Inside the range of $\Omega \in (0, 6.3)$ there are some simple stick motions. The detailed mapping structures for periodic motions are tabulated in Table 1.

Table 1: Summary of driving frequency for periodic motions ($m_1 = 0.01, m_2 = 1, k = 10, c = 6, e_1 = 0.9, e_2 = 0.7, h = 1, Q = 0.5$).

	Mapping Structure		Excitation Frequency		Mapping Structure		Excitation Frequency
1	$P_{(321)^2}$	P($2T$)	(6.33, 6.56)	16	$P_{(32121)^2}$	P($4T$)	(11.01, 11.05) (11.35, 11.40)
2	P_{321}	P(T)	(6.56, 7.01)	17	P_{32121}	P($2T$)	(11.05, 11.16) (11.40, 11.52)
3	$P_{(333321)^2}$	P($2T$)	(7.41, 7.43)	18	$P_{(321)^4}$	P($8T$)	(11.75, 11.76)
4	$P_{3(321)^2}$	P($3T$)	(7.60, 7.80) (7.91, 7.93)	19	$P_{(321)^2}$	P($4T$)	(11.76, 11.77)
5	$P_{(33(321)^2)^2}$	P($6T$)	(7.87, 7.89)	20	P_{3^2}	P($4T$)	(11.77, 11.79)
6	$P_{33(321)^2}$	P($3T$)	(8.05, 8.07) (8.18, 8.20)	21	P_{3^4}	P($8T$)	(11.79, 11.80)
7	$P_{3(3(321)^2)^2}$	P($6T$)	(8.27, 8.29)	22	$P_{(21)^4}$	P($4T$)	(19.15, 19.40)
8	$P_{(33321)^2}$	P($6T$)	(8.62, 8.64)	23	$P_{(21)^2}$	P($2T$)	(19.40, 21.25)
9	$P_{(3(3321)^2)^2}$	P($8T$)	(8.76, 8.78)	24	P_{21}	P(T)	(21.25, 35.20)
10	$P_{3(3321)^2}$	P($4T$)	(8.94, 8.96)	25	$P_{(21)^4}$	P($8T$)	(56.42, 56.92)
11	$P_{(3321)^4}$	P($8T$)	(9.06, 9.07)	26	$P_{(21)^2}$	P($4T$)	(56.92, 59.35)
12	$P_{(3321)^2}$	P($4T$)	(9.07, 9.09) (9.81, 9.83)	27	P_{21}	P($2T$)	(59.35, 70.23)
14	$P_{(321)^2 33(21)^2}$	P($6T$)	(10.42, 10.44)	28	Chaos & Complex Motion	—	(7.01, 19.15) (19.15, 35.20) (70.23, 80.00)
15	$P_{(32121)^4}$	P($8T$)	(11.00, 11.01) (11.34, 11.35)				

5.2. Analytical Prediction

The analytical prediction of periodic motions is based on the corresponding mapping structures presented in Section 4 to solve the switching points of the periodic motions. In other words, for a specified mapping structure, a set of nonlinear algebraic equations can be developed, and such nonlinear algebraic equations are solved by the Newton-Raphson method to obtain the switching points. The local stability and bifurcation analysis of the periodic motion are also completed as presented in Section 4.2. In this section, the analytical prediction of periodic motions will be presented via two examples, and the corresponding eigenvalue analysis will be given.

Consider $m_1 = 0.01$, $m_2 = 1$, $k = 10$, $c = 6$, $e_1 = 0.9$, $e_2 = 0.7$, $h = 1$, $Q = 0.5$. The analytical prediction of the periodic motions pertaining to mapping $P_{(21)^k}$ $(k = 1, 2, 2^2 \cdots)$ is presented in Figure 11, and the corresponding eigenvalues of the periodic motions are shown in Figure 12. The solid curves represent stable solutions, which can be observed in the bifurcation scenario. The dashed curves denote unstable solutions that may exist in the region of chaotic motion in the bifurcation scenario. The acronyms PD, SN, and GB represent the period-doubling, saddle-node, and grazing bifurcations, respectively. From this analytical prediction of periodic motions, the stable solution disappears at the saddle-node bifurcation with $\Omega = 35.20$. The unstable solution of mapping structure P_{21} disappears at the grazing bifurcation with $\Omega = 5.88$. The unstable solutions of mapping structure $P_{(21)^2}$ and $P_{(21)^4}$ vanish at the grazing bifurcations with $\Omega = 9.39$. The saddle-node bifurcation of the periodic motion

Figure 11: Analytical prediction of periodic motion for $\Omega \in (21, 35)$: (a) switching displacement and (b) switching velocity of the particle; (c) switching displacement and (d) switching velocity of the piston; (e) switching phase. Solid and dashed curves represent the stable and unstable solutions, respectively, ($m_1 = 0.01,\ m_2 = 1,\ k = 10,\ c = 6,\ e_1 = 0.9,\ e_2 = 0.7,\ h = 1,\ Q = 0.5$.)

of P_{21} occurs at $\Omega = 35.20$. This is because one of the eigenvalues equals 1 and all the others are inside a unit circle, as shown in Figure 12. At $\Omega = 19.1, 19.4,$ and 21.2, one of the eigenvalues is equal to -1 and all the others are inside a unit circle, and the period-doubling bifurcations of the corresponding motion of $P_{(21)^4}$, $P_{(21)^2}$, and P_{21} occur. At $\Omega = 5.88$ and 9.39, the grazing bifurcations of periodic motions of P_{21} and ($P_{(21)^4}$, $P_{(21)^2}$) occur, and such periodic motions are unstable. Such a grazing bifurcation should be determined from the normal vector fields and

Figure 12: Eigenvalues varying with excitation frequency for $\Omega \in (19.15, 35.20)$: (a) magnitude and (b) real part. Solid and dashed curves represent the stable and unstable solutions, respectively. ($m_1 = 0.01$, $m_2 = 1$, $k = 10$, $c = 6$, $e_1 = 0.9$, $e_2 = 0.7$, $h = 1$, $Q = 0.5$).

the corresponding grazing conditions are given in (3.26) and (3.28). The grazing bifurcations of unstable solutions imply that the corresponding motion relative to a specific mapping structure will disappear in the region of a chaotic motion. The chaotic motion will be switched into another motion with a sudden switching of motion. Both grazing and stick bifurcations give a catastrophe mechanism of the motion switching in the Fermi-acceleration oscillator.

The methodology presented in Section 4 can give the analytical prediction of any periodic motions. For a further demonstration of this technique, consider the same set of parameters except for the range of excitation frequency for the analytical prediction of periodic motions relative to the mapping structures of $P_{(132)^k}$ $(k = 1, 2, 2^2 \cdots)$. With varying excitation frequency, the switching displacements, switching velocities and switching phases of periodic motions of particles, and piston are analytically predicted and presented in Figure 13. The periodic motion of P_{132} is stable for $\Omega \in (3.24, 4.38) \cup (6.57, 7.00)$ and unstable for $\Omega \in (4.38, 6.57)$. The corresponding period-doubling bifurcations occur at $\Omega = 4.38$ and 6.57 and the saddle-node bifurcations occur at $\Omega = 3.24$ and 7.00. The periodic motion of $P_{(132)^2}$ is unstable for $\Omega \in (4.80, 6.31)$ and stable for $\Omega \in (4.38, 4.80) \cup (6.31, 6.57)$. The period-doubling bifurcations of the periodic motion of $P_{(132)^2}$ are at $\Omega = 4.80$ and 6.31, and the corresponding saddle-node bifurcations are at $\Omega = 4.38$ and 6.57. In the vicinity of $\Omega = 4.80$ and 6.31, there are many periodic motions relative to $P_{(132)^4}$, $P_{(132)^8}$, and chaos. Since the ranges of excitation frequency for periodic motions are very small, herein such analytical results will not be presented. For this analytical prediction of periodic motions relative to P_{132} and $P_{(132)^2}$, the corresponding stability and bifurcations of the periodic motions are presented through the magnitudes and real parts of eigenvalues, as shown in Figure 14. For such periodic motions, no grazing and stick bifurcation can be observed. The motion in the range of $\Omega \in (4.80, 6.31)$ is complex periodic motions and chaotic motions.

For a better understanding of the switching dynamics of the Fermi-acceleration oscillator, a parameter map for periodic motions and chaos should be developed from the analytical prediction, and the corresponding local stability and bifurcation conditions with the grazing and stick conditions will be adopted. The parameter map about excitation amplitude and frequency is presented in Figure 15 for parameters ($m_1 = 0.01$, $m_2 = 1$, $k = 10$, $c = 6$, $e_1 = 0.9$, $e_2 = 0.7$, $h = 1$). The "forbidden area" in the parameter map exists because the displacement of piston cannot pass over the location of the fixed wall. In

Figure 13: Analytical prediction of periodic motion relative to mapping structure P_{123} for $\Omega \in (3, 7.5)$: (a) switching displacement and (b) switching velocity of the particle; (c) switching displacement and (d) switching velocity of the piston; (e) switching phase. Solid and dashed curves represent the stable and unstable solutions, respectively, ($m_1 = 0.01,\ m_2 = 1,\ k = 10,\ c = 6,\ e_1 = 0.9,\ e_2 = 0.7,\ h = 1,\ Q = 0.5$).

Figure 15(a), the parameter map of (Ω, Q) for $\Omega \in (0, 200)$ and $Q \in (0, 35)$ is presented. Periodic motions of $P_{(12)^k}$ $(k = 1, 2, 2^2, 2^3, \ldots)$, $P_{(312)}$, and $P_{(312)^2}$ with complex and chaotic motions are presented. The small regions of periodic motions cannot be clearly illustrated. To illustrate such regions, a zoomed view of the parameter map in such regions is given in Figure 15(b). The regions for periodic motion of $P_{(21)^4}$, P_{312}, and $P_{(312)^2}$ are presented clearly.

Figure 14: Eigenvalues varying with excitation frequency for $\Omega \in (19.15, 35.20)$: (a) magnitude and (b) real part. Solid and dashed curves represent the stable and unstable solutions, respectively, ($m_1 = 0.01,\ m_2 = 1,\ k = 10,\ c = 6,\ e_1 = 0.9,\ e_2 = 0.7,\ h = 1,\ Q = 0.5$).

Figure 15: Parameter maps: (a) overall view, (b) zoomed view, ($m_1 = 0.01,\ m_2 = 1,\ k = 10,\ c = 6,\ e_1 = 0.9,\ e_2 = 0.7,\ h = 1$).

5.3. Simulations

From the previous section, periodic motions were analytically predicted. It is significant to illustrate motions for a better understanding of the dynamics of the Fermi-acceleration oscillator. To illustrate motions in the generalized Fermi-acceleration oscillator, the closed-form solutions of particle and piston in the appendix are used again for numerical simulations. As in the section of bifurcation scenario, the corresponding conditions for motion switching should be embedded in the computer program. The time histories of displacement and velocity of both the particle and piston will be presented for periodic motions. In addition, phase planes of both particle and piston will be illustrated. For chaotic motions, the

Poincare mapping sections consisting of the switching points on the switching boundaries will be presented. In numerical illustrations, the switching points are labeled by circular symbols. The large and small circular symbols are for the particle and piston in the Fermi-acceleration oscillator, respectively.

Consider a set of parameters ($m_1 = 0.01$, $m_2 = 1$, $k = 10$, $c = 6$, $e_1 = 0.9$, $e_2 = 0.7$, $h = 1$, $Q = 0.5$) with $\Omega = 6.7$. From the analytical prediction, one obtains the initial condition (i.e., $t_0 = 0.0741633079$, $x_0^{(1)} = 1.0$, $\dot{x}_0^{(1)} = 5.4104888431$, $x_0^{(2)} = 0.3280350633$, $\dot{x}_0^{(2)}$=1.1504459376) for a periodic motion of P_{213}. Using such an initial condition, a numerical simulation gives phase plane and the time histories of displacements and velocities of particle and piston, as shown in Figure 16. The initial conditions are marked by green circular symbols. The arrows give the directions of motions for the particle and piston. The switching points are marked by the hollow circular symbols. The corresponding mappings are labeled by ${}^{(1)}P_\lambda$, ${}^{(2)}P_\lambda$ ($\lambda = 1, 2, 3$). In Figure 16(a), the switching displacements of particle and piston at the impacting points are of the same. However, the switching velocities for both of them are different because impacts cause the velocity to be discontinuous, as shown in Figure 16(b). In Figures 16(a) and 16(b), the solid and dashed curves depict the motions of particle and piston, respectively. When the particle impacts with the fixed wall, the piston motion will not be influenced because the piston moves by itself. Thus, the piston velocity will not be changed. When the impact between the particle and piston occurs, the velocity change of the piston is very small because the particle's mass is much smaller than the piston's mass. If the particle's mass is with the same quantitative level of the piston's mass, the velocity change before and after impacts can be clearly observed. In Figure 16(c), a trajectory of particle for such a periodic motion of P_{213} in phase plane is plotted. The dashed curves are the switching boundary for the particle motion, which is given by the displacement of piston. The periodic motion of particle consists of three pieces of trajectories. Similarly, a trajectory of piston with the corresponding switching boundary for the periodic motion is plotted in Figure 16(d).

As discussed before, there are two kinds of motions for motion switching: (i) stick motion and nonstick motion. In other words, the nonstick motion means that the particle does not stay on the piston, and the stick motion means that the particle stays on the piston and moves together. In Figure 16, the impacting motion of the particle and piston without stick are presented. The stick motion of particle and piston is an important phenomenon. Thus, consider parameters ($m_1 = 0.001$, $m_2 = 1$, $k = 20$, $c = 8$, $e_1 = 0.9$, $e_2 = 0.01$, $h = 1$, $Q = 0.5$) with $\Omega = 6.5$ to demonstrate a periodic motion with stick. From the analytical prediction, one obtains an initial condition ($t_0 = 0.70442282$, $x_0^{(1)} = x_0^{(2)} = -0.38755043$, $\dot{x}_0^{(1)} = \dot{x}_0^{(2)}$=1.97682140) for a periodic motion with stick, which is relative to a mapping structure of P_{03^2}. When such an initial condition is used for a numerical simulation, the corresponding results are presented in Figure 17. Before the stick motion is formed, there are two times of the impact chattering between the particle and piston. In numerical computations, the relative velocity tolerance of the particle and piston at the impact boundary is set to be $|\dot{x}_0^{(1)} - \dot{x}_0^{(2)}| < \varepsilon = 10^{-4}$. Under such a condition, it is assumed that the velocities of particle and piston are of the same. If the chosen impact restriction coefficient is relatively large, this set of parameters is very difficult to observe the periodic motion with stick. The detailed discussion of such impact chattering can be referred to Luo and O'Connor [46, 47]. To observe the stick motion of particle and piston in periodic motions, the acceleration and jerk of particle and piston are very important from the analytical conditions for motion switching in Section 3.3. So the time histories of the displacements, velocities, accelerations and jerks of particle and piston are presented in Figure 17, and the trajectories of particle and piston in phase plane

Figure 16: Periodic motion of $P_{(213)}$ ($\Omega = 6.7$ and $e_2 = 0.7$): (a) displacement history, (b) velocity history, (c) phase plane of particle, and (d) phase plane of piston. Solid and dashed curves indicate the motion of particle and piston, respectively, ($m_1 = 0.01$, $m_2 = 1$, $k = 10$, $c = 6$, $e_1 = 0.9$, $h = 1$, $Q = 0.5$, $t_0 = 0.0741633079$, $x_0^{(1)} = 1.0$, $\dot{x}_0^{(1)} = 5.4104888431$, $x_0^{(2)} = 0.3280350633$, $\dot{x}_0^{(2)}$=1.1504459376).

are plotted in Figure 18. For the nonstick part of motion, the discussion is the same as in Figure 16. For the part of stick motion, the displacement and velocity for particle and piston should be $x^{(1)} = x^{(2)}$ and $\dot{x}^{(1)} = \dot{x}^{(2)}$, as shown in Figures 17(a) and 17(b). Such a stick motion is labeled by $({}^{(1)}P_0, {}^{(2)}P_0)$. The solid and dashed curves depict the motions of particle and piston, respectively. In addition, the condition of stick motion in (3.19) should be satisfied, which is expressed by the relative acceleration (i.e., $g^{(i)} = \ddot{x}^{(i)} - \ddot{x}^{(\bar{i})}$, $i \in \{1,2\}$) with $a^{(i)} \equiv \ddot{x}^{(i)}$, $a^{(\bar{i})} \equiv \ddot{x}^{(\bar{i})}$. The condition of stick motion (i.e., $g^{(1)} < 0$ or $g^{(2)} > 0$) gives the absolute acceleration relation (i.e., $-g = \ddot{x}^{(1)} < \ddot{x}^{(2)}$) under which the stick motion will be formed. Such a condition of stick motion is observed in the shaded area in Figure 17(c). The vanishing condition of stick motion is given by $g^{(i)} = \ddot{x}^{(i)} - \ddot{x}^{(\bar{i})} = 0$ with the relative jerks $((d/st)g^{(i)} = J^{(i)} - J^{(\bar{i})})$ with $J^{(i)} \equiv \dddot{x}^{(i)}$, $J^{(\bar{i})} \equiv \dddot{x}^{(\bar{i})}$. In other words, $(d/dt)g^{(1)} = J^{(1)} - J^{(2)} > 0$ or $(d/dt)g^{(2)} = J^{(2)} - J^{(1)} < 0$. From such conditions, one obtains $J^{(2)} < J^{(1)} = 0$ from which the stick motion will disappear. Such a condition can be observed in Figures 17(c) and 17(d). For the nonstick motion, the acceleration and jerk for particle are constants (i.e., $a^{(1)} = -g$ and $J^{(1)} = 0$). The dashed curves in Figures 17(c) and 17(d) give the acceleration and jerks for the piston. For a further observation of the analytical condition of stick, the phase plane, acceleration, and

Figure 17: Periodic motion of P_{03^4} ($\Omega = 6.5$ and $e_2 = 0.01$): (a) displacement time-history, (b) velocity time-history, (c) phase plane of particle, and (d) phase plane of piston. Solid and dashed curves represents the motions of particle and piston, respectively, ($m_1 = 0.01,\ m_2 = 1,\ k = 10,\ c = 6,\ e_1 = 0.9,\ h = 1,\ Q = 0.5$) ($t_0 = 0.70442282,\ x_0^{(1)} = -0.38755043,\ \dot{x}_0^{(1)} = 1.90527812,\ x_0^{(2)} = -0.38755043,\ \dot{x}_0^{(2)} = 1.96895970$).

jerk versus displacement for particle and piston are presented in Figure 18. In Figures 18(a) and 18(d), the trajectories of periodic motion with stick for particle and piston are presented, and the dashed curves are the displacement boundaries for motion switching. The arrows still show the directions of flows for such a periodic motion. The stick motion portions are shaded by the gray color. It is observed that the displacement and velocity for the stick motion are overlapped. To look into the condition for stick motion, in Figures 18(b), 18(c), 18(e), and 18(f), the vertical arrows show the motion switching at the switching boundaries. At the initial condition (or stick starting point), the acceleration of particle is less than the piston's acceleration (i.e., $a^{(2)} > a^{(1)} = -9.8$) with the same displacement and velocity. So the stick motion will be formed, because the jerk of particle is zero, and the jerk of piston is negative. With increasing time, the acceleration of piston becomes small. Until $a^{(2)} = a^{(1)} = -9.8$, the stick motion should still exist. However, for $a^{(2)} = a^{(1)} = -9.8$, if the jerk of the piston is still negative, then the acceleration of piston will be less than the particle's acceleration. Thus, the stick motion will disappear and the nonstick motion is formed. Such a switching mechanism of the stick motion is clearly observed in Figures 18(b), 18(c), 18(e), and 18(f). Finally, the trajectories in phase plane for a set of periodic motions are presented in Figures 19(a)–19(f) with parameters ($m_1 = 0.001,\ m_2 = 1,\ k = 20,\ c = 8,\ e_1 = 0.9,\ h = 1,\ Q = 0.5$). The mapping

Figure 18: Periodic motion of P_{3^3} ($\Omega = 6.5$ and $e_2 = 0.01$). Particle: (a) phase plane, (b) acceleration and (c) jerks versus displacement of particle. Piston: (d) phase plane, (e) acceleration, and (f) jerks versus displacement of piston, ($m_1 = 0.01$, $m_2 = 1$, $k = 10$, $c = 6$, $e_1 = 0.9$, $h = 1$, $Q = 0.5$) ($t_0 = 0.70442282$, $x_0^{(1)} = -0.38755043$, $\dot{x}_0^{(1)} = 1.90527812$, $x_0^{(2)} = -0.38755043$, $\dot{x}_0^{(2)} = 1.96895970$).

structures, other parameters, and initial conditions are listed in Table 2 for periodic motions. The green points are the starting points (initial points), and the solid and dashed curves represent the motions of particle and piston, respectively. The other periodic motions can be similarly illustrated.

Figure 19: Phase planes. Solid and dashed curves represent motions of particle and piston, respectively, ($m_1 = 0.01$, $m_2 = 1$, $k = 10$, $c = 6$, $e_1 = 0.9$, $Q = 0.5$): (a) P_{03^2} ($\Omega = 6.5$, $e_2 = 0.01$) ($t_0 = 0.70442282$, $x_0^{(1)} = x_0^{(2)} = -0.38755043$, $\dot{x}_0^{(1)} = 1.90527812$, $\dot{x}_0^{(2)}$=1.96895970), (b) P_{3^2} ($\Omega = 11.8$, $e_2 = 0.01$) ($t_0 = 0.51035890$, $x_0^{(1)} = x_0^{(2)} = 0.01422235$, $\dot{x}_0^{(1)} = 0.87719897$, $\dot{x}_0^{(2)}$=2.83304464), (c) $P_{(213)}$ ($\Omega = 6.7$, $e_2 = 0.7$) ($t_0 = 0.07416331$, $x_0^{(1)} = 1.0$, $\dot{x}_0^{(1)} = 5.41048884$, $x_0^{(2)} = 0.32803506$, $\dot{x}_0^{(2)}$=1.15044594), (d) $P_{(213)^2}$ ($\Omega = 6.5$, $e_2 = 0.7$) ($t_0 = 0.08347153$, $x_0^{(1)} = 1.0$, $\dot{x}_0^{(1)} = 5.54878710$, $x_0^{(2)} = 0.34730036$, $\dot{x}_0^{(2)}$=0.98801503), (e) P_{12} ($\Omega = 22.7$, $e_2 = 0.7$) ($t_0 = 0.08699328$, $x_0^{(1)} = 1.0$, $\dot{x}_0^{(1)} = 8.11298196$, $x_0^{(2)} = 0.04059727$, $\dot{x}_0^{(2)}$= − 1.64929089), and (f) $P_{(12)^2}$ ($\Omega = 20.4$, $e_2 = 0.7$) ($t_0 = 0.39975776$, $x_0^{(1)} = 1.0$, $\dot{x}_0^{(1)} = 8.17571963$, $x_0^{(2)} = 0.07102098$, $\dot{x}_0^{(2)}$= − 1.44790328).

Table 2: Initial conditions for periodic motions ($m_1 = 0.01$, $m_2 = 1$, $k = 10$, $c = 6$, $e_1 = 0.9$, $h = 1$, $Q = 0.5$).

Mapping	(Ω, e_2)	Initial time t_0	$(x_0^{(1)}, \dot{x}_0^{(1)})$	$(x_0^{(2)}, \dot{x}_0^{(2)})$
$P_{3^2 0}$	(6.5, 0.01)	0.70490614	(−0.38755043, 1.97682140)	(−0.38755043, 1.97682140)
P_{3^2}	(11.8, 0.01)	0.51035890	(0.01422235, 0.87719897)	(0.01422235, 2.83304464)
$P_{(213)}$	(6.7, 0.7)	0.07416331	(1.0, 5.41048884)	(0.32803506, 1.15044594)
$P_{(213)^2}$	(6.5, 0.7)	0.08347153	(1.0, 5.54878710)	(0.34730036, 0.98801503)
P_{12}	(22.7, 0.7)	0.08699328	(1.0, 8.11298196)	(0.04059727, −1.64929089)
$P_{(12)^2}$	(20.4, 0.7)	0.39975776	(1.0, 8.17571963)	(0.07102098, −1.44790328)

Finally, a chaotic motion relative to $P_{\cdots 21212121333}$ is illustrated for parameters ($m_1 = 0.01$, $m_2 = 1$, $k = 10$, $c = 6$, $e_1 = 0.9$, $e_2 = 0.7$, $h = 1$, $Q = 0.5$, $\Omega = 12$) and the initial conditions ($t_0 = 0.40024561$, $x_0^{(1)} = x_0^{(2)} = -0.22472018$, $\dot{x}_0^{(1)} = 2.58913681$, $\dot{x}_0^{(2)}$=1.19198278) are used. The time histories of displacement and velocity of particles and piston are presented in Figures 20(a) and 20(b), respectively. In addition, the trajectories of particle and piston in phase plane are plotted in Figures 20(c) and 20(d), accordingly. The particle motion is very chaotic and such a chaotic motion of particle takes a big region in phase plane. However, the trajectory of piston is in a small region of ring in phase plane. This is because the particle's mass is much smaller than the piston's mass. The impact effect on the particle motion is much bigger than on the piston motion. To further looking into the chaotic behavior of this motion, the corresponding Poincare mapping sections for such chaotic motion are presented in Figure 21. The Poincare mapping sections consist of the switching points on the switching boundaries. In Figure 21(a) and **??**, the switching displacement ($x_k^{(1)}$) and velocity ($\dot{x}_k^{(1)}$) versus switching phase ($\mathrm{mod}\,(\Omega t_k, 2\pi)$) for the particle are presented. The switching displacement and velocity of piston versus switching phase ($\mathrm{mod}\,(\Omega t_k, 2\pi)$) are also illustrated in Figures 21(c) and 21(d), respectively. The Poincare mapping sections in phase space for the particle and piston are shown in Figures 21(e) and 21(f), respectively. Again, because the particle's mass is chosen to be much smaller than the piston's mass, the impact effect on the piston is not very obvious. Therefore, one often assumed the piston have a fixed sinusoidal motion. Once the particle's mass is not much less than the piston's mass, the impact effect on the piston motion will become significant. The aforementioned assumption should not be adequate. For a further investigation of chaos in such a Fermi-acceleration oscillator, the Lyapunov exponent can be used, and the scaling properties of chaotic motions can be studied. This paper mainly focuses on the motion switching of the multimotion states. Therefore, the other investigations on nonlinear dynamics of the generalized Fermi-acceleration oscillator can be performed in sequel.

6. Conclusion

In this paper, the mechanism of motion switching of a particle in such a generalized Fermi-oscillator was studied through the theory of discontinuous dynamical systems, and the corresponding analytical conditions for the motion switching were developed. From the solutions of linear systems in each domain, the generic mappings are introduced. Further, the

Figure 20: Chaotic motion ($\Omega = 12$): (a) displacement time-history, (b) velocity time-history, (c) phase plane for particle, and (d) phase plane for piston. Solid and dashed curves represent the motion of particle and piston, respectively, ($m_1 = 0.01$, $m_2 = 1$, $k = 10$, $c = 6$, $e_1 = 0.9$, $e_2 = 0.7$, $h = 1$, $Q = 0.5$) and the initial condition ($t_0 = 0.40024561$, $x_0^{(1)} = x_0^{(2)} = -0.22472018$, $\dot{x}_0^{(1)} = 2.58913681$, $\dot{x}_0^{(2)}$=1.19198278).

mapping structures for periodic motions were developed, and such periodic motions in the Fermi-acceleration oscillator were predicted analytically. The corresponding local stability and bifurcation are carried out. From the analytical prediction, parameter maps of regular and chaotic motions were achieved for a global view of motions in the Fermi-acceleration oscillator. Illustrations of periodic and chaotic motions in such an oscillator were done. This methodology will provide a useful way to determine dynamical behaviors in the Fermi-acceleration oscillator.

Appendix

Equation of motion for the particle in the generalized Fermi-oscillator is

$$\ddot{x}^{(1)} = -g. \tag{A.1}$$

Figure 21: Poincare mapping sections of chaotic motion ($\Omega = 12$). (a) Switching displacement of particle, (b) switching velocity of particle, (c) switching displacement of piston, (d) switching velocity of piston, (e) Poincare section of particle, and (f) Poincare section of piston. ($m_1 = 0.01$, $m_2 = 1$, $k = 10$, $c = 6$, $e_1 = 0.9$, $e_2 = 0.7$, $h = 1$, $Q = 0.5$) and initial condition ($t_0 = 0.40024561$, $x_0^{(1)} = x_0^{(2)} = -0.22472018$, $\dot{x}_0^{(1)} = 2.58913681$, $\dot{x}_0^{(2)}$=1.19198278).

The solution for the particle motion is

$$\dot{x}^{(1)} = \dot{x}_k^{(1)} - g(t - t_k),$$
$$x^{(1)} = x_k^{(1)} + \dot{x}_k^{(1)}(t - t_k) - \frac{1}{2}g(t - t_k)^2. \tag{A.2}$$

Solution for Piston

The equation of motion for piston and stick motion in the extended Fermi-oscillator is

$$\ddot{x}^{(\alpha)} + 2d^{(\alpha)}\dot{x}^{(\alpha)} + \left(\omega^{(\alpha)}\right)^2 x^{(\alpha)} = \left(\omega^{(\alpha)}\right)^2 Q\cos\Omega t - 2d^{(\alpha)}Q\Omega\sin\Omega t. \tag{A.3}$$

Superscript indices ($\alpha = 0, 2$) denote stick and nonstick motions of piston, respectively. The subscripts k and $k+1$ represent the initial and final states.

Case 1 ($d^{(\alpha)} = \omega^{(\alpha)}$).

$$\begin{aligned}
x^{(\alpha)} &= \left[C_1^{(\alpha)} + C_2^{(\alpha)}(t - t_k)\right]e^{r_1^{(\alpha)}(t-t_k)} + A^{(\alpha)}\cos\Omega t + B^{(\alpha)}\sin\Omega t,\\
\dot{x}^{(\alpha)} &= C_2^{(\alpha)}e^{r_1^{(\alpha)}(t-t_k)} + r_1^{(\alpha)}\left[C_1^{(\alpha)} + C_2^{(\alpha)}(t - t_k)\right]e^{r_1^{(\alpha)}(t-t_k)}\\
&\quad - A^{(\alpha)}\Omega\sin\Omega t + B^{(\alpha)}\Omega\cos\Omega t,\\
\ddot{x}^{(\alpha)} &= 2r_1^{(\alpha)}C_2^{(\alpha)}e^{r_1^{(\alpha)}(t-t_k)} + \left(r_1^{(\alpha)}\right)^2\left[C_1^{(\alpha)} + C_2^{(\alpha)}(t - t_k)\right]e^{r_1^{(\alpha)}(t-t_k)}\\
&\quad - A^{(\alpha)}\Omega^2\cos\Omega t - B^{(\alpha)}\Omega^2\sin\Omega t,
\end{aligned} \tag{A.4}$$

where

$$\begin{aligned}
C_1^{(\alpha)} &= x_k^{(\alpha)} - A^{(\alpha)}\cos\Omega t_k - B^{(\alpha)}\sin\Omega t_k,\\
C_2^{(\alpha)} &= \dot{x}_k^{(\alpha)} + A^{(\alpha)}\Omega\sin\Omega t_k - B^{(\alpha)}\Omega\cos\Omega t_k - r_1^{(\alpha)}C_1^{(\alpha)},\\
A^{(\alpha)} &= \frac{\left(\omega^{(\alpha)}\right)^2 Q\left[\left(\omega^{(\alpha)}\right)^2 - \Omega^2\right] + 4\left(d^{(\alpha)}\right)^2\Omega^2 Q}{4\left(d^{(\alpha)}\right)^2\Omega^2 + \left[\left(\omega^{(\alpha)}\right)^2 - \Omega^2\right]^2},\\
B^{(\alpha)} &= \frac{2d^{(\alpha)}\Omega^3 Q}{4\left(d^{(\alpha)}\right)^2\Omega^2 + \left[\left(\omega^{(\alpha)}\right)^2 - \Omega^2\right]^2},\\
r_1^{(\alpha)} &= -d^{(\alpha)}.
\end{aligned} \tag{A.5}$$

Case 2 ($d^{(\alpha)} > \omega^{(\alpha)}$).

$$\begin{aligned}
x^{(\alpha)} &= C_1^{(\alpha)}e^{r_1^{(\alpha)}(t-t_k)} + C_2^{(\alpha)}e^{r_2^{(\alpha)}(t-t_k)} + A^{(\alpha)}\cos\Omega t + B^{(\alpha)}\sin\Omega t,\\
\dot{x}^{(\alpha)} &= C_1^{(\alpha)}r_1^{(\alpha)}e^{r_1^{(\alpha)}(t-t_k)} + C_2^{(\alpha)}r_2^{(\alpha)}e^{r_2^{(\alpha)}(t-t_k)} - A^{(\alpha)}\Omega\sin\Omega t + B^{(\alpha)}\Omega\cos\Omega t,\\
\ddot{x}^{(\alpha)} &= C_1^{(\alpha)}\left(r_1^{(\alpha)}\right)^2 e^{r_1^{(\alpha)}(t-t_k)} + C_2^{(\alpha)}\left(r_2^{(\alpha)}\right)^2 e^{r_2^{(\alpha)}(t-t_k)} - A^{(\alpha)}\Omega^2\cos\Omega t - B^{(\alpha)}\Omega^2\sin\Omega t,
\end{aligned} \tag{A.6}$$

where

$$
\begin{aligned}
C_1^{(\alpha)} &= \frac{r_2^{(\alpha)}\left(x_k^{(\alpha)} - A^{(\alpha)}\cos\Omega t_k - B^{(\alpha)}\sin\Omega t_k\right) - A^{(\alpha)}\Omega\sin\Omega t_k + B^{(\alpha)}\Omega\cos\Omega t_k - \dot{x}_k^{(\alpha)}}{r_2^{(\alpha)} - r_1^{(\alpha)}},\\
C_2^{(\alpha)} &= x_k^{(\alpha)} - A^{(\alpha)}\cos\Omega t_k - B^{(\alpha)}\sin\Omega t_k - C_1^{(\alpha)},\\
A^{(\alpha)} &= \frac{\left(\omega^{(\alpha)}\right)^2 Q\left[\left(\omega^{(\alpha)}\right)^2 - \Omega^2\right] + 4\left(d^{(\alpha)}\right)^2\Omega^2 Q}{4\left(d^{(\alpha)}\right)^2\Omega^2 + \left[\left(\omega^{(\alpha)}\right)^2 - \Omega^2\right]^2},\\
B^{(\alpha)} &= \frac{2d^{(\alpha)}\Omega^3 Q}{4\left(d^{(\alpha)}\right)^2\Omega^2 + \left[\left(\omega^{(\alpha)}\right)^2 - \Omega^2\right]^2},\\
r_{1,2}^{(\alpha)} &= -d^{(\alpha)} \pm \sqrt{\left(d^{(\alpha)}\right)^2 - \left(\omega^{(\alpha)}\right)^2}.
\end{aligned}
\tag{A.7}
$$

Case 3 ($d^{(\alpha)} < \omega^{(\alpha)}$).

$$
\begin{aligned}
x^{(\alpha)} &= e^{-d^{(\alpha)}(t-t_k)}\left[C_1^{(\alpha)}\cos\omega_d^{(\alpha)}(t-t_k) + C_2^{(\alpha)}\sin\omega_d^{(\alpha)}(t-t_k)\right] + A^{(\alpha)}\cos\Omega t + B^{(\alpha)}\sin\Omega t,\\
\dot{x}^{(\alpha)} &= e^{-d^{(\alpha)}(t-t_k)}\left(-d^{(\alpha)}C_2^{(\alpha)} - C_1^{(\alpha)}\omega_d^{(\alpha)}\right)\sin\omega_d^{(\alpha)}(t-t_k)\\
&\quad + e^{-d^{(\alpha)}(t-t_k)}\left(-d^{(\alpha)}C_1^{(\alpha)} + C_2^{(\alpha)}\omega_d^{(\alpha)}\right)\cos\omega_d^{(\alpha)}(t-t_k)\\
&\quad - A^{(\alpha)}\Omega\sin\Omega t + B^{(\alpha)}\Omega\cos\Omega t,\\
\ddot{x}^{(\alpha)} &= e^{-d^{(\alpha)}(t-t_k)}\left\{\left[\left(d^{(\alpha)}\right)^2 - \left(\omega_d^{(\alpha)}\right)^2\right]C_2^{(\alpha)} + 2d^{(\alpha)}\omega_d^{(\alpha)}C_1^{(\alpha)}\right\}\sin\omega_d^{(\alpha)}(t-t_k)\\
&\quad + e^{-d^{(\alpha)}(t-t_k)}\left\{\left[\left(d^{(\alpha)}\right)^2 - \left(\omega_d^{(\alpha)}\right)^2\right]C_1^{(\alpha)} - 2d^{(\alpha)}\omega_d^{(\alpha)}C_2^{(\alpha)}\right\}\cos\omega_d^{(\alpha)}(t-t_k)\\
&\quad - A^{(\alpha)}\Omega^2\cos\Omega t - B^{(\alpha)}\Omega^2\sin\Omega t,
\end{aligned}
\tag{A.8}
$$

where

$$
\begin{aligned}
C_1^{(\alpha)} &= x_k^{(\alpha)} - A^{(\alpha)}\cos\Omega t_k - B^{(\alpha)}\sin\Omega t_k,\\
C_2^{(\alpha)} &= \frac{\dot{x}_k^{(\alpha)} + A^{(\alpha)}\Omega\sin\Omega t_k - B^{(\alpha)}\Omega\cos\Omega t_k + d^{(\alpha)}C_1^{(\alpha)}}{\omega_d^{(\alpha)}},\\
A^{(\alpha)} &= \frac{\left(\omega^{(\alpha)}\right)^2 Q\left[\left(\omega^{(\alpha)}\right)^2 - \Omega^2\right] + 4\left(d^{(\alpha)}\right)^2\Omega^2 Q}{4\left(d^{(\alpha)}\right)^2\Omega^2 + \left[\left(\omega^{(\alpha)}\right)^2 - \Omega^2\right]^2},\\
B^{(\alpha)} &= \frac{2d^{(\alpha)}\Omega^3 Q}{4\left(d^{(\alpha)}\right)^2\Omega^2 + \left[\left(\omega^{(\alpha)}\right)^2 - \Omega^2\right]^2},\\
\omega_d^{(\alpha)} &= \sqrt{\left(\omega^{(\alpha)}\right)^2 - \left(d^{(\alpha)}\right)^2}.
\end{aligned}
\tag{A.9}
$$

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Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 409857, 13 pages
doi:10.1155/2009/409857

Research Article

Can Drag Force Suppress Fermi Acceleration in a Bouncer Model?

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Received 14 April 2009; Accepted 20 July 2009

Recommended by Alexander Loskutov

Some dynamical properties for a bouncer model—a classical particle of mass m falling in the presence of a constant gravitational field g and hitting elastically a periodically moving wall—in the presence of drag force that is assumed to be proportional to the particle's velocity are studied. The dynamics of the model is described in terms of a two-dimensional nonlinear mapping obtained via solution of the second Newton's law of motion. We characterize the behavior of the average velocity of the particle as function of the control parameters as well as the time. Our results show that the average velocity starts growing at first and then bends towards a regime of constant value, thus confirming that the introduction of drag force is a sufficient condition to suppress Fermi acceleration in the model.

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1. Introduction

The bouncer model consists basically of a classical particle of mass m that moves under the presence of a gravitational field g and suffers elastic collisions with an infinitely heavy and periodically moving wall. It is well known in literature [1–3] that, depending on both the combination of control parameters and initial conditions, the particle can accumulate, along the orbit, large values of energy thus exhibiting a phenomenon called as Fermi acceleration [4].

The bouncer model has been studied along last decades under different versions [5, 6] and considering several approaches including theoretical and experimental. For the theoretical approach, results like fixed point stability [7, 8], manifold crossings [9], period doubling cascades, and fully chaotic behavior were discussed [10] both as function of the

initial conditions and control parameters. For the case of experimental approach, many results have been obtained considering the presence of inelastic collisions [11–13] including the observance of locking phenomenon.

There exist many situations where one can observe Fermi acceleration in one-dimensional models. One of them is assuming that the perturbation is no longer periodic but rather stochastic [14]. This perturbation leads to a growth in the average velocity according to square root of the number of collisions of the particle with the randomly moving wall. It is more often to be studied, in such models, conditions where Fermi acceleration (FA) is feasible. Our approach in this paper is rather different. Instead of looking at conditions to produce FA, we will consider a condition to suppress such a phenomenon. Although many results are known in literature concerning the bouncer model, our approach is original in the sense we look at conditions to break down the unlimited energy growth.

In this paper, we consider the bouncer model with the particle experiencing a drag force that is proportional to its velocity. As usual in literature, we have described the model in terms of a two-dimensional nonlinear mapping for the variables velocity of the particle and time at the instant of the collisions with the moving wall. The mapping was obtained by analytical solution of Newton's law of motion. We show that the dissipation suppresses the phenomenon of Fermi acceleration.

The organization of the paper is as follows. In Section 2 we describe all the details used for the mapping construction. Section 3 presents and discusses our numerical results while final remarks and conclusions are drawn in Section 4.

2. The Model and the Mapping

In this section we describe with full details the model we are considering and the procedure used to obtain the nonlinear mapping. The system consists of a classical particle of mass m which experiences a drag force that is proportional and contrary to the particle's velocity. The particle is suffering the action of a constant gravitational field g, and it collides elastically a time varying wall whose equation of motion is given by $y_w(t) = \epsilon\cos(\omega t)$. Here ϵ is the amplitude of oscillation and ω denotes the angular frequency of the wall. As it is so common in literature [15–18] the dynamics of the model is described in terms of a two-dimensional mapping $T(v_n, t_n) = (v_{n+1}, t_{n+1})$ where v_n and t_n denote the posthit velocity of the particle and the instant of the nth collision with the moving wall, respectively.

To obtain the mapping that fully describes the dynamics of the system, we must first write down Newton's law of motion, which is given by $md\vec{v}/dt = m\vec{g} - \eta'\vec{v}$ where positive η' is the coefficient of viscosity. Defining $\eta = \eta'/m$ we can rewrite the differential equation in scalar form as $dv/dt = -g - \eta v$ where the ascendent motion has positive v, and g is the modulus of gravitational acceleration. Integrating with respect to the time this equation assuming that at the time $t = t_n$ which is the time that the particle hits the moving wall, the particle is at the position $y_p(t_n) = \epsilon\cos(\omega t_n)$ with initial velocity $v(t_n) = v_n > 0$, and then we find that

$$v(t) = \left[v_n + \frac{g}{\eta}\right]e^{-\eta(t-t_n)} - \frac{g}{\eta}, \quad t \geq t_n. \tag{2.1}$$

Since the expression of the velocity of the particle is known, we must integrate with respect to the time again the relation $dy(t)/dt = v(t)$. Then we have that the position of the

particle is given by

$$y_p(t) = \epsilon\cos(\omega t_n) + \frac{1}{\eta}\left(v_n + \frac{g}{\eta}\right)\left(1 - e^{-\eta(t-t_n)}\right) - \frac{g}{\eta}(t - t_n), \quad t \geq t_n. \tag{2.2}$$

To find the instant of the collisions of the particle with the moving wall, we must take into account two different kinds of collisions that can occur, namely, (1) successive collisions—those collisions that the particle experiences coming from inside the collision zone $y \in [-\epsilon, \epsilon]$; (2) indirect collisions—those collisions where the particle comes from outside the collision zone.

Let us discuss the equations of the mapping for case (1) first. Matching the condition $y_p(t_c) = y_w(t_c)$ we obtain the instant of the collision ($t_c = t_{n+1}$) of the particle with the moving wall. Such condition gives rise to the following transcendental equation:

$$0 = G(t_c) = \epsilon\cos[\omega(t_n + \Delta t_c)] - \epsilon\cos(\omega t_n) + \frac{g}{\eta}\Delta t_c - \frac{1}{\eta}\left(v_n + \frac{g}{\eta}\right)\left(1 - e^{-\eta\Delta t_c}\right), \tag{2.3}$$

where the interval of collision ($\Delta t_c = t_c - t_n$) $\in (0, 2\pi/\omega]$, while exist, is the smallest solution of the equation $G(t_c) = 0$ along $\Delta t_c \in (0, 2\pi/\omega]$. At the instant of the collision, we assume a noninertial system in which the moving wall is at rest, then the exchange of momentum can be obtained. Finally we return to the original inertial system, leading to a mapping of the type

$$T_M : \begin{cases} v_{n+1} = -\left[v_n e^{-\eta\Delta t_c} + \dfrac{g}{\eta}\left(e^{-\eta\Delta t_c} - 1\right)\right] - 2\epsilon\omega\sin(\omega t_{n+1}), \\ t_{n+1} = t_n + \Delta t_c. \end{cases} \tag{2.4}$$

We now discuss case (2), that is, indirect collisions. In the approach we have considered there are three steps to obtain the time the particle spent between collisions in the bouncer model. The first one is the elapsed time from the last collision up to the maximum height where the particle acquires null velocity. The second one is the time the particle spent from the maximum height up to the entrance of the collision zone. Finally the third time is the time the particle spent from the entrance of the collision zone up to experiences a collision with the moving wall again. With these steps in mind, we must first find the maximum height that the particle reaches, which is obtained by considering the case $v(t_u) = 0$ where $\Delta t_u = t_u - t_n$ corresponds to the time the particle spent from the last collision up to the maximum height. It is given by $\Delta t_u = -\ln[g/(\eta v_n + g)]/\eta$. At this time, the position of the particle is

$$y_p(t_u) = h_{\max} = \epsilon\cos(\omega t_n) + \frac{v_n}{\eta} + \frac{g}{\eta^2}\ln\left(\frac{g}{\eta v_n + g}\right). \tag{2.5}$$

From this maximum height ($h_{\max}$), the expression that describes the particle's position in the downwards motion is given by

$$y_p(t) = \epsilon\cos(\omega t_n) + \frac{v_n}{\eta} + \frac{g}{\eta^2}\ln\left(\frac{g}{\eta v_n + g}\right) + \frac{g}{\eta^2}\left(1 - e^{-\eta(t-t_u)}\right) - \frac{g}{\eta}(t - t_u), \quad \text{if } t \geq t_u. \tag{2.6}$$

The next step is to obtain the time the particle spends until the entrance of the collision zone. The instant t_d is obtained by matching the condition $y_p(t_d) = \epsilon$, leading then to the following transcendental function $Z(t) = y_p(t) - \epsilon$ that must be solved:

$$0 = Z(t_d) = \epsilon\cos(\omega t_n) + \frac{v_n}{\eta} + \frac{g}{\eta^2}\ln\left(\frac{g}{\eta v_n + g}\right) + \frac{g}{\eta^2}\left(1 - e^{-\eta\Delta t_d}\right) - \frac{g}{\eta}\Delta t_d - \epsilon, \tag{2.7}$$

where $\Delta t_d = t_d - t_u$. Since we obtain the solution of the above equation, then we found that the velocity of the particle at the entrance of the collision zone is $v_e = v(t_d)$, that is written as $v_e = g/\eta(\exp(-\eta\Delta t_d) - 1)$. From there, the expression of the particle velocity is given by

$$v(t) = e^{-\eta(t-t_d)}\left(v_e + \frac{g}{\eta}\right) - \frac{g}{\eta} \quad \text{if } t \geq t_d. \tag{2.8}$$

To find the instant of the collision of the particle with the moving wall, we must solve the following equation $F(t_c) = y_w(t_c) - y_p(t_c) = 0$ where $y_w(t)$ corresponds to the moving wall position and $y_p(t)$ is the particle position obtained by integration of (2.8). Then we found that

$$0 = F(t_c) = \frac{1}{\eta}\left(v_e + \frac{g}{\eta}\right) - \frac{g}{\eta}\Delta t_c - \frac{1}{\eta}\left(v_e + \frac{g}{\eta}\right)e^{-\eta\Delta t_c} - \epsilon\cos(\omega(t_n + \Delta t_u + \Delta t_d + \Delta t_c)) + \epsilon, \tag{2.9}$$

where $\Delta t_c = t_c - t_d$ is the smallest positive root of the above equation for the interval $\Delta t_c \in [0, 2\pi/\omega)$. Thus, in the instant of the collision we have

$$T_I = \begin{cases} v_{n+1} = -\left[e^{-\eta\Delta t_c}\left(v_e + \dfrac{g}{\eta}\right) - \dfrac{g}{\eta}\right] - 2\epsilon\omega\sin(\omega t_{n+1}), \\ t_{n+1} = t_n + \Delta t_u + \Delta t_d + \Delta t_c, \end{cases} \tag{2.10}$$

where Δt_c is obtained from the smallest solution of $F(t_c) = 0$.

We can see that there is a relatively large number of control parameters, 4 in total, namely, ϵ, η, ω, g and that the dynamics of the system does not depend on all of them. It is convenient to define dimensionless and more appropriate variables. Thus we define $V = (v_n\omega)/g$, $\delta = \omega/\eta$, $a = (\epsilon\omega^2)/g$, and measure the time in terms of the number of oscillations

of the moving wall, $\phi = \omega t$. With this new set of variables, the mapping that describes the dynamics of the model is given by

$$T = \begin{cases} V_{n+1} = -\left[(V_n^* + \delta)e^{-\Delta\phi_c/\delta} - \delta\right] - 2a\sin(\phi_{n+1}), \\ \phi_{n+1} = \left[\phi_n + \Delta\phi\right] \quad \mathrm{mod}\ (2\pi), \end{cases} \tag{2.11}$$

where the variables V_n^* and $\Delta\phi$ depend on the kind of collisions that occurs. For successive collisions we have that $V_n^* = V_n$ and $\Delta\phi = \Delta\phi_c$ where $\Delta\phi_c$ is obtained as the smallest positive solution of $G(\phi_c) = 0$ for $\Delta\phi_c \in (0, 2\pi]$, with

$$\begin{aligned} 0 = G(\phi_c) &= a\cos(\phi_n) + \delta(V_n + \delta)\left(1 - e^{-\Delta\phi_c/\delta}\right) \\ &\quad - \delta\Delta\phi_c - a\cos(\phi_n + \Delta\phi_c). \end{aligned} \tag{2.12}$$

For the case of indirect collisions, we have that $V_n^* = V_e$ where $V_e = \delta[\exp(-\Delta\phi_d/\delta) - 1]$ and $\Delta\phi = \Delta\phi_u + \Delta\phi_d + \Delta\phi_c$ with $\Delta\phi_u = -\delta\ln(\delta/(V_n + \delta))$ corresponding to the upward elapsed time, and $\Delta\phi_d$ is obtained as the smallest positive root of $Z(\phi_d) = 0$ with

$$\begin{aligned} 0 = Z(\phi_d) &= a\cos(\phi_n) + \delta V_n + \delta^2\ln\left(\frac{\delta}{V_n + \delta}\right) \\ &\quad + \delta^2\left(1 - e^{-\Delta\phi_d/\delta}\right) - \delta\Delta\phi_d - a, \end{aligned} \tag{2.13}$$

where $\Delta\phi_d$ is the elapsed time along the downward direction until the entrance of the collision zone. Finally $\Delta\phi_c$ is the smallest positive solution of $F(\phi_c) = 0$, with

$$\begin{aligned} 0 = F(\phi_c) &= a + \delta(V_e + \delta) - \delta\Delta\phi_c - \delta(V_e + \delta)e^{-\Delta\phi_c/\delta} \\ &\quad - a\cos(\phi_n + \Delta\phi_u + \Delta\phi_d + \Delta\phi_c). \end{aligned} \tag{2.14}$$

After some straightforward algebra we found that the determinant of the Jacobian matrix is written as

$$\det\begin{pmatrix} \dfrac{\partial V_{n+1}}{\partial V_n} & \dfrac{\partial V_{n+1}}{\partial \phi_n} \\ \dfrac{\partial \phi_{n+1}}{\partial V_n} & \dfrac{\partial \phi_{n+1}}{\partial \phi_n} \end{pmatrix} = A\left(\frac{V_n + a\sin(\phi_n)}{V_{n+1} + a\sin(\phi_{n+1})}\right), \tag{2.15}$$

where A depends on the kind of collisions. For multiple collisions $A = \exp(-\phi_c/\delta)$ and for indirect collisions $A = \delta\exp(-[\phi_c + \phi_d]/\delta)/(V_n + \delta)$. We note that when $\delta \to \infty$, which is equivalent to $\eta \to 0$, then $A \to 1$ and therefore all results for the non dissipative case are recovered.

Figure 1: (a) represents a chaotic attractor for the dissipative bouncer model. The control parameters used were $\delta = 60$ and $a = 10$. (b) corresponds to the convergence of the positive Lyapunov exponent for five different initial conditions for the same control parameters used in (a).

3. Numerical Results

In this section we will present and discuss our numerical results for the dissipative bouncer model. It is shown in Figure 1(a) the behavior of a chaotic attractor in phase space for the control parameters $\delta = 60$ and $a = 10$. To confirm that the orbit shown in Figure 1(a) is really chaotic, we should then evaluate the Lyapunov exponent and check whether the larger value is positive [19]. It is well known in literature that the Lyapunov exponent is a good tool that can be used to quantify the sensitivity to the initial conditions. They are defined as

$$\lambda_j = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln\left|\Lambda_j^i\right|, \quad j = 1, 2, \tag{3.1}$$

Figure 2: Positive Lyapunov exponent as function of the control parameter: (a) a and (b) δ.

where Λ_j^i are the eigenvalues of the matrix $M = \prod_{i=1}^{n} J_i(V_i, \phi_i)$ with J_i representing the Jacobian matrix of the mapping. In the appendix we show all the expressions of the Jacobian matrix for both multiple and indirect collisions.

After implementing the algorithm in a computer code, we found that the Lyapunov exponent for the attractor shown in Figure 1(a) is positive as can be seen in Figure 1(b). We have evolved in time five different initial conditions for the control parameters $\delta = 60$ and $a = 10$ and saw that both converge for long time for a constant value of $\overline{\lambda} = 2.443 \pm 0.003$.

We have also obtained the behavior of the positive Lyapunov exponent for the dissipative bouncer model as a function of the control parameters a and δ, as we can see in Figures 2(a) and 2(b). Each point in Figure 2 was obtained as an average of five different initial conditions along the chaotic attractor and evolved up to $6 \cdot 10^6$ collisions of the particle with the moving wall. We can see that the positive Lyapunov exponent grows as a function of a as well as a function of δ.

Let us now discuss the behavior of the average velocity for chaotic orbits. To obtain the average velocity we consider two different kinds of averages: (1) average over the orbit

Figure 3: Behavior of $\overline{V} \times n$ for different control parameter, as shown in the figure.

(sometimes also called as Birkoff's average) and (2) average over an ensemble of initial conditions. The average over the orbit is obtained by evolving for a long time a single initial condition so that we have

$$V_i = \frac{1}{n}\sum_{j=1}^{n} V_{i,j}, \tag{3.2}$$

where j corresponds to the number of collisions and i corresponds to a sample in an ensemble of M different initial conditions. After that, we can average over different initial conditions

leading to

$$\overline{V} = \frac{1}{M}\sum_{i=1}^{M} V_i, \tag{3.3}$$

where M is the number of different initial conditions used. It is then shown in Figures 3(a) and 3(b) the behavior of $\overline{V} \times n$ for different control parameters. We can see that the velocity grows at first and then approaches a regime of saturation for long enough time (similar results for nondissipative dynamics have also been observed [20–22]). We consider now the behavior of the velocity for long time, thus corresponding to the constant plateau. Such plateaus are clear evidence that the phenomenon of Fermi acceleration has been suppressed [14].

After a careful analysis of Figures 3(a) and 3(b) we can see that as both the control parameters rise, the value of the average velocity for long time, we call it now as $\overline{V}_{\text{sat}}$, also rises. Such a behavior allows us to suppose that

$$\overline{V}_{\text{sat}} \propto \delta^{\alpha_1} a^{\alpha_2}, \tag{3.4}$$

where α_1 and α_2 are critical exponents that must be obtained.

If we obtain the behavior of $\overline{V}_{\text{sat}}$ as function of both a and δ, we can then obtain the critical exponents α_1 and α_2. It is shown in Figures 4(a) and 4(b) the behavior of $\overline{V}_{\text{sat}}$ as a function of the control parameter a and δ. After adjusting a power law fitting, we obtain that the critical exponents are $\alpha_1 = 0.299 \pm 0.003$ and $\alpha_2 = (0.747 \pm 0.007) \cong 3/4$.

Let us now discuss the consequences of the results obtained above. If we increase the value of the control parameter a, it corresponds to enlarge the ratio of accelerations, then allowing the particle to acquire more energy upon collision with the moving wall. This result is then confirmed since the critical exponent obtained was $\alpha_2 = 0.747 \pm 0.007$. Similar result can be obtained as function of the control parameter δ. However, we must emphasize that as the control parameter δ rises, the straight of the dissipation decreases since $\delta \propto 1/\eta$. So in the limit of very large values of δ, almost no dissipation is present. For such a regime, it is then expected that Fermi acceleration should holds. Our results confirm such a supposition since the critical exponent $\alpha_1 = 0.299 \pm 0.003$ leading then to the phenomenon of Fermi acceleration for the regime of $\delta \to \infty$. However for finite values of δ, suppression of Fermi acceleration is observed. This result enlarges the condition where FA is suppressed since it is already known [14] that inelastic collision also suppress Fermi acceleration.

4. Conclusions

We have studied the bouncer model with the presence of a drag force that is proportional to particle's velocity. The model was described by using a two-dimensional nonlinear map, obtained by the solution of a Newton's law of motion. We show that chaotic attractors are present in the system and we have characterized them by use of the positive Lyapunov exponent. We have shown that the introduction of a drag force leads the particle to experience a limited energy gain. Then as to answer the question proposed in the title, the introduction of a drag force proportional to the particle's velocity can indeed be assumed as a mechanism to suppress Fermi acceleration. Such suppression is topologically demostrated by the determinant, of the Jacobian matrix. Since the pre-factor A in the determinant of the

Figure 4: Behavior of $\overline{V}_{\text{sat}}$ as function of (a) a and (b) δ. After a power law fitting we obtain that $\alpha_1 = 0.299 \pm 0.003$ and $\alpha_2 = (0.747 \pm 0.007) \cong 3/4$.

Jacobian matrix is less than the unity, attractors are present in the phase space; thus unlimited energy growth is not feasible anymore.

Appendix

We present in this appendix all the expressions of the partial derivatives of the Jacobian matrix. We start with the expressions obtained for the multiple collisions

$$\frac{\partial V_{n+1}}{\partial V_n} = e^{-\Delta\phi_c/\delta}\left[\frac{\partial \phi_c}{\partial V_n}\left(1+\frac{V_n}{\delta}\right)-1\right] - 2a\cos(\phi_{n+1})\frac{\partial \phi_{n+1}}{\partial V_n},$$

$$\frac{\partial V_{n+1}}{\partial \phi_n} = e^{-\Delta\phi_c/\delta}\frac{\partial \Delta\phi_c}{\partial \phi_n}\left(1+\frac{V_n}{\delta}\right) - 2a\cos(\phi_{n+1})\frac{\partial \phi_{n+1}}{\partial \phi_n},$$

$$\frac{\partial \phi_{n+1}}{\partial V_n} = \frac{\partial \phi_c}{\partial V_n},$$
$$\frac{\partial \phi_{n+1}}{\partial \phi_n} = \frac{\partial \phi_c}{\partial \phi_n} + 1, \tag{A.1}$$

where $\partial\phi_c/\partial V_n$ and $\partial\phi_c/\partial\phi_n$ are obtained, respectively, via implicit derivative of $G(\Delta\phi_c)$ with respect to V_n and ϕ_n. Their expressions are

$$\frac{\partial \phi_c}{\partial V_n} = \frac{\delta\left(e^{-\Delta\phi_c/\delta} - 1\right)}{e^{-\Delta\phi_c/\delta}(V_n + \delta) - \delta + a\sin(\phi_{n+1})},$$
$$\frac{\partial \phi_c}{\partial \phi_n} = \frac{a\sin(\phi_n) - a\sin(\phi_{n+1})}{(V_n + \delta)e^{-\Delta\phi_c/\delta} - \delta + a\sin(\phi_n + 1)}. \tag{A.2}$$

We now obtain the expression of the Jacobian terms for the indirect collisions

$$\frac{\partial V_{n+1}}{\partial V_n} = e^{-\Delta\phi_c/\delta}\left[\frac{\partial \Delta\phi_c}{\partial V_n}\left(1 + \frac{V_e}{\delta}\right) - \frac{\partial V_e}{\partial V_n}\right] - 2a\cos(\phi_{n+1})\frac{\partial \phi_{n+1}}{\partial V_n},$$
$$\frac{\partial V_{n+1}}{\partial \phi_n} = e^{-\Delta\phi_c/\delta}\left[\frac{\partial \Delta\phi_c}{\partial \phi_n}\left(1 + \frac{V_e}{\delta}\right) - \frac{\partial V_e}{\partial \phi_n}\right] - 2a\cos(\phi_{n+1})\frac{\partial \phi_{n+1}}{\partial \phi_n},$$
$$\frac{\partial \phi_{n+1}}{\partial V_n} = \frac{\partial \Delta\phi_u}{\partial V_n} + \frac{\partial \Delta\phi_d}{\partial V_n} + \frac{\partial \Delta\phi_c}{\partial V_n},$$
$$\frac{\partial \phi_{n+1}}{\partial \phi_n} = 1 + \frac{\partial \Delta\phi_d}{\partial \phi_n} + \frac{\partial \Delta\phi_c}{\partial \phi_n}, \tag{A.3}$$

where $\partial\phi_d/\partial V_n$, $\partial\phi_d/\partial\phi_n$, $\partial\phi_c/\partial V_n$, and $\partial\phi_c/\partial\phi_n$ are obtained by implicit derivative of $Z(\Delta\phi_c)$ and $F(\Delta\phi_c)$ with respect to V_n and ϕ_n. Their expressions are

$$\frac{\partial \Delta\phi_d}{\partial V_n} = \frac{\partial \Delta\phi_u/\partial V_n - 1}{e^{-\Delta\phi_d/\delta} - 1},$$
$$\frac{\partial \Delta\phi_d}{\partial \phi_n} = \frac{a\sin(\phi_n)}{\delta\left(e^{-\Delta\phi_d/\delta} - 1\right)},$$
$$\frac{\partial \Delta\phi_c}{\partial V_n} = \frac{\delta(\partial V_e/\partial V_n)\left(1 - e^{-\Delta\phi_c/\delta}\right) + a\sin(\phi_{n+1})\left((\partial\Delta\phi_u/\partial V_n)(\partial\Delta\phi_d/\partial V_n)\right)}{\delta - e^{-\Delta\phi_c/\delta}(V_e + \delta) - a\sin(\phi_{n+1})},$$
$$\frac{\partial \Delta\phi_c}{\partial \phi_n} = \frac{\delta(\partial V_e/\partial \phi_n)\left(1 - e^{-\Delta\phi_c/\delta}\right) + a\sin(\phi_{n+1})(\partial\Delta\phi_d/\partial\phi_n + 1)}{\delta - e^{-\Delta\phi_c/\delta}(V_e + \delta) - a\sin(\phi_{n+1})}, \tag{A.4}$$

where $\partial\Delta\phi_u/\partial V_n$, $\partial V_e/\partial V_n$, and $\partial V_e/\partial\phi_n$ are the implicit derivative of $\Delta\phi_u$ with respect to V_n and the the implicit derivative of V_e with respect to V_n and ϕ_n, respectively. They are written as

$$\begin{aligned}
\frac{\partial\Delta\phi_u}{\partial V_n} &= \frac{\delta}{V_n+\delta},\\
\frac{\partial V_e}{\partial V_n} &= -e^{-\Delta\phi_d/\delta}\frac{\partial\Delta\phi_d}{\partial V_n},\\
\frac{\partial V_e}{\partial\phi_n} &= -e^{-\Delta\phi_d/\delta}\frac{\partial\Delta\phi_d}{\partial\phi_n}.
\end{aligned} \tag{A.5}$$

Acknowledgments

F. A. de Souza and L. E. A. Simões thank FAPESP, Brazilian agency. E. D. Leonel gratefully acknowledges support to CNPq, FAPESP and FUNDUNESP, Brazilian agencies.

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Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 513023, 9 pages
doi:10.1155/2009/513023

Research Article

Dissipation Effects in the Ratchetlike Fermi Acceleration

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Received 23 April 2009; Accepted 29 June 2009

Recommended by Edson Denis Leonel

Ac-driven asymmetric pulses can be used to control the Fermi acceleration between three different motions, **A**: the *accelerated* mode, **D**: the *decelerated* mode, and **H**: the *hyperaccelerated* mode. In this paper we show that dissipation strongly affects the particles velocity, reducing the possibility for an accurate control of the dynamics. The saturation time, where the mean velocity starts to be constant due to dissipation, decays with a power law $\sim \gamma^{-\beta}$, where γ is the dissipation parameter and β is close to 1. The value of the saturated mean velocity also decays with a power law with exponent $\beta \sim 0.6$ for case **H**, and $\beta \sim 0.3$ for case **A**. In case **D** this velocity is almost constant for small dissipations.

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Fermi acceleration is a topic which got attention in various areas of physics, ranging from nonlinear physics [1–8], atom optics [9–11], plasma physics [12, 13] to astrophysics [14–16]. After the first model proposed by Fermi [17], essentially two different versions became common in the literature. In the first one, the Fermi-Ulam (FU) model, a bouncing particle moves between a fixed surface and a parallel oscillating surface [18]. In this case the regular islands in the phase space prevent the Fermi acceleration. A simplified version of this model was proposed to improve simulations [1], called the static wall model. It essentially ignores the displacement of the moving wall but keeps the information for the momentum transfer as the wall was oscillating. The dynamics of the static model was studied in different aspects [1, 3–5] and for different models [6], and the relevant result for the purpose of the present work is that invariant curves in the phase space, found for higher velocities, prevent the particle to increase its kinetic energy without bounds. Recently the hopping wall approximation was proposed [7, 8] which takes into account the effect of the wall displacement and allows the analytical estimation of the particle mean velocity. Compared to the simplified static model, the particle acceleration is enhanced. The second kind of Fermi accelerated model was proposed in 1977 by Pustylńikov [19], who considered a particle on

a periodically oscillating horizontal surface in the presence of a gravitational field. Different from the FU model, for some initial conditions and control parameters, the particle energy can grow indefinitely.

In this paper we analyze the effect of dissipation in the simplified FU model when an ac-driven asymmetric pulse controls the Fermi acceleration (deceleration) [20]. Small dissipation is inevitable in real systems and its influence on the dynamics of conservative systems is of great interest [21–23] since elliptic periodic orbits become small sinks and attractors start to exists [24, 25]. In the context of Fermi models dissipation effects have been inserted in two ways: frictional force [26] and inelastic collisions at the walls [27–29]. Here we use the second approach and the pulse is a *deformed* sawtooth driving law for the moving wall. This Ratchetlike pulse differs from the ac-driven asymmetric pulses (*symmetric* sawtooth) used for the Fermi acceleration in the early work of Lieberman and Lichtenberg [1] and proposed recently to control the motion of magnetic flux quanta [30] and to analyse the relative efficiency of mechanism leading to increased acceleration in the hopping wall approximation [8]. In the simplified Fermi model [1] the particle is free to move between the elastic impacts with the walls. Consider that the moving wall oscillates between two extrema with amplitude v_0. The gravitational force is considered zero. The system is described by a two-dimensional map $M_{1(2)}(V_n, \phi_n) = (V_{n+1}, \phi_{n+1})$ which gives, respectively, the velocity of the particle, and the phase of the moving wall, immediately after the particle suffers a collision with the wall. Considering dimensionless variables the dissipative FU map with the *deformed* sawtooth wall is written as

$$M_1 : \begin{cases} V_{n+1} = \left| (1-\gamma) V_n + \dfrac{v_0}{\eta_1} (\phi_n - \eta_1) \right|, \\ \phi_{n+1} = \phi_n + \mu \dfrac{(\eta_1 + \eta_2)}{V_{n+1}} \mod(\eta_1 + \eta_2), \end{cases} \tag{1}$$

for $\phi_n < \eta_1$, and

$$M_2 : \begin{cases} V_{n+1} = \left| (1-\gamma) V_n + \dfrac{v_0}{\eta_2} (\phi_n - \eta_1) \right|, \\ \phi_{n+1} = \phi_n + \mu \dfrac{(\eta_1 + \eta_2)}{V_{n+1}} \mod(\eta_1 + \eta_2), \end{cases} \tag{2}$$

for $\phi_n \geq \eta_1$, where n is the iteration number and μ is the maximum distance between the walls. For $\gamma \neq 0$ we have dissipation effects. Since in this simplified model the displacement of the moving wall is ignored, the modulus function is used to avoid errors due to successive collisions which may occur in the original model. In other words, if after a collision with the wall the particle continues to have a negative velocity (a successive collision will occur in the original model), the particle moves beyond the wall. The modulus for the velocity injects the particle back and fixes the problem.

The time asymmetry of the oscillating wall in (1) and (2) is controlled by varying the parameters (η_1, η_2). The *deformed* sawtooth (Ratchetlike) is obtained when $\eta_1 \neq \eta_2$. Figures 1(a)–1(c) show the time behaviour of the oscillating wall (the pulse) for different values of the asymmetry $\Delta\eta = \eta_2 - \eta_1$. The deformed sawtooth pulse is obtained when $\Delta\eta \neq 0.0$. Such pulses can be easily obtained from pulse generators. For the case of no dissipation three different

motions were obtained and controlled [20]. They are **A**: for $\Delta\eta = 0$ (symmetric sawtooth) the *accelerated* case (see Figure 2(a)); **D**: for $\Delta\eta < 0$ (asymmetric sawtooth) the *decelerated* case (see Figure 3(a)); **H**: for $\Delta\eta > 0$ (asymmetric sawtooth) the *hyperaccelerated* case (see Figure 4(a)). This classification, different from [31], is based on how fast the average velocity grows or decreases. At next we discuss separately the effects of dissipation in each case. To do this we show the corresponding phase space dynamics and determine the mean particles velocity at a given time n from

$$\langle V \rangle(n) = \frac{1}{n+1}\sum_{i=0}^{n}\frac{1}{\xi}\sum_{j=1}^{\xi}V_{n,j}, \tag{3}$$

where the index i refers to the ith iteration of the sample j, and ξ is the number of initial conditions. We iterate the map (1) or (2) for times $n = 1 \times 10^8$ and 3000 initial conditions in the interval $0 < \phi \leq \eta_1 + \eta_2$ and $0 \leq V \leq 10^{-3}$.

First we consider the accelerated mode which is similar to the simplified model studied by Lichtenberg and Lieberman [1, 2], where the harmonic force was considered. This case is shown in Figure 2. As the particle velocity increases, regular islands are observed and $\langle V \rangle$ increases slowly until ~10 (see Figures 2(a) and 2(c)). The regular islands prevent the particle velocity to increase very fast. We mention that all initial conditions start inside the chaotic region at low velocities. The growth rate of $\langle V \rangle$ depends on the number of regular islands inside the phase space. For very small dissipations ($\gamma = 10^{-6}$) the regular islands are transformed into sinks, as can be observed by comparing Figures 2(a) and 2(b) (observe the dark regions), and attract the chaotic trajectories which pass nearby. As a consequence the mean velocity cannot increase as before (compare Figures 2(c) and 2(d)). This effect increases for higher values of the dissipation parameter, as can be observed in Figures 2(e)–2(f). Therefore in case **A** the dissipation decreases the mean velocity (to a constant plateau) for any values of γ when compared to the nondissipative case. There are two additional information we can get from Figures 2(d)–2(f), the saturation time (t_s), where the mean velocity starts to be constant in time, and the saturated mean velocity (v_s). Plotting both quantities as a function of γ in the interval $10^{-6} \leq \gamma \leq 10^{-1}$ we obtain the decay $t_c \sim \gamma^{-1.0}$ and $v_c \sim \gamma^{-0.3}$.

The second case to discuss is the dissipation in the *hyperaccelerated* case **H** (see Figure 3(a)). For a very small asymmetry $\Delta\eta = 0.01$ the phase space is totally filled and no regular islands are observed (see Figure 3(a)). As shown in [20], the Lyapunov exponents approach zero for this case. The corresponding $\langle V \rangle$ increases very fast until $\langle V \rangle \sim 3 \times 10^4$ (see Figure 3(c)), showing that the accelerated mode is enhanced when compared to case **A**. By adding small dissipation ($\gamma = 10^{-6}$) into the system we observe in Figure 3(b) that again many sinks appear. In this case however, since no regular islands were observed for $\gamma = 0$, we are not able to relate the sinks to regular islands. On the other hand, we can say that the dissipation is able to find attracting points which were not visible in the conservative limit. Also here small dissipation prevents the hyperacceleration of particles. The mean particles velocity also decreases (to a constant value) due to the small sinks which appear in the phase space shown in Figure 3(b). For higher values of the dissipation parameter the maximal mean velocity decreases. Observe Figures 3(d)–3(f) for $\gamma = 0.0, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-1}$. For the hyperaccelerated case we found that the saturated quantities decay like $t_c \sim \gamma^{-1.1}$ and $v_c \sim \gamma^{-0.6}$.

Figure 1: (a)–(c) The shape of the pulses used in the simulations. The deformed sawtooth effect is obtained when $\Delta\eta = \eta_2 - \eta_1 \neq 0.0$. For (a) $\Delta\eta = 0$ we have the symmetric *accelerated* case **A**, (b) $\Delta\eta < 0$ we have the deformed *decelerated* case **D**, and (c) $\Delta\eta > 0$ the deformed *hyperaccelerated* case **H**.

Now we discuss the decelerated case **D** ($\Delta\eta = -0.01$). Here many regular islands appear in phase space (see Figure 4(a)) which again prevent the acceleration to increase without bounds, as in case **A**. The regular islands are very large so that the maximal instantaneous velocity is around ~2.5, while for case **A** it was ~70.0. The corresponding $\langle V \rangle$ remains here almost constant (see Figure 4(c)). As in the other cases, when small dissipation is added ($\gamma = 10^{-6}$) into the system, regular islands are transformed into sinks. However here the mean velocity *increases* (see Figure 4(d)). This increasing in the mean velocity is easy to explain. Since the first (from below) four islands from the conservative system are very large (see Figure 4(a)), almost all initial conditions cannot reach velocities higher than ~1.1. It is a kind of upper partial barrier for the velocities. However, when small dissipation is added, these four islands are transformed into sinks so that the chaotic trajectories can penetrate the islands and the upper barrier does not exist anymore. This makes the mean velocity to increase because the positions of the sinks are higher (in velocity) than those of the upper

Figure 2: Evolution of 500 chaotic orbits on the phase space $V \times \phi$ for the parameters $\mu = 10$, $v_0 = 0.2$ for case **A** ($\Delta\eta = 0.0$, $\eta_1 = \eta_2 = 1.00$). For (a) $\gamma = 0.0$ and (b) $\gamma = 10^{-6}$. The mean values of the velocity, calculated over 3000 trajectories, are shown in (c)–(f) and are related to the cases $\gamma = 0.0, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-1}$.

Figure 3: Evolution of 500 chaotic orbits on the phase space $V \times \phi$ for the parameters $\mu = 10$, $v_0 = 0.2$ for case **H** ($\Delta\eta = 0.01$, $\eta_1 = 1.00$, $\eta_2 = 1.01$). For (a) $\gamma = 0.0$ and (b) $\gamma = 10^{-6}$. The mean values of the velocity, calculated over 3000 trajectories are shown in (c)–(f) and are related to the cases $\gamma = 0.0, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-1}$.

Figure 4: Evolution of 500 chaotic orbits on the phase space $V \times \phi$ for the parameters $\mu = 10$, $v_0 = 0.2$ for case **D** ($\Delta\eta = -0.01$, $\eta_1 = 1.01$, $\eta_2 = 1.00$). For (a) $\gamma = 0.0$ and (b) $\gamma = 10^{-6}$. The mean values of the velocity, calculated over 3000 trajectories, are shown in (c)–(f) and are related to the cases $\gamma = 0.0, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$.

barrier. Such increase in the mean velocity when dissipation is present was also observed in [25]. For the decelerated case we found for the saturated time $t_c \sim \gamma^{-1.2}$ while v_c is constant in the interval $10^{-6} \leq \gamma \leq 10^{-2}$ and decreases for $\gamma = 0.1$.

To conclude, for a long time the Fermi acceleration has been studied in different models and applications [1–19, 30], but just recently [20] a deformed sawtooth (Ratchetlike) pulse was proposed to control the Fermi de(acceleration). With changing the asymmetry parameter from the Ratchetlike pulse it was possible to get Fermi hyperacceleration and deceleration. By switching the pulse between hyperacceleration and deceleration modes an accurate control of the particles velocity was achieved. Beside the remarkable control of velocities obtained in the dissipation free problem, we observe here that dissipation effects transform regular islands into sinks, change the mean velocity and reduce the ability to control accurately the particle dynamics. The saturation time decays with a power law $t_c \sim \gamma^{-\beta}$, where β is close to 1.0 (± 0.01) for all considered cases. On the other hand, the saturated mean velocity also obeys the power law decay, but with exponents $\beta \sim 0.6$ for case **H**, and $\beta \sim 0.3$ for case **A**. In the decelerated case the saturated velocity is almost constant for small dissipations. Our results motivate further analysis related to the control of Fermi de(acceleration) for larger Ratchetlike asymmetries of the ac-driven pulse in the presence of dissipation [32] and in the hopping wall approximation. In the later case we expect to obtain a better efficiency to control the particles velocity since the energy gain per collision is not underestimated as in the present model. In addition, it would be interesting to analyse the implementation of the Ratchetlike ac-driven pulse applied to the walls of quasi-one-dimensional billiards which are coupled to thermal baths at different temperatures. In such cases it is desirable to optimize and achieve the control of directed heat conduction [33], which could leave to an important increase of the thermal efficiency in physical devices such as rectifiers and thermal transistors [34].

Acknowledgment

The authors thank CNPq and FINEP (under project CTINFRA-1) for financial support.

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Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 531024, 10 pages
doi:10.1155/2009/531024

Research Article

Time-Dependent Circular Billiard

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Received 6 May 2009; Revised 8 July 2009; Accepted 15 July 2009

Recommended by Edson Denis Leonel

We investigate a time-dependent circular billiard with a two-frequency driving function and derive a new simplified form for the map, which is a symplectic nontwist map. Stability boundaries and reconnection thresholds are derived for fixed points and period-two vortex pairs. An island interspersal condition is derived such that neighboring island chains of the first frequency are exactly separated by those of the second. The results show a strong enhancement of the KAM barrier to chaotic diffusion with increasing second frequency content and overall amplitude.

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1. Introduction

There is much current interest in the construction of billiard maps for systems with moving walls. For an introduction to the physics of particle confinement with moving walls, see Loskutov et al. [1]. For single frequency motion, it is well known that Fermi acceleration does not occur in the Fermi map with smooth wall motion, but does occur for a sawtooth profile [2]. In the case of the circular billiard Kamphorst and de Carvalho [3] (KdC) have proven that if the wall motion $R(t)$ is T-periodic and C^7 then Fermi acceleration does not occur, which suggests a need for numerical experiments. Two-frequency techniques were first used as a means of enhancing plasma wave heating [4]. Indeed, the two-frequency Fermi map was studied as a simplified model for plasma wave heating [4]. Each frequency carries its own series of resonances, and parameters are chosen such that the resonant islands are interspersed at a desired location. For a nearly integrable system each island chain possesses a surrounding chaotic layer, and it is these layers that "overlap", destroying local invariant circles and thereby enhancing local chaoticity. Later on, similar techniques were applied to the important physical problem of microwave ionization of Rydberg atoms [5]. Here the system is 3/2 degrees of freedom and one must make several canonical transformations in order to identify local wave-particle resonances. An experiment was designed and successfully run [6, 7], which verified a predicted enhancement in the ionization curve.

Figure 1: Geometry of the circular billiard map.

For billiards the motivating question is the existence or nonexistence of Fermi acceleration, that is, unbounded energy gain. KdC [3] showed that, based on an unpublished theorem of Douady [8], space-spanning invariant circles exist in a 2D map for harmonic oscillations of a circular billiard. On the other hand it has recently been shown that the four-dimensional time-dependent elliptic billiard exhibits Fermi accleration [9]. Here we modulate a simplified version of the 2D KdC map with a second frequency in such a way that the total energy of the two components is preserved, a direct generalization of earlier work [10]. An island interspersal condition is derived such that the second frequencies' islands fall exactly halfway in-between those of the first. The results show significant raising of the KAM barrier to global diffusion with increasing harmonic content. An important difference from previous examples is the presence of nontwist islands [11], which occur in pairs and can complicate the island overlap criterion.

2. The Circular Billiard Map

Following KdC we consider a breathing billiard of radius $R = R_0(1 + A\sin(2\pi t))$, where A is the oscillation amplitude. In the spirit of the classical Fermi map [2] we fix $R = R_0$ and ignore multiple collisions with the wall. For specular collisions the angular momentum is conserved and we map (v_r, t) from collision to collision. Figure 1 illustrates the geometry, where the angular momentum v_θ is preserved. As in [10], we neglect variations in R (which we fix at unity) but retain $V(t) = \dot{R}$ to obtain

$$T : \begin{cases} v_r' = v_r - 2V(t), \\ t' = t + \dfrac{2v_r'}{v_r'^2 + v_\theta^2} \quad (\mathrm{mod}\,1), \end{cases} \tag{2.1}$$

where v_r is the particle's radial velocity and $V(t)$ is the specified wall velocity. It has generating function

$$F_2(t', v_r) = t'v_r - \ln\left(v_r^2 + v_\theta^2\right) - 2\int^{t'} V(t')\,dt' \tag{2.2}$$

Figure 2: Comparison of circular billiard map with Fermi map for $M = 2$ and (a) $v_\theta = 19.94$ and (b) $v_\theta = 19.997$. The island centers have been shifted to $\phi = 0$.

and is therefore symplectic. The first equation follows from Figure 1, from which the half-chord $C = (1/2)v'\Delta t = R\sin\phi = Rv_r/v'$, where ϕ is the angle between v and v_θ. It has unit Jacobian and when $v_\theta \mapsto 0$, reduces to the area preserving Fermi map. The factors of two arise from the fact that there is no "there and back" for the billiard map. The same map can also be obtained by setting R = constant in KdC's somewhat more complex version. It agrees well with KdC's full map.

Before analyzing this map we first transform it so that direct comparison with the Fermi map is possible. First let $V(t) = V_0\sin(\omega t)$ and introduce the phase $\phi = \omega t$. Then, with $v_r \to 2V_0 v_r, v_\theta \to 2V_0 v_\theta$ and taking $M = \omega/2\pi V_0$ gives

$$T_2 : \begin{cases} v_r' = v_r - \sin\phi, \\ \phi' = \phi + (2\pi M)\dfrac{v_r'}{v_r'^2 + v_\theta^2} \quad (\mathrm{mod}\, 2\pi). \end{cases} \tag{2.3}$$

In the limit as $v_\theta \to 0$ we recover the Fermi map [2], for which $\phi' = \phi + 2\pi M/v_r$. Figure 2 compares the map T_2 with the Fermi map. For $v_\theta = 0$ the maps are identical but for $v_\theta = 16.5$ nontwist features appear at the bottom of the billiard map.

2.1. Stability of Fixed Points

The fixed points are given by $\phi = (0, \pi)$ and

$$\frac{Mv_r}{v_r^2 + v_\theta^2} = k \tag{2.4}$$

which occur in pairs

$$v_r^{\pm} = \frac{1}{2}\left(\frac{M}{k} \pm \sqrt{\left(\frac{M}{k}\right)^2 - 4{v_\theta}^2}\right). \tag{2.5}$$

They are linearly stable for $|\tau| = |\mathrm{Tr}\,\mathcal{J}| < 2$, where $\mathcal{J}$ is the Jacobian matrix of the map [2, 12]. We find, for $k = 1$,

$$0 < 2\pi M \cos\phi \frac{v_\theta^2 - v_r^2}{\left(v_r^2 + v_\theta^2\right)^2} < 1. \tag{2.6}$$

With the help of the equilibrium condition (2.4) this yields the stability boundaries.

Saddle-Node Bifurcation ($\tau = 2$)

We have either $v_r = 0$ or

$$v_r = v_\theta = \frac{M}{2}. \tag{2.7}$$

Period-Doubling Bifurcation ($\tau = -2$)

There is

$$\frac{\pi}{2M}\cos\phi\left(v_\theta^2 - v_r^2\right) = v_r^2. \tag{2.8}$$

For $\phi = 0$ we find either $v_r = 0$ or

$$\begin{aligned} v_r &= \frac{M\pi}{2(\pi + M)}, \\ v_\theta &= v_r\sqrt{\frac{\pi + 2M}{\pi}}, \end{aligned} \tag{2.9}$$

and for $\phi = \pi$, either $v_r = 0$ or

$$\begin{aligned} v_r &= \frac{M\pi}{2(\pi - M)}, \\ v_\theta &= v_r\sqrt{\frac{\pi - 2M}{\pi}}. \end{aligned} \tag{2.10}$$

Note that the second case is unlikely to occur, since it requires $M < (1/2)\pi$.

2.2. Reconnection

Reconnection is a global bifurcation wherein fixed points do not change type, but the separatrix layers surrounding them undergo a topological rearrangement [11]. As the effect is generic and described in detail elsewhere [11], we will be succinct. Near a resonance we have the approximate Hamiltonian [2]

$$\begin{aligned} H(t, v_r, n) &= \pi M \ln\left(v_r^2 + v_\theta^2\right) + \cos\phi\,\delta_1(n) \\ &\approx \pi M \ln\left(v_r^2 + v_\theta^2\right) - 2\pi v_r + \cos\phi, \end{aligned} \tag{2.11}$$

where $\delta_1(n) = \sum_{m=-\infty}^{\infty} \delta(n-m)$. In deriving (2.11) we Fourier analyzed $\delta_1(n)$ and averaged over higher-order resonances. We also Taylor expanded H in the vertical direction.

The reconnection threshold is given by equating the value of $\overline{H}$ on neighboring separatrices. With this definition, we calculate $\overline{H}_{\text{lower}} = 1$ and

$$\overline{H}_{\text{upper}} = \pi M_e \ln\left(\frac{(v_r^+)^2 + v_\theta{}^2}{(v_r^-)^2 + v_\theta^2}\right) - 2\pi\left(v_r^+ - v_r^-\right) - 1, \tag{2.12}$$

where $v_r^\pm = (1/2)(M_e \pm \sqrt{M_e^2 - 4v_\theta^2})$. Equating $H_{\text{upper}} = H_{\text{lower}}$ then gives

$$\frac{1}{\pi M_e} = \ln\left(\frac{1 + \sqrt{1-\tilde{v}^2}}{\tilde{v}}\right) - \sqrt{1-\tilde{v}^2}, \tag{2.13}$$

where $\tilde{v} = 2v_\theta/M_e$. Figure 3 shows a typical reconnection with $M = 20$. Equation (2.14) shows that this occurs for $v_\theta = 9.355$, in excellent agreement with the figure. Essentially perfect agreement occurs for larger M, where the chaotic layer is thinner.

2.3. Vortex Pairs

Vortex pairs also exist for this map. Figure 4 shows such a pair for $M = 20$ and $v_\theta = 19.94$ and 19.997. To understand this structure, consider a general period-two orbit, for which $\sin\phi + \sin\phi' = 0$, so that either $\phi' = -\phi$ or $\phi' = \phi - \pi$. The former orbits do not exist for all parameters, being born in a pitchfork bifurcation. The latter orbits turn out to have $\phi = \pi$, $\phi' = 0$, with $2Mv_r/(v_r^2+v_\theta^2) = 1$. They are stable for $\pi M_e|f'(v_r)| < 1$, where $f(v_r) = v_r/(v_r^2+v_\theta^2)$. It follows that $v_r = v_r'$ and $v_r^2 - 2Mv_r + v_\theta^2 = 0$, so that

$$v_r = M \pm \sqrt{M^2 - v_\theta^2}. \tag{2.14}$$

Thus, vortex pairs annihilate when $v_\theta = M$, in agreement with Figure 4. They are formed when x-points of adjacent island chains merge. Note that vortex pair configuration is extremely ephemeral and therefore probably of little importance in particle transport. For details see [11].

Figure 3: Circular Billiard map for $M = 20$ and (a) $v_\theta = 9.4$, (b) $v_\theta = 9.3$, illustrating reconnection.

Figure 4: Vortex pair formation for circular billiard map for $M = 20$ and (a) $v_\theta = 19.99$, (b) $v_\theta = 19.997$.

3. A Two-Frequency Map

Following the program of [10] we employ the bichromatic wall function

$$V(\phi) = \frac{V_a \sin(\omega_a t) + V_b \sin(\omega_b t)}{\sqrt{V_a^2 + V_b^2}}, \tag{3.1}$$

where V_a and V_b are the amplitudes, and ω_a and ω_b are the frequencies of the two components. The condition

$$V_a^2 + V_b^2 = \text{const} \tag{3.2}$$

maintains constant energy in the two waves, while the frequencies are chosen such that

$$\frac{\omega_a}{\omega_b} = \frac{a}{b} \tag{3.3}$$

where a and b are coprime integers, in order to maintain periodicity.

Defining the phase $\phi = \omega_a t/a$, the map T_2 can be written

$$T_3 : \begin{cases} v_r' = v_r - V(\phi), \\ \phi' = \phi + \left(\dfrac{4\pi M}{a+b}\right)\dfrac{v_r'}{v_r'^2 + v_\theta^2} \quad (\text{mod}\, 2\pi), \end{cases} \tag{3.4}$$

where $M = \overline{\omega}/\pi$ and

$$\frac{\omega_a}{a} = \frac{\omega_b}{b} = \frac{\overline{\omega}}{a+b} \tag{3.5}$$

with $\overline{\omega} = (1/2)(\omega_a + \omega_b)$. Thus (3.1) becomes

$$V(\phi) = \frac{[\sin(a\phi) + \mu \sin(b\phi)]}{\sqrt{1+\mu^2}}, \tag{3.6}$$

where $\mu = V_b/V_a$.

3.1. Island Interspersal

For the map T_3 we may distinguish three distinct families: *common* resonances, which are shared by both a-and b-fold frequencies and defined by $\phi \mapsto \phi + 2\pi k$,

$$\frac{v_r}{v_r^2 + v_\theta^2} = \frac{k}{M_e} \tag{3.7}$$

with effective $M_e = 2M/(a+b)$ or

$$v_r^2 - \frac{M_e}{k} v_r + v_\theta^2 = 0 \tag{3.8}$$

with solution

$$v_{rk}^{\pm} = \frac{1}{2}\left(\frac{M_e}{k} \pm \sqrt{\left(\frac{M_e}{k}\right)^2 - 4v_\theta^2}\right). \tag{3.9}$$

The angle ϕ_0 is given by

$$\sin a\phi + \mu \sin b\phi = 0. \tag{3.10}$$

In addition, when $\mu = 0$ there exists a set of fixed points given by $a\phi \mapsto a\phi + 2\pi k$, which includes the common set (3.7):

$$\frac{av_{rk}}{v_{rk}^2 + v_\theta^2} = \frac{k}{M_e}. \tag{3.11}$$

These fixed points have period P, where P/Q is a/b reduced to the lowest terms. Similarly, when $\mu \mapsto \infty$, setting $b\phi \mapsto b\phi + 2\pi k$ yields period-b fixed points, given by

$$\frac{bv_{rk}}{v_{rk}^2 + v_\theta^2} = \frac{k}{M_e}, \tag{3.12}$$

which also includes (3.7) as a subset. At intermediate values of μ the a- and b-fold islands at $\phi_0 = 0$ and π are unperturbed, while the other fixed points move according to (3.10), which locates $\phi_0(\mu)$.

Figure 5 compares the two-frequency Billiard map for $a = 3$, $b = 2$, $v_\theta = 16$ for $\mu = 0.0$ and $\mu = 0.25$. The general effect follows closely that for the Fermi map [10]. For $\mu = 0.25$ one can see the groups of three a-fold islands interspersed with groups of two b-fold islands. One can also see chains of five islands, due to nonlinear mixing of the 2 and 3-fold chains. Increasing the admixture μ causes the amplitude of the b-fold islands to increase to the point where two neighboring a-chains "overlap." The net effect is to raise the KAM barrier; maximal chaos is attained for $\mu \approx 0.5$. Figure 6 shows the result of iterating a single orbit with intial condition $\phi = 0$, $v_r = 10$. For $\mu = 0.0$ the orbit hangs up on the twistless torus near $v_r = 20$, but for $\mu = 0.5$ it breaks through and reaches a new KAM barrier near $v_r = 70$. Interesting complications can arise due to pairs of islands and their bifurcations. Using the "two-thirds rule," one can be quite quantitative about overlap, [10] but that is not our present goal, which is to demonstrate that the KAM barrier can indeed be raised substantially.

4. Discussion

We have derived a simplified version of the circular billiard map, which promises to be very useful in parametric studies. As an example we have studied the effects of two frequencies and demonstrated the raising of the KAM barrier to diffusion. The map can be "straightened"

Figure 5: A Two-frequency Fermi map for $M = 600,\ a = 3,\ b = 2,\ v_\theta = 16$ for (a) $\mu = 0.0$, (b) $\mu = 0.25$.

Figure 6: Single orbits; 10^6 iterations for $M = 600,\ a = 3,\ b = 2,\ v_\theta = 16$ (a) with $\mu = 0$, (b) $\mu = 0.5$.

by adding appropriate terms to the map. It was surprising to encounter a nontwist map and one wonders how this fact would manifest itself in the 4D elliptic billiard. We emphasize that the chaos enhancement effect of using two frequencies is *not* Fermi acceleration; the motion is still firmly bounded by the existence of space-spanning invariant circles.

Acknowledgments

It is a pleasure to acknowledge the contributions of Professor J. D. Meiss, A. Loskutov, S. M. Soskin and S. Kamphorst. A preliminary version of this paper was given at the Conference Billiards09, in Aquas de Lindóia, Brazil. The author is grateful to Professor E. Leonel for inviting him to this conference.

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Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 345947, 11 pages
doi:10.1155/2009/345947

Research Article

Simulation of Inhomogeneous Columns of Beads under Vertical Vibration

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Received 27 November 2008; Revised 21 January 2009; Accepted 9 February 2009

Recommended by Edson Denis Leonel

We present results from computational experiments of one-dimensional inhomogeneous granular column. With characteristic properties lying between those of liquids and solids, this class of material consists of assemblies of solid particles which interact via mechanical forces (contact and friction) and are maintained together by a gravitational field. The system is composed of a vertical column of spherical beads driven by a sinusoidally vibrating plate. Results for 10 bead experiments show the fluidization and condensation phenomena, both dependent on the amplitude and frequency of the driving plate. Some scaling properties arising from the dynamics of the beads are also shown.

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1. Introduction

Granular materials are present in everyday life playing a critical role in several fields of applications such as pharmaceutical, cosmetic, and housing industries. Knowlegde of granular materials is essential to improve procedures involving extraction, transport, storage, and mixing processes. Each process can present desired and undesired phenomena occasioned by intrinsic features of the granular dynamic. Moreover, understanding these proprieties turns out to be important to minimize handling costs such as waste of energy. Granular flow systems can present clogging such that mixture procedures may produce segregation when handling the material in an inadequate manner [1]. Due to the large and practical applicability, this field was first restricted to engineering studies. However, as granular systems can also exhibit period doubling followed by chaotic behavior, they have being investigated by physicists and computational mathematicians.

Earlier studies [2–4] have investigated the collective behavior of a single column of beads composed by identical spheres, with the same radius and mass. These works rely on

Figure 1: Model of a vertical column of spheres under gravity and driven by a vibrating plate.

a simple model with minimum degree of details, namely, neglecting rotational movement and tangential components to describe specific features of the granular material. Results display different regimes similar to solids and fluids just by adjusting a single parameter. Other analyses quantify dilation of the column and energy transfer to the system, which are also important to model the behavior of the granular column.

Later works focused on describing the granular dynamic by using more realistic models. Some authors extended one-dimensional studies to two-dimensional columns and included rotational movement in the model [5, 6] and verified through a two-dimensional study and in the absence of gravity clustering formation using a velocity dependent coefficient of restitution [7]. Other improvements include applying more realistic interaction contact models taking into account different tangential forces such as the ones proposed by Cundall and Strack [8], Walton and Braun [9].

In this paper, we investigate the occurance of fluidization and condensation regimes in a column of beads having an inhomogeneous distribution of mass. In the fluidized regime, the motion of the individual particles looks erratic, and the beads are well separated, while in the condensed regime, the motion becomes more regular with the beads moving as a whole and locked on the excitation frequency of the drive source [2]. As is widely found in granular mixtures of particles with different densities of mass, a change in mass distribution significantly impacts the dynamics of the system upon the inclusion of an additional control parameter, namely, the beads mass ratio. Since clustering and condensation regimes of nearby particles consist of a collective motion with low or nearly zero relative velocities, changes in the velocity equations can modify the onset conditions of such processes.

The present paper is organized as follows. Section 2 explains both the model and the numerical procedures used to simulate a column of beads driven by a sinusoidally vibrating plate. Results are given in Section 3 which addresses discussions on fluidization and condensation states as well as scaling laws related to the center of mass of the column. Conclusions are presented in Section 4.

2. Numerical Procedure

The simulation system consists of a vertical column of spheres with radius r and mass m_i, $i = 1, 2, \ldots, N$ driven by a sinusoidally oscillating plate of infinitely large mass, as shown in Figure 1. Spheres are allowed to move only in the vertical direction and, therefore, they cannot interchange positions. The coefficient of restitution is assumed to be constant and, therefore, dissipation only occurs during collisions between particles with a coefficient of

restitution $0 < e < 1$. Dissipation may also occur between the lowest sphere and the plate, according to coefficient of restitution e_w.

Collisions between spheres are instantaneous and head-to-head since there is no angular momentum transfer between particles. The post collisions velocities follow the relations

$$v_1 = \frac{m_2 u_2 (1+e) + u_1 (m_1 - m_2 e)}{m_1 + m_2},$$
$$v_2 = \frac{m_1 u_1 (1+e) + u_2 (m_2 - m_1 e)}{m_1 + m_2}, \tag{2.1}$$

where u_i and v_i are the corresponding velocities of the ith particle before and after collision. A similar treatment quantifies the interaction of the lowest sphere with the base, which is driven by a sinusoidal signal $A\sin(\omega t)$, where A is the amplitude and $\omega = 2\pi f$ the angular frequency. The instant of collision t^* between the bottom sphere and the base has to be determined by solving a system of equations of motion doing $z(t^*) = S(t^*)$ as follows:

$$z(t) = z_0 + v_0 t + \frac{g(t)^2}{2},$$
$$S(t) = A\sin(\omega t). \tag{2.2}$$

This nonlinear model is often referred to as the bouncing ball model studied due to the richness of nonlinear and chaotic phenomena [10–12]. In order to avoid numerical problems in the computation, it is more convenient to use dimensionless variables. They are obtained according to the scheme:

$$\tilde{t} \equiv \tau = \omega t, \qquad \tilde{z} = \frac{z}{g/\omega^2}, \qquad \tilde{v} = \frac{v}{g/\omega}, \qquad \Gamma = \frac{Aw^2}{g}, \tag{2.3}$$

such that (2.2) reduces to

$$\tilde{z}(\tau) = \tilde{z}_0 + \tilde{v}_0 \tau - \frac{\tau^2}{2},$$
$$\tilde{s}(\tau) = \Gamma \sin(\tau), \tag{2.4}$$

where Γ is the nondimensionalized acceleration.

By using numerical methods [2], the next collision time τ^* is calculated, and the postcollision velocity is updated according to following collision rule:

$$\begin{pmatrix} \tilde{v}_0 \\ \tilde{v}_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 + e_w & -e_w \end{pmatrix} \begin{pmatrix} \tilde{u}_0 \\ \tilde{u}_1 \end{pmatrix}, \tag{2.5}$$

where $\tilde{u}_0$ and $\tilde{v}_0$ are the velocities of the base, and e_w is the coefficient of restitution between the sphere and the base. The collision time is calculated through Newton's method, from

which the crossings of a sinusoidal function with a parabolic curve are determined. The nondimensionalization procedure ensures finding only two roots, one negative and the other positive, where the later is the solution sought.

A similar procedure holds for collisions between the upper spheres, $2 \leq i \leq N$, which follow between events (a collision or a sudden change of momentum) the equation of motion

$$\tilde{z}_i(\Delta\tau) = \tilde{z}_{i,0} + \tilde{v}_{i,0}\Delta\tau - \frac{1}{2}(\Delta\tau)^2 \tag{2.6}$$

with $\Delta\tau$ denoting the time span between two consecutive collisions, the first of which at the time τ_0 assigning the initial position $z_{i,0}$ and initial velocity $v_{i,0}$ of a generic sphere. The second event or collision between two spheres, is identified by $\tilde{z}_i(\Delta\tau) = \tilde{z}_{i-1}(\Delta\tau)$ which gives

$$\Delta\tau = -\frac{\tilde{z}_{i,0} - \tilde{z}_{i-1,0}}{\tilde{v}_{i,0} - \tilde{v}_{i-1,0}}, \tag{2.7}$$

where $\tilde{z}$ and $\tilde{v}$ are the position and the velocity of ith particle, respectively. A negative solution $\Delta\tau < 0$ means a collision in the past (at a time prior to τ_0) or, equivalently, that the two spheres are moving away from each other, while a solution $\Delta\tau = 0$ implies that the spheres remain in contact. Solutions of interest are those $\Delta\tau > 0$ thus meaning the occurance of a forward collision. Once $\Delta\tau$ is determined, the position and velocity of all spheres are updated, and the system is released thereafter to evolve according (2.6) until a next event is identified [2]. All calculated values are stored, and the smallest positive τ^* determines the next collision time, which, in turn, is numerically iterated to calculate the initial values for the forward collision. In between events, we assume that all particles describe trajectories following the classical laws of mechanics. Positions and velocities of all particles must be updated in each event, and the ordered list of time events is managed by adding the latest event times.

3. Results

First, setting the parameters $N = 2$ spheres, $\Gamma = 1.0$, $e_w = 1$, $0.6 < e < 0.9$ (e is the restitution coefficient for collision between spheres), and $m_b/m_t = 2$ (indices b and t correspond, resp., to bottom and top) leads to a steady-state motions in which both of the balls execute 1-periodic motions which are out of phase by half a period. This is a nice example of a ball bouncing over another bouncing ball, which impacts on the harmonically driven table. In performing the simulation, the two balls are dropped from arbitrary heights and after the transient is finished, the balls synchronize mutually by executing steady-state motions that consist of equal jumps shifted by a half oscillation period.

This periodic mode is stable and presents a large basin of attraction for dissipation values, as we found periodic convergence within the interval of $0.6 < e < 0.9$. For matter of simplicity, we keep e_w unit to quantify the overall dissipation by using a single parameter. Obviously, the amount of dissipated energy decreases when the restitution coefficient increases within the range $0.6 < e < 0.9$. This 2-ball system, in which the balls stay apart during their flights, bears a similarity to the segregation process in a binary granular mixture, wherein two granular materials differing by a physical property (be it, e.g., density, shape, or size) exhibit an inexorable propensity for segregation. In fact, this result is not achievable

Figure 2: At $\Gamma = 1.0$, $e = 0.9$, $e_w = 0.95$, $m_b/m_t = 2$, we obtain an extended solution for the 1-mode periodic motion of the bouncing ball model. The lower ball hits the base and the upper ball at the same phase of oscillation.

Figure 3: At $\Gamma = 1.5$, $m_t = m_b$, $e = 0.92$, and $e_w = 1.0$, the homogeneous string of ten balls is locked in the condensation regime. The frequency spectrum of the center-of-mass motion shows subharmonic responses from the driving source.

when we set a unit mass ratio, $m_b/m_t = 1$. We note that convection streams play a key role in the segregation of a two-dimensional granular system [13, 14], while in our one-dimensional model (Figure 2) segregation is accounted for timing and synchronization effects.

Using $N = 10$ spheres of the identical masses, $\Gamma = 1.5$, $e = 0.92$, and $e_w = 1$, we obtain the condensation regime with collective motion of particles for which the center of mass also describes a periodic motion (Figure 3). All the beads remain clustered together and remain collectively in phase with the driving plate. In a fashion quite identical to that of an inelastic ball, the beads move in the form of a single block (but not necessarily in contact), and the corresponding frequency spectrum of the center-of-mass motion displays subharmonics of the driving source, as shown by the inset in Figure 3. This regime no longer appears if we introduce an inhomogeneous mass distribution; motion of the beads is erratic, much like particles in a gas, with the center of mass describing a chaotic trajectory. Results for $m_i = 2$,

Figure 4: At $\Gamma = 1.5$, $e = 0.92$, and $e_w = 1.0$ (the same as in Figure 3), but placing 5 heavier spheres (mass m_b) on the bottom and 5 spheres (mass m_t) on the top of column with mass ratio $m_b/m_t = 2$, the spheres follow chaotic trajectories, as indicated in frequency spectrum of the center-of-mass motion.

Figure 5: At $\Gamma = 1.5$, $e = 0.92$, and $e_w = 1.0$ (the same as in Figure 4) and placing 5 heavier spheres (mass m_b) on the bottom and 5 spheres (mass m_t) on the top of the column with mass ratio $m_b/m_t = 100$ makes the condensation and fluidization regimes to coexist (an intermediate situation between Figures 3 and 4). The inset shows the frequency spectrum of the center-of-mass motion.

$i = 1, \ldots, 5$ and $m_j = 1$, $j = 6, \ldots, 10$ are given in Figure 4. Decreasing Γ makes the particles come to a permanent contact with the plate. It has been verified also that a double mass ratio does not lead to the condensation regime.

Increasing the mass ratio by setting $m_i = 10$, $i = 1, 2, \ldots, 5$, and $m_j = 0.1$, $j = 6, 7, \ldots, 10$ produces a condensation regime coexisting with a fluidized regime as shown in Figure 5. The mass ratio is sufficiently large so that we can consider the column as two isolated systems. The center of mass, however, is essentially determined by the heavier spheres, which together with the center of the mass follows a periodic motion. Yet, we note that the center of mass fails to describe the behavior of the system accurately because the frequency spectrum does not show up the coexistence of the two regimes.

Figure 6: At $\Gamma = 1.5$, $e = 0.92$, $e_w = 1.0$, and mass ratio $m_b/m_t = 100$ (the same as in Figure 5) the two groups of particles enter in the condensed regime upon adjusting the restitution coefficients to $e_b = 0.92$ (heavier spheres) and $e_t = 0.82$ (lighter spheres).

From the previously assembled column in Figure 5 and by decreasing the coefficient of restitution to increase energy dissipation rate, we can drive the upper layer into the condensation regime. We set a coefficient of restitution for each layer making $e_1 = 0.92$ for the heavier particles and $e_2 = 0.82$ for the lighter ones. This strategy is useful when adjusting energy in each layer without interfering with another.

With this column so assembled, Figure 6 shows two groups of particles in the condensed regime. This result extends the result in Figure 3 for two layers moving in the form of two blocks; each one with its own characteristic weight.

Next, using the mass ratio $m_b/m_t = 100$ as in Figure 5, but with $e = 0.67$, and $e_w = 1$, we increase the acceleration parameter to a sufficiently large value ($\Gamma = 10$) to fluidize the lower portion of the column (with five heavy spheres on the lower part). Results are shown in Figure 7, where the lower half part turns out fluidized, while the upper part (five light spheres) is in the condensed state because fewer energy (after being attenuated and absorbed by the bottom portion) rests available to fluidize the upper part. The upper layer (the lighter one), however, is more likely to be affected by smaller values of the restitution coefficient. For instance, decreasing this parameter to $e = 0.5$, the light spheres become tightly clustered, like a single inelastic object, yet the lower spheres still remain in the fluidized state.

In order to quantify the degree of dilation, we use a convenient parameter defined by [2]

$$\lambda = \frac{\langle z_n - z_1 \rangle}{A\Gamma}, \tag{3.1}$$

which averages over time the column's height.

Then, we assembled two different columns with a specific global mass distributed among 10 spheres, which are arrayed into distinct arrangements. With a total mass of 20 m.u., the first arrangement includes five light spheres (1 m.u.) placed on the lower portion of the column and five heavy spheres (3 m.u.) on the top. Simulations with light spheres on the upper portion (with heavy ones on the bottom) as well as arrangements with light and heavy spheres alternated were performed. Also compared with homogenous columns

Figure 7: At $\Gamma = 10$, $e = 0.67$, $e_w = 1.0$, and mass ratio $m_b/m_t = 100$, the lower portion of the column becomes fluidized, while the upper half part remains in the condensed regime.

Figure 8: Plotted against the acceleration parameter Γ, time-averaged dilations of a homogeneous, and three inhomogeneous columns, 20 m.u. heavy, assembled with two sets of five spheres of mass ratio 1:3.

(10 identical spheres, 2 m.u. each) in Figure 8, results indicate different degrees of dilation for each arrangement. We see that columns with light particles on the top show larger dilation, as linear momentum transfer from heavy particles to light ones allows the upper particles to jump higher. Arrangements with alternated spheres present the same dilation as that of the uniform distribution. Differently from the homogeneous column (green dots), the array with alternated spheres does not exhibit sawtooth oscillations at low values of the acceleration parameter (Γ). Configurations with heavy spheres on top have the smallest dilations since the collision energy is attenuated as it propagates up, thus making it impossible to fluidize the upper portion of the column. In all cases, and consistent with previous results [2], the column dilation remains constant when increasing the shaking parameter $\Gamma > 8$.

Figure 9: Plotted against the acceleration parameter Γ, time-averaged dilations of a homogeneous, and three inhomogeneous columns, 30 m.u. heavy, assembled with two sets of five spheres of mass ratio 1:5.

Figure 10: Displayed using the reduced variable $X = N(1-e)$, the center of mass elevation $h_{cm} - h_{cm,0}$ (from the center of mass at rest) of 10-sphere columns of the same height assembled with spheres of different mass ratios.

Results for the heavier column of 30 m.u., assembled with 5 m.u. and 1 m.u. spheres, are given in Figure 9, showing again that for homogeneous and alternating-sphere columns, the degrees of dilation are nearly the same as those for the 20 m.u. column (Figure 8). We see in Figure 9 that the heavier column (30 m.u.) with light spheres on the top exhibits a larger dilation than that obtained with the 20 m.u. column (Figure 8), since as expected, the heavier spheres on the bottom of the 30 m.u. column absorb energy from the table which, in turn, is transferred to the lighter spheres hovering on the top. For all cases, the dilation values

converge to a constant value at large acceleration ($\Gamma > 10$), meaning that the length of the dilated (or energized) column is proportional to $A^2\omega^2$, which is the time-averaged square velocity of the driving table.

The amount of energy absorbed by the column is quantified in Figure 10, which compares the up shift of the center of mass of five columns assembled with ten spheres. The mass distribution of inhomogeneous columns is labeled by the mass ratio $m_t : m_b$ indicating that five spheres of relative mass m_b are placed on the lower half part of the column, and the remaining five spheres (of mass m_t) are stacked on the upper part of the column. Simulations have been performed at the 20 Hz frequency, so that the dimensionalized center-of-mass up shift is expressed as $\Delta h(\text{cm}) = 6.20 \times 10^{-2}\tilde{h}$. Lower elevations are associated with the 2:1 column (with heavier spheres on the top and when the mass ratio m_t/m_b is largest), while the converse holds for the 1:10 columns. Moreover, a distinctive feature of column with $m_t/m_b < 1$ relates to the existence of an inflection point on the corresponding curves, which turn out concave down toward reaching a stagnation zone at about the critical value of the reduced variable X. The physical process behind the coalescence of the curves is simply the occurrence of condensation at values $X_c \approx 3.0$, meaning that the whole column never becomes completely fluidized [2]. We also note that at around $X_c \approx 3.0$, the center-of-mass up shift is close to zero, thus reflecting that the motion of the spheres is collective. For still larger values of X, the dissipation becomes very high, which makes the column act like a compact block with dynamic properties of a single inelastic object. On the other hand, at lower X values, the columns become fully fluidized, with each curve reaching a saturation value.

4. Conclusions

The present paper has given, to our knowledge, the first exploratory study of the dynamics of inhomogeneous granular columns under vertical vibration. Relying on numerical methods and computer simulations, a number of features have been revealed, namely, the occurrence of fluidized and condensed states (resp., characterized by the acceleration parameter Γ and by the square velocity $A^2\omega^2$ of the of the driving source) and the convergence of the dilation curves to a region defined by a critical value of the reduced variable $N(1 - e) \approx 3.0$. Settings of control parameters (normalized acceleration Γ, coefficient of elastic restitution e, mass ratio m_b/m_t) have been determined so as to drive the system in specific states, either through tailoring the column into separate arrays to oscillate collectively in phase or through fluidizing the whole column. In what seems to be the first reported example, a two-ball bouncing system has been discussed in connection with its control parameters. Driven in the 1-periodic oscillation mode, the system is quite stable over the range $0.6 < e < 0.9$, thus deserving a thorough analytical study by extension of the methods used in its one-ball counterpart [12, 15], upon including five equations in the dynamical mapping, in addition to a further control parameter, namely, the mass ratio m_b/m_t that arises from the mass inhomogeneity of the column.

Acknowledgments

This work has been supported by the National Council for Scientific and Technological Development (CNPq), Brazil. M. V. Carneiro also thanks the Physics Group of Federal University of Lavras (UFLA), MG, Brazil for computing facilities.

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Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 290356, 11 pages
doi:10.1155/2009/290356

Research Article

Multistability and Self-Similarity in the Parameter-Space of a Vibro-Impact System

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Received 22 April 2009; Accepted 23 June 2009

Recommended by Edson Denis Leonel

The dynamics of a dissipative vibro-impact system called impact-pair is investigated. This system is similar to Fermi-Ulam accelerator model and consists of an oscillating one-dimensional box containing a point mass moving freely between successive inelastic collisions with the rigid walls of the box. In our numerical simulations, we observed multistable regimes, for which the corresponding basins of attraction present a quite complicated structure with smooth boundary. In addition, we characterize the system in a two-dimensional parameter space by using the largest Lyapunov exponents, identifying self-similar periodic sets.

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1. Introduction

In order to study cosmic ray accelerated to high energy, Enrico Fermi proposed an accelerator model as a dynamical system [1], consisting of a classical particle interacting with a time dependent magnetic field. The original model was later modified and studied under different approaches. For example, the dynamics of the well-known Fermi-Ulam model has been investigated [2]. This model consists of a point mass moving between a rigid fixed wall and an oscillatory wall. In the recent years, the Fermi-Ulam model has attracted a significant attention to understand the dynamics of systems with limited energy gain, for certain parameters and initial conditions, in contrast with the Fermi acceleration [3–6].

In the engineering context, mathematical models describing mechanical impacts similar to those considered in the Fermi-Ulam model have been intensively studied, like gearbox model [7–10] and impact damper [11, 12]. These systems, called vibro-impact or impact oscillator, appear in a wide range of practical problems, such as percussive drilling

Figure 1: Schematic view of an impact-pair system.

tools [13], print hammers [14], and vibro-impact moling systems [15], just to mention a few. For these systems a better understanding of their dynamics may help to reduce the negative effects of impacts and ultimately to improve practical designs. Although the results obtained for the impact oscillator have different interpretations, the mathematical analyses are similar to those used to investigate the Fermi-Ulam model.

In this paper, we consider a prototypical vibro-impact system, known as impact-pair. This system is comprised of a ball moving between two oscillatory walls. Numerical studies have shown a rich dynamical behavior with several nonlinear phenomena observed, such as bifurcations, chaotic regimes [16, 17], crises [18], and basin hoppings [19]. In addition, control of chaotic dynamics can be applied to stabilize unstable periodic orbits embedded in the chaotic attractor [20]. We aim here to explore some dynamical properties with emphasis on characterization of basins of attraction with complicated smooth boundaries and identification of self-similar periodic sets called shrimps [21].

This paper is organized as follows. In Section 2 we present the model and the equations of motion for the impact-pair system. In Section 3, we investigate coexistence of different regimes and their basins of attraction. In Section 4, we characterize the impact-pair system in the two-parameter space. The last section contains our conclusions.

2. Mathematical Description

In this section we present the basic equations of the impact-pair system [16, 17] shown schematically in Figure 1. In addition, following the mathematical description of [19], we describe how to obtain an impact map, also called transcendental map. The map is useful to calculate the Lyapunov exponents [18].

The impact-pair system is composed of a point mass m, whose displacement is denoted by x, and a one-dimensional box with a gap of length v. The mass m is free to move inside the gap and the motion of the box is described by a periodic function, $\alpha \sin(\omega t)$.

In an absolute coordinate systems, equation of motion of the point mass m is given by

$$\ddot{x} = 0. \tag{2.1}$$

On denoting the relative displacement of the mass m by y, we have

$$x = y + \alpha \sin(\omega t), \tag{2.2}$$

Figure 2: (a) Bifurcation diagram showing coexisting attractors. Velocity $\dot{y}_T$, for a stroboscopic map, as a function of the amplitude excitation, α, for $r = 0.8$. (b) The Lyapunov exponents, $\lambda_{1,2}$, for the attractors plotted in blue.

such that, on substituting (2.2) into (2.1), the equation of motion in the relative coordinate system is

$$\ddot{y} = \alpha\omega^2 \sin(\omega t), \qquad \left(-\frac{\nu}{2} < y < \frac{\nu}{2}\right). \tag{2.3}$$

Integrating (2.3) and for the initial conditions $y(t_0) = y_0$ and $\dot{y}(t_0) = \dot{y}_0$, the displacement y and the velocity $\dot{y}$, between impacts, are

$$\begin{gathered} y(t) = y_0 + \alpha \sin(\omega t_o) - \alpha \sin(\omega t) + \left[\dot{y}_0 + \alpha\omega \cos(\omega t_0)\right](t - t_0), \\ \dot{y}(t) = \dot{y}_0 + \alpha\omega \cos(\omega t_0) - \alpha\omega \cos(\omega t). \end{gathered} \tag{2.4}$$

An impact occurs wherever $y = \nu/2$ or $-\nu/2$. After each impact, we apply into (2.4) the new set of initial conditions (Newton's law of impact)

$$t_0 = t, \qquad y_0 = y, \qquad \dot{y}_0 = -r\dot{y}, \tag{2.5}$$

where r is a constant restitution coefficient.

Therefore, the dynamics of the impact-pair system is obtained from (2.4) and (2.5). In this case, the system depends on control parameters ν, r, α, and ω.

Since there is an analytical solution for the motion between impacts, we can obtain an impact map. First, we define the discrete variables y_n, $\dot{y}_n$, t_n as the displacement, the velocity, and the time (modulo 2π) collected just the nth impact. Substituting the Newton law of impact into (2.4), we have a two-dimensional map that is given by

$$\begin{gathered} y_{n+1} = y_n + \alpha \sin(\omega t_n) - \alpha \sin(\omega t_{n+1}) + \left[-r\dot{y}_n + \alpha\omega \cos(\omega t_n)\right](t_{n+1} - t_n), \\ \dot{y}_{n+1} = -r\dot{y}_n + \alpha\omega \cos(\omega t_n) - \alpha\omega \cos(\omega t_{n+1}), \end{gathered} \tag{2.6}$$

where $y_n = \nu/2$ or $-\nu/2$.

As mentioned before, this map, obtained by considering the analytical solutions and the sequence of impact times, is only used to evaluate the Lyapunov exponents [18]. The system dynamics is directly analyzed from the analytical solutions of (2.4) and (2.5). The Lyapunov exponents are computed through

$$\lambda_i = \lim_{n\to\infty} \frac{1}{n} \ln|\Lambda_i(n)| \quad (i = 1, 2), \tag{2.7}$$

where $\Lambda_i(n)$ are the eigenvalues of the matrix $A = J_1 \cdot J_2, \ldots, J_n$ with J_n being the Jacobian matrix of the map (2.6), computed at time n. For systems without analytical solutions between the impacts, the Lyapunov exponents can be calculated by using the method proposed by Jin et al. [22].

3. Multistability with Complex Basins of Attraction

The dynamics was investigated using bifurcation diagrams, phase portraits, Lyapunov exponents, basins of attraction, uncertainty exponent, and parameter space diagrams. We fix the control parameters at $\nu = 2.0$ (length of the gap), $\omega = 1.0$ (excitation frequency) and vary the parameters r (restitution coefficient) and α (excitation amplitude).

In order to obtain a representative example of the kind of dynamics generated by the impact-pair system, we use a bifurcation diagram for the velocity, $\dot{y}_T$, versus the amplitude excitation, α. The dynamical variable $\dot{y}_T$ is obtained from a stroboscopic map (Time-2π). To characterize the nature of the behavior observed, we calculate the Lyapunov exponents. If the largest Lyapunov exponent is positive the attractor is chaotic, if not the attractor is periodic.

In Figure 2(a), we present a bifurcation diagram showing multiple coexisting attractors plotted in different colors. For example, we can note two period-1 orbits at $\alpha = 3.2$. In this case, these attractors are symmetric and appear due to the pitchfork bifurcation for $\alpha \approx 3.0878$. For $\alpha \approx 3$, the diagram shows two coexisting attractors depicted in green and blue. The Lyapunov exponents for attractors plotted in blue are shown in Figure 2(b).

In Figures 3(a) and 3(b), we show the phase portraits of the two symmetrical coexisting periodic attractors for $\alpha = 3.2$. For the same set of parameters, we identify two more other possible solutions, namely, two equilibrium points shown in Figures 3(c) and 3(d).

Figure 3: Phase portrait of velocity versus displacement of two coexisting periodic attractors (a) and (b). Time histories of two equilibrium points (c) and (d). For the control parameters $\alpha = 3.2$ and $r = 0.8$.

The corresponding basins of attraction of the four possible solutions are depicted in Figure 4(a). This figure is constructed using a grid of equally spaced 1000 × 1000 cells as set of initial conditions for velocity *versus* time (*modulo* 2π) with initial position fixed at $y_0 = 0$. The basins of the periodic attractors are plotted in blue and red, and the equilibrium points in white and green.

As can be seen, the structure of the basins is quite complex and the basin boundary between periodic attractors is convoluted and apparently fractal (Figures 4(b) and 4(c)). However, in contrast to fractal basin boundaries, we cannot observe an infinitely fine scaled structure with magnifications of the boundary region, as shown in Figure 4(d). Therefore, the basin boundary here is composed of smooth curve with dimension one. In this case, the complexity of basins structure is generated by band accumulations [23–25]. Moreover, the observed smooth basin boundary was also observed for other parameter regions with coexisting periodic or chaotic attractors.

As a consequence of the complex basins of attraction observed, uncertainty in initial conditions leads to uncertainty in the final state. To evaluate the final state sensitivity we can calculate the uncertainty exponent, β, which was proposed by Grebogi and his collaborators [26] and has been used to characterize fractal basin boundaries [10, 12].

Figure 4: Basins of attraction and successive magnifications for the parameters $\alpha = 3.2$ and $r = 0.8$, varying initial conditions $\dot{y}_0$ and t_0 with $y_0 = 0$. Basins in red and blue for the two coexisting attractors were shown in Figures 3(a) and 3(b) basins, in green and white for the two equilibrium points were shown in Figures 3(c) and 3(d).

To obtain the uncertainty exponent, we choose randomly a large number of set initial conditions $A : (\tau_0, \dot{y}_0)$ for y_0 constant. We keep $\dot{y}_0$ constant and vary the other coordinate by a small amount ϵ. We also choose the slightly displaced initial conditions $B : (\tau_0 + \epsilon, \dot{y}_0)$ and $C : (\tau_0 - \epsilon, \dot{y}_0)$. If the trajectory starting from initial condition A goes to one of the basins, and either one (or both) of the displaced initial conditions, B or C, go to the other basin, we call A as an ϵ-uncertain initial condition. By considering a large number of such points, we can estimate the fraction of ϵ-uncertain points, $f(\epsilon)$. This number scales with the uncertain radius as a power-law $f(\epsilon) \sim \epsilon^{\beta}$, where β is called uncertainty exponent [24, 26].

For the basins of attraction shown in Figure 4(c), we obtain the uncertainty exponent $\beta = 0.490 \pm 0.001$, as indicated in Figure 5. In this case, if we want to gain a factor 2 in the ability to predict the final state of the system, it is necessary to increase the accuracy of initial condition by a factor approximately 4 ($2^{1/0.49} \approx 4$). In general a noninteger uncertainty exponent is a consequence of a fractal basin boundary. However, in this case the considered boundary is smooth with a dimension $d_B = 1$; but even so the uncertainty to predict the final state remains.

Figure 5: Uncertain fraction versus uncertainty radius ϵ for basins of attraction showed in Figure 4(c) for $\alpha = 3.2$ and $r = 0.8$. The solid curve is a linear regression fit with slope $\beta = 0.490 \pm 0.001$.

4. Self-Similar Periodic Sets in Parameter Space

In order to obtain a further insight into the influence of the restitution coefficient r and amplitude excitation α on the dynamics of the impact-pair system, we construct parameter space diagrams, depicted in Figures 6(a) to 6(d). To obtain these diagrams, we use a grid of 1000×1000 cells with the initial conditions fixed at $(y_0, \dot{y}_0, \tau_0) = (0, 1, 0)$. For each point the largest Lyapunov exponent is calculated and plotted with the appropriately allocated color. Chaotic attractors ($\lambda_1 > 0$) are plotted in blue and periodic attractors ($\lambda_1 < 0$) according to a color scale ranging from minimum value in red and maximum in green. Zero Lyapunov exponents (bifurcation points) are plotted in blue.

To clarify how the parameter space diagram was constructed, we fix the restitution coefficient at $r = 0.683$ for Figure 6(c) and vary the amplitude excitation α determining bifurcation diagram and the corresponding largest Lyapunov exponents in Figures 7(a) and 7(b), respectively. Comparing the Lyapunov exponents with parameter space diagram (Figure 6(c)), we can note that red lines correspond with the local minimum of exponents and blue lines embedded in green region (periodic region) correspond with bifurcation points.

On examining the parameter space diagram (Figure 6(b)), we note a main periodic structure existing embedded in chaotic region. Around this structure a vast quantity of self-similar periodic sets is found. For instance, we can see in Figure 6(c) a very regular network of self-similar structures for a magnification of small box of Figure 6(b). These self-similar structures have been observed before [27–31] and have been referred to as shrimps [32–34]. In our work, the shrimps organize themselves along a very specific direction in parameter space with a series of accumulations and fractality can be observed with successive magnifications of the parameter space, as shown in Figure 6(d) for a magnification of a shrimp of Figure 6(c).

Figure 6: (a) Parameter space diagram with periodic structures. (b)–(d) Successive magnifications of parameter space diagram.

To finalize, in Figures 8(a) and 8(b) we depict two successive magnifications of the parameter space diagram (Figure 6(b)). In this case, we can observe a high concentration of periodic structures with different shapes (Figure 8(a)). These structures cross each other indicating coexistence of periodic attractors. In Figure 8(b), we identify a periodic structure that appears abundantly in parameter space, for the system considered, consisting of three shrimp shapes connected. In addition, we again observe crossings between periodic sets.

5. Conclusions

In this paper, we considered the impact-pair system studying its dynamics by a means of numerical simulations. Initially, we discussed the coexistence of attractors with smooth basin boundary, but with complicated and evoluted basins structure. According to the uncertainty exponent evaluated for a certain region in phase space, this type basins of attraction is associated with effect of obstruction on predictability of time-asymptotic final state (attractor). In other words, the uncertainty in initial conditions leads to uncertainty in the final state.

Figure 7: Bifurcation diagram of velocity $\dot{y}_T$ as a function of the amplitude excitation α for $r = 0.683$. (b) The largest Lyapunov exponents, λ_1, for attractors of the bifurcation diagram.

Figure 8: (a) Parameter space diagram with periodic structures. (b) Magnification of a portion of the previous figure.

In the end, we explored the dynamics in the two-dimensional parameter space by using the largest Lyapunov exponents. We identified several periodic sets embedded in chaotic region. Some of them, known as shrimps, present self-similar structures and organize themselves along a very specific direction in parameter space with a series of accumulations.

In addition, the periodic sets cross each other indicating a large quantity of coexisting periodic attractors.

Acknowledgments

This work was made possible by partial financial support from the following Brazilian government agencies: FAPESP and CNPq.

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Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 902506, 30 pages
doi:10.1155/2009/902506

Research Article

Quantum Energy Expectation in Periodic Time-Dependent Hamiltonians via Green Functions

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Received 10 November 2008; Revised 29 January 2009; Accepted 21 February 2009

Recommended by Edson Denis Leonel

Let U_F be the Floquet operator of a time periodic Hamiltonian $H(t)$. For each positive and discrete observable A (which we call a *probe energy*), we derive a formula for the Laplace time average of its expectation value up to time T in terms of its eigenvalues and Green functions at the circle of radius $e^{1/T}$. Some simple applications are provided which support its usefulness.

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1. Introduction

Consider a general periodically driven quantum hamiltonian system

$$H(t) = H_0 + V(t) \tag{1.1}$$

with period τ acting in a separable Hilbert space $\mathscr{H}$, and let U_F denote its Floquet operator, so that if ξ is the initial state (at time zero) of the system, then $U_F^m \xi$ is this state at time $m\tau$. Typically, the unperturbed hamiltonian H_0 is assumed to have purely point spectrum so that the same is true for $e^{-i\tau H_0}$. What happens when H_0 is perturbed by $V(t)$? A natural physical question is if the expectation values of the unperturbed energy H_0 remain bounded when $V(t) \neq 0$. This question is formulated based on many physical models, in particular on the Fermi accelerator in which a particle can acquire unbounded energy from collisions with a heavy periodically moving wall. Here quantum mechanics is considered and, more precisely, if

$$\sup_{m\in\mathbb{N}} \left| \left\langle U_F^m \xi, H_0 U_F^m \xi \right\rangle \right| \tag{1.2}$$

is finite or not, where $\xi \in \operatorname{dom} H_0 \subset \mathscr{H}$, the domain of H_0.

Motivated by models with hamiltonians as above $H(t) = H_0 + V(t)$, one is suggested to probe quantum (in)stability through the behavior of an "abstract energy operator" which we call a *probe operator* and will be represented by a positive, unbounded, self-adjoint operator $A : \text{dom } A \subset \mathscr{H} \to \mathscr{H}$ and with discrete spectrum,

$$A\varphi_n = \lambda_n \varphi_n, \tag{1.3}$$

$0 \leq \lambda_n < \lambda_{n+1}$, such that if $U_F^m \xi \in \text{dom } A$ for all $m \in \mathbb{N}$, then, for each m, the expectation value $E_\xi^A(m) = \langle U_F^m \xi, A U_F^m \xi \rangle$ is finite. It is convenient to write $E_\xi^A(m) = +\infty$ if $U_F^m \xi$ does not belong to dom A.

We say the system is *A-dynamically stable* when $E_\xi^A(m)$ is a bounded function of time m, and *A-dynamically unstable* otherwise (usually we say just *(un)stable*). If the function $E_\xi^A(m)$ is not bounded, one can ask about its asymptotic behavior, that is, how does $E_\xi^A(m)$ behave as m goes to infinity? Usually this is a very difficult question and sometimes the temporal average of $E_\xi^A(m)$ is considered, as we will do in this work.

Quantum systems governed by a time periodic hamiltonian have their dynamical stability often characterized in terms of the spectral properties of the corresponding Floquet operator. As in the autonomous case, the presence of continuous spectrum is a signature of unstable quantum systems; this is a rigorous consequence of the famous RAGE theorem [1], firstly proved for the autonomous case and then for time-periodic hamiltonians [2, 3]. At first sight a Floquet operator with purely point spectrum would imply stability, but one should be alerted by examples with purely point spectrum and dynamically unstable [4–6] in the autonomous case and, recently, also a time-periodic model with energy instability [7] was found.

Dynamical stability of time-dependent systems was studied, for example, in references [2, 8–19]. In [14] it was proved that the applicability of the KAM method gives a uniform bound at the expectation value of the energy for a class of time-periodic hamiltonians considered in [20].

For hamiltonians $H(t) = H_0 + V(t)$, not necessarily periodic, with H_0 a positive self-adjoint operator whose spectrum consists of separated bands $\{\sigma_j\}_{j=1}^\infty$ such that $\sigma_j \subset [\lambda_j, \Lambda_j]$, upper bounds of the type

$$\langle U(t,0)\psi, H_0 U(t,0)\psi \rangle \leq \text{cte } t^{(1+\alpha)/n\alpha} \tag{1.4}$$

were obtained in [10] if the gaps $\lambda_{j+1} - \Lambda_j$ grow like j^α, with $\alpha > 0$, and if $V(t)$ is strongly C^n with $n \geq [(1+\alpha)/2\alpha] + 1$. The proof is based on adiabatic techniques that require smooth time dependence and therefore do not apply to kicked systems. In [11, 13] upper bounds complementary to those of [10] described above are obtained.

In [2, 8, 9, 15] stability results are obtained through topological properties of the orbits $\xi(t) = U(t,0)\xi$ for $\xi \in \mathscr{H}$, while in [16–19] lower bounds for averages of the type

$$\frac{1}{T}\sum_{m=1}^{T} \langle U_F^m \xi, H_0 U_F^m \xi \rangle \geq C T^\gamma \tag{1.5}$$

are obtained for periodic hamiltonians $H(t) = H_0 + V(t)$ through dimensional properties of the spectral measure μ_ξ associated with U_F and ξ (the exponent γ depends on the measure μ_ξ).

In this work we study (in)stability of periodic time-dependent systems. As for tight-binding models (see [21] and references therein) we consider the Laplace-like average of $\langle U_F^m \xi, A U_F^m \xi \rangle$, that is,

$$\frac{2}{T} \sum_{m=0}^{\infty} e^{-2m/T} \langle U_F^m \xi, A U_F^m \xi \rangle, \tag{1.6}$$

where A is a probe energy, ξ is an element of dom A, and U_F is the Floquet operator. The main technical reason for working with this expression for the time average is that it can be written in terms of (see Theorem 2.3) the eigenvalues of A, that is, $A\varphi_j = \lambda_j \varphi_j$, and the matrix elements $\langle \varphi_j, R_z(U_F)\xi \rangle$ of the resolvent operator $R_z(U_F) = (U_F - z\mathbf{1})^{-1}$ (with $z = e^{-iE} e^{1/T}$) with respect to the orthonormal basis $\{\varphi_j\}$ of the Hilbert space (here $\mathbf{1}$ denotes the identity operator). Lemma 2.2 relates the long run of Laplace-like average with the usual Cesàro average. In Section 2 we shall prove this abstract results and present some applications in Section 3.

Since our main results are for temporal Laplace averages of expectation values of probe energies (see Section 2), in practice we will think of (in)stability in terms of (un)boundedness of such averages. Note that unbounded Laplace averages imply unboundedness of expectation values of probe energies themselves.

2. Average Energy and Green Functions

Consider a time-dependent hamiltonian $H(t)$ with $H(t+\tau) = H(t)$ for all $t \in \mathbb{R}$, acting in the separable Hilbert space $\mathscr{H}$. Suppose the existence of the propagators $U(t,s)$, so that the Floquet operator $U_F = U(\tau, 0)$ is at our disposal. Let A be a probe energy and λ_j, φ_j as in the introduction. Also, $\{\varphi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathscr{H}$.

The main interest is in the study of the expectation values, herein defined by

$$E_\xi^A(m) := \begin{cases} \langle U_F^m \xi, A U_F^m \xi \rangle, & \text{if } U_F^m \xi \in \operatorname{dom} A, \\ +\infty, & \text{if } U_F^m \xi \in \mathscr{H} \setminus \operatorname{dom} A, \end{cases} \tag{2.1}$$

as function of time $m \in \mathbb{N}$. Another quantity of interest is the time dependence of the moment of this probe energy, which takes values in $[0, +\infty]$ and is defined by

$$M_\xi^A(m) := \sum_{j=1}^{\infty} \lambda_j \left| \langle \varphi_j, U_F^m \xi \rangle \right|^2. \tag{2.2}$$

Our first remark is the equivalence of both concepts (under certain circumstances).

Proposition 2.1. *If $U_F^m \xi \in \operatorname{dom} A$ for all m, then*

$$E_\xi^A(m) = M_\xi^A(m), \quad \forall m. \tag{2.3}$$

This holds, in particular, if dom A *is invariant under the time evolution U_F^m and $\xi \in \operatorname{dom} A$.*

Proof. Since $\operatorname{dom} A \subset \operatorname{dom} A^{1/2}$ [1], one has $U_F^m \xi \in \operatorname{dom} A^{1/2}$, for all m, and so

$$\begin{aligned} M_\xi^A(m) &= \sum_{j=1}^{\infty} \left| \left\langle \lambda_j^{1/2} \varphi_j, U_F^m \xi \right\rangle \right|^2 \\ &= \sum_{j=1}^{\infty} \left| \left\langle A^{1/2} \varphi_j, U_F^m \xi \right\rangle \right|^2 \\ &= \sum_{j=1}^{\infty} \left| \left\langle \varphi_j, A^{1/2} U_F^m \xi \right\rangle \right|^2 \\ &= \left\| A^{1/2} U_F^m \xi \right\|^2 \\ &= \langle A^{1/2} U_F^m \xi, A^{1/2} U_F^m \xi \rangle \\ &= \langle U_F^m \xi, A U_F^m \xi \rangle \\ &= E_\xi^A(m), \end{aligned} \tag{2.4}$$

which is the stated result. □

We introduce the temporal Laplace average of E_ξ^A (see also the appendix) by the following function of $T > 0$, which also takes values in $[0, +\infty]$:

$$L_\xi^A(T) := \frac{2}{T} \sum_{m=0}^{\infty} e^{-2m/T} E_\xi^A(m). \tag{2.5}$$

Under certain conditions, the next result shows that the upper β^+ and lower β^- growth exponents of this average, that is, roughly they are the best exponents so that for large T there exist $0 \le c_1 \le c_2 < \infty$ with

$$c_1 T^{\beta^-} \le L_\xi^A(T) \le c_2 T^{\beta^+}, \tag{2.6}$$

and the corresponding exponents for the temporal Cesàro average

$$C_\xi^A(T) = \frac{1}{T} \sum_{m=0}^{T} E_\xi^A(m) \tag{2.7}$$

are closely related; this follows at once by Lemma 2.2, which perhaps could be improved to get equality also between lower exponents. Note that, although not indicated, these exponents depend on the initial condition ξ.

Lemma 2.2. *If $(h(m))_{m=0}^{\infty}$ is a nonnegative sequence and $h(m) \leq Cm^n$ for some $C > 0$ and $n \geq 0$, then $\beta_e^+ = \beta_d^+$ and $\beta_e^- \leq \beta_d^-$, where*

$$\begin{aligned} \beta_e^+ &= \limsup_{T\to\infty} \frac{\log\left(\sum_{m=0}^{T} h(m)\right)}{\log T}, \qquad \beta_e^- = \liminf_{T\to\infty} \frac{\log\left(\sum_{m=0}^{T} h(m)\right)}{\log T}, \\ \beta_d^+ &= \limsup_{T\to\infty} \frac{\log\left(\sum_{m=0}^{\infty} e^{-2m/T} h(m)\right)}{\log T}, \qquad \beta_d^- = \liminf_{T\to\infty} \frac{\log\left(\sum_{m=0}^{\infty} e^{-2m/T} h(m)\right)}{\log T}. \end{aligned} \tag{2.8}$$

Proof. Note that for $0 \leq m \leq T$ we have $e^{-2} \leq e^{-2m/T} \leq 1$, and so

$$\sum_{m=0}^{T} h(m) \leq \sum_{m=0}^{T} e^2 e^{-2m/T} h(m) \leq e^2 \sum_{m=0}^{\infty} e^{-2m/T} h(m). \tag{2.9}$$

Hence $\beta_e^{\pm} \leq \beta_d^{\pm}$.

On the other hand, for each $\epsilon > 0$, denoting by $\lceil x \rceil$ the smallest integer larger or equal to x, one has

$$\begin{aligned} \sum_{m=0}^{\infty} e^{-2m/T} h(m) &= \sum_{m=0}^{\lceil T^{1+\epsilon} \rceil} e^{-2m/T} h(m) + \sum_{m=\lceil T^{1+\epsilon} \rceil+1}^{\infty} e^{-2m/T} h(m) \\ &\leq \sum_{m=0}^{\lceil T^{1+\epsilon} \rceil} h(m) + C \sum_{m=\lceil T^{1+\epsilon} \rceil+1}^{\infty} e^{-2m/T} m^n. \end{aligned} \tag{2.10}$$

Now, for T large enough $nT/2 < T^{1+\epsilon} \leq \lceil T^{1+\epsilon} \rceil$. Thus

$$\sum_{m=\lceil T^{1+\epsilon} \rceil+1}^{\infty} e^{-2m/T} m^n \leq \int_{\lceil T^{1+\epsilon} \rceil}^{\infty} e^{-2t/T} t^n dt. \tag{2.11}$$

Therefore, for each $\epsilon > 0$ and T large enough

$$\begin{aligned} \sum_{m=0}^{\infty} e^{-2m/T} h(m) &\leq \sum_{m=0}^{\lceil T^{1+\epsilon} \rceil} h(m) + C \int_{\lceil T^{1+\epsilon} \rceil}^{\infty} e^{-2t/T} t^n dt \\ &\leq \sum_{m=0}^{\lceil T^{1+\epsilon} \rceil} h(m) + \widetilde{C} e^{-2T^{\epsilon}} T^n. \end{aligned} \tag{2.12}$$

Since $e^{-2T^{\epsilon}}T^{n} \to 0$ as $T \to \infty$, it follows that

$$\begin{aligned}
\beta_d^+ &= \limsup_{T\to\infty} \frac{\log \sum_{m=0}^{\infty} e^{-2m/T} h(m)}{\log T} \\
&\le \limsup_{T\to\infty} \frac{\log \sum_{m=0}^{\lceil T^{1+\epsilon}\rceil} h(m)}{\log T} \\
&= \limsup_{T\to\infty} \frac{\log \sum_{m=0}^{\lceil T^{1+\epsilon}\rceil} h(m)}{\log \lceil T^{1+\epsilon}\rceil} \frac{\log \lceil T^{1+\epsilon}\rceil}{\log T} \\
&\le \limsup_{T\to\infty} \frac{\log \sum_{m=0}^{\lceil T^{1+\epsilon}\rceil} h(m)}{\log \lceil T^{1+\epsilon}\rceil} \frac{\log (T+1)^{1+\epsilon}}{\log T} \\
&= (1+\epsilon) \limsup_{T\to\infty} \frac{\log \sum_{m=0}^{\lceil T^{1+\epsilon}\rceil} h(m)}{\log \lceil T^{1+\epsilon}\rceil} \\
&\le (1+\epsilon)\beta_e^+.
\end{aligned} \tag{2.13}$$

As $\epsilon > 0$ was arbitrary, $\beta_d^+ \le \beta_e^+$. □

Recall that the Green functions $G_z^{\xi}(j)$ associated with the operators A, U_F at $\xi \in \mathcal{H}$ and $z \in \mathbb{C}, |z| \neq 1$, are defined by the matrices elements of the resolvent operator $R_z(U_F) = (U_F - z\mathbf{1})^{-1}$ along the orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$, that is,

$$G_z^{\xi}(j) := \langle \varphi_j, R_z(U_F)\xi \rangle. \tag{2.14}$$

Note that $G_z^{\xi}(j)$ is always well defined since for $|z| \neq 1$ that resolvent operator is bounded. Theorem 2.3 is the main reason for considering the temporal averages $L_{\xi}^{A}(T)$. It presents a formula that translates the Laplace average of wavepackets at time T into an integral of the Green functions over "energies" in the circle of radius $e^{1/T}$ in the complex plane (centered at the origin). As T grows, the integration region approaches the unit circle where the spectrum of U_F lives and $R_z(U_F)$ takes singular values, so that (hopefully) A-(in)stability can be quantitatively detected.

Theorem 2.3. *Assume that $U_F^m \xi \in \operatorname{dom} A$ for all $m \ge 0$. Then*

$$L_{\xi}^{A}(T) = \frac{1}{\pi e^{-2/T}} \frac{1}{T} \sum_{j=1}^{\infty} \lambda_j \int_0^{2\pi} \left| G_z^{\xi}(j) \right|^2 dE, \quad z = e^{-iE+1/T}. \tag{2.15}$$

Before the proof of this theorem, we underline that this formula, that is, the expression on the right-hand side of (2.15), is a sum of positive terms and so it is well defined for all $\xi \in \mathcal{H}$ if we let it take values in $[0, +\infty]$; hence, in principle it can happen that this formula is

finite even for vectors $U_F^m\xi$ not in the domain of A, where $L_\xi^A(T) = +\infty$. The general case, that is, $\forall\xi \in \mathscr{H}$, can then be gathered in the following inequality:

$$L_\xi^A(T) \geq \frac{1}{\pi e^{-2/T}} \frac{1}{T} \sum_{j=1}^{\infty} \lambda_j \int_0^{2\pi} \left| G_z^\xi(j) \right|^2 dE, \quad z = e^{-iE+1/T}, \tag{2.16}$$

so that lower bound estimates for this formula always imply lower bound estimates for the Laplace average.

Proof of Theorem 2.3. First note that, by hypothesis, $U_F^m\xi \in \operatorname{dom} A^{1/2}$ for each $m \in \mathbb{N}$. Denote by μ_j the spectral measure of U_F associated with the pair (φ_j, ξ) and by $\mathcal{F}$ the Fourier transform $\mathcal{F} : \mathrm{L}^2[0, 2\pi] \to l^2(\mathbb{Z})$. By the spectral theorem for unitary operators

$$\langle \varphi_j, U_F\xi \rangle = \int_0^{2\pi} e^{-iE'} d\mu_j(E'). \tag{2.17}$$

For each j let $a^{(j)} = (a^{(j)}(m))_{m\in\mathbb{Z}}$ be the sequence

$$a^{(j)}(m) = \begin{cases} 0, & \text{if } m < 0, \\ e^{-m/T} \int_0^{2\pi} e^{-iE'm} d\mu_j(E'), & \text{if } m \geq 0. \end{cases} \tag{2.18}$$

Since $a^{(j)} \in l^1(\mathbb{Z}) \cap l^2(\mathbb{Z})$ and $\mathcal{F}$ is a unitary operator, it follows that $\|a^{(j)}\|_{l^2(\mathbb{Z})} = \|\mathcal{F}^{-1}a^{(j)}\|_{\mathrm{L}^2[0,2\pi]}$ and also

$$\begin{aligned}
\left(\mathcal{F}^{-1}a^{(j)}\right)(E) &= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{iEm} a^{(j)}(m) \\
&= \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} e^{iEm} e^{-m/T} \int_0^{2\pi} e^{-iE'm} d\mu_j(E') \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \left(\sum_{m=0}^{\infty} e^{im(E-E')+i/T} \right) d\mu_j(E') \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{1}{1 - e^{i(E-E'+i/T)}} d\mu_j(E') \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{d\mu_j(E')}{e^{i(E+i/T)}\left(e^{-i(E+i/T)} - e^{-iE'}\right)} \\
&= -\frac{1}{\sqrt{2\pi}\ e^{i(E+i/T)}} \int_0^{2\pi} \frac{d\mu_j(E')}{e^{-iE'} - e^{-i(E+i/T)}} \\
&= -\frac{1}{\sqrt{2\pi}\ e^{i(E+i/T)}} \langle \varphi_j, R_z(U_F)\xi \rangle \\
&= -\frac{1}{\sqrt{2\pi}\ e^{iE} e^{-1/T}}\ G_z^\xi(j),
\end{aligned} \tag{2.19}$$

with $z = e^{-iE+1/T}$. Therefore

$$\left|\mathcal{F}^{-1}a^{(j)}\right|^2(E) = \frac{1}{2\pi e^{-2/T}}\left|G_z^{\xi}(j)\right|^2, \tag{2.20}$$

and so

$$\left\|\mathcal{F}^{-1}a^{(j)}\right\|^2_{L^2[0,2\pi]} = \frac{1}{2\pi e^{-2/T}}\int_0^{2\pi}\left|G_z^{\xi}(j)\right|^2 dE. \tag{2.21}$$

From such relation it follows that

$$\begin{aligned}
L_{\xi}^{A}(T) &= \sum_{m=0}^{\infty}\frac{2}{T}e^{-2m/T}M_{\xi}^{A}(m) \\
&= \sum_{j=1}^{\infty}\lambda_j\sum_{m=0}^{\infty}\frac{2}{T}e^{-2m/T}\left|\langle\varphi_j, U_F^m\xi\rangle\right|^2 \\
&= \sum_{j=1}^{\infty}\lambda_j\frac{2}{T}\sum_{m=0}^{\infty}\left|e^{-m/T}\int_0^{2\pi}e^{-iE'm}d\mu_j(E')\right|^2 \\
&= \sum_{j=1}^{\infty}\lambda_j\frac{2}{T}\left\|a^{(j)}\right\|^2_{l^2(\mathbb{Z})} \\
&= \sum_{j=1}^{\infty}\lambda_j\frac{2}{T}\left\|\mathcal{F}^{-1}a^{(j)}\right\|^2_{L^2[0,2\pi]} \\
&= \frac{1}{\pi e^{-2/T}}\frac{1}{T}\sum_{j=1}^{\infty}\lambda_j\int_0^{2\pi}\left|G_z^{\xi}(j)\right|^2 dE,
\end{aligned} \tag{2.22}$$

which is exactly the stated result. □

Theorem 2.3 clearly remains true if the eigenvalues λ_j of A have finite multiplicity. In this case, for each λ_j consider the corresponding orthonormal eigenvectors $\varphi_{j_1}, \ldots, \varphi_{j_k}$, and one obtains

$$L_{\xi}^{A}(T) = \frac{1}{\pi e^{-2/T}}\frac{1}{T}\sum_{j=1}^{\infty}\lambda_j\left(\sum_{n=1}^{k}\int_0^{2\pi}\left|\langle\varphi_{j_n}, R_z(U_F)\xi\rangle\right|^2 dE\right), \tag{2.23}$$

with z as before.

In case the initial condition is $\xi = \varphi_1$, put $\eta^{(z)} := R_z(U_F)\varphi_1$. Thus, $(U_F - z)\eta^{(z)} = \varphi_1$ and so $U_F\eta^{(z)} = z\eta^{(z)} + \varphi_1$. Hence

$$\langle \varphi_j, U_F\eta^{(z)} \rangle = z\langle \varphi_j, \eta^{(z)} \rangle + \delta_{j,1} \tag{2.24}$$

and by denoting

$$G_z(j) := G_z^{\varphi_1}(j), \tag{2.25}$$

one concludes.

Lemma 2.4.

$$G_z(j) = \begin{cases} \dfrac{1}{z}\left(\langle \varphi_1, U_F\eta^{(z)} \rangle - 1\right), & \text{if } j = 1, \\ \dfrac{1}{z}\langle \varphi_j, U_F\eta^{(z)} \rangle, & \text{if } j > 1. \end{cases} \tag{2.26}$$

In Section 3 we discuss some Floquet operators that are known in literature and analyze their Green functions through the equation

$$(U_F - z\mathbf{1})\eta^{(z)} = \varphi_1. \tag{2.27}$$

3. Applications

This section is devoted to some applications of the formula obtained in Theorem 2.3. In general it is not trivial to get expressions and/or bounds for the Green functions of Floquet operators, and so one of the main goals of the applications that follow is to illustrate how to approach the method we have just proposed.

3.1. Time-Independent Hamiltonians

As a first example and illustration of the formula proposed in Theorem 2.3, we consider the special case of autonomous hamiltonians. In this case $H(t) = H_0$ for all t, and we assume that H_0 is a positive, unbounded, self-adjoint operator and with simple discrete spectrum, $H_0\varphi_j = \chi_j\varphi_j$, so that $\{\varphi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$ and $0 \leq \chi_1 < \chi_2 < \chi_3 < \cdots$ with $\chi_j \to \infty$. For $q > 0$ we can consider H_0^q as our abstract energy operator A, so that its eigenvalues are $\lambda_j = \chi_j^q$ (since A and H_0 have the same eigenfunctions, we are justified in using the notation φ_j for the eigenfunctions of H_0). We take $U_F = e^{-iH_0}$ (time $t = 1$) and for $\xi \in \mathcal{H}$

$$G_z^{\xi}(j) = \langle \varphi_j, R_z(H_0)\xi \rangle = \langle R_{\bar{z}}(H_0)\varphi_j, \xi \rangle = \frac{\langle \varphi_j, \xi \rangle}{e^{-i\chi_j} - z}. \tag{3.1}$$

Since dom H_0^q is invariant under the time evolution e^{-itH_0}, then for $z = e^{-iE}e^{1/T}$ and $\xi \in$ dom H_0^q we have

$$\begin{aligned} L_\xi^q(T) &:= L_\xi^{H_0^q}(T) \\ &= \frac{1}{\pi e^{-2/T}} \frac{1}{T} \sum_{j=1}^{\infty} \chi_j^q \int_0^{2\pi} \left| G_z^\xi(j) \right|^2 dE \\ &= \frac{1}{\pi e^{-2/T}} \frac{1}{T} \sum_{j=1}^{\infty} \chi_j^q \int_0^{2\pi} \frac{\left| \langle \varphi_j, \xi \rangle \right|^2}{\left| e^{-i\chi_j} - z \right|^2} dE \\ &= \frac{1}{\pi e^{-2/T}} \frac{1}{T} \sum_{j=1}^{\infty} \chi_j^q \left| \langle \varphi_j, \xi \rangle \right|^2 \int_0^{2\pi} \frac{dE}{\left| e^{-i\chi_j} - z \right|^2}. \end{aligned} \tag{3.2}$$

Thus we need to calculate the integral $I_j := \int_0^{2\pi} (dE/|e^{-i\chi_j} - z|^2)$. Let γ be the closed path in $\mathbb{C}$ given by $\gamma(E) = e^{iE}$ with $0 \le E \le 2\pi$, $\alpha_j = e^{1/T} e^{i\chi_j}$ and $\beta_j = e^{-1/T} e^{i\chi_j}$, then

$$\begin{aligned} I_j &= \int_0^{2\pi} \frac{dE}{\left(e^{-i\chi_j} - z\right)\left(e^{i\chi_j} - \overline{z}\right)} \\ &= \int_0^{2\pi} \frac{dE}{\left(e^{-i\chi_j} - e^{-iE}e^{1/T}\right)\left(e^{i\chi_j} - e^{iE}e^{1/T}\right)} \\ &= \int_0^{2\pi} \frac{dE}{e^{2/T}\left(e^{-1/T}e^{-i\chi_j} - e^{-iE}\right)\left(e^{-1/T}e^{i\chi_j} - e^{iE}\right)} \\ &= -\frac{1}{e^{2/T}} \int_0^{2\pi} \frac{dE}{e^{-iE}e^{-1/T}e^{-i\chi_j}\left(e^{iE} - \alpha_j\right)\left(e^{iE} - \beta_j\right)} \\ &= -\frac{1}{e^{1/T}e^{-i\chi_j}} \frac{1}{i} \int_0^{2\pi} \frac{ie^{iE}dE}{\left(e^{iE} - \alpha_j\right)\left(e^{iE} - \beta_j\right)} \\ &= \frac{i}{e^{1/T}e^{-i\chi_j}} \int_\gamma \frac{dw}{\left(w - \alpha_j\right)\left(w - \beta_j\right)}. \end{aligned} \tag{3.3}$$

As $|\alpha_j| > 1$ and $|\beta_j| < 1$, β_j is the unique pole in the interior of γ. Thus, by using residues,

$$I_j = \frac{i}{e^{1/T}e^{-i\chi_j}} 2\pi i \frac{1}{\left(\beta_j - \alpha_j\right)} = \frac{2\pi}{e^{2/T} - 1} \tag{3.4}$$

and I_j is independent of χ_j.

Therefore by (3.2) it follows that

$$\begin{aligned} L_{\xi}^{q}(T) &= \frac{1}{\pi e^{-2/T}} \frac{1}{T} \sum_{j=1}^{\infty} \chi_j^q \left|\langle \varphi_j, \xi \rangle\right|^2 \frac{2\pi}{e^{2/T}-1} \\ &= \frac{2}{e^{-2/T}} \frac{1}{T} \frac{1}{\left(e^{2/T}-1\right)} \sum_{j=1}^{\infty} \chi_j^q \left|\langle \varphi_j, \xi \rangle\right|^2 \\ &= \frac{2}{\left(1-e^{-2/T}\right)} \frac{1}{T} \left\| H_0^{q/2} \xi \right\|^2. \end{aligned} \tag{3.5}$$

Since $(1-e^{-2/T}) = 2/T + \mathcal{O}(1/T^2)$, for large T it is found that

$$L_{\xi}^{q}(T) \approx \left\| H_0^{q/2} \xi \right\|^2, \tag{3.6}$$

with (for $\xi \in \operatorname{dom} H_0^q$)

$$\lim_{T\to\infty} L_{\xi}^{q}(T) = \langle \xi, H_0^q \xi \rangle. \tag{3.7}$$

Then we conclude that the function

$$\mathbb{N} \ni m \longmapsto \left\langle e^{-iH_0 m}\xi, H_0^q e^{-iH_0 m}\xi \right\rangle \tag{3.8}$$

is bounded for $\xi \in \operatorname{dom} H_0^q$, which is (of course) an expected result (see Proposition 2.1).

3.2. Lower-Bounded Green Functions

As a first theoretical application we get dynamical instability from some lower bounds of the Green functions. See [21] for a similar result in the one-dimensional discrete Schrödinger operators context; there, a relation to transfer matrices allows interesting applications to nontrivial models, what is not available in the unitary setting yet (and it constitutes of an important open problem). As before, λ_j denote the increasing sequence of positive eigenvalues of the abstract energy operator A, the ones we use to probe (in)stability.

Let $[\cdot]$ denotes the integer part of a real number, and $|\cdot|$ indicates Lebesgue measure.

Theorem 3.1. *Suppose that there exist $K > 0$ and $\alpha > 0$ such that for each $2N > 0$ large enough there exists a nonempty Borel set $J(N) \subset S^1$ such that*

$$\left| G_z^{\xi}(j) \right| \geq \frac{K}{N^{\alpha}}, \quad N \leq j \leq 2N \tag{3.9}$$

holds for all $z = e^{-iE+1/T}$ *with* $E \in J_T(N) = \{E'' \in S^1 : \exists\, E' \in J(N); |E'' - E'| \leq 1/T\}$ *(the* $(1/T)$*-neighborhood of* $J(N)$*). Let* $\delta > 0$*, then for* T *large enough such that* $N(T) = [T^\delta]$*, one has*

$$L_\xi^A(T) \geq \text{cte}\, \lambda_{[T^\delta]} T^{\delta(1-2\alpha)-2}. \tag{3.10}$$

Moreover, if $\lambda_j \geq \text{cte}\, j^\gamma$, $\gamma \geq 0$*, then*

$$L_\xi^A(T) \geq \text{cte}\, T^{\delta(\gamma-2\alpha+1)-2}. \tag{3.11}$$

Proof. By the formula in Theorem 2.3, or its more general version (2.16),

$$\begin{aligned}
L_\xi^A(T) &\geq \frac{1}{\pi e^{-2/T}} \frac{1}{T} \sum_{j=1}^{\infty} \lambda_j \int_0^{2\pi} \left| G_z^\xi(j) \right|^2 dE \\
&\geq \frac{\text{cte}}{T} \sum_{j=N(T)}^{2N(T)} \lambda_j \int_0^{2\pi} \left| G_z^\xi(j) \right|^2 dE \\
&\geq \frac{\text{cte}}{T} \lambda_{N(T)} \sum_{j=N(T)}^{2N(T)} \int_{J_T(N)} \left| G_z^\xi(j) \right|^2 dE \\
&\geq \frac{\text{cte}}{T} \lambda_{N(T)} \sum_{j=N(T)}^{2N(T)} \frac{K^2}{N(T)^{2\alpha}} |J_T(N)| \\
&= \frac{\text{cte}}{T} |J_T(N)| \lambda_{N(T)} \frac{K^2}{N(T)^{2\alpha-1}} \\
&= \frac{\text{cte}}{T} |J_T(N)| \lambda_{[T^\delta]} \frac{1}{[T^\delta]^{2\alpha-1}} \\
&\geq \text{cte}\, \lambda_{[T^\delta]} T^{\delta(1-2\alpha)-2};
\end{aligned} \tag{3.12}$$

and we have used that $|J_T(N)| \geq 1/T$. If $\lambda_j \geq \text{cte}\, j^\gamma$, then

$$L_\xi^A(T) \geq \text{cte}\, T^{\delta\gamma} T^{\delta(1-2\alpha)-2} = \text{cte}\, T^{\delta(\gamma-2\alpha+1)-2}. \tag{3.13}$$

The proof is complete. □

The above theorem becomes appealing when the exponent of T is greater than zero and instability is obtained, for instance, when $\delta(\gamma - 2\alpha + 1) > 2$ in case $\lambda_j \geq \text{cte}\, j^\gamma$. However, up to now we have not yet been able to find explicit estimates in models of interest; in any event, we think that the future applications will be useful, and so we point out some speculations. First, note that it applies even if the set $J(N)$ is a single point! Nevertheless, we expect that Theorem 3.1 will be applied to models whose Floquet operators have some kind of "fractal spectrum" (usually singular continuous or uniformly Hölder continuous spectral measures), and, somehow, α should be related to dimensional properties of those spectra; indeed, this

was our first motivation for the derivation of this result, and, in our opinion, such applications are among the most interesting open problems left here.

3.3. Rank-One Kicked Perturbations

Now consider

$$H(t) = H_0 + \kappa P_\phi \sum_n \delta(t - n2\pi), \tag{3.14}$$

with H_0 as in Section 3.1, with eigenvectors $\{\varphi_j\}_{j=1}^{\infty}$ and χ_j the corresponding eigenvalues; $P_\phi(\cdot) = \langle \phi, \cdot \rangle \phi$ where $\kappa \in \mathbb{R}$ and ϕ is a normalized cyclic vector for H_0, in the sense that $\|\phi\| = 1$ and the closed subspace spanned by $\{H_0^m \phi : m \in \mathbb{N}\}$ equals $\mathscr{H}$. Let

$$\phi = \sum_j b_j \varphi_j. \tag{3.15}$$

In this case (see [22–24])

$$U_F = U_0 (\mathbf{1} + \alpha P_\phi), \tag{3.16}$$

with $U_0 = e^{-i2\pi H_0}$ and $\alpha = (e^{-i2\pi\kappa} - 1)$. Note that $\phi \in \operatorname{dom} H_0^q$, $\forall q > 0$, and so for $\xi \in \operatorname{dom} H_0^q$,

$$U_F \xi = U_0 \xi + \alpha \langle \phi, \xi \rangle \, U_0 \phi \tag{3.17}$$

also belongs to $\operatorname{dom} H_0^q$; a simple iteration process shows that $U_F^m \xi \in \operatorname{dom} H_0^q$ for all $m \geq 0$, and we are justified in using the formula in Theorem 2.3 to estimate Laplace averages.

We are interested in $\eta^{(z)} = R_z(U_F)\varphi_1$. As $|z| \neq 1$, it follows that $\eta^{(z)}$ belongs to the Hilbert space, and so one can write

$$\eta^{(z)} = \sum_{j=1}^{\infty} a_j \varphi_j. \tag{3.18}$$

Note that $a_j = G_z(j)$, and we have

$$U_F \eta^{(z)} - z\eta^{(z)} = \varphi_1. \tag{3.19}$$

By the relation

$$\begin{aligned} U_F\eta^{(z)} &= U_0\eta^{(z)} + \alpha U_0 P_\phi \eta^{(z)} \\ &= \sum_{j=1}^{\infty} a_j U_0 \varphi_j + \alpha U_0 \langle \phi, \eta^{(z)} \rangle \phi \\ &= \sum_{j=1}^{\infty} a_j e^{-i2\pi\chi_j} \varphi_j + \alpha \langle \phi, \eta^{(z)} \rangle \sum_{j=1}^{\infty} b_j e^{-i2\pi\chi_j} \varphi_j \\ &= \sum_{j=1}^{\infty} \left(a_j + \alpha \langle \phi, \eta^{(z)} \rangle b_j \right) e^{-i2\pi\chi_j} \varphi_j, \end{aligned} \tag{3.20}$$

and (3.19) it follows that

$$\sum_{j=1}^{\infty} \left(a_j + \alpha \langle \phi, \eta^{(z)} \rangle b_j \right) e^{-i2\pi\chi_j} \varphi_j - z \sum_{j=1}^{\infty} a_j \varphi_j = \varphi_1, \tag{3.21}$$

that is,

$$\sum_{j=1}^{\infty} \left[a_j \left(e^{-i2\pi\chi_j} - z \right) + \alpha \langle \phi, \eta^{(z)} \rangle b_j e^{-i2\pi\chi_j} \right] \varphi_j = \varphi_1, \tag{3.22}$$

and we get the equations

$$\begin{aligned} a_1 \left(e^{-i2\pi\chi_1} - z \right) + \alpha \langle \phi, \eta^{(z)} \rangle b_1 e^{-i2\pi\chi_1} &= 1, \\ a_j \left(e^{-i2\pi\chi_j} - z \right) + \alpha \langle \phi, \eta^{(z)} \rangle b_j e^{-i2\pi\chi_j} &= 0 \quad \text{for } j > 1. \end{aligned} \tag{3.23}$$

Thus

$$\begin{aligned} a_1 &= \frac{1 - \alpha \langle \phi, \eta^{(z)} \rangle b_1 e^{-i2\pi\chi_1}}{e^{-i2\pi\chi_1} - z}, \\ a_j &= -\frac{\alpha \langle \phi, \eta^{(z)} \rangle b_j e^{-i2\pi\chi_j}}{e^{-i2\pi\chi_j} - z}, \quad j > 1. \end{aligned} \tag{3.24}$$

For the trivial case $\alpha = 0$ or, equivalently, $\kappa \in \mathbb{Z}$, one has

$$\begin{aligned} a_1 &= \frac{1}{e^{-i2\pi\chi_1} - z}, \\ a_j &= 0, \quad j > 1, \end{aligned} \tag{3.25}$$

and $\eta^{(z)} = \varphi_1/(e^{-i2\pi\chi_1} - z)$. In this case the analysis of $L^q_{\varphi_1}(T)$ is reduced to

$$\int_0^{2\pi} |a_1|^2 dE = \int_0^{2\pi} \frac{dE}{|e^{-i2\pi\chi_1} - z|^2} = \frac{2\pi}{e^{2/T} - 1} \tag{3.26}$$

as calculated in Section 3.1. Thus $L^q_{\varphi_1}(T) \approx \|H^q_0 \varphi_1\|$ for large T, as expected.

Returning to the general case $\alpha \neq 0$, note that

$$\begin{aligned}
\langle \phi, \eta^{(z)} \rangle &= \sum_{j=1}^{\infty} \overline{b}_j a_j \\
&= \overline{b}_1 \left(\frac{1 - \alpha \langle \phi, \eta^{(z)} \rangle b_1 e^{-i2\pi\chi_1}}{e^{-i2\pi\chi_1} - z} \right) + \sum_{j=2}^{\infty} \overline{b}_j \frac{(-\alpha) \langle \phi, \eta^{(z)} \rangle b_j e^{-i2\pi\chi_j}}{e^{-i2\pi\chi_j} - z} \\
&= \frac{\overline{b}_1}{e^{-i2\pi\chi_1} - z} - \langle \phi, \eta^{(z)} \rangle \sum_{j=1}^{\infty} \frac{\alpha |b_j|^2 e^{-i2\pi\chi_j}}{e^{-i2\pi\chi_j} - z}.
\end{aligned} \tag{3.27}$$

So

$$\langle \phi, \eta^{(z)} \rangle = \frac{\overline{b}_1}{(e^{-i2\pi\chi_1} - z)} \left[1 + \sum_{j=1}^{\infty} \frac{\alpha |b_j|^2 e^{-i2\pi\chi_j}}{e^{-i2\pi\chi_j} - z} \right]^{-1}. \tag{3.28}$$

By denoting

$$\tau(z) = 1 + \sum_{j=1}^{\infty} \frac{\alpha |b_j|^2 e^{-i2\pi\chi_j}}{e^{-i2\pi\chi_j} - z}, \tag{3.29}$$

by (3.24) we finally obtain the relations

$$\begin{aligned}
a_1 &= \frac{1}{e^{-i2\pi\chi_1} - z} - \frac{\alpha |b_1|^2 e^{-i2\pi\chi_1} \tau(z)^{-1}}{(e^{-i2\pi\chi_1} - z)^2}, \\
a_j &= -\frac{\alpha b_j \overline{b}_1 e^{-i2\pi\chi_j} \tau(z)^{-1}}{(e^{-i2\pi\chi_1} - z)(e^{-i2\pi\chi_j} - z)}, \quad j > 1.
\end{aligned} \tag{3.30}$$

3.3.1. A Harmonic Oscillator

Now we present an application of the above relations to a kicked harmonic oscillator with natural frequency equals to 1; we will write $L_{\xi}^{q} = L_{\xi}^{H_0^q}$.

Proposition 3.2. *Let H_0 be a harmonic oscillator hamiltonian with appropriate parameters so that its eigenvalues are integers j, $j \geq 1$, and $U_F = U_0(\mathbf{1} + \alpha P_\phi)$ as aforementioned. Then for any $\kappa \in \mathbb{R}$ and cyclic vector ϕ for H_0, there exists $C > 0$ so that, for T large enough,*

$$L_{\varphi_1}^{q}(T) \leq C, \tag{3.31}$$

where φ_1 is the harmonic oscillator ground state. Hence one has H_0^q-dynamical stability.

Proof. We use the above notation; note that $\varphi_1 \in \operatorname{dom} H_0^q$, $\forall q > 0$ and Theorem 2.3 can be applied. In this case we have

$$\tau(z) = 1 + \sum_{j=1}^{\infty} \frac{\alpha |b_j|^2}{1-z} = 1 + \frac{\alpha}{1-z} \|\phi\|^2 = \frac{1-z+\alpha}{1-z}, \tag{3.32}$$

and so

$$\begin{aligned} a_1 &= \frac{1}{1-z} - \frac{\alpha |b_1|^2}{(1-z)(e^{-i2\pi\kappa} - z)}, \\ a_j &= -\frac{\alpha b_j \bar{b}_1}{(1-z)(e^{-i2\pi\kappa} - z)}, \quad j > 1. \end{aligned} \tag{3.33}$$

Now we evaluate $I_j := \int_0^{2\pi} |a_j|^2 dE$. For $j > 1$ and $\gamma(E) = e^{iE}$, $0 \leq E \leq 2\pi$,

$$\begin{aligned} \int_0^{2\pi} |a_j|^2 dE &= \int_0^{2\pi} \left| \frac{\alpha b_j \bar{b}_1}{(1-z)(e^{-i2\pi\kappa} - z)} \right|^2 dE \\ &= |\alpha|^2 |b_j|^2 |\bar{b}_1|^2 \int_0^{2\pi} \frac{dE}{\left|(1 - e^{-iE} e^{1/T})(e^{-i2\pi\kappa} - e^{-iE} e^{1/T})\right|^2} \\ &= \frac{|\alpha|^2 |b_j|^2 |\bar{b}_1|^2}{i e^{2/T} e^{-i2\pi\kappa}} \int_\gamma \frac{w\, dw}{(w-\beta_1)(w-\beta_2)(w-\beta_3)(w-\beta_4)}, \end{aligned} \tag{3.34}$$

where $\beta_1 = e^{1/T}$, $\beta_2 = e^{-1/T}$, $\beta_3 = e^{1/T}e^{i2\pi\kappa}$, and $\beta_4 = e^{-1/T}e^{i2\pi\kappa}$; only β_2 and β_4 are poles in the interior of γ. By residue, for $j > 1$,

$$\begin{aligned} I_j &= \frac{2\pi|\alpha|^2|b_j|^2|\overline{b}_1|^2}{e^{2/T}e^{-i2\pi\kappa}}\left(\frac{\beta_2}{(\beta_2-\beta_1)(\beta_2-\beta_3)(\beta_2-\beta_4)} + \frac{\beta_4}{(\beta_4-\beta_1)(\beta_4-\beta_2)(\beta_4-\beta_3)}\right) \\ &= \frac{2\pi\alpha|b_j|^2|\overline{b_1}|^2}{(e^{2/T}-1)(e^{-i2\pi\kappa}-e^{2/T})} - \frac{2\pi\alpha|b_j|^2|\overline{b_1}|^2 e^{i2\pi\kappa}}{(e^{2/T}-1)(e^{i2\pi\kappa}-e^{2/T})} \\ &= \frac{2\pi\alpha|b_j|^2|\overline{b}_1|^2}{(e^{2/T}-1)}\left(\frac{1}{e^{-i2\pi\kappa}-e^{2/T}} - \frac{e^{i2\pi\kappa}}{e^{i2\pi\kappa}-e^{2/T}}\right), \end{aligned} \tag{3.35}$$

and for $j = 1$

$$\begin{aligned} I_1 &= \int_0^{2\pi}\left|\frac{1}{1-z} - \frac{\alpha|b_1|^2}{(1-z)(e^{-i2\pi\kappa}-z)}\right|^2 dE \\ &= \int_0^{2\pi}\frac{dE}{(1-z)(1-\overline{z})} - \overline{\alpha}|b_1|^2\int_0^{2\pi}\frac{dE}{(1-z)(1-\overline{z})(e^{i2\pi\kappa}-\overline{z})} \\ &\quad - \alpha|b_1|^2\int_0^{2\pi}\frac{dE}{(1-z)(1-\overline{z})(e^{-i2\pi\kappa}-z)} \\ &\quad + |\alpha|^2|b_1|^4\int_0^{2\pi}\frac{dE}{(1-z)(1-\overline{z})(e^{-i2\pi\kappa}-z)(e^{i2\pi\kappa}-\overline{z})}, \end{aligned} \tag{3.36}$$

and evaluating the integrals we obtain

$$\begin{aligned} I_1 &= \frac{2\pi}{(e^{2/T}-1)} - \frac{2\pi|b_1|^2}{(e^{2/T}-1)} - \frac{2\pi|b_1|^2}{(e^{i2\pi\kappa}-e^{2/T})} - \frac{2\pi\alpha|b_1|^2}{(e^{2/T}-1)(e^{-i2\pi\kappa}-e^{2/T})} \\ &\quad + \frac{2\pi\alpha|b_1|^4}{(e^{2/T}-1)}\left(\frac{1}{e^{-i2\pi\kappa}-e^{2/T}} - \frac{e^{i2\pi\kappa}}{e^{i2\pi\kappa}-e^{2/T}}\right), \end{aligned} \tag{3.37}$$

and after inserting this in the expression of the average energy we get

$$\begin{aligned} L^q_{\varphi_1}(T) &= \frac{2}{(1-e^{-2/T})T}\left(1 - |b_1|^2 - \frac{\alpha|b_1|^2}{(e^{-i2\pi\kappa}-e^{2/T})}\right) \\ &\quad - \frac{2|b_1|^2}{e^{-2/T}(e^{i2\pi\kappa}-e^{2/T})T} \\ &\quad + \frac{2\alpha|b_1|^2}{(1-e^{-2/T})T}\left(\frac{1}{e^{-i2\pi\kappa}-e^{2/T}} - \frac{e^{i2\pi\kappa}}{e^{i2\pi\kappa}-e^{2/T}}\right)\langle\phi, H^q_0\phi\rangle. \end{aligned} \tag{3.38}$$

Therefore, for large T there is a constant $C(\kappa, b_1) > 0$ so that

$$L^q_{\varphi_1}(T) \leq C(\kappa, b_1)\left(1 + \langle \phi, H_0^q \phi \rangle + \frac{1}{T}\right). \tag{3.39}$$

This completes the proof. □

For harmonic oscillators with eigenvalues ωj, $\omega \neq 1$, the evaluations of the resulting integrals are more intricate and were not carried out.

3.4. Kicked Perturbations by a V in $\mathrm{L}^2(S^1)$

3.4.1. Kicked Linear Rotor

Consider

$$H(t) = \omega p + V(x) \sum_{n \in \mathbb{Z}} \delta(t - n2\pi), \tag{3.40}$$

where $p = -i(d/dx)$, $\omega \in \mathbb{R}$, and $V \in \mathrm{L}^2(S^1)$. The Hilbert space is $\mathrm{L}^2(S^1)$; this model was considered in [12, 25, 26] and references therein. The Floquet operator is

$$U_F = U_V = e^{-i2\pi\omega p} e^{-iV(x)}. \tag{3.41}$$

Denote $\varphi_j(x) = e^{ijx}/\sqrt{2\pi}$, $0 \leq x < 2\pi$, and $j \in \mathbb{Z}$ to be the eigenvectors of p^2 whose eigenvalues are the square of integers j^2; all eigenvalues have multiplicity 2 (the corresponding eigenvectors are φ_j and φ_{-j}), except for the null eigenvalue which is simple.

Consider the case $\omega = 1$; then

$$\left((U_F - z)^{-1}\varphi_0\right)(x) = \frac{1}{\sqrt{2\pi}\left(e^{-iV(x)} - z\right)}, \tag{3.42}$$

and so

$$G_z^{\varphi_0}(j) = \langle \varphi_j, R_z(U_F)\varphi_0 \rangle = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-ijx}}{e^{-iV(x)} - z} dx. \tag{3.43}$$

Denote $I_j := \int_0^{2\pi} |G_z^{\varphi_0}(j)|^2 dE$. It follows that

$$\begin{aligned} I_j &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\left|\int_0^{2\pi} \frac{e^{-ijx}}{e^{-iV(x)} - z} dx\right|^2 dE \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} e^{-ijx} e^{ijy}\left(\int_0^{2\pi} \frac{dE}{\left(e^{-iV(x)} - z\right)\left(e^{iV(y)} - \overline{z}\right)}\right) dx dy. \end{aligned} \tag{3.44}$$

For $x, y \in S^1$ fixed denote $I_{xy} := \int_0^{2\pi} (dE/(e^{-iV(x)} - z)(e^{iV(y)} - \overline{z}))$. If $\gamma(E) = e^{iE}$, $0 \le E \le 2\pi$, one has

$$\begin{aligned} I_{xy} &= \int_0^{2\pi} \frac{dE}{\left(e^{-iV(x)} - e^{-iE}e^{1/T}\right)\left(e^{iV(y)} - e^{iE}e^{1/T}\right)} \\ &= \int_0^{2\pi} \frac{dE}{e^{-iE}e^{-iV(x)}\left(e^{iE} - e^{iV(x)}e^{1/T}\right)e^{1/T}\left(e^{-1/T}e^{iV(y)} - e^{iE}\right)} \\ &= -\frac{1}{e^{1/T}e^{-iV(x)}}\frac{1}{i}\int_\gamma \frac{dw}{\left(w - e^{iV(x)}e^{1/T}\right)\left(w - e^{-1/T}e^{iV(y)}\right)}, \end{aligned} \tag{3.45}$$

and by residues

$$I_{xy} = -\frac{2\pi}{e^{1/T}e^{-iV(x)}\left(e^{-1/T}e^{iV(y)} - e^{iV(x)}e^{1/T}\right)} = \frac{2\pi}{\left(e^{2/T} - e^{-iV(x)}e^{iV(y)}\right)}. \tag{3.46}$$

Hence

$$\begin{aligned} I_j &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} e^{-ijx}e^{ijy}\frac{2\pi}{\left(e^{2/T} - e^{-iV(x)}e^{iV(y)}\right)}dxdy \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-ijx}\left(\int_0^{2\pi}\frac{e^{ijy}dy}{\left(e^{2/T} - e^{-iV(x)}e^{iV(y)}\right)}\right)dx \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-ijx}}{e^{-iV(x)}}\left(\int_0^{2\pi}\frac{e^{ijy}dy}{\left(e^{2/T}e^{iV(x)} - e^{iV(y)}\right)}\right)dx. \end{aligned} \tag{3.47}$$

The analytical evaluation of these integrals is not a simple task. As an illustration, consider the particular potential $V(x) = x$; since by Cauchy's integral formula

$$\int_0^{2\pi}\frac{e^{ijy}dy}{\left(e^{2/T}e^{ix} - e^{iy}\right)} = -\frac{1}{i}\int_\gamma \frac{w^{j-1}dw}{\left(w - e^{2/T}e^{ix}\right)} = 0, \quad \text{if } j \ge 1 \tag{3.48}$$

and by residue theorem

$$\int_0^{2\pi}\frac{e^{ijy}dy}{\left(e^{2/T}e^{ix} - e^{iy}\right)} = -\frac{1}{i}\int_\gamma \frac{dw}{w^{1-j}\left(w - e^{2/T}e^{ix}\right)} = \frac{2\pi}{\left(e^{2/T}e^{ix}\right)^{1-j}}, \quad \text{if } j \le 0, \tag{3.49}$$

it is found that

$$\begin{gathered} I_j = 0 \quad \text{if } j \ge 1, \\ I_j = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-ijx}}{e^{-ix}}\frac{2\pi}{\left(e^{2/T}e^{ix}\right)^{1-j}}dx = \frac{1}{e^{(2/T)(1-j)}}\int_0^{2\pi}dx = \frac{2\pi}{e^{(2/T)(1-j)}}, \quad \text{if } j \le 0. \end{gathered} \tag{3.50}$$

Therefore, by (2.16) it follows that for any $q > 0$

$$L_{\varphi_0}^{p^{2q}}(T) \geq \frac{1}{\pi e^{-2/T}} \frac{1}{T} \sum_{j=1}^{\infty} j^{2q} I_{-j} = \frac{2}{T} \sum_{j=1}^{\infty} j^{2q} e^{-(2/T)j}, \tag{3.51}$$

and we conclude that (see the appendix)

$$L_{\varphi_0}^{p^{2q}}(m) \geq \text{cte}\, m^{2q} \tag{3.52}$$

and also that the sequence $m \mapsto \langle U_F^m \varphi_0, p^{2q} U_F^m \varphi_0 \rangle$ is unbounded. This behavior is expected since the spectrum of U_F is absolutely continuous in this case [25], but here we got the result explicitly without passing through spectral arguments, although in a rather involved way; indeed, a much simpler derivation is possible by direct calculating $U_F^m \varphi_0$ and the corresponding expectation values.

For $V(x) = kx$ with integer $k \geq 2$, similar results are obtained, that is

$$I_j = \begin{cases} 0, & \text{if } j = lk,\ l \geq 1, \\ \dfrac{2\pi}{e^{2/T}(1-l)}, & \text{if } j = lk,\ l \leq 0 \end{cases} \tag{3.53}$$

and so

$$L_{\varphi_0}^{p^{2q}}(T) \geq \frac{2k^{2q}}{T} \sum_{l=1}^{\infty} l^{2q} e^{-(2/T)l}. \tag{3.54}$$

Therefore, we have the following lower bound for the Laplace average:

$$L_{\varphi_0}^{p^{2q}}(m) \geq C(k,q)\ m^{2q} \tag{3.55}$$

(see appendix). The same is valid if $V(x) = kx$ with k denoting any negative integer number.

3.4.2. Power-Kicked Systems

Due to the difficulty in evaluating the integrals in (3.47), in order to estimate $L_{\varphi_0}^{p^{2q}}(T)$ in some situations we take an alternative way.

Consider the Kicked models in $L^2(S^1)$ with Floquet operator

$$U_F = U_V = e^{-i2\pi\omega f(p)} e^{-iV(x)}, \tag{3.56}$$

corresponding to the hamiltonian

$$H(t) = \omega f(p) + V(x) \sum_{n \in \mathbb{Z}} \delta(t - 2\pi n), \tag{3.57}$$

with p, V, φ_j as before and $f(p) = p^N$ for some $N \in \mathbb{N}$. Let $\mathcal{F} : \mathrm{L}^2(S^1) \to l^2(\mathbb{Z})$ be the Fourier transform. Then $\mathcal{F}U_V\mathcal{F}^{-1} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ and

$$\mathcal{F}U_V\mathcal{F}^{-1} = \mathcal{F}e^{-i2\pi\omega f(p)}e^{-iV(x)}\mathcal{F}^{-1} = \mathcal{F}e^{-i2\pi\omega f(p)}\mathcal{F}^{-1}\mathcal{F}e^{-iV(x)}\mathcal{F}^{-1}, \tag{3.58}$$

where $\mathcal{F}e^{-i2\pi\omega f(p)}\mathcal{F}^{-1}$ is represented by a diagonal matrix D whose elements are

$$D(m,n) = e^{-i2\pi\omega f(n)}\delta_{mn}, \tag{3.59}$$

and $\mathcal{F}e^{-iV(x)}\mathcal{F}^{-1}$ is represented by a matrix W whose elements are

$$W(m,n) = (\mathcal{F}\rho)(m-n) = \widehat{\rho}(m-n), \tag{3.60}$$

where $\rho(x) = (1/\sqrt{2\pi})e^{-iV(x)}$. Denote $B = DW$; so

$$B(m,n) = e^{-i2\pi\omega f(n)}\widehat{\rho}(m-n), \tag{3.61}$$

$$U_V = \mathcal{F}^{-1}B\mathcal{F}. \tag{3.62}$$

Put $\eta^{(z)} = R_z(U_V)\varphi_0$; then

$$U_V\eta^{(z)} - z\eta^{(z)} = \varphi_0, \tag{3.63}$$

and using (3.62) we obtain

$$B\mathcal{F}\eta^{(z)} - z\mathcal{F}\eta^{(z)} = \mathcal{F}\varphi_0. \tag{3.64}$$

Thus, for each $n \in \mathbb{Z}$,

$$\left(B\mathcal{F}\eta^{(z)}\right)(n) - \left(z\mathcal{F}\eta^{(z)}\right)(n) = (\mathcal{F}\varphi_0)(n), \tag{3.65}$$

so that

$$e^{-i2\pi\omega f(n)}\sum_{j\in\mathbb{Z}}\widehat{\rho}(n-j)G_z^{\varphi_0}(j) - zG_z^{\varphi_0}(n) = \delta_{n0}. \tag{3.66}$$

Tridiagonal Case

In order to deal with the above equations, we try to simplify them by supposing that V is such that $\widehat{\rho}(m-n) = 0$ if $|m-n| > 1$. Then, for each $n \in \mathbb{Z}$ fixed (3.66) becomes

$$e^{-i2\pi\omega f(n)}\sum_{|n-j|\le 1}\widehat{\rho}(n-j)G_z^{\varphi_0}(j) - zG_z^{\varphi_0}(n) = \delta_{n0}, \tag{3.67}$$

and $\mathcal{F}^{-1}U_V\mathcal{F} = B$ is tridiagonal and has the structure

$$B = \begin{pmatrix} \ddots & & & & & & \\ & g(-1)\widehat{\rho}(0) & g(-1)\widehat{\rho}(-1) & & & & \\ & \widehat{\rho}(1) & \widehat{\rho}(0) & \widehat{\rho}(-1) & & & \\ & & g(1)\widehat{\rho}(1) & g(1)\widehat{\rho}(0) & g(1)\widehat{\rho}(-1) & & \\ & & & g(2)\widehat{\rho}(1) & g(2)\widehat{\rho}(0) & & \\ & & & & & \ddots & \end{pmatrix}, \tag{3.68}$$

where $g(n) = e^{-i2\pi\omega f(n)}$.

Now, a tridiagonal unitary operator U on $l^2(\mathbb{Z})$ is either unitarily equivalent to a (bilateral) shift operator or an infinite direct sum of 2×2 and 1×1 unitary matrices, as shown in [27, Lemma 3.1]. For proving this result it was only used that U is unitary and $Ue_k = \alpha_k e_{k-1} + \beta_k e_k + \gamma_k e_{k+1}$, where $\{e_k\}$ is the canonical basis of $l^2(\mathbb{Z})$, that is,

$$U = \begin{pmatrix} \ddots & \alpha_{k-1} & & & \\ & \beta_{k-1} & \alpha_k & & \\ & \gamma_{k-1} & \beta_k & \alpha_{k+1} & \\ & & \gamma_k & \beta_{k+1} & \\ & & & \gamma_{k+1} & \ddots \end{pmatrix}. \tag{3.69}$$

It then follows that for all $k \in \mathbb{Z}$

$$\begin{gathered} |\alpha_k|^2 + |\beta_k|^2 + |\gamma_k|^2 = 1, \\ \gamma_{k-1}\overline{\beta_{k-1}} + \beta_k\overline{\alpha_k} = 0, \\ \alpha_k\overline{\gamma_k} = 0. \end{gathered} \tag{3.70}$$

Applying these relations to $B = \mathcal{F}^{-1}U_V\mathcal{F}$ we obtain the following.

(i) If $\widehat{\rho}(-1) \neq 0$, then $\widehat{\rho}(1) = \widehat{\rho}(0) = 0$ and $|\widehat{\rho}(-1)| = 1$.

(ii) If $\widehat{\rho}(1) \neq 0$, then $\widehat{\rho}(-1) = \widehat{\rho}(0) = 0$ and $|\widehat{\rho}(1)| = 1$.

(ii) If $\widehat{\rho}(0) \neq 0$, then $\widehat{\rho}(1) = \widehat{\rho}(-1) = 0$ and $|\widehat{\rho}(0)| = 1$.

The next step is to investigate these cases. If $\widehat{\rho}(0) \neq 0$, it reduces to the autonomous case $H(t) = H_0$ previously considered.

The cases $\widehat{\rho}(-1) \neq 0$ and $\widehat{\rho}(1) \neq 0$ are similar, so we only discuss that $\widehat{\rho}(1) \neq 0$. For $n \in \mathbb{Z}$ fixed, (3.67) takes the form

$$e^{-i2\pi\omega f(n)}\widehat{\rho}(1)G_z^{\varphi_0}(n-1) - zG_z^{\varphi_0}(n) = \delta_{n0}, \tag{3.71}$$

and so we can write $G_z^{\varphi_0}(n)$ in terms of $G_z^{\varphi_0}(0)$ and $G_z^{\varphi_0}(-1)$ for all $n \in \mathbb{Z}$. More precisely

$$\begin{aligned} G_z^{\varphi_0}(n) &= \frac{e^{-i2\pi\omega\left(f(n)+\cdots+f(1)\right)}\widehat{\rho}(1)^n}{z^n} G_z^{\varphi_0}(0), \quad n \geq 1, \\ G_z^{\varphi_0}(-n) &= \frac{z^{n-1}}{e^{-i2\pi\omega\left(f(-n+1)+\cdots+f(-1)\right)}\widehat{\rho}(1)^{n-1}} G_z^{\varphi_0}(-1), \quad n \geq 2; \end{aligned} \tag{3.72}$$

moreover, for $n = 0$ in (3.71) we obtain $\widehat{\rho}(1)G_z^{\varphi_0}(-1) - zG_z^{\varphi_0}(0) = 1$, and so for $z = e^{-iE}e^{1/T}$ and $T > 1$

$$\begin{aligned} 1 &\leq \left|G_z^{\varphi_0}(-1)\right| + |z|\left|G_z^{\varphi_0}(0)\right| \\ &= \left|G_z^{\varphi_0}(-1)\right| + e^{1/T}\left|G_z^{\varphi_0}(0)\right| \\ &\leq e\left(\left|G_z^{\varphi_0}(-1)\right| + \left|G_z^{\varphi_0}(0)\right|\right), \end{aligned} \tag{3.73}$$

and there exists $d > 0$ so that

$$\left|G_z^{\varphi_0}(-1)\right|^2 + \left|G_z^{\varphi_0}(0)\right| \geq d > 0. \tag{3.74}$$

Therefore, by (2.16), for $T > 1$ one has

$$\begin{aligned} L_{\varphi_0}^{p^{2q}}(T) &\geq \frac{1}{\pi e^{-2/T}} \frac{1}{T} \sum_{n=1}^{\infty} n^{2q} \left(\int_0^{2\pi} \left|G_z^{\varphi_0}(n)\right|^2 dE + \int_0^{2\pi} \left|G_z^{\varphi_0}(-n)\right|^2 dE \right) \\ &= \frac{1}{\pi e^{-2/T}} \frac{1}{T} \sum_{n=1}^{\infty} n^{2q} \left(\frac{1}{e^{2n/T}} \int_0^{2\pi} \left|G_z^{\varphi_0}(0)\right|^2 dE \right. \\ &\qquad \left. + e^{2(n-1)/T} \int_0^{2\pi} \left|G_z^{\varphi_0}(-1)\right|^2 dE \right) \\ &\geq \frac{1}{\pi e^{-2/T}} \frac{1}{T} \sum_{n=1}^{\infty} n^{2q} e^{-2n/T} \int_0^{2\pi} \left(\left|G_z^{\varphi_0}(0)\right|^2 + \left|G_z^{\varphi_0}(-1)\right|^2 \right) dE \\ &\geq d \frac{2}{T} \sum_{n=0}^{\infty} (n+1)^{2q} e^{-2n/T}, \end{aligned} \tag{3.75}$$

so that, by the discussion at the end of the appendix,

$$L_{\varphi_0}^{p^{2q}}(m) \geq C(m+1)^{2q}, \tag{3.76}$$

and $\langle U_V^m \varphi_0, p^{2q} U_V^m \varphi_0 \rangle$ is unbounded. Hence we have instability.

Pentadiagonal Case

Suppose now that V is such that $\widehat{\rho}(m-n) = 0$ if $|m-n| > 2$. Then for each $n \in \mathbb{Z}$ fixed, equation (3.66) becomes

$$e^{-i2\pi\omega f(n)} \sum_{|n-j|\leq 2} \widehat{\rho}(n-j) G_z^{\varphi_0}(j) - z G_z^{\varphi_0}(n) = \delta_{n0}, \tag{3.77}$$

and $\mathcal{F}^{-1} U_V \mathcal{F}$ is pentadiagonal and has a structure similar to the corresponding operator in the previous case, just adding the elements whose distance to the diagonal is 2. The elements in the new upper diagonal are $e^{-i2\pi\omega f(n)}\widehat{\rho}(-2)$, and those in the new lower diagonal are $e^{-i2\pi\omega f(n)}\widehat{\rho}(2)$.

For not repeating the tridiagonal case we suppose that either $\widehat{\rho}(2)$ or $\widehat{\rho}(-2)$ is different from zero. If U is a pentadiagonal unitary operator in $l^2(\mathbb{Z})$, that is, $Ue_k = \zeta_k e_{k-2} + \alpha_k e_{k-1} + \beta_k e_k + \gamma_k e_{k+1} + \theta_k e_{k+2}$, one gets the matrix representation

$$U = \begin{pmatrix} \ddots & & & & & & \\ & \beta_{k-2} & \alpha_{k-1} & \zeta_k & & & \\ & \gamma_{k-2} & \beta_{k-1} & \alpha_k & \zeta_{k+1} & & \\ & \theta_{k-2} & \gamma_{k-1} & \beta_k & \alpha_{k+1} & \zeta_{k+2} & \\ & & \theta_{k-1} & \gamma_k & \beta_{k+1} & \alpha_{k+2} & \\ & & & \theta_k & \gamma_{k+1} & \beta_{k+2} & \\ & & & & & & \ddots \end{pmatrix}. \tag{3.78}$$

From this we obtain the following relations, for each $k \in \mathbb{Z}$,

$$\begin{gathered} |\zeta_k|^2 + |\alpha_k|^2 + |\beta_k|^2 + |\gamma_k|^2 + |\theta_k|^2 = 1, \\ \overline{\zeta_k}\alpha_{k-1} + \overline{\alpha_k}\beta_{k-1} + \overline{\beta_k}\gamma_{k-1} + \overline{\gamma_k}\theta_{k-1} = 0, \\ \overline{\beta_{k-1}}\theta_{k-1} + \overline{\alpha_k}\gamma_k + \overline{\zeta_{k+1}}\beta_{k+1} = 0, \\ \overline{\alpha_{k-1}}\theta_{k-1} + \overline{\zeta_k}\gamma_k = 0, \\ \overline{\zeta_k}\theta_k = 0. \end{gathered} \tag{3.79}$$

Suppose that $\widehat{\rho}(2) \neq 0$. The case $\widehat{\rho}(-2) \neq 0$ is similar. Then by the above relations we obtain $\widehat{\rho}(-2) = \widehat{\rho}(-1) = \widehat{\rho}(0) = \widehat{\rho}(1) = 0$, and so (3.77) becomes

$$e^{-i2\pi\omega f(n)}\widehat{\rho}(2) G_z^{\varphi_0}(n-2) - z G_z^{\varphi_0}(n) = \delta_{n0}. \tag{3.80}$$

For $n = 0$ one gets $\widehat{\rho}(2)G_z^{\varphi_0}(-2) - zG_z^{\varphi_0}(0) = 1$ and analogously to the previous case

$$\left|G_z^{\varphi_0}(-2)\right|^2 + \left|G_z^{\varphi_0}(0)\right|^2 \geq d > 0, \tag{3.81}$$

with $z = e^{-iE}e^{1/T}$ and $T > 1$. Since for $n \geq 1$

$$\begin{aligned} G_z^{\varphi_0}(2n) &= \frac{e^{-i2\pi\omega\left(f(2n)+f(2n-2)+\cdots+f(2)\right)}\widehat{\rho}(2)^n}{z^n}G_z^{\varphi_0}(0), \\ G_z^{\varphi_0}(-2n) &= \frac{z^{n-1}G_z^{\varphi_0}(-2)}{\widehat{\rho}(2)^{n-1}e^{-i2\pi\omega(f(-2(n-1))+\cdots+f(-2))}}, \end{aligned} \tag{3.82}$$

we obtain

$$\begin{aligned} L_{\varphi_0}^{p^{2q}}(T) &\geq \frac{1}{\pi e^{-2/T}}\frac{1}{T}\sum_{n=1}^{\infty}(2n)^{2q}\left(\int_0^{2\pi}\left|G_z^{\varphi_0}(2n)\right|^2 dE + \int_0^{2\pi}\left|G_z^{\varphi_0}(-2n)\right|^2 dE\right) \\ &= \frac{1}{\pi e^{-2/T}}\frac{1}{T}\sum_{n=1}^{\infty}(2n)^{2q}\left(\frac{1}{e^{2n/T}}\int_0^{2\pi}\left|G_z^{\varphi_0}(0)\right|^2 dE\right. \\ &\qquad \left. + e^{2(n-1)/T}\int_0^{2\pi}\left|G_z^{\varphi_0}(-2)\right|^2 dE\right) \\ &\geq \frac{1}{\pi e^{-2/T}}\frac{1}{T}\sum_{n=1}^{\infty}(2n)^{2q}e^{-2n/T}\int_0^{2\pi}\left(\left|G_z^{\varphi_0}(0)\right|^2 + \left|G_z^{\varphi_0}(-2)\right|^2\right)dE \\ &\geq d\frac{2}{m}\sum_{n=0}^{\infty}(2(n+1))^{2q}e^{-2n/T}, \end{aligned} \tag{3.83}$$

hence

$$L_{\varphi_0}^{p^{2q}}(T) \geq C(2(m+1))^{2q}, \tag{3.84}$$

and $\langle U_V^m\varphi_0, p^{2q}U_V^m\varphi_0\rangle$ is unbounded.

N-Diagonal Case

If V satisfies $\widehat{\rho}(m-n) = 0$ for $|m-n| > N$, we suppose that either $\widehat{\rho}(N)$ or $\widehat{\rho}(-N)$ is different from zero. In case $\widehat{\rho}(N) \neq 0$, by unitarity and the structure of $\mathcal{F}^{-1}U_V\mathcal{F}$ we obtain that $\widehat{\rho}(N-1) = \cdots\widehat{\rho}(0) = \widehat{\rho}(-1) = \cdots = \widehat{\rho}(-N) = 0$, thus (3.66) becomes, for each $n \in \mathbb{Z}$,

$$e^{-i2\pi\omega f(n)}\widehat{\rho}(N)G_z^{\varphi_0}(n-N) - zG_z^{\varphi_0}(n) = \delta_{n0}, \tag{3.85}$$

and so

$$\left|G_z^{\varphi_0}(-N)\right|^2 + \left|G_z^{\varphi_0}(0)\right|^2 \geq d > 0, \tag{3.86}$$

with $z = e^{-iE}e^{1/T}$, $T > 1$. Moreover, for $n \geq 1$

$$\begin{aligned} G_z^{\varphi_0}(nN) &= \frac{e^{-i2\pi\omega(f(nN)+f((n-1)N)+\cdots+f(N))}\widehat{\rho}(N)^n}{z^n} G_z^{\varphi_0}(0), \\ G_z^{\varphi_0}(-nN) &= \frac{z^{n-1}G_z^{\varphi_0}(-N)}{\widehat{\rho}(N)^{n-1}e^{-i2\pi\omega(f(-N(n-1))+\cdots+f(-N))}}. \end{aligned} \tag{3.87}$$

Similarly to the previous cases we conclude that

$$L_{\varphi_0}^{p^{2q}}(T) \geq d\frac{2}{T}\sum_{n=0}^{\infty}(N(n+1))^{2q}e^{-2n/T}. \tag{3.88}$$

Therefore we can state the following result.

Theorem 3.3. *For Kicked systems in* $\mathrm{L}^2(S^1)$ *with*

$$U_V = e^{-i2\pi\omega f(p)}e^{-iV(x)} \tag{3.89}$$

as in (3.56)*, one can obtain that* $\mathcal{F}U_V\mathcal{F}^{-1} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ *is represented by the matrix* B *with elements* $B(m,n) = e^{-i2\pi\omega f(n)}\widehat{\rho}(m-n)$*, where* $\rho(x) = (2\pi)^{-1/2}e^{-iV(x)}$*. If* V *satisfies* $\widehat{\rho}(m-n) = 0$ *for* $|m-n| > N \in \mathbb{N}^*$ *and either* $\widehat{\rho}(N)$ *or* $\widehat{\rho}(-N)$ *is different from zero, then* $V(x) = \pm Nx + \theta$*, for some* $\theta \in \mathbb{R}$*, and* $\mathcal{F}U_V\mathcal{F}^{-1}$ *is unitarily equivalent to* T^N *(the* N*th power of* T*) where* T *is the bilateral shift. Furthermore,*

$$L_{\varphi_0}^{p^{2q}}(T) \geq d\frac{2}{T}\sum_{n=0}^{\infty}(N(n+1))^{2q}e^{-2n/T}. \tag{3.90}$$

Proof. It is enough to prove that $\mathcal{F}U_V\mathcal{F}^{-1}$ is unitarily equivalent to T^N. Suppose that $\widehat{\rho}(N) \neq 0$ (the case for $\widehat{\rho}(-N) \neq 0$ is similar); then by the above discussion we obtain

$$B(m,n) = \begin{cases} 0, & \text{if } m \neq n+N, \\ e^{-i2\pi\omega f(n)}\widehat{\rho}(N), & \text{if } m = n+N, \end{cases} \tag{3.91}$$

that is, $Be_n = e^{-i2\pi\omega f(n)}\widehat{\rho}(N)e_{n+N}$ where $\{e_n\}$ is the canonical basis of $l^2(\mathbb{Z})$. Since $|\widehat{\rho}(N)| = 1$, write $\widehat{\rho}(N) = e^{-i\theta}$. Let W be the unitary operator defined by

$$We_n = e^{i\vartheta_n}e_n, \quad n \in \mathbb{Z}, \tag{3.92}$$

where ϑ_n are elements in $[0, 2\pi)$. If ϑ_n satisfies for all $n \in \mathbb{Z}$

$$\vartheta_{n+N} - \vartheta_n = 2\pi\omega f(n) + \theta, \tag{3.93}$$

it follows that $W^{-1}BW = T^N$. Equation (3.93) is satisfied taking, for example, $\vartheta_0 = \vartheta_1 = \cdots = \vartheta_{N-1} = 0$ and another ϑ_n obeying (3.93). □

Although Theorem 3.3 gives a nice illustration of the potential applications of our expression for the Laplace average, since it is one of the few instances that such average can be explicitly estimated from below, again it can be derived by more direct methods and one can also conclude [25] that the spectrum of the corresponding Floquet operators is absolutely continuous.

4. Conclusions

Although most of our applications of Theorem 2.3 give expected results (sometimes known results that can be derived in simpler ways), we believe that that formula is interesting and has a potential to be applied to more sophisticated models as the Fermi accelerator. The difficulty is to get expressions or estimates for the Green functions, since calculating the resolvent of an operator is not always an easy task; sometimes we have the expressions for resolvent operators (e.g., for kicked systems), but the resulting integrals can be too involved. We have not tried any numerical approach to formula (2.15), which might be useful for some specific models.

In the case of one-dimensional discrete Schrödinger operators, where the hamiltonian is $H_V : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ defined by

$$(H_V\xi)(n) = \xi(n+1) + \xi(n-1) + V(n)\xi(n), \tag{4.1}$$

V a bounded sequence, a similar formula can be handled in some cases by relating the resolvent $R_{E+i/T}(H_V)$ to transfer matrices. Then adequate upper bounds of such transfer matrices, on some set of energies E, result in lower estimates for the corresponding Green functions, and then transport properties are obtained for interesting models (see [21] and references therein).

In [27] a class of Floquet operators displaying a pentadiagonal structure was introduced; for these models there is a transfer matrix formalism. However, such transfer matrices are too complicated, and analytical estimates seem far from trivial.

Anyway, the technique here is quite general; it asks no particular regularity of the time-dependence and can be virtually applied to any time-periodic system as soon as the time evolution is well posed. As already said, the chief difficulty is related to suitable bounds of matrix elements of the resolvents of unitary (Floquet) operators, a task harder than we initially envisaged. Herein we put forward for consideration the challenge of getting additional applications for the formula (2.15) deduced for the Laplace averages, including an application of Theorem 3.1 to physical models. It is also worth mentioning the question left open in Lemma 2.2, that is, is it true that $\beta_e^- = \beta_d^-$?

Appendix

Laplace Transform of Sequences

Let $a = (a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers. The Laplace transform of a, denoted by f_a, is the function defined by

$$f_a(s) = \sum_{n=0}^{\infty} e^{-sn} a(n), \tag{A.1}$$

for s in a subset of $\mathbb{R}$. It will also be denoted by $f_a(s) = \mathcal{L}(a)$.

We say that the Laplace transform of $a = (a_n)$ exists if the series in (A.1) converges for some s. For example, if $a(n) = e^{n^2}$, then the sum in (A.1) diverges for all $s \in \mathbb{R}$.

Examples

(1) For the constant sequence $a(n) = 1$ it follows that

$$f_a(s) = \sum_{n=0}^{\infty} e^{-sn} = \frac{1}{(1-e^{-s})}, \tag{A.2}$$

for $s > 0$. By using Taylor expansion, for small s one finds that $1/(1-e^{-s}) \approx 1/s$.

(2) Since $\sum_{n=0}^{\infty} z^n = 1/(1-z)$, for $z \in \mathbb{C}, |z| < 1$, it follows that

$$\sum_{n=0}^{\infty} (n+k)(n+k-1)\cdots(n+1)z^n = \frac{k!}{(1-z)^{k+1}}, \tag{A.3}$$

for $k = 1, 2, 3, \ldots$, and z as above. Thus, the Laplace transform of $a^k(n) = (n+k)(n+k-1)\cdots(n+1)$ is

$$f_a^k(s) = \sum_{n=0}^{\infty} e^{-sn} a^k(n) = \frac{k!}{(1-e^{-s})^{k+1}}, \qquad s > 0. \tag{A.4}$$

For small s, $f_a^k(s) \approx k!/s^{k+1}$.

A sequence of complex numbers $a = (a_n)$ is said to be exponential of order σ_0 (real) if there exists $M > 0$ so that $|a(n)| \le Me^{\sigma_0 n}$, $\forall n$. That is, $a(n)$ does not increase faster than $e^{\sigma_0 n}$ as $n \to \infty$. If $a = (a_n)$ is exponential of order $\sigma_0 > 0$, then

$$f_a(s) = \sum_{n=0}^{\infty} e^{-sn} a(n) \tag{A.5}$$

is convergent for any $s > \sigma_0$.

Let $\mathcal{V}$ denote the set of positive sequences of exponential order σ_0. The Laplace transform $\mathcal{L}$ satisfies

$$\mathcal{L}(ca) = c\mathcal{L}(a), \qquad \mathcal{L}(a+b) = \mathcal{L}(a) + \mathcal{L}(b), \tag{A.6}$$

where c is a positive number, and a and b are sequences in $\mathcal{V}$. Moreover, if $a \in \mathcal{V}$ and $\mathcal{L}(a) = 0$, then $\sum_{n=0}^{\infty} e^{-sn} a(n) = 0$ and so $a(n) = 0$ for all n, that is, $a = 0$. Thus $\mathcal{L}$ is injective on $\mathcal{V}$.

The Laplace average (2.5) is related to the Laplace transform of $E_\xi^A(n)$ by

$$L_\xi^A(T) = \frac{2}{T}\sum_{n=0}^{\infty} e^{-2n/T} E_\xi^A(n) = \frac{2}{T} f_{E_\xi^A}\left(\frac{2}{T}\right). \tag{A.7}$$

If $a(n) = 1$ for all n, then

$$\frac{2}{T} f_a\left(\frac{2}{T}\right) = \frac{2}{T}\frac{1}{\left(1 - e^{-2/T}\right)} \approx \frac{2}{T}\frac{1}{2/T} = 1 \tag{A.8}$$

for T large enough. If $a(n) = (n+k)(n+k-1)\cdots(n+1) \approx n^k$, then

$$\frac{2}{T} f_a\left(\frac{2}{T}\right) = \frac{2}{T}\frac{k!}{\left(1 - e^{-2/T}\right)^{k+1}} \approx \frac{2}{T}\frac{k!}{(2/T)^{k+1}} = k!\left(\frac{T}{2}\right)^k \tag{A.9}$$

for large T. Hence, if $E_\xi^A(T)$ grows like T^k, then the same law holds for its average Laplace transform. We have a restricted converse, that is, if $L_\xi^A(n)$ grows with a positive power of n then, by Lemma 2.2, its Cesàro average is unbounded (with a rather similar behavior at large times) and so is $E_\xi^A(n)$. These properties are repeatedly used in the text.

One should be aware that there are special situations of unbounded positive sequences $a(n)$ with bounded average Laplace transforms (so that $\beta_e^+ = \beta_d^+ = 0$); an explicit example is $a(n^2) = n$ and $a(n) = 0$ for $n \notin \{k^2 : k \in \mathbb{N}\}$. The same phenomenon is well known for Cesàro averages, and, by Lemma 2.2, such phenomena are connected.

Acknowledgments

The authors thank the anonymous referees for careful readings and suggestions. C. R. de Oliveira thanks the warm hospitality of PIMS (Vancouver) and the support by CAPES (Brazil).

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Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 246438, 8 pages
doi:10.1155/2009/246438

Research Article

Acceleration of Material Waves in Fermi Accelerator

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Received 1 May 2009; Revised 30 June 2009; Accepted 6 July 2009

Recommended by Edson Denis Leonel

Cold Atoms bouncing on modulated atomic mirror exhibits acceleration and dynamical localization subject to modulated strength. We explain characteristics of acceleration and define control parameters in terms of effective Planck's constant. We show that the effective Planck's constant plays a vital role in limiting classical-like overall linear growth of the variance of accelerated atoms with time. For large values of the effective Planck's constant the atomic quantum acceleration is seized as localization window overlaps the accelerated window.

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1. Introduction

Fermi Accelerator is a system well investigated to study Hamiltonian chaos and its manifestations in quantum mechanical systems [1, 2]. In 1961, Ulam suggested the accelerator as a two-wall system, one fixed and the other oscillating with a frequency, named as Fermi-Ulam Accelerator. At the end of the seventies, Pustylńikov replaced fixed wall by gravity and discussed unbounded acceleration for it. The classical dynamics in Fermi accelerator is described by the Standard Map which explains that the stochasticity in the phase space increases with the driving strength, and when the latter is sufficiently strong global diffusion takes place. In latter accelerator model, for particular set of initial data in phase space and modulation strength [3], onset of unbounded acceleration modes takes place [4–7]. In this paper we explain Fermi-Pustylńikov accelerator and show that by increasing effective Planck's constant the acceleration of a bouncing particle in the accelerator system is modified, due to the onset of dynamical localization [8].

A classical system subject to time-periodic modulation, in general, becomes globally chaotic for increasing modulation strengths and absorbs energy from the external field in a diffusive way. However, in corresponding quantum domain the diffusive dynamics may be suppressed by quantum interference effect. It is a manifestation of dynamical localization phenomenon in the system, which is an analogous to Anderson localization of solid state

Figure 1: A cloud of atoms is trapped and cooled in a magneto-optical trap upto a few micro-Kelvin. The magneto-optical trap (MOT) is placed at a height at the start of the experiment. On switching off the MOT, atoms move with constant gravitational acceleration towards the exponential decaying field.

physics. The phenomenon has been discussed in model systems in quantum chaos, such as, kicked rotator [9], modulated quantum bouncer [10], atoms in modulated standing wave fields [11], ion in a Paul trap [12, 13], and molecular systems in the presence of electric and magnetic fields [14]. Dynamical localization is a general phenomenon in periodically driven systems [15, 16]. The *delocalization* in such quantum systems is a purely quantum effect since the long-time unbounded propagation is not related to the corresponding classical diffusion [2, 17].

In Section 2, we describe the physical system and develop its Hamiltonian. In Section 3, we discuss the diffusion versus acceleration in Fermi accelerator with the help of acceleration windows. In Section 4, we discuss quantum control on accelerator dynamics, discuss results, and explain numerical data based on analytical calculations in Section 5.

2. The Experimental System

Thirty years after the first suggestion of Fermi, Pustylńikov provided detailed study of the accelerator model, which we call Fermi Accelerator or modulated quantum bouncer [4, 18] in this paper. In his work, Pustylńikov proved that a particle bouncing in the accelerator system attains modes, where it always gets unbounded acceleration. The feature makes the Fermi-Pustylńikov model richer in its dynamical beauty. In the atomic Fermi accelerator, atoms move under the influence of gravitational field towards an atomic mirror made up of an evanescent wave field. The atomic mirror is provided a spatial modulation by means of an acousto-optic modulator which provides intensity modulation to the incident laser light field [10]. Hence, an ultra-cold two-level atom, after a normal incidence with the modulated atomic mirror, bounces off and travels in the gravitational field, as shown in Figure 1. In order to avoid any atomic momentum along the plane of the mirror the laser light which undergoes total internal reflection is reflected back. Therefore, we find a standing wave in the plane of the mirror which avoids any specular reflection [19]. The periodic modulation in the intensity of the evanescent wave optical field may lead to the spatial modulation of the atomic mirror as

$$\bar{I}\left(\bar{z},\bar{t}\right) = I_0 e^{-2\kappa\bar{z}+a\cos(\omega\bar{t})}. \tag{2.1}$$

Thus, the motion of the atom in $\overline{z}$-direction follows effectively the Hamiltonian

$$\overline{H} = \frac{\overline{p}_z^{\,2}}{2m} + mg\overline{z} + \hbar\Omega_{\text{eff}} e^{-2\kappa\overline{z}+a\cos(\omega\overline{t})}, \tag{2.2}$$

where Ω_{eff} denotes the effective Rabi frequency [5]. Moreover, a and ω express the amplitude and the frequency of the external modulation, respectively.

3. Diffusion versus Acceleration in Fermi Accelerator

In case the decay constant κ of the evanescent wave field is large, simplified Hamiltonian of our system in moving coordinates becomes

$$\overline{H} = \frac{\overline{p}_z^{\,2}}{2m} + mg\tilde{z} + V\tilde{z}\cos\omega\overline{t}, \quad \tilde{z} \geq 0, \tag{3.1}$$

where $\tilde{z} = \overline{z} - (\epsilon/2\kappa)\cos\omega\overline{t}$. The Hamiltonian given in (3.1) describes a particle of mass m bouncing off an oscillating hard surface in the presence of gravitational field. We proceed onward by introducing the dimensionless position and momentum coordinates. Here, we define $H = \overline{H}(\omega^2/mg^2)$ as dimensionless Hamiltonian, $V = \hbar\omega^2\Omega/4mg^2$ potential of the external field and then other parameters for this Hamiltonian system as, $\epsilon = \omega^2 a/2\kappa g$, $t = \omega\overline{t}$, and dimensionless Planck's constant, $\bar{k} = \hbar(\omega^3/mg^2)$. Hence, the Hamiltonian takes the dimensionless form as

$$H(z,p,t) = \frac{p^2}{2} + z + \epsilon z\cos t, \quad z \geq 0. \tag{3.2}$$

The Hamiltonian system is integrable in the absence of time dependent term. We may express the time development of the particle moving in time dependent system by the impact map which gives the evolution immediately after a bounce in terms of immediately after the previous bounce [1, 2], that is,

$$\begin{aligned} \wp_{i+1} &= \wp_i + K\sin(\phi_i), \\ \phi_{i+1} &= \phi_i + \wp_{i+1}. \end{aligned} \tag{3.3}$$

The map obtained in (3.3) is the Standard Map, where $\wp_i = 2p_i$ and $\phi_i = \omega t_i$. The onset of diffusive excitation in the system takes place as the chaos parameter $K = 4\epsilon$ takes a value larger than $K_{cr} \approx 0.96$, or when the perturbation amplitude exceeds the critical value $\epsilon_{cr} = 0.24$ [20], while the quantum evolution remains localized until a larger value λ_u of the modulation. Above that point both the classical and the quantum dynamics are diffusive. However, for specific set of initial conditions that originate within phase space disks of radius ρ, accelerating modes appear for values of the modulation strength ϵ within the windows [4–6]

$$s\pi\left(\frac{1-\xi}{1+\xi}\right) \leq \epsilon < \sqrt{1 + (s\pi)^2\left(\frac{1-\xi}{1+\xi}\right)^2}, \tag{3.4}$$

Figure 2: We plot the trajectories of a bouncing particle in time domain for three different phase points (z, p), which represent stochastic near acceleration and acceleration regions. For these three phase points we take $V = 60$ and $\epsilon = 1.7$. (a) Our particle is in a chaotic region with phase point $(23.239, 0)$, so there is no consistent acceleration of the particle. (b) Particle originates from the phase point $(21.898, 0)$ is on the boundary of the accelerated region of phase space. Initially the position of the particle increase gradually but after some time it decreases as we move in time domain. (c) Classical particle is in accelerated region of phase space with phase point $(19.739, 0)$. The position of the particle increases consistently as we move in time domain.

where, s can take integer and half-integer values for the sinusoidal modulation of the reflecting surface considered here. As the fundamental requirement for the acceleration is met by choosing a modulation strength within the acceleration windows and the bouncing particle from supporting areas of phase the particle displays acceleration discussed in Figure 2. We found numerically that for a modulation strength outside the windows of (3.4) the dynamics is dominantly diffusive as shown in Figure 3(a). However, for the ensemble originating from areas of phase space of radius ρ, exhibit unbounded acceleration for modulation strength from acceleration windows, as shown in Figure 3(b). A small diffusive background results from a small part of the initial distribution which is residing outside the area of phase space supporting acceleration. This coherent acceleration restricts the momentum space variance Δp which then remains very small indicating the absence of diffusive dynamics.

4. Quantum Control on Acceleration Dynamics

As a matter of fact, the variables $\wp$ and ϕ in (3.3) do not form a conjugate pair in the full Hamiltonian formulation of the model. The variable conjugate to the phase ϕ is the quantity $N = E$, where E is unperturbed energy [16] and its value can be determined as $E = (3\pi I)^{2/3}/2$. By introducing $\wp_{i+1} = \sqrt{2(N_{i+1} - z)}$ and $\wp_i = \sqrt{2(N_i - z)}$ in (3.3), the impact map in conjugate variables (N, ϕ) takes the form

$$\begin{aligned} N_{i+1} &= N_i + 2\epsilon\sqrt{2N_i}\sin\phi_i, \\ \phi_{i+1} &= \phi_i + 2\sqrt{2N_{i+1}} + O(\epsilon), \end{aligned} \tag{4.1}$$

here we neglected higher-order terms. The preservation of phase space volume for Hamiltonian systems has a consequence that there are no attractors, that is, no subregions of lower phase-space dimension to which the motion is confined asymptotically. The map given

Figure 3: The above diagram of the phase space evolution of a classical ensemble of particles initially in a narrowly peaked Gaussian distribution originating from the area of phase space that supports accelerated trajectories. The initial distribution, centered at $\overline{z} = 2\pi^2$ and $\overline{p} = 0$ with $\Delta p(0) = \Delta z(0) = 0.1$, is propagated for $\epsilon = 1$ (a) and $\epsilon = 1.7$ (b) for time $t = 1000$. The numerical calculations correspond to Cesium atoms of mass $m = 2.2 \times 10^{-25}$ Kg bouncing off an atomic mirror with an intensity modulation of $a = 0.55$. The modulation frequencies extend to the megahertz range, and $\kappa^{-1} = 0.55\,\mu$m.

in (4.1) is the analogue of the Kepler map which was found very helpful in the hydrogen atom problem [21]. A classical analysis of mapping given in (4.1) leads to predict the onset of chaos under the same conditions found for the Standard Map description. Above the chaotic threshold a diffusive growth of N is observed. Consider the phase space defined in conjugate coordinates (N, ϕ) and introduce the initial distribution of phase-space points $f(N, \phi, t = 0)$. The time evolution of f is described by a Fokker-Planck equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial}{\partial N}\left(D_N \frac{\partial f}{\partial N}\right), \tag{4.2}$$

where t is time measured in the number of iterations of the map, that is, in the numbers of the bounces. The Fokker-Planck (4.2) can be solved by the method of characteristics [22]. We consider the conjugate pair mapping as in (4.1), switching parameters and with the help of (4.2), we get the interesting relation between, ΔN, and $\hbar$, that is,

$$\Delta N^2 = \frac{4Nj\epsilon^2}{\hbar}. \tag{4.3}$$

Here, ΔN is the dimensionless energy and j describes the number of bounces. The diffusion coefficient D_N is

$$D_N = \frac{4\epsilon^2 N}{\hbar}. \tag{4.4}$$

Figure 4: We present ΔN versus time for different values of effective Planck's constant $\hbar$: $\hbar = 1$ (circle and line), $\hbar = 7$ (star and line), and $\hbar = 16$ (triangle and line). We note that as we increase the value of $\hbar$ the dispersion decreases. The remaining parameters are the same as in Figure 2.

We plot ΔN versus time for different values of $\hbar$, shown in Figure 4, which illustrates that, as we increase the value of the $\hbar$, we expand the dynamical localization window, which is $0.24 < \epsilon < \sqrt{\hbar}/2$. In this window, the lower boundary $\epsilon_l = 0.24$ is set by classical dynamics, we can find the ϵ_l by evaluating Lyapunov exponents and from the standard map. For a modulation amplitude $\epsilon < 0.24$, the Lyapunov exponent converges to zero in a vast range of initial conditions, expect in small regions near separatrices. In the simulation, we initially take our wave packet as Gaussian distribution. A comparison between quantum mechanically and classical values of ΔN versus $\hbar$ is shown in Figure 5. We take the average value of ΔN for last 400 to 1000 times. For different values of effective Planck's constant $\hbar$, we extend localization window towards accelerated window. We note that ΔN displaying a decaying behavior for the higher values of effective Planck's constant, $\hbar$.

5. Results and Discussions

In this contribution, we derive a quantum mechanical map for an atom bouncing off a modulated mirror under the influence of gravity by reducing standard map into conjugate pair map. We write the standard mapping as conjugate pair mapping in quantum domain which provides dispersion law and its dependence on system parameters, such as modulation strength, energy, effective Planck's constant, and number of bounces. Our numerical result show a very good agreement with analytically obtained results. We show the dependence of ΔN on $\hbar$ by numerically and by deriving the Fokker Planck equation for our system as shown in Figure 5.

As discussed in Section 4 the localization occurs in the quantum modulated bouncer in a window defined by classical and quantum dispersion laws. The latter is a function of effective Planck's constant and follows a square root law. Whereas the accelerated dynamics takes place in the system for another window on modulation strength in the presence of initial areas in phase space [2]. As we increase the value of the $\hbar$, we extend localization window towards the accelerated window. For larger value of $\hbar$ variance in energy decreases and the

Figure 5: We plot ΔN versus $\bar{k}$ and compare analytical results obtained by (4.3) with the numerical results. Here, $\epsilon = 1.7$, $N = 1235$ and $j = 15$. Black squared line expresses numerical results of ΔN whereas, red line expresses corresponding analytical results obtained in Section 3.

two windows may overlap, where we do not find any accelerated dynamics, but dynamical localization as shown in Figure 4.

Acknowledgments

Authors thank Rameez-Ul-Islam, M. Ayub, and T. Abbas for stimulating discussions and useful suggestions. F. Saif thanks Higher Education Commission for financial support under Grant no. R&D03143. F. Saif submits his thanks to Professor E. D. Leonel for his invitation to contribute in the special issue of Mathematical Problem in Engineering.

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